Advanced Macro
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Lecture 15: A firm-level model of innovation

Outline

- Today we study the Klette and Kortum (2004, JPE) model of firm-level innovation.
- The model incorporates
  - firm-level innovation
  - endogenous aggregate growth
  - realistic firm demographics (birth, growth, death)
  - endogenous firm size distribution

Outline

- We will stop short of full GE.
- We only study industry equilibrium:
  - There is interaction among firms, entry and exit.
  - But total demand, interest rate and wages are not endogenized.

Firm demographics vs size distribution

- In a Markovian world, these are intricatelly linked.
- Remember that
$$\dot{\pi}(t) = \pi(t)\Lambda$$
characterizes the dynamics of the *probabilities* of each state.
- When there are "many firms," they also correspond to the *fraction* of firms being in each state.

Stylized facts

Stylized facts

- The model is consistent with a range of facts about
  1. firm-level innovation
  2. firm dynamics
  3. firm size distribution

Stylized facts

rank size
1 S_1
2 $S_1/2$
3 $S_1/3$

Zipf's law

1. R&D intensity is independent of firm size.
2. The size distribution of firms is highly skewed.
3. Smaller (younger) firms are more likely to exit. Survivors grow faster.
4. Among larger firms, growth is unrelated to size (Gibrat's law).
5. Small (young) firms are more volatile.
6. The market share of a cohort declines as it ages.

Setup

Environment

- There is a large number of firms.
- Each firm has a number of products, each product is only produced by one firm.
- Overall, there is a large number of products.

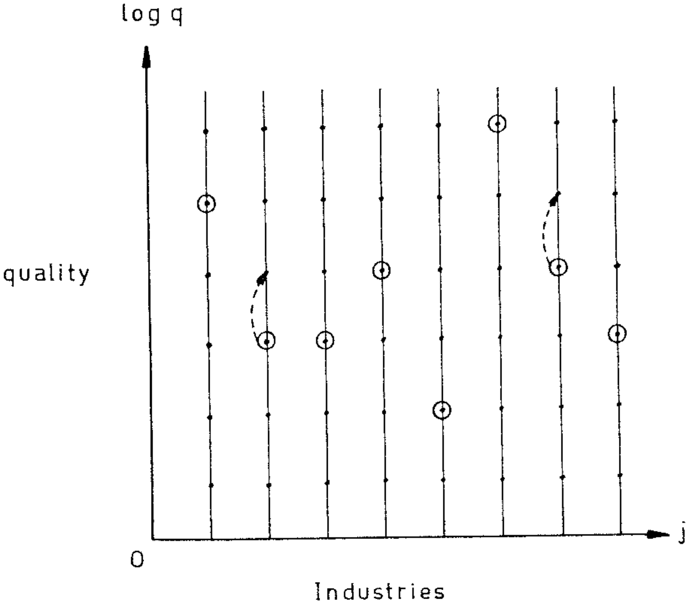
Environment

- There is a continuum of firms.
- Each firm has a countable number of products, each product is only produced by one firm.
- Overall, there is a continuum of products.

Continuum $\times$ countable =

Continuum

Products of different quality



Aside: Cardinality of the continuum

- The *cardinality* of a set is a measure of the "number of elements" in it.
  - $|A| = |B|$ if there is a bijection between the elements of A and B.
  - $|A| > |B|$ if $|a| = |B|$ for a proper subset $a \subset A$, but not $|A| = |B|$.
- *Continuum* is the cardinality of the real line.
  - $|\mathbb{R}| = |[0,1]| = |\mathbb{R}^2|$
  - $|\mathbb{R}| > |\mathbb{N}|$

$$\frac{\exp(+x)}{1+\exp(+x)}$$

- The cardinality of the continuum is *infinite*. What does this mean?

Notes

Why continuum?

- We need an infinite number of firms to invoke the LOLN – we want to rule out aggregate uncertainty.
- Any fraction of firms can be characterized by a real numbers – just like a probability. $k \sim \mathrm{Binom}(N, p)$
- For practical purposes, we can simply think of the continuum case as $N \to \infty$.

$$\frac{k}{N} \xrightarrow{N \to \infty} p$$

$$\Sigma \longrightarrow \int$$

Basic setup

- Firms invest in R&D to develop better versions of products (vertical product innovation).
- These products are "stolen" from other firms.
- Entrants also innovate.

