ECBS6001: Advanced Macroeconomics

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Lecture 18: A model of new products

Outline

- Today we apply continuous-time dynamic programming to a model of endogenous growth.
- We will use
  - dynamic programming with a jump process
  - differential equations
  - steady-state properties
- The application is also heavy in general equilibrium.
  - aggregation of individual decisions
  - resource constraints
  - endogenous prices

Outline

- Today we will study a model where growth occurs through an increase in the number of products.
- Product innovation is a source of technical progress.
  - Most of income differences across countries are due to differences in productivity.
- We will make use of the Dixit–Stiglitz model of monopolistic competition.

Innovation

- Innovation is a conscious economic activity.
  - In contrast to exogenous technical progress.
  - Responds to profit incentives of innovators.
- Process innovation reduces production costs of existing products. Product innovation entails coming up with new products.
  - We focus on product innovation.
- Horizontal innovation leads to products with new functions. Vertical innovation serves similar function at higher quality.
- Innovation ("idea") may or may not be replicable.
  - We begin with a setup with fully private benefits.
  - Then discuss the case of knowledge spillovers.

Product innovation

- Firms spend on R&D to come up with blueprints of products.
- Only products for which blueprints exist can be produced.
- The holder of a blueprint obtains a monopoly over producing that product.
  - Patent protection
  - Any small cost of imitation prevents it in equilibrium.
  - Later we will study imitation more generally.
- Firms do two things:
  1. develop blueprints
  2. produce from existing blueprints

Static equilibrium

Structure of the economy

- Firms produce goods based on existing blueprints.
  - using labor
- They also employ researchers to develop new blueprints
  - same type of labor
  - demand for funds
- Because of IPR protection, firms make profits.
- Workers earn wages and hold a portfolio of all firms (stock market).
- They decide how much to consume and how much to save.
  - supply of funds

Static equilibrium

- We begin by characterizing the equilibrium at a given point in time.
  - production of existing products
- We then move on to dynamic decisions
  - development of new products

Consumers

- Consumers value all existing products symmetrically.
  - No obsolescence.
- Suppose $N(t)$ products exist at time t.
- Composite consumption good:

$$D = \left[\sum_{i=1}^{N(t)} x_i^{\alpha} \right]^{1/\alpha} \qquad 0 < \alpha < 1$$

- This is a constant-elasticity-of-substitution utility function a la Dixit and Stiglitz.
- Elasticity of substitution is $\varepsilon = 1/(1-\alpha) > 1$.
  - What does $\varepsilon > 1$ mean?

Love of variety

- Suppose each variety costs $p_i = p$.
- Total spending on n goods: $Npx = E$, so that $x = E/(pN)$.
- What utility does the consumer achieve?

$$D = \left[\sum_{i=1}^{N} x^{\alpha}\right]^{1/\alpha} = \frac{E}{p}\left[\sum_{i=1}^{N} N^{-\alpha}\right]^{1/\alpha} = \frac{E}{p} N^{(1-\alpha)/\alpha}$$

- For given income E and prices p, utility increases in N.
- Because x_is are imperfect substitutes of one another, it is better to have a little of each than much of one.
- Consumption of non-existent varieties is 0. The convexity of preferences dislikes zeros.
- This is the love-of-variety feature of preferences.

Love of variety

- Alternatively, we can express love of variety in the expenditure function.
- The minimum cost of obtaining one unit of utility,

$$P = \left[\sum_{i=1}^{N} p^{1-\varepsilon} \right]^{1/(1-\varepsilon)} = pN^{1/(1-\varepsilon)}$$

  is decreasing in N.
- We can think of the price of non-existent varieties as ∞.
- When the product becomes available, its price falls from ∞ to p.
- An increase in N then reduces the aggregate price index.
- Hence indirect utility,

$$u = \frac{E}{P} = \frac{E}{p} N^{(1-\alpha)/\alpha}$$

  is increasing in N.

