

# ECBS6001: Advanced Macroeconomics

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## Lecture 18: A model of new products

# Outline

- ▶ Today we apply continuous-time dynamic programming to a model of endogenous growth.
- ▶ We will use
  - ▶ dynamic programming with a jump process
  - ▶ differential equations
  - ▶ steady-state properties
- ▶ The application is also heavy in general equilibrium.
  - ▶ aggregation of individual decisions
  - ▶ resource constraints
  - ▶ endogenous prices

# Outline

- ▶ Today we will study a model where growth occurs through an increase in the number of products.
- ▶ Product innovation is a source of technical progress.
  - ▶ Most of income differences across countries are due to differences in productivity.
- ▶ We will make use of the Dixit–Stiglitz model of monopolistic competition.

# Innovation

- ▶ Innovation is a conscious economic activity.
  - ▶ In contrast to exogenous technical progress.
  - ▶ Responds to profit incentives of innovators.
- ▶ Process innovation reduces production costs of existing products. Product innovation entails coming up with new products.
  - ▶ We focus on product innovation.
- ▶ Horizontal innovation leads to products with new functions. Vertical innovation serves similar function at higher quality.
- ▶ Innovation (“idea”) may or may not be replicable.
  - ▶ We begin with a setup with fully private benefits.
  - ▶ Then discuss the case of knowledge spillovers.

# Product innovation

- ▶ Firms spend on R&D to come up with blueprints of products.
- ▶ Only products for which blueprints exist can be produced.
- ▶ The holder of a blueprint obtains a monopoly over producing that product.
  - ▶ Patent protection
  - ▶ Any small cost of imitation prevents it in equilibrium.
  - ▶ Later we will study imitation more generally.
- ▶ Firms do two things:
  1. develop blueprints
  2. produce from existing blueprints

# Static equilibrium

# Structure of the economy

- ▶ Firms produce goods based on existing blueprints.
  - ▶ using labor
- ▶ They also employ researchers to develop new blueprints
  - ▶ same type of labor
  - ▶ demand for funds
- ▶ Because of IPR protection, firms make profits.
- ▶ Workers earn wages and hold a portfolio of all firms (stock market).
- ▶ They decide how much to consume and how much to save.
  - ▶ supply of funds



# Static equilibrium

- ▶ We begin by characterizing the equilibrium at a given point in time.
  - ▶ production of existing products
- ▶ We then move on to dynamic decisions
  - ▶ development of new products

# Consumers

# Consumers

- ▶ Consumers value all existing products symmetrically.
  - ▶ No obsolescence.
- ▶ Suppose  $N(t)$  products exist at time  $t$ .
- ▶ Composite consumption good:

$$D = \left[ \sum_{i=1}^{N(t)} x_i^\alpha \right]^{1/\alpha} \quad 0 < \alpha < 1$$

- ▶ This is a constant-elasticity-of-substitution utility function a la Dixit and Stiglitz.
- ▶ Elasticity of substitution is  $\varepsilon = 1/(1 - \alpha) > 1$ .
  - ▶ What does  $\varepsilon > 1$  mean?

Love of variety

## Love of variety

- ▶ Suppose each variety costs  $p_i = p$ .
- ▶ Total spending on  $n$  goods:  $Npx = E$ , so that  $x = E/(pN)$ .
- ▶ What utility does the consumer achieve?

$$D = \left[ \sum_{i=1}^N x^\alpha \right]^{1/\alpha} = \frac{E}{p} \left[ \sum_{i=1}^N N^{-\alpha} \right]^{1/\alpha} = \frac{E}{p} N^{(1-\alpha)/\alpha}$$

- ▶ For given income  $E$  and prices  $p$ , utility increases in  $N$ .
- ▶ Because  $x_i$ s are imperfect substitutes of one another, it is better to have a little of each than much of one.
- ▶ Consumption of non-existent varieties is 0. The convexity of preferences dislikes zeros.
- ▶ This is the love-of-variety feature of preferences.

