

# ECBS6001: Advanced Macroeconomics

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## Lecture 15: Dynamic programming in continuous time

# Outline

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- ▶ Today we start using our tools for dynamic programming.
  - ▶ no uncertainty
  - ▶ Markov chains
- ▶ We review some simple applications.

From last time

## Solutions to examples

- ▶ Both CEU email servers had the following stationary distribution:

$$\pi_n^* = 1/N$$

for  $n = 0, 1, \dots, N - 1$  and 0 thereafter.

- ▶ It was easiest to see for server 2, because the Kolmogorov equation gave  $\pi_n^* = \pi_{n-1}^*$  and  $\pi_0^* = \pi_{N-1}^*$ .

# Solutions to the teamwork assignments

- For server 1,

$$\lambda\pi_{N-1}^* = (\lambda + \eta)\pi_N^*$$

so the stationary distribution is

$$\pi_n^* = \begin{cases} \frac{\lambda + \eta}{N(\lambda + \eta) + \lambda} & \text{if } n < N \\ \frac{\lambda}{N(\lambda + \eta) + \lambda} & \text{if } n = N \end{cases}$$

- Take  $\eta \rightarrow \infty$  to get the result.

## Discrete time



# Dynamic programming with no uncertainty

- ▶ Suppose you have a vector of state variables  $x_t$ , and control variables  $c_t$ .
- ▶ The equation of motion for the state is

$$\Delta x_{t+1} = F(x_t, c_t).$$

- ▶ Per-period utility is

$$u(x_t, c_t)$$

- ▶ The sequential problem is

$$\max_{\{c_t\}} \sum_{t=1}^{\infty} \beta^t u(x_t, c_t) \text{ s.t. } \Delta x_{t+1} = x_t + F(x_t, c_t)$$

# The recursive formulation

- ▶ The corresponding Bellman equation is

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V(x_{t+1})\}$$

- ▶ or substituting in the equation of motion

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

# Solution

- ▶ The solution is a value function  $V(x)$  that maps the state into the PDV of utility.
- ▶ Equivalently, the solution can be given as a policy function  $c(x)$ .
- ▶ What would change if time periods were days instead of years?

## Continuous time

## Moving to continuous time

- ▶ Let time periods be  $\Delta$  apart.
- ▶ As before, we want to characterize the time series as  $\Delta$  becomes smaller and smaller.
- ▶ We take the limit as  $\Delta \rightarrow 0$ .
  - ▶ We will have to rescale flows, but not stocks.

# Differential equations

- Now dynamics are characterized by the differential equation:

$$\dot{x}(t) = \lim_{\Delta t \rightarrow 0} \frac{F(x_t, c_t, \Delta t)}{\Delta t} \equiv f(x_t, c_t).$$

# Dynamic programming

- ▶ Back to our discrete-time Bellman:

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

- ▶ Which of the objects here depend on the length of the time period?
  - ▶  $u$ , because utility is a *flow*: shorter time periods yield less utility
  - ▶  $\beta$ , because shorter time periods are discounted less
  - ▶  $F$  as we have seen above
- ▶ Let us make this period dependence explicit:

$$V(x_t) = \max_{c_t} \left\{ u(x_t, c_t)\Delta + \frac{1}{1 + \rho\Delta} V[x_t + f(x_t, c_t)\Delta] \right\}$$

- ▶ Now  $u$  is the *per-period* utility,  $\rho$  is the *per-period* discount rate,  $f$  is the *per-period* growth rate.

## Infinitesimal periods

As you might expect, we take  $\Delta$  to 0.

First multiply by  $(1 + \rho\Delta)$ :

$$(1 + \rho\Delta)V(x_t) = \max_{c_t} \{u(x_t, c_t)\Delta(1 + \rho\Delta) + V[x_t + f(x_t, c_t)\Delta]\}$$

Then subtract  $V(x_t)$ :

$$\rho\Delta V(x_t) = \max_{c_t} \{u(x_t, c_t)\Delta(1 + \rho\Delta) + V[x_t + f(x_t, c_t)\Delta] - V(x_t)\}$$

Then divide by  $\Delta$ :

$$\rho V(x_t) = \max_{c_t} \left\{ u(x_t, c_t)(1 + \rho\Delta) + \frac{V[x_t + f(x_t, c_t)\Delta] - V(x_t)}{\Delta} \right\}$$

Now we're ready to take the limit.



# The Hamilton-Jacobi-Bellman equation

$$\rho V(x_t) = \max_{c_t} [u(x_t, c_t) + V'(x_t)f(x_t, c_t)]$$

- ▶ What is different:
  - ▶ we have  $\rho V$ , not  $V$  on the LHS
  - ▶ we have  $\dot{V}$ , not new value on RHS
- ▶ Intuition:
  - ▶ the per-period discount loss from my value should be compensated by
  - ▶ flow utility
  - ▶ and (expected) gains in future value

## Example: eat-the-pie problem

- ▶ You have wealth  $W$ , accruing interest  $r$  per unit of time.
- ▶ You maximize

$$\int_{t=0}^{\infty} \exp(-\rho t) \ln c(t) dt.$$

1. Write down the Bellman equation.
2. Guess that  $V(W) = a + b \ln W$  and solve for  $a$  and  $b$ .
3. What is the optimal consumption policy,  $c(W)$ ?

# Application to the Ramsey model

► Let

$$\begin{aligned}x_t &= k_t, \\u(x_t, c_t) &= c_t^{1-\theta} / (1 - \theta), \\f(x_t, c_t) &= f(k_t) - \delta k_t - c_t\end{aligned}$$

► The Bellman equation is now

$$\rho V(k) = \max_c \left\{ \frac{c^{1-\theta}}{1-\theta} + V'(k)[f(k) - \delta k - c] \right\}$$

## Deriving the Euler equation

- ▶ The FOC for optimal  $c$ :

$$c^{-\theta} = V'(k).$$

- ▶ Taking logs and differentiating wrt time

$$-\theta \hat{c} = \frac{V''(k)}{V'(k)} [f(k) - \delta k - c]$$

( $\hat{x}$  denotes  $\dot{x}/x$ )

- ▶ Differentiating through the Bellman to express  $V'(k)$ :

$$\rho V'(k) = V''(k) [f(k) - \delta k - c] + V'(k) [f'(k) - \delta]$$

- ▶ Substituting in  $V''(k)$ :

$$\hat{c} = \frac{1}{\theta} [f'(k) - \delta - \rho] \equiv \frac{1}{\theta} [r(k) - \rho]$$

## The consumption rule

$$\hat{c} = \frac{1}{\theta} [f'(k) - \delta - \rho] \equiv \frac{1}{\theta} [r(k) - \rho]$$

- ▶ Agents like to smooth consumption (especially with high  $\theta$ ).
- ▶ Consumption grows if  $r > \rho$ : the market return on my saving (late consumption) is higher than the private return from early consumption.
- ▶ Consumption growth is high if  $k$  is low ( $r$  is high).