Novelties

- Firms are modelled as a collection of product lines, $n = 0, 1, 2, 3, ...$
- Innovation technology is convex – we can pin down innovation for each firm.

Decisions of the firm

Firm profits

- Each product brings a total profit of π. This could be motivated by
  - fixed aggregate spending
  - fixed aggregate number of products
  - constant markup
- Products are symmetric.

Decisions of the firm

- Each firm is characterized by the number of products it has, $n = 0, 1, 2, 3, ...$
  - $n = 0$ is a dead firm
- Total flow profits are $n\pi$.
- Once we know per-product profits, the only decision is about changing n.
- The firm invests in R&D to increase n.

Firm dynamics

- After successful R&D, n increases by 1.
- If a product is stolen by a rival, n decreases by 1.
- Each product is stolen with arrival rate μ (to be endogenized later).

Inputs of innovation

- Innovation takes a flow R&D expenditure R
  - researchers
  - materials
  - Econometrica subscription
- and stock of firm-specific knowledge
  - previous patents
  - previous ideas
  - previous papers

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  - previous papers
- The knowledge stock is assumed proportional to n.

Output of innovation

- The output of innovation is a new product.
- As usual, the success of innovation is random.
- A new product arrives with Poisson arrival rate I.

The innovation production function

$$I = G(R, n)$$

- standard neoclassical production function
- increasing and concave in R

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  - Bigger firms are better at R&D – they do more.
- linear homogenous in R and n
  - Bigger firms do proportionally more R&D.

Intensive form

Because of CRS,

$$I = G(R, n) = nG(R/n, 1) \equiv ng(R/n),$$

or written as a cost function

$$R/n = g^{-1}(I/n) \equiv c(I/n).$$

Intensive form

Because of CRS,

$$I = G(R, n) = nG(R/n, 1) \equiv ng(R/n),$$

$g'(0) = \infty$

$g'(\infty) = 0$

or written as a cost function

$$R/n = g^{-1}(I/n) \equiv c(I/n).$$

c is

- increasing
- convex

$c'(0) = 0$

$c'(\infty) = \infty \leftarrow c'(\mu) = \infty$

The Bellman equation

We are now ready for the Bellman equation:

$$
\begin{aligned}
rV(n) = \max_I \{ n\pi - nc(I/n) \\
+ I[V(n+1) - V(n)] \\
+ \mu n[V(n-1) - V(n)] \}
\end{aligned}
$$

The Bellman equation

We are now ready for the Bellman equation:

$$\begin{aligned} rV(n) = \max_I \{ & n\pi - nc(I/n) \\ & + I[V(n+1) - V(n)] \\ & + \mu n[V(n-1) - V(n)]\} \end{aligned}$$

- The opportunity cost of time is traded off against
- operating profits minus R&D expenditures
- capital gains from developing a new product
- capital losses from losing any of the n products

Teamwork

$$c'\left(\frac{I}{n}\right) = v \qquad \frac{I}{n} = f(\underset{+}{v})$$

$$r v \cancel{n} = \pi \cancel{n} - \cancel{n}\, c\left(\frac{I}{n}\right) + \cancel{n}\,\frac{I}{n} v - \mu \cancel{n} v$$

1. Guess and verify that $V(n) = vn$.
2. Solve for v and optimal innovation I.
3. How does R&D expenditure depend on size?

Solution

- Guess that the value function is

$$V(n) = vn.$$

- In that case,

$$rvn = \max_I \{\pi n - nc(I/n) + Iv - \mu nv\}.$$

- The FOC:

$$c'(I/n) = v.$$

- Denoting $I/N \equiv \lambda$, and dividing by n,

$$rv = \pi - c(\lambda) + \lambda v - \mu v$$

Solution

- v and λ jointly solve
$$c'(\lambda) = v$$
and
$$v = \frac{\pi - c(\lambda)}{r + \mu - \lambda}.$$
- R&D intensity, $R/n = c(\lambda)$ is independent of firm size.

Solution

Firm dynamics

Firm dynamics

- Firm dynamics is characterized by a Markov chain over $n = 0, 1, 2, 3., ...$
- Exit $(n = 0)$ is an *absorbing state*.
- Forecast the size of a firm born with $n = 1$.