Love of variety

## Love of variety

- ▶ Alternatively, we can express love of variety in the expenditure function.
- ▶ The minimum cost of obtaining one unit of utility,

$$P = \left[ \sum_{i=1}^N p^{1-\varepsilon} \right]^{1/(1-\varepsilon)} = pN^{1/(1-\varepsilon)}$$

is decreasing in  $N$ .

- ▶ We can think of the price of non-existent varieties as  $\infty$ .
- ▶ When the product becomes available, its price falls from  $\infty$  to  $p$ .
- ▶ An increase in  $N$  then reduces the aggregate price index.
- ▶ Hence indirect utility,

$$u = \frac{E}{P} = \frac{E}{p} N^{(1-\alpha)/\alpha}$$

is increasing in  $N$ .

# Firms



# Firms

- ▶ Each product is produced with labor only:  $x_i = l_i$ .
- ▶ The unitary labor restrictive requirement is not restrictive. Why?
- ▶ The demand for product  $i$ :

$$x_i = E \frac{p_i^{-\varepsilon}}{\sum_{j=1}^N p_j^{1-\varepsilon}}$$

# Market structure

# Market structure

- ▶ Each firm has a monopoly over its blueprint.
- ▶ There are many firms so that each firm takes aggregate prices and quantities as given.
- ▶ Monopolistic competition.
- ▶ Demand for product  $i$  is isoelastic with elasticity  $\varepsilon$ :

$$x_i = E \frac{p_i^{-\varepsilon}}{\sum_{j=1}^N p_j^{1-\varepsilon}} = A p_i^{-\varepsilon}$$

- ▶ The firm takes  $A$  as given but maximizes over  $p_i$ .
- ▶ Markup pricing

$$p_i = \frac{\varepsilon}{\varepsilon - 1} w = \frac{w}{\alpha}$$

## Choice of numeraire

- ▶ We normalize aggregate expenditure to  $E \equiv 1$ .
- ▶ This is a weird choice of numeraire but will prove convenient later.
  - ▶ We are free to fix any one price in each time period to whatever value we want.
  - ▶ The price need not be 1, need not even be constant.
  - ▶ So we pick  $P = 1/D$  so that  $E = PD = 1$ .

## Firm profits

## Firm profits

- ▶ A fraction  $\alpha$  of revenue goes to labor,  $(1 - \alpha)$  goes to profits.

$$\pi_i = (1 - \alpha)p_i x_i$$

- ▶ In symmetric equilibrium, each firm sells  $1/N$ .

$$\pi_i = \frac{1 - \alpha}{N}.$$

- ▶ Note that flow profits tend to zero as  $N$  increases without bound.
- ▶ Will R&D be sustained in the long run?

# Determining wages

# Determining wages

- ▶ The wagebill of the firm is

$$wl_i = \frac{\alpha}{N}.$$

- ▶ Aggregating across all  $N$  firms,

$$wL_m = \alpha,$$

where  $L_m$  is the amount of labor in production.

- ▶ So wages are

$$w = \frac{\alpha}{L_m}.$$



# Checklist

So far we have determined

- ▶ how consumers value variety
- ▶ how firms price their products
- ▶ what is the total labor demand in production

# Dynamic decisions

## Supply of funds

# Consumption smoothing

# Consumption smoothing

- ▶ Consumer has log utility

$$\mathcal{U} = \int_{t=0}^{\infty} \exp(-\rho t) \ln D(t) dt.$$

- ▶ Can save and borrow at interest rate  $r$ :

$$\dot{a} = ra + y - c.$$

- ▶ The corresponding Bellman:

$$\rho V(a, y) = \max_c \{ \ln c + V_a(a, y)(ra + y - c) + V_y(a, y)\dot{y} \}$$

# Deriving the Euler equation

# Deriving the Euler equation

- ▶ The FOC for optimal  $c$ :

$$\frac{1}{c} = V_a(a, y).$$

- ▶ Taking logs and differentiating wrt time

$$-\hat{c} = \frac{V_{aa}}{V_a}(ra + y - c)$$

( $\hat{x}$  denotes  $\dot{x}/x$ )

