ECBS6001: Advanced Macroeconomics

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Lecture 13: Continuous time dynamics

Learning outcomes

You will have applicable knowledge of

- discrete-state Markov processes in continuous time
- dynamic programming in continuous time
  - without uncertainty
  - with discrete-state uncertainty
- using dynamic programming in general equilibrium
- aggregation of heterogeneous agent models

Applications

We will cover four applications:

1. The expanding variety model of growth (Grossman and Helpman)
2. The rising product quality model of growth (Aghion and Howitt)
3. A firm-level model of innovation (Klette and Kortum)
4. Growth and volatility (Koren and Tenreyro)

Outline

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- Today we review Markov processes.
- We show how they work in continuous time.
- We consider two cases:
  - no uncertainty
  - discrete states

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- Hamiltonians and calculus of variations
  - Hamiltonians are an engineer's way of thinking.
  - The recursive formulation is much more intuitive from the point-of-view of a decision maker in an ever-changing environment.
- Wiener processes and Brownian motions
  - They are very useful in finance.
  - but they require a special set of tools.

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- Time *is* continuous, only measurement is discrete.
  - Q1 GDP measures all the value added in the economy between January 1, 2009, 12am and March 31, 2009, 11.59.59pm.
  - Prices are measured monthly, unemployment is reported weekly.
  - Full-population census is usually done every 10 years.

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- It is often useful to think about the "true" model first and then ask how it is measured.
- Often, continuous-time math is simpler.

Continuous time

- In continuous time, $t \in \mathbb{R}$.
- There are no special points or intervals, all t are similar.
  - We can define arbitrary intervals as we wish.
- Continuous time forces you to think about flows and stocks carefully.

Example 1: “Bill Gates could buy Costa Rica”

- *Forbes* reports that Bill Gates’ net worth, \$50 billion, is higher than the GDP of Costa Rica, hence “Bill Gates could buy Costa Rica”.
- We, economists, know this is totally stupid: net worth is a *stock*, GDP is a *flow*.
- But just in case:
  - In continuous time, the two actually have different units: GDP is measured in \$/year (or second), net worth is measured in \$.
  - They cannot be added, subtracted or compared.
  - Even the math does not let you commit this silly mistake.

Example 2: Cash-in-advance models

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- In cash-in-advance models, all your period consumption has to be financed by your cash in hand:
$$C_t \leq M_t.$$
- But C_t is a flow, M_t is a stock!
- This comparison does not make sense until you know how often you can replenish your stock of M.
$$C(t, t+\Delta) \equiv \int_t^{t+\Delta} c(s)ds \leq M(t)$$
- The choice of time period, Δ, is crucial.

Example 2 (continued)

- In practice, there are very few actual flows (maybe your electricity consumption).
- Income, production, consumption etc mostly happen in chunks (you rarely buy a new computer).
- We will also learn tools to deal with these rare occurrences.

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- Use continuous time if
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  - you want to think about your problem at different intervals
- Use discrete time instead if
  - you want to simulate your model in a computer (for a computer, nothing is continuous)
  - you want to estimate your model on data measured at discrete intervals (years, quarters, months)
  - your model assigns a special role to certain points or intervals in time (e.g. trading day).

Markov processes

Markov processes

- A Markov process is a stochastic process for which conditional on the present state of the system, its past and future are independent.
- Time homogeneous Markov processes:

$$\Pr\big[X(t+h) = y \mid X(t) = x\big] = \Pr\big[X(h) = y \mid X(0) = x\big]$$

- Many processes have a Markovian representation.

Example: An AR(1) process

- Suppose GDP follows an AR(1) process:

$$y_t = \rho y_{t-1} + u_t$$

- Knowing y_t helps you predict y_{t+1}, y_{t+2}, etc.

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- Suppose GDP follows an AR(1) process:

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- Knowing y_t helps you predict y_{t+1}, y_{t+2}, etc.
- But the key is that *nothing else does*.

Markovian representations

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  - Say, technological developments are reported in Science and this affects future GDP growth.
  - Again, we can include a state variable (or vector) to account for these news.
  - This works as long the news themselves have Markovian dynamics.
- One can use Markovian tools for even inherently forward-looking phenomena.
  - Say, the continuation value in a dynamic contract can be a state variable. (Sargent calls this "dynamic programming squared".)

No uncertainty

Discrete time review

First-order difference equations

Cobweb plot

Continuous time

Moving to continuous time

- Let time periods be Δ apart.
- How can we characterize the time series as Δ becomes smaller and smaller?
- We take the limit as $\Delta \to 0$.
  - Often, we will have to rescale changes in the variable for the limit to make sense.

Differential equations

Infinitesimal changes

Steady state

- The steady state of this system is x^* such that

$$f(x^*) = 0.$$

Stability

- The local stability of the steady state depends on the derivative (*gradient*) of f.

Example 1: The Solow model

Phase diagram

Example 2: The Ramsey model

Phase diagram

Appendix

Big-O, small-o

Big-O

A function $f(x)$ is $O(g(x))$ for a known function $g(x)$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} < \infty$$

Small-o

A function $f(x)$ is $o(g(x))$ for a known function $g(x)$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

Examples

- $f(x) = x^2$ is both $O(x)$ and $o(x)$. It is also $O(x^2)$.
- $f(x) = x^2 + 2x$ is $O(x)$ but not $o(x)$.
- $f(x) = x^2 + 2x + 4$ is not $O(x)$.