Firms

- Each product is produced with labor only: $x_i = l_i$.
- The unitary labor restrictivequirement is not restrictive. Why?
- The demand for product i:

$$x_i = E \frac{p_i^{-\varepsilon}}{\sum_{j=1}^{N} p_j^{1-\varepsilon}}$$

Market structure

- Each firm has a monopoly over its blueprint.
- There are many firms so that each firm takes aggregate prices and quantities as given.
- Monopolistic competition.
- Demand for product i is isoelastic with elasticity ε:

$$x_i = E\frac{p_i^{-\varepsilon}}{\sum_{j=1}^{N} p_j^{1-\varepsilon}} = Ap_i^{-\varepsilon}$$

  - The firm takes A as given but maximizes over p_i.
- Markup pricing

$$p_i = \frac{\varepsilon}{\varepsilon - 1}w = \frac{w}{\alpha}$$

Choice of numeraire

- We normalize aggregate expenditure to $E \equiv 1$.
- This is a weird choice of numeraire but will prove convenient later.
  - We are free to fix any one price in each time period to whatever value we want.
  - The price need not be 1, need not even be constant.
  - So we pick $P = 1/D$ so that $E = PD = 1$.

Firm profits

- A fraction α of revenue goes to labor, $(1-\alpha)$ goes to profits.

$$\pi_i = (1-\alpha)p_i x_i$$

- In symmetric equilibrium, each firm sells $1/N$.

$$\pi_i = \frac{1-\alpha}{N}.$$

- Note that flow profits tend to zero as N increases without bound.
- Will R&D be sustained in the long run?

Determining wages

- The wagebill of the firm is
$$wl_i = \frac{\alpha}{N}.$$
- Aggregating across all N firms,
$$wL_m = \alpha,$$
where L_m is the amount of labor in production.
- So wages are
$$w = \frac{\alpha}{L_m}.$$

Checklist

So far we have determined

- how consumers value variety
- how firms price their products
- what is the total labor demand in production

Dynamic decisions

Supply of funds

Consumption smoothing

- Consumer has log utility

$$\mathcal{U} = \int_{t=0}^{\infty} \exp(-\rho t) \ln D(t) dt.$$

- Can save and borrow at interest rate r:

$$\dot{a} = ra + y - c.$$

- The corresponding Bellman:

$$\rho V(a, y) = \max_c \{\ln c + V_a(a, y)(ra + y - c) + V_y(a, y)\dot{y}\}$$

Deriving the Euler equation

- The FOC for optimal c:
$$\frac{1}{c} = V_a(a, y).$$
- Taking logs and differentiating wrt time
$$-\hat{c} = \frac{V_{aa}}{V_a}(ra + y - c)$$
($\hat{x}$ denotes $\dot{x}/x$)
- Differentiating through the Bellman to express V_a:
$$\rho V_a = V_{aa} \cdot [ra + y - c] + V_a \cdot r$$
- Substituting in V_{aa}:
$$\hat{c} = r - \rho$$

Nominal vs real interest rate?

- All this referred to an economy in which there is no inflation.
- If there is inflation,

$$\dot{a} = ra + y - pc.$$

- And the FOC becomes

$$\frac{1}{c} = pV_a(a, y, p).$$

Exercise

- Derive the Euler equation in this economy.
- What is the optimal rate of growth for consumption?
- What do you need to know about $p(t)$?

The Euler equation

- The Euler equation
$$\hat{D} = r - \hat{P} - \rho.$$
- $\hat{D} + \hat{P}$ is the growth rate of expenditure, $E = PD$.
- But expenditure is constant, $E = 1$.
- The interest rate equals the discount rate
$$r = \rho.$$
- This completely characterizes the supply side of financial markets.

Demand for funds

The decision to innovate

- Take a firm with n products.
  - n is a firm-level state variable.
- The firm takes aggregates, N and L_m as given.
- The firm can raise capital at rate $r = \rho$.
- We next analyze the R&D decision.

Research and development

- R&D is costly and random.
  - Successes arrive with a Poisson process.
  - The arrival rate depends on R&D expenditure.
- So that a new product arrives with rate λ, the firm has to hire $a\lambda$ workers.
  - Again, λ is an instantaneous arrival rate.
  - R&D expenditure $a\lambda$ is a flflow.
  - Past expenditure and past success do not matter.
- Let us write down the Bellman equation for the value of the firm.