- ▶ Differentiating through the Bellman to express  $V_a$ :

$$\rho V_a = V_{aa} \cdot [ra + y - c] + V_a \cdot r$$

- ▶ Substituting in  $V_{aa}$ :

$$\hat{c} = r - \rho$$

## Nominal vs real interest rate?

- ▶ All this referred to an economy in which there is no inflation.
- ▶ If there is inflation,

$$\dot{a} = ra + y - pc.$$

- ▶ And the FOC becomes

$$\frac{1}{c} = pV_a(a, y, p).$$



# Exercise

- ▶ Derive the Euler equation in this economy.
- ▶ What is the optimal rate of growth for consumption?
- ▶ What do you need to know about  $p(t)$ ?

# The Euler equation

# The Euler equation

- ▶ The Euler equation

$$\hat{D} = r - \hat{P} - \rho.$$

- ▶  $\hat{D} + \hat{P}$  is the growth rate of expenditure,  $E = PD$ .
- ▶ But expenditure is constant,  $E = 1$ .
- ▶ The interest rate equals the discount rate

$$r = \rho.$$

- ▶ This completely characterizes the supply side of financial markets.

# Demand for funds

# The decision to innovate

- ▶ Take a firm with  $n$  products.
  - ▶  $n$  is a firm-level state variable.
- ▶ The firm takes aggregates,  $N$  and  $L_m$  as given.
- ▶ The firm can raise capital at rate  $r = \rho$ .
- ▶ We next analyze the R&D decision.

# Research and development

- ▶ R&D is costly and random.
  - ▶ Successes arrive with a Poisson process.
  - ▶ The arrival rate depends on R&D expenditure.
- ▶ So that a new product arrives with rate  $\lambda$ , the firm has to hire  $a\lambda$  workers.
  - ▶ Again,  $\lambda$  is an instantaneous arrival rate.
  - ▶ R&D expenditure  $a\lambda$  is a flow.
  - ▶ Past expenditure and past success do not matter.
- ▶ Let us write down the Bellman equation for the value of the firm.

# The value of a firm

# The value of a firm

- ▶ The Bellman equation

$$\rho V(n, N, L_m) = \max_n \left\{ \pi_i n - a\lambda w + \lambda[V(n+1, N, L_m) - V(n, N, L_m)] + V_N \dot{N} + V_{L_m} \dot{L}_m \right\}$$
$$\max_n \left\{ \frac{1-\alpha}{N} n - a\lambda \frac{\alpha}{L_m} + \lambda[V(n+1, N, L_m) - V(n, N, L_m)] + V_N \dot{N} + V_{L_m} \dot{L}_m \right\}$$

- ▶ The state variables are  $n$ ,  $N$  and  $L_m$ . Only  $n$  is affected by the firm.
- ▶ The opportunity (“financing”) cost of the firm equals
  - ▶ flow profits:
    - ▶ operative profits
    - ▶ minus the cost of R&D
  - ▶ capital gains:
    - ▶ the invention of a new product
    - ▶ change in the value of existing products



# Solution

# Solution

- ▶ Guess that the value is linear in  $n$ ,  $V(n, N, L_m) = nv(N, L_m)$ .

$$\rho nv(N, L_m) = \max_{\lambda} \left\{ \frac{1-\alpha}{N}n - a\lambda \frac{\alpha}{L_m} + \lambda v(N, L_m) + n\dot{v} \right\}$$

- ▶ The FOC for  $\lambda$  is

$$\frac{\alpha a}{L_m} \geq v(N, L_m),$$

with equality if  $\lambda > 0$ .

- ▶ Simplify to

$$\rho v(N, L_m) = \frac{1-\alpha}{N} + \dot{v}.$$

- ▶ This links  $N$ ,  $L_m$ , and  $v$ .

