Advanced Macro I
Fall 2009

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Lecture 12: Dynamic programming in continuous time

Outline

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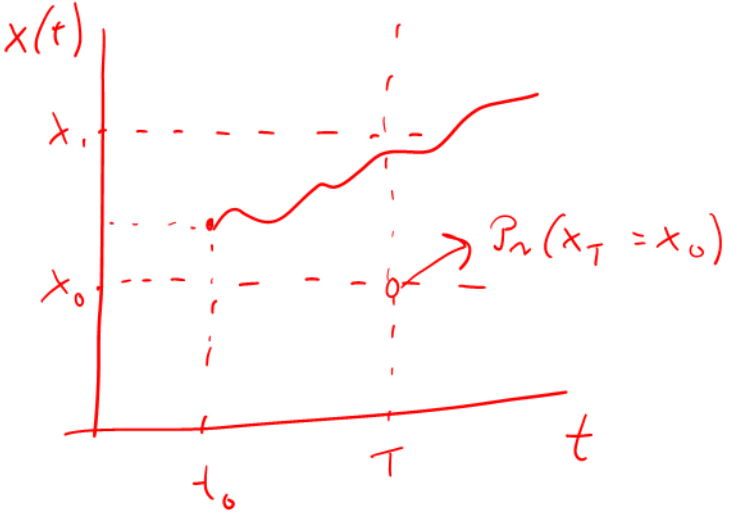
- Today we finish reviewing continuous-time Markov chains.
  - steady-state distribution
  - Poisson process
- Then move on to dynamic programming
  - no uncertainty
  - Markov chains

Forecasting with Markov chains



Example 1: Employment and unemployment

- Suppose you start out employed at $t = 0$, $\pi_0(0) = 1$.
- What is the probability that you will be employed at $t + \Delta$?

$$\pi_0(t + \Delta) = (1 - \delta\Delta)\pi_0(t) + \lambda\Delta\pi_1(t)$$

$$Pr(0|0)\,\pi_0 + Pr(0|1)\,\pi_1$$

  - with prob $(1 - \delta\Delta)$ you remained in state 0
  - with prob $\lambda\Delta$ you exited from state 1

Example 1: Employment and unemployment

$$\pi_0(t+\Delta) = (1-\delta\Delta)\pi_0(t) + \lambda\Delta\pi_1(t)$$

- Subtract $\pi_0(t)$ from both sides and divide by Δ.

$$\frac{\pi_0(t+\Delta)-\pi_0(t)}{\Delta} = -\delta\pi_0(t)+\lambda\pi_1(t) = -\delta\pi_0(t)+\lambda[1-\pi_0(t)]$$

- Take $\Delta \to 0$

$$\dot{\pi}_0(t) = \lambda - (\lambda+\delta)\pi_0(t)$$

- This is a first-order ordinary differential equation, known as the Kolmogorov forward equation.

The Kolmogorov equation

$$\dot{\pi}_0(t) = \lambda - (\lambda + \delta)\pi_0(t)$$

- To get the likelihood of each state at any t, we can solve the Kolmogorov equation forward, starting from an initial value at $t = 0$.
- The solution to this ODE

$$\pi_0(t) = \frac{\lambda}{\lambda + \delta}\left[1 - Ce^{-(\lambda+\delta)t}\right],$$

where C is a constant of integration (pinned down by the boundary condition).

The steady-state distribution

- The steady-state distribution of ~~un~~employment is

$$\lim_{t\to\infty} \pi_0(t) = \frac{\lambda}{\lambda+\delta}.$$

- (We will also derive with another method.)

More generally

- More generally, the Kolmogorov equation is

$$\begin{pmatrix}\dot{\pi}_0 \\ \dot{\pi}_1\end{pmatrix}' = \dot{\pi}(t) = \pi(t)\Lambda. = \begin{pmatrix}\pi_0 & \pi_1\end{pmatrix}\begin{bmatrix}-\lambda_{11} & +\lambda_{11} \\ \lambda_{22} & -\lambda_{22}\end{bmatrix}$$

- Given an initial $\pi(0)$ and a transition rate matrix Λ, we can calculate the probability of each state in any future t.
  - Often there is no analytical solution for this ODE.
  - However, in dynamic programming it is sufficient to only look at the *immediate future*.
  - The transition rates will be sufficient to do recursive optimization.

$$\pi_{t+\Delta} = \pi_t P(\Delta) \qquad \frac{\pi_{t+\Delta} - \pi_t}{\Delta} = \pi_t \frac{P(\Delta) - I}{\Delta}$$

The stationary distribution

- A stationary distribution π^* satisfies $\dot{\pi} = 0$, so

$$\pi^* \Lambda = 0.$$

Example 1: Employment and unemployment

- Remember the transition rate matrix:

$$\Lambda = \begin{bmatrix} -\delta & \delta \\ \lambda & -\lambda \end{bmatrix}$$

- We are looking for π_0^* and $\pi_1^* = 1 - \pi_0^*$ such that $\pi^* \Lambda = 0$.
- That is,

$$-\delta \pi_0^* + \lambda(1 - \pi_0^*) = 0.$$

Example 1: Employment and unemployment

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- This gives us
$$\pi_0^* = \frac{\lambda}{\lambda + \delta}.$$
- The same as $\pi_0(\infty)$ (not a coincidence).