The value of a firm

- The Bellman equation

$$\rho V(n, N, L_m) =$$

$$\max_n \left\{ \pi_i n - a\lambda w + \lambda[V(n+1, N, L_m) - V(n, N, L_m)] + V_N \dot{N} + V_{L_m} \dot{L}_m \right\} =$$

$$\max_n \left\{ \frac{1-\alpha}{N} n - a\lambda \frac{\alpha}{L_m} + \lambda[V(n+1, N, L_m) - V(n, N, L_m)] + V_N \dot{N} + V_{L_m} \dot{L}_m \right\}$$

- The state variables are n, N and L_m. Only n is affected by the firm.
- The opportunity ("financing") cost of the firm equals
  - flow profits:
    - operative profits
    - minus the cost of R&D
  - capital gains:
    - the invention of a new product
    - change in the value of existing products

Solution

- Guess that the value is linear in n, $V(n, N, L_m) = nv(N, L_m)$.

$$\rho nv(N, L_m) = max_\lambda \left\{ \frac{1-\alpha}{N} n - a\lambda \frac{\alpha}{L_m} + \lambda v(N, L_m) + n\dot{v} \right\}$$

- The FOC for λ is

$$\frac{\alpha a}{L_m} \geq v(N, L_m),$$

  with equality if $\lambda > 0$.
- Simplify to

$$\rho v(N, L_m) = \frac{1-\alpha}{N} + \dot{v}.$$

- This links N, L_m, and v.

Innovation and growth

- Note that the FOC did not pin down λ.
  - This because of the linearity of both the benefit and the cost of innovation.
  - Would change with convex costs of innovation.
- Suppose firm i innovates with rate λ_i, using $a\lambda_i$ R&D workers.
- This leads to a new product with arrival rate λ_i.

Aggregate innovation

- The arrival rate of the first new product across all firms is

$$\sum_i \lambda_i \equiv \Lambda.$$

- The overall number of R&D workers is

$$\sum_i a\lambda_i = a\Lambda.$$

- Even if λ_i is indeterminate, aggregate innovation Λ will be pinned down in equilibrium.
- Because new products arrive with λ, the growth rate of N is

$$\frac{EdN/dt}{N} = \frac{\Lambda}{N}.$$

Dynamic equilibrium

- We now characterize the dynamic equilibrium.
- The key is to pin down the allocation of labor to its two uses:
  1. production: L_m
  2. R&D: $a\Lambda$
- Resource constraint for labor

$$L_m + a\Lambda = L,$$

- or

$$\dot{N} = \frac{L - Lm}{a}.$$

- Note that whenever $\dot{N} > 0$, $L_m = \alpha a / v$, so that

$$\dot{N} = \frac{L}{a} - \frac{\alpha}{v}.$$

Long-run growth

- We first show that there is no long-run growth in this economy.
- This is because the incentive to innovate disappears as N grows large.
- Suppose

$$N > \bar{N} \equiv \frac{(1-\alpha)L}{\alpha a \rho}$$

  and there is no R&D.
- Then both N and L_m are constant, so is $v(N, L_m)$.
- From the Bellman equation,

$$v(N, L) = \frac{1-\alpha}{\rho N}.$$

- But because $N > \tilde{N}$, this is indeed smaller than the cost of innovation $\alpha a / L$.
- So no innovation is a unique equilibrium.

Steady state

The steady-state N (and hence steady-state productivity) is

- increasing in country size L
- increasing in profit share $(1-\alpha)$
- decreasing in R&D cost a
- decreasing in discount rate ρ

$$\frac{D}{L_m} = N^{(1-\alpha)/\alpha} \to \left(\frac{(1-\alpha)L}{\alpha a \rho}\right)^{(1-\alpha)/\alpha}$$

Phase diagram

Recipe

3. Constant interest rate: $r = \rho$
4. Symmetric profits: $\pi = (1 - \alpha)/N$
5. Wage equation: $w = \alpha/L_m$
6. Firm valuation: $\rho v = \pi + \dot{v}$
7. Optimal R&D: $v \leq aw$
8. Resource constraint: $L_m + a\dot{N} = L$

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Characterizing the dynamics

We collapse the six equations of the "recipe" into two.