# Innovation and growth

- ▶ Note that the FOC did not pin down  $\lambda$ .
  - ▶ This because of the linearity of both the benefit and the cost of innovation.
  - ▶ Would change with convex costs of innovation.
- ▶ Suppose firm  $i$  innovates with rate  $\lambda_i$ , using  $a\lambda_i$  R&D workers.
- ▶ This leads to a new product with arrival rate  $\lambda_i$ .

# Aggregate innovation

# Aggregate innovation

- ▶ The arrival rate of the first new product across all firms is

$$\sum_i \lambda_i \equiv \Lambda.$$

- ▶ The overall number of R&D workers is

$$\sum_i a\lambda_i = a\Lambda.$$

- ▶ Even if  $\lambda_i$  is indeterminate, aggregate innovation  $\Lambda$  will be pinned down in equilibrium.
- ▶ Because new products arrive with  $\lambda$ , the growth rate of  $N$  is

$$\frac{EdN/dt}{N} = \frac{\Lambda}{N}.$$

# Dynamic equilibrium

# Dynamic equilibrium

- ▶ We now characterize the dynamic equilibrium.
- ▶ The key is to pin down the allocation of labor to its two uses:
  1. production:  $L_m$
  2. R&D:  $a\Lambda$
- ▶ Resource constraint for labor

$$L_m + a\Lambda = L,$$

- ▶ or

$$\dot{N} = \frac{L - L_m}{a}.$$

- ▶ Note that whenever  $\dot{N} > 0$ ,  $L_m = \alpha a/v$ , so that

$$\dot{N} = \frac{L}{a} - \frac{\alpha}{v}.$$

# Long-run growth



## Long-run growth

- ▶ We first show that there is no long-run growth in this economy.
- ▶ This is because the incentive to innovate disappears as  $N$  grows large.
- ▶ Suppose

$$N > \bar{N} \equiv \frac{(1 - \alpha)L}{\alpha a \rho}$$

and there is no R&D.

- ▶ Then both  $N$  and  $L_m$  are constant, so is  $v(N, L_m)$ .
- ▶ From the Bellman equation,

$$v(N, L) = \frac{1 - \alpha}{\rho N}.$$

- ▶ But because  $N > \tilde{N}$ , this is indeed smaller than the cost of innovation  $\alpha a / L$ .
- ▶ So no innovation is a unique equilibrium.

# Steady state

The steady-state  $N$  (and hence steady-state productivity) is

- ▶ increasing in country size  $L$
- ▶ increasing in profit share  $(1 - \alpha)$
- ▶ decreasing in R&D cost  $a$
- ▶ decreasing in discount rate  $\rho$

$$\frac{D}{L_m} = N^{(1-\alpha)/\alpha} \rightarrow \left( \frac{(1-\alpha)L}{\alpha a \rho} \right)^{(1-\alpha)/\alpha}$$

# Phase diagram

# Recipe

1. Constant interest rate:  $r = \rho$
2. Symmetric profits:  $\pi = (1 - \alpha)/N$
3. Wage equation:  $w = \alpha/L_m$
4. Firm valuation:  $\rho v = \pi + \dot{v}$
5. Optimal R&D:  $v \leq aw$
6. Resource constraint:  $L_m + a\dot{N} = L$

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# Characterizing the dynamics

We collapse the six equations of the “recipe” into two.

Labor market clearing + wage equation + optimal R&D

$$\dot{N} = \max \left\{ 0, \frac{L}{a} - \frac{\alpha}{v} \right\}$$

Bellman equation

$$\dot{v} = \rho v - \frac{1 - \alpha}{N}$$



## Steady state

- ▶ The steady state is such that both  $\dot{N}$  and  $\dot{v}$  are zero.
- ▶ Steady-state  $N$  is

$$N_{ss} = \frac{(1 - \alpha)L}{\alpha \rho a}.$$

- ▶ Steady-state productivity (output per worker) is

$$\frac{D}{L_m} = N^{(1-\alpha)/\alpha} \rightarrow \left( \frac{(1 - \alpha)L}{\alpha a \rho} \right)^{(1-\alpha)/\alpha}.$$