Firm dynamics

- Firm dynamics is characterized by a Markov chain over $n = 0, 1, 2, 3., ...$
- Exit $(n = 0)$ is an *absorbing state*.
- Forecast the size of a firm born with $n = 1$.
  - This is a birth-and-death process, but with different rates than in the problem set.

The Kolmogorov equation

$$\dot{\pi}_0(t) = \mu\pi_1(t)$$

$$\dot{\pi}_n(t) = \overbrace{\lambda(n-1)}^{I_{n-1}}\pi_{n-1}(t) - (\lambda+\mu)n\pi_n(t) + \mu(n+1)\pi_{n+1}(t)$$

Boundary condition:

$$\pi_1(0) = 1$$

Solution

- This system of differential equations has the solution

$$\begin{aligned}
\pi_0(t) &= \frac{\mu}{\lambda}\gamma(t) \\
\pi_1(t) &= [1 - \pi_0(t)][1 - \gamma(t)] \\
&\vdots \\
\pi_n(t) &= \pi_{n-1}(t)\gamma(t),
\end{aligned}$$

where

$$\gamma(t) = \frac{\lambda - \lambda e^{-(\mu-\lambda)t}}{\mu - \lambda e^{-(\mu-\lambda)t}}.$$

- Conditional on survival ($n > 0$), this is a geometric distribution with parameter $\gamma(t)$.

Firm demographics

Hazard of exit

$$\frac{\dot{\pi}_0(t)}{1-\pi_0(t)} = \mu - \lambda\pi_0(t)$$

decreasing in t – consistent with younger firms exiting more often.

Firm demographics

Unconditional expected growth

$$\frac{Edn(t)}{n(t)} \equiv \lim_{\Delta \to 0} \frac{En(t+\Delta) - n(t)}{n(t) \cdot \Delta} = \lambda - \mu$$

is independent of firm size and age.

Firm demographics

$$n(t+\Delta) = \begin{cases} n(t)+1 & \text{w/ Pr } \lambda \Delta n \\ n(t)-1 & \mu n \Delta \\ n(t) & \text{otherwise} \end{cases}$$

Unconditional expected growth

$$\frac{E dn(t)/dt}{n(t)} \equiv \lim_{\Delta \to 0} \frac{E n(t+\Delta) - n(t)}{n(t) \cdot \Delta} = \lambda - \mu$$

is independent of firm size and age.

- This includes the possibility of exit, $n(t+\Delta) = 0$.
- For large, mature firms, whose exit is unlikely, this is consistent with data.

$$E\left[n(t+\Delta) - n(t)\right] = +1 \times \lambda \Delta n - 1 \mu n \Delta$$

Firm demographics

$$Var(x) = E(x^2) - E(x)^2$$

$$E\left(\frac{\Delta n}{n}\right) = O((\lambda - \mu)\Delta)$$

Volatility

$$\frac{\mathrm{Var}\, dn(t)}{n(t)} \equiv \lim_{\Delta \to 0} \frac{E\left[n(t+\Delta) - n(t)\right]^2}{n(t) \cdot \Delta} = \frac{\lambda + \mu}{n}$$

is decreasing in firm size

$$E\left(\frac{dn^2}{dt}\right) = (\lambda + \mu) n$$

$$Var\left(\frac{dn}{n}\right)/dt = \frac{1}{n^2} Var(dn)/dt$$

Industry equilibrium

Industry equilibrium

- We want to endogenize μ, the rate at which products are stolen.
- Part of μ is driven by innovation by *all* the incumbents.
  - When Google comes up with Android phones, they will steal the market from Apple's iPhone.
- Each time an incumbent is successful in innovation, the new product is selected randomly from the product space $[0, 1]$.
  - undirected innovation

Industry equilibrium

- The hazard of your product being stolen by an incumbent is

$$\Lambda = \int_0^1 \lambda(j)dj = \int_0^1 \lambda dj = \lambda.$$

- There are also new entrants, who develop better versions of the existing products.
- Suppose this happens at rate η. Then

$$\mu = \lambda + \eta.$$

Entry

- There are infinite number of firms wishing to enter.
- If they spend $F\eta$ on R&D, they can improve on a randomly chose product with arrival rate η.
- Their Bellman equation:

$$\rho \cdot 0 = \max_{\eta} \{-F\eta + \eta[V(1) - 0]\}$$

Entry

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- If they spend $F\eta$ on R&D, they can improve on a randomly chose product with arrival rate η.
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$$\rho \cdot 0 = \max_{\eta} \{-F\eta + \eta[V(1) - 0]\}$$

  - Anybody can be an entrant, so they should derive no value from it.
  - Success brings about an increase in value (incumbents enjoy rents).