Labor market clearing + wage equation + optimal R&D

$$\dot{N} = \max\left\{0, \frac{L}{a} - \frac{\alpha}{v}\right\}$$

Bellman equation

$$\dot{v} = \rho v - \frac{1-\alpha}{N}$$

Steady state

- The steady state is such that both $\dot{N}$ and $\dot{v}$ are zero.
- Steady-state N is
$$N_{ss} = \frac{(1-\alpha)L}{\alpha\rho a}.$$
- Steady-state productivity (output per worker) is
$$\frac{D}{L_m} = N^{(1-\alpha)/\alpha} \to \left(\frac{(1-\alpha)L}{\alpha a \rho}\right)^{(1-\alpha)/\alpha}.$$
    - (This only includes manufacturing, not R&D.)
- Both are
    - increasing in country size L
    - increasing in profit share $(1-\alpha)$
    - decreasing in R&D cost a
    - decreasing in discount rate ρ

Knowledge spillovers

Knowledge spillovers

- Now suppose that R&D has external benefits to other researchers.
- In particular, let the cost of R&D decrease with the number of existing products, N, a/N.
- This changes
  1. the incentive to innovate
  2. the resource requirements of innovation

The new Bellman

- The new Bellman equation:

$$\rho n v(N, L_m) = \max_{\lambda} \left\{ \frac{1-\alpha}{N} n - \frac{\alpha a}{N L_m} \lambda + \lambda v(N, L_m) + n \dot{v} \right\}$$

- First-order condition

$$\frac{\alpha a}{N L_m} = v$$

- The rate of return on innovation

$$r_{R\&D} = \frac{1-\alpha}{\alpha} \frac{L_m}{a} - \hat{N} - \hat{L}_m.$$

  - Profits per fixed cost are now independent of N.
  - There are capital losses because innovation becomes ever cheaper.

The new resource constraint

If aggregate innovation is Λ, it takes up $\Lambda a/N$ workers.

$$L_m + \Lambda \frac{a}{N} = L$$

Balanced growth

- Suppose that this economy attains a balanced growth path with constant growth rate $g = \hat{N}$.
- "Balanced" means that labor allocations (L_m) are constant.
- We then verify that this is an equilibrium.

Balanced growth

- Suppose that this economy attains a balanced growth path with constant growth rate $g = \hat{N}$.
- "Balanced" means that labor allocations (L_m) are constant.
- We then verify that this is an equilibrium.
- From the Bellman equation for firm value

$$\rho = \frac{1-\alpha}{\alpha}\frac{L_m}{a} - g.$$

- From the resource constraint,

$$L_m + ag = L.$$

Solution

$$L_m = \alpha(L + a\rho)$$
$$g = \frac{1-\alpha}{a}L - \alpha\rho$$

- Indeed, as long as $(1-\alpha)L/a > \alpha\rho$, balanced growth is an equilibrium.
- The growth rate is
  - increasing in the profit share $(1-\alpha)$
  - increasing in country size L (more on this later)
  - decreasing in the cost of R&D a
  - decreasing in the discount rate ρ

Other ways to generate growth

Other ways to generate growth

- Knowledge spillovers reduce the cost of innovation so that profit *per cost* do not vanish.
- We have other ways to generate growth:
  - In a different demand system / competition, profits may not vanish. (See quality competition later.)
  - If innovation costs are in the final good rather than in labor units, they "mechanically" get lower with development:

$$P = N^{(\alpha-1)/\alpha}.$$

  - (This is also an external benefit of R&D but it is *pecuniary*.)
  - If population grows, firms keep doing R&D. This may even lead to growth in output *per capita*. (See later.)