- ▶ (This only includes manufacturing, not R&D.)
- ▶ Both are
  - ▶ increasing in country size  $L$
  - ▶ increasing in profit share  $(1 - \alpha)$
  - ▶ decreasing in R&D cost  $a$
  - ▶ decreasing in discount rate  $\rho$

# Knowledge spillovers

# Knowledge spillovers

- ▶ Now suppose that R&D has external benefits to other researchers.
- ▶ In particular, let the cost of R&D decrease with the number of existing products,  $N$ ,  $a/N$ .
- ▶ This changes
  1. the incentive to innovate
  2. the resource requirements of innovation

# The new Bellman

# The new Bellman

- ▶ The new Bellman equation:

$$\rho nv(N, L_m) = \max_{\lambda} \left\{ \frac{1 - \alpha}{N} n - \frac{\alpha a}{NL_m} \lambda + \lambda v(N, L_m) + n\dot{v} \right\}$$

- ▶ First-order condition

$$\frac{\alpha a}{NL_m} = v$$

- ▶ The rate of return on innovation

$$r_{R\&D} = \frac{1 - \alpha}{\alpha} \frac{L_m}{a} - \hat{N} - \hat{L}_m.$$

- ▶ Profits per fixed cost are now independent of  $N$ .
- ▶ There are capital losses because innovation becomes ever cheaper.

# The new resource constraint

## The new resource constraint

If aggregate innovation is  $\Lambda$ , it takes up  $\Lambda a/N$  workers.

$$L_m + \Lambda \frac{a}{N} = L$$

# Balanced growth



## Balanced growth

- ▶ Suppose that this economy attains a balanced growth path with constant growth rate  $g = \hat{N}$ .
- ▶ “Balanced” means that labor allocations ( $L_m$ ) are constant.
- ▶ We then verify that this is an equilibrium.
- ▶ From the Bellman equation for firm value

$$\rho = \frac{1 - \alpha}{\alpha} \frac{L_m}{a} - g.$$

- ▶ From the resource constraint,

$$L_m + ag = L.$$

# Solution

## Solution

$$L_m = \alpha(L + a\rho)$$
$$g = \frac{1 - \alpha}{a}L - \alpha\rho$$

- ▶ Indeed, as long as  $(1 - \alpha)L/a > \alpha\rho$ , balanced growth is an equilibrium.
- ▶ The growth rate is
  - ▶ increasing in the profit share  $(1 - \alpha)$
  - ▶ increasing in country size  $L$  (more on this later)
  - ▶ decreasing in the cost of R&D  $a$
  - ▶ decreasing in the discount rate  $\rho$

## Other ways to generate growth

## Other ways to generate growth

- ▶ Knowledge spillovers reduce the cost of innovation so that profit *per cost* do not vanish.
- ▶ We have other ways to generate growth:
  - ▶ In a different demand system / competition, profits may not vanish. (See quality competition later.)
  - ▶ If innovation costs are in the final good rather than in labor units, they “mechanically” get lower with development:

$$P = N^{(\alpha-1)/\alpha}.$$

- ▶ (This is also an external benefit of R&D but it is *pecuniary*.)
- ▶ If population grows, firms keep doing R&D. This may even lead to growth in output *per capita*. (See later.)

# Policy and welfare

# Policies

- ▶ We want to see if policy has an effect on growth.
- ▶ We consider two policies:
  1. an R&D subsidy
  2. a production subsidy

## R&D subsidy

- ▶ The government pays a fraction  $\phi$  of research expenses.
- ▶ This is financed by a lump-sum tax.
- ▶ This changes the incentive to innovate,

$$\frac{\alpha a(1 - \phi)}{NL_m} = v,$$

- ▶ and the Bellman equation

$$\rho = \frac{(1 - \alpha)L_m}{\alpha a(1 - \phi)} - g.$$



# Solution

## Solution

$$L_m = \alpha[L + a(1 - \phi)\rho] < L_m(\text{no subsidy})$$

$$g = \frac{1 - \alpha}{a(1 - \phi)}L - \alpha\rho > g(\text{no subsidy})$$

- An R&D subsidy increases growth and decreases manufacturing employment/output.