Example 1: Employment and unemployment

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- This gives us
$$\pi_0^* = \frac{\lambda}{\lambda + \delta}.$$
- The same as $\pi_0(\infty)$ (not a coincidence).
- The steady-state probability of employment
    - increases in the job finding rate
    - decreases in the firing rate

Example 3: A faulty email server

- New emails arrive with rate λ:
  - $n \rightarrow n+1$
- With arrival rate η, all emails are lost
  - $n \rightarrow 0$

Example 4: The CEU email server

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Example 4: The CEU email server

- New emails arrive with rate λ:
  - $n \to n+1$
- After n reaches N, all emails are lost *immediately*
- What does *immediately* mean in this setup?

n
1. $\cancel{N} \to 0$ with arrival rate η, where $\eta \to \infty$
2. N is never reached from $N-1$

Questions

Questions

Take the 3 different email servers and

1. Write down the transition rate matrix.
2. Write down the Kolmogorov forward equation.
3. Solve for the steady-state distribution.

Example 3: A faulty email server

- The transition rate matrix: $\begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \eta & -\lambda-\eta & \lambda & \cdots \\ \eta & 0 & -\lambda-\eta & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Example 3: A faulty email server

- What is the stationary distribution of this process?
- For all $n \geq 0$:

$$\lambda\pi_n^* - (\lambda + \eta)\pi_{n+1}^* = 0$$

- This defines π_{n+1}^* recursively as

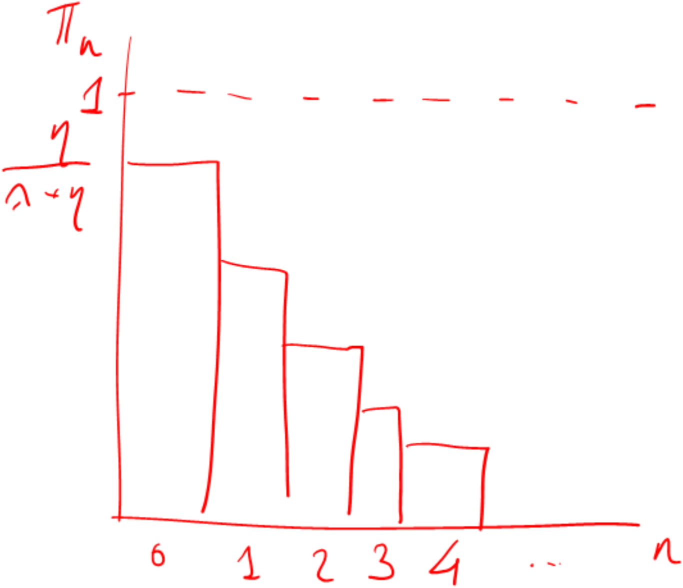
$$\pi_{n+1}^* = \frac{\lambda}{\lambda + \eta}\pi_n^*.$$

  (A geometric distribution.)
- In turn,

$$\pi_0^* = \frac{\eta}{\lambda + \eta}.$$

  (To make sure that π_n^*s sum to 1.)

A geometric distribution



The Poisson process

The Poisson process

- The possible states are $n = 0, 1, 2, ...$
- The Poisson process is characterized by an *arrival rate* λ (aka hazard rate).
- The transition rate matrix is

$$\begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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- The Poisson process is used to characterize *rare, memoryless* events.

Examples

- Phone calls to emergency center.
- Particles emitted via radioactive decay.
- Views of the CEU website.

Characterizing the Poisson process

The two key characteristics of the Poisson process

1. No two events happen at the same time ("rare events").
2. The future arrival of events is independent of past events ("memoryless").

Characterizing the Poisson process

The Poisson process may arise

- from a truly memoryless process
  - radioactive decay
- from the law of small numbers
  - view of the CEU website from California

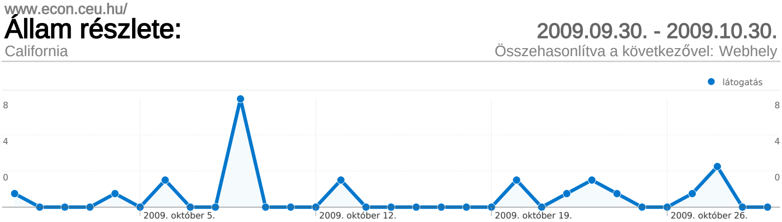
$k \sim \text{Binom}(n, p) \rightarrow$ LOLN $n \to \infty$ p fix