Policy and welfare

Policies

- We want to see if policy has an effect on growth.
- We consider two policies:
    1. an R&D subsidy
    2. a production subsidy

R&D subsidy

- The government pays a fraction ϕ of research expenses.
- This is financed by a lump-sum tax.
- This changes the incentive to innovate,

$$\frac{\alpha a(1-\phi)}{NL_m} = v,$$

- and the Bellman equation

$$\rho = \frac{(1-\alpha)L_m}{\alpha a(1-\phi)} - g.$$

Solution

$$L_m = \alpha[L + a(1-\phi)\rho] < L_m(\text{no subsidy})$$
$$g = \frac{1-\alpha}{a(1-\phi)}L - \alpha\rho > g(\text{no subsidy})$$

- An R&D subsisdy increases growth and decreases manufacturing employment/output.

Production subsidy

- Manufacturers receive an ad valorem subsidy of ϕ_x.
- Their aggregate revenue is hence $1 + \phi_x$,
- profit per variety is
$$(1 + \phi_x)(1 - \alpha)/N.$$
- This seems to raise the profitability of R&D.

Production subsidy

- But note that the wage equation changes as well,

$$w = (1 + \phi_x)\alpha / L_m.$$

- So the *returns* to R&D are unchanged:

$$\frac{(1 + \phi_x)(1 - \alpha)L_m}{(1 + \phi_x)a\alpha N}.$$

- Production subsidy raises prices and wages proportionally, so does not lead to any reallocation.

Welfare

- Is the equilibrium growth rate g *optimal*?
- We see that R&D subsidies can increase the growth rate – should they be employed?

Welfare

- Is the equilibrium growth rate g *optimal*?
- We see that R&D subsidies can increase the growth rate – should they be employed?
- We answer that by solving the benevolent social planners problem.
- The social planner maximizes discounted utility subject to technology contraints.
  - (Prices and markets do not matter.)
- We begin with the case without knowledge spillovers.

Static optimum

- First note that the static equilibrium is optimal – despite imperfect competition.
- Given L_m workers and N existing varieties, the social planner would like to allocate L_m/N workers to each – just as in equilibrium.
- Because markups are symmetric, they do not involve any distrotion – relative prices across firms are unchanged.

Dynamic optimum

- Aggregate output of the final good is
$$D = N^{(1-\alpha)/\alpha} L_m.$$
- Per-period utility is
$$\ln D = \frac{1-\alpha}{\alpha} \ln N + \ln L_m.$$
- The one state variable is N.
- Choice variable is L_m, or, equivalently, aggregate innovation Λ.

Bellman equation

- The Bellman equation of the policy maker

$$\rho V(N) = \max_{L_m} \left\{ \frac{1-\alpha}{\alpha} \ln N + \ln L_m \right.$$
$$\left. + \frac{L - L_m}{a} \left[V(N+1) - V(N) \right] \right\}$$

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$$\left. + \frac{L - L_m}{a} [V(N+1) - V(N)] \right\}$$

- We make the approximation

$$V(N+1) - V(N) \approx V'(N)$$

so that we can use the envelope theorem. (Works for large N.)

Solution

- The first-order condition for L_m is

$$\frac{1}{L_m} = \frac{V'(N)}{a}.$$

- Using the envelope theorem to determine V',

$$\rho V'(N) = \frac{1-\alpha}{\alpha N} + \dot{N} V''(N).$$

- Now introduce $v \equiv \alpha V'(N)$. The two equations can be rewritten as

$$L_m = \frac{a\alpha}{v}$$
$$\rho v = \frac{1-\alpha}{N} + \dot{v}$$

Solution

- Substituting in the resource constraint,

$$\dot{N} = \frac{L}{a} - \frac{\alpha}{v}.$$

- And the Bellman equation

$$\dot{v} = \rho v - \frac{1-\alpha}{N}.$$

Solution

- Substituting in the resource constraint,

$$\dot{N} = \frac{L}{a} - \frac{\alpha}{v}.$$

- And the Bellman equation

$$\dot{v} = \rho v - \frac{1-\alpha}{N}.$$

- But notice that these are the same as the equilibrium conditions.
- The equilibrium is hence efficient.