## Production subsidy

- ▶ Manufacturers receive an ad valorem subsidy of  $\phi_x$ .
- ▶ Their aggregate revenue is hence  $1 + \phi_x$ ,
- ▶ profit per variety is

$$(1 + \phi_x)(1 - \alpha)/N.$$

- ▶ This seems to raise the profitability of R&D.

## Production subsidy

## Production subsidy

- ▶ But note that the wage equation changes as well,

$$w = (1 + \phi_x)\alpha/L_m.$$

- ▶ So the *returns* to R&D are unchanged:

$$\frac{(1 + \phi_x)(1 - \alpha)L_m}{(1 + \phi_x)a\alpha N}.$$

- ▶ Production subsidy raises prices and wages proportionally, so does not lead to any reallocation.

# Welfare

- ▶ Is the equilibrium growth rate  $g$  *optimal*?
- ▶ We see that R&D subsidies can increase the growth rate – should they be employed?
- ▶ We answer that by solving the benevolent social planners problem.
- ▶ The social planner maximizes discounted utility subject to technology constraints.
  - ▶ (Prices and markets do not matter.)
- ▶ We begin with the case without knowledge spillovers.

# Static optimum

- ▶ First note that the static equilibrium is optimal – despite imperfect competition.
- ▶ Given  $L_m$  workers and  $N$  existing varieties, the social planner would like to allocate  $L_m/N$  workers to each – just as in equilibrium.
- ▶ Because markups are symmetric, they do not involve any distortion – relative prices across firms are unchanged.

# Dynamic optimum



# Dynamic optimum

- ▶ Aggregate output of the final good is

$$D = N^{(1-\alpha)/\alpha} L_m.$$

- ▶ Per-period utility is

$$\ln D = \frac{1-\alpha}{\alpha} \ln N + \ln L_m.$$

- ▶ The one state variable is  $N$ .
- ▶ Choice variable is  $L_m$ , or, equivalently, aggregate innovation  $\Lambda$ .

# Bellman equation

# Bellman equation

- ▶ The Bellman equation of the policy maker

$$\rho V(N) = \max_{L_m} \left\{ \frac{1-\alpha}{\alpha} \ln N + \ln L_m + \frac{L - L_m}{a} [V(N+1) - V(N)] \right\}$$

- ▶ We make the approximation

$$V(N+1) - V(N) \approx V'(N)$$

so that we can use the envelope theorem. (Works for large  $N$ .)

# Solution

## Solution

- ▶ The first-order condition for  $L_m$  is

$$\frac{1}{L_m} = \frac{V'(N)}{a}.$$

- ▶ Using the envelope theorem to determine  $V'$ ,

$$\rho V'(N) = \frac{1 - \alpha}{\alpha N} + \dot{N} V''(N).$$

- ▶ Now introduce  $v \equiv \alpha V'(N)$ . The two equations can be rewritten as

$$L_m = \frac{a\alpha}{v}$$
$$\rho v = \frac{1 - \alpha}{N} + \dot{v}$$

# Solution

## Solution

- ▶ Substituting in the resource constraint,

$$\dot{N} = \frac{L}{a} - \frac{\alpha}{v}.$$

- ▶ And the Bellman equation

$$\dot{v} = \rho v - \frac{1 - \alpha}{N}.$$

- ▶ But notice that these are the same as the equilibrium conditions.
- ▶ The equilibrium is hence efficient.

# The case with knowledge spillovers



## The case with knowledge spillovers

- ▶ The social planner's Bellman now

$$\rho V(N) = \max_{L_m} \left\{ \frac{1-\alpha}{\alpha} \ln N + \ln L_m + \frac{(L - L_m)N}{a} V'(N) \right\}$$

- ▶ Guess that the value function is of the form

$$V(N) = b_0 + b_1 \ln N.$$

# Solution

## Solution

- ▶ Guess that the value function is of the form

$$V(N) = b_0 + b_1 \ln N.$$

- ▶ Then  $V'(N) = b_1/N$  and

$$L_m = \frac{a}{V'(N)N} = \frac{a}{b_1}$$

is constant.