Solution

- The FOC:
$$F = v,$$
which implies $F = c'(\lambda)$.
- Because new entrants technology is linear, η is not determined by this FOC.

Solution

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which implies $F = c'(\lambda)$.
- Because new entrants technology is linear, η is not determined by this FOC.
- Recall the incumbents' optimum:
$$v = F = \frac{\pi - c(\lambda)}{r + \mu - \lambda} = \frac{\pi - c[c'^{-1}(F)]}{r + \eta}$$
- This pins down η.
  - If entry is slower, incumbents enjoy larger profits, v is higher than F.

Solution

- This completes the characterization of industry equilibrium.
- We can do simple comparative statics.
  - What happens if F increases?

The firm-size distribution

The firm-size distribution

- Denote the mass of firms with n products at time t by $M_n(t)$.
- How can we characterize $M_1(t), M_2(t), ...$?
- Is there a steady-state distribution, $M_1, M_2, ...$?

The firm-size distribution

- Denote the mass of firms with n products at time t by $M_n(t)$.
- How can we characterize $M_1(t), M_2(t), ...$?
- Is there a steady-state distribution, $M_1, M_2, ...$?
- These questions sound similar to Markov chain forecasting.
- Indeed we will be using the same tools.

Incumbents

- The evolution of incumbents is characterized by the Markov chain.
- Because there are a continuum of firms, and shocks are independent across firms, the *fraction* of firms moving from state n to k is identical to the *probability* of moving from n to k.
- We can use the Kolmogorov equation to characterize the *flows* between firms of different sizes.

$$\dot{M}_n(t) = -(\lambda+\mu)nM_n(t)+\lambda(n-1)M_{n-1}(t)+\mu(n+1)M_{n+1}(t)$$

Entrants

- The flow of entrants adds to the stock of size-1 firms:

$$\dot{M}_1(t) = -(\lambda + \mu)M_1(t) + 2\mu M_2(t) + \eta$$

Solving for the steady-state distribution

- In steady state, $\dot{M}_n = 0$.

$$0 = -(\lambda + \mu)M_1 + 2\mu M_2 + \eta 0 \quad = -(\lambda + \mu)nM_n + \lambda(n-1)M_{n-1}$$

- This can be solved to yield

$$M_n = \frac{\lambda^{n-1}\eta}{n\mu^n}.$$

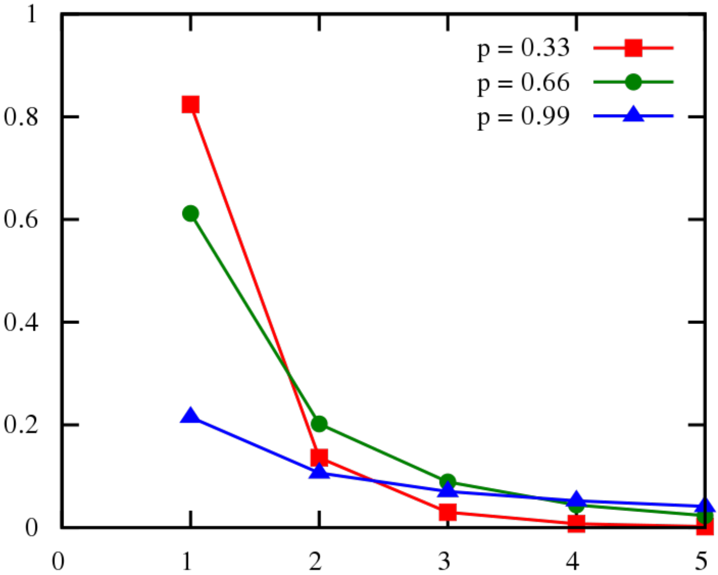
The firm-size distribution

- The firm-size distribution is

$$\frac{M_n}{\sum_{i=1}^{\infty} M_i} = \frac{(1+\theta)^{-n}}{n \ln(1+1/\theta)},$$

  where $\theta = \eta/\lambda$.
- This is the logarithmic distribution, which is very skewed.
  - In line with the stylized facts.

The logarithmic distribution



Appendix

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