The case with knowledge spillovers

- The social planner's Bellman now

$$\rho V(N) = \max_{L_m} \left\{ \frac{1-\alpha}{\alpha} \ln N + \ln L_m + \frac{(L - L_m)N}{a} V'(N) \right\}$$

The case with knowledge spillovers

- The social planner's Bellman now

$$\rho V(N) = \max_{L_m} \left\{ \frac{1-\alpha}{\alpha} \ln N + \ln L_m \right.$$
$$\left. + \frac{(L - L_m)N}{a} V'(N) \right\}$$

- Guess that the value function is of the form

$$V(N) = b_0 + b_1 \ln N.$$

Solution

- Guess that the value function is of the form
$$V(N) = b_0 + b_1 \ln N.$$
- Then $V'(N) = b_1/N$ and
$$L_m = \frac{a}{V'(N)N} = \frac{a}{b_1}$$
is constant.
- So is the growth rate
$$\frac{\dot{N}}{N} = \frac{L - L_m}{a} = \frac{L}{a} - b_1.$$
- Verify that the Bellman equation holds for
$$b_1 = \frac{1-\alpha}{\alpha\rho}$$
and some (ugly) b_0.

Optimal growth

- Substituting in b_1, optimal growth is

$$g^* = \frac{L}{a} - \frac{\alpha\rho}{1-\alpha}.$$

- Notice that

$$g^* = \frac{g}{1-\alpha} > g.$$

- Equilibrium growth is *inefficiently low*.
- What is the intuition?

Discussion

- We have endogenized *technology*: companies invest in new technology just as they invested in physical capital in the Solow/Ramsey model.
- But it has proven difficult to endogenize *growth*: R&D can also be subject to decreasing returns to scale.
- We had to assume spillovers: the social returns to R&D are higher than the private returns.
- This model is not necessarily about *endogenous growth*, but certainly about *endogenous innovation* and technology.
- Innovation (and potentially growth) responds to taste and policy parameters and, notably, *country size*.
- Equilibrium growth is lower that optimal, there is room for policy.

Appendix

CES review

- Take the following CES utility function:

$$u(x_1, x_2) = \left[x_1^\alpha + x_2^\alpha\right]^{1/\alpha},$$

  and define $\varepsilon = 1/(1-\alpha)$, $\alpha = 1 - 1/\varepsilon$
- Maximize utility subject to prices p_1 and p_2:

$$p_1 x_1 + p_2 x_2 = E$$

- What is the relative demand for x_1 and x_2?

Utility maximization

- The marginal rate of substitution

$$\frac{u_1}{u_2} = \frac{x_1^{\alpha-1}}{x_2^{\alpha-1}} = \left(\frac{x_1}{x_2}\right)^{-1/\varepsilon}$$

- In the optimum, this equals the relative price, p_1/p_2:

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{-\varepsilon}$$

- The relative demand is loglinear in relative prices.
    - The elasticity of substitution is constant at ε.

Cost minimization

- In parallel, we can solve the cost minimization problem.
- Minimize $E = p_1 x_2 + p_2 x_2$ subject to $u(x_1, x_2) = u_0$.
  - FOC:

$$p_i = \lambda x_i^{\alpha-1}$$

$$E = u_0 \left[p_1^{1-\varepsilon} + p_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)}$$

- The term

$$P \equiv \left[p_1^{1-\varepsilon} + p_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)}$$

  is the *ideal price index*.

Markup pricing

- Take a demand function $D(p)$ and a cost function $C(Q)$.
- Maximize profit
$$pD(p) - C[D(p)]$$
- First-order condition
$$D(p) + pD'(p) - C'[D(p)]D'(p) = 0$$
- Divide by pD' and rearrange
$$\frac{p - C'[D(p)]}{p} = \frac{D(p)}{-pD'(p)} \equiv \frac{1}{\varepsilon}.$$
- Price–cost markup
$$\frac{p}{C'[D(p)]} = \frac{\varepsilon}{\varepsilon - 1}.$$