- ▶ So is the growth rate

$$\frac{\dot{N}}{N} = \frac{L - L_m}{a} = \frac{L}{a} - b_1.$$

- ▶ Verify that the Bellman equation holds for

$$b_1 = \frac{1 - \alpha}{\alpha \rho}$$

and some (ugly)  $b_0$ .

# Optimal growth

# Optimal growth

- ▶ Substituting in  $b_1$ , optimal growth is

$$g^* = \frac{L}{a} - \frac{\alpha\rho}{1-\alpha}.$$

- ▶ Notice that

$$g^* = \frac{g}{1-\alpha} > g.$$

- ▶ Equilibrium growth is *inefficiently low*.
- ▶ What is the intuition?

# Discussion

- ▶ We have endogenized *technology*: companies invest in new technology just as they invested in physical capital in the Solow/Ramsey model.
- ▶ But it has proven difficult to endogenize *growth*: R&D can also be subject to decreasing returns to scale.
- ▶ We had to assume spillovers: the social returns to R&D are higher than the private returns.
- ▶ This model is not necessarily about *endogenous growth*, but certainly about *endogenous innovation* and technology.
- ▶ Innovation (and potentially growth) responds to taste and policy parameters and, notably, *country size*.
- ▶ Equilibrium growth is lower than optimal, there is room for policy.

# Appendix

## CES review

- ▶ Take the following CES utility function:

$$u(x_1, x_2) = [x_1^\alpha + x_2^\alpha]^{1/\alpha},$$

and define  $\varepsilon = 1/(1 - \alpha)$ ,  $\alpha = 1 - 1/\varepsilon$

- ▶ Maximize utility subject to prices  $p_1$  and  $p_2$ :

$$p_1x_1 + p_2x_2 = E$$

- ▶ What is the relative demand for  $x_1$  and  $x_2$ ?



# Utility maximization

- ▶ The marginal rate of substitution

$$\frac{u_1}{u_2} = \frac{x_1^{\alpha-1}}{x_2^{\alpha-1}} = \left( \frac{x_1}{x_2} \right)^{-1/\varepsilon}$$

- ▶ In the optimum, this equals the relative price,  $p_1/p_2$ :

$$\frac{x_1}{x_2} = \left( \frac{p_1}{p_2} \right)^{-\varepsilon}$$

- ▶ The relative demand is loglinear in relative prices.
  - ▶ The elasticity of substitution is constant at  $\varepsilon$ .

# Cost minimization

- ▶ In parallel, we can solve the cost minimization problem.
- ▶ Minimize  $E = p_1x_1 + p_2x_2$  subject to  $u(x_1, x_2) = u_0$ .
  - ▶ FOC:

$$p_i = \lambda x_i^{\alpha-1}$$

$$E = u_0 [p_1^{1-\varepsilon} + p_2^{1-\varepsilon}]^{1/(1-\varepsilon)}$$

- ▶ The term

$$P \equiv [p_1^{1-\varepsilon} + p_2^{1-\varepsilon}]^{1/(1-\varepsilon)}$$

is the *ideal price index*.

## Markup pricing

- ▶ Take a demand function  $D(p)$  and a cost function  $C(Q)$ .
- ▶ Maximize profit

$$pD(p) - C[D(p)]$$

- ▶ First-order condition

$$D(p) + pD'(p) - C'[D(p)]D'(p) = 0$$

- ▶ Divide by  $pD'$  and rearrange

$$\frac{p - C'[D(p)]}{p} = \frac{D(p)}{-pD'(p)} \equiv \frac{1}{\varepsilon}.$$

- ▶ Price-cost markup

$$\frac{p}{C'[D(p)]} = \frac{\varepsilon}{\varepsilon - 1}.$$