$\downarrow$

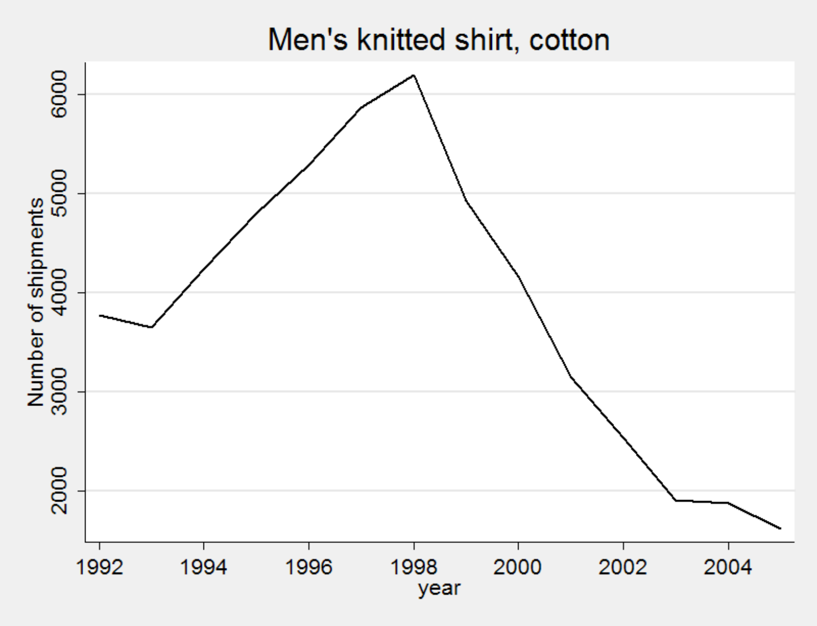
LOSN: $n \to \infty$ $p \to 0$ $np \to \text{constant}$

$k \sim \text{Poisson}(np)$

Visits to econ.ceu.hu from California



Export shipments of shirts from the U.S.



Export shipments of shirts from the U.S. to Iceland

Counterexamples

- Emergency phone calls during a natural disaster.
- Arrival of guests at a restaurant.
- Your phone calls to your mother.

Properties of the Poisson process

- The waiting time between the $n-1$sth and nth arrival is T_n.
- T_n is random, exponentially distributed with parameter λ:

$$\Pr(T_n \leq t) = 1 - \exp(-\lambda t).$$

- Waiting times are independent.

$$\Pr(T \leq t \mid T > s) = 1 - \exp(-\lambda(t-s))$$

Properties of the Poisson process

- Let $N = n(t+h) - n(t)$ denote the number of arrivals between t and $t+h$.
- N is a Poisson-distributed random variable with parameter λh.
- It takes on values $0, 1, 2, \ldots$ with pdf

$$\Pr(n = k) = \frac{\exp(-\lambda h)(\lambda h)^k}{k!}$$

Properties of Poisson processes (continued)

- Take two independent Poisson processes with arrival λ_1 and λ_2.
  - The sum is also a Poisson process with arrival $\lambda_1 + \lambda_2$.
  - The waiting time for the first arrival is exponential with parameter $\lambda_1 + \lambda_2$. $T_{1+2} = \min\{T_1, T_2\}$

Properties of Poisson processes (continued)

- Take two independent Poisson processes with arrival λ_1 and λ_2.
  - The sum is also a Poisson process with arrival $\lambda_1 + \lambda_2$.
  - The waiting time for the first arrival is exponential with parameter $\lambda_1 + \lambda_2$.
- Take a Poisson processes with arrival λ and a probability p.
- Kill each arrival with probability $1 - p$.
  - The new process is Poisson with arrival $p\lambda$.

Poisson representation of Markov chains

- Think of a Markov chain with N states.
- Starting in any given state, only $N - 1$ things can happen (or nothing).
- Each $N - 1$ jump has its own arrival rate.
- The first jump occurs with a Poisson arrival $\lambda_1 + ... + \lambda_{n-1}$ (see above).

Poisson representation of Markov chains (continued)

- Once there *is* a jump, which one is it?
- It could be any one of the $1, ..., n-1$.
- The probability of jump 1 is

$$\frac{\lambda_1}{\lambda_1 + ... + \lambda_{n-1}}.$$

Poisson representation of Markov chains (continued)

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- This is a good old probability $\in [0, 1]$.
- This looks more like a discrete transition matrix.

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- This is a good old probability $\in [0, 1]$.
- This looks more like a discrete transition matrix.
- I find it useful to think about Markov chains as the sum of Poisson processes.

Checklist

By now you should understand

1. continuous-time Markov chain
2. arrival rate matrix
3. forward Kolmogorov equation
4. stationary distribution
5. Poisson process
6. Poisson distribution