Advanced Macro I
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Lecture 13: Dynamic programming in continuous time

Outline

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- Today we start using our tools for dynamic programming.
  - no uncertainty
  - Markov chains
- We review some simple applications.

From last time

Solutions to the teamwork assignments

- Both CEU email servers had the following stationary distribution:
$$\pi_n^* = 1/N$$
for $n = 0, 1, ..., N - 1$ and 0 thereafter.
- It was easiest to see for server 2, because the Kolmogorov equation gave $\pi_n^* = \pi_{n-1}^*$ and $\pi_0^* = \pi_{N-1}^*$.

Solutions to the teamwork assignments

- For server 1,
$$\lambda\pi^*_{N-1} = (\lambda+\eta)\pi^*_N$$
so the stationary distribution is
$$\pi^*_n = \begin{cases} \frac{\lambda+\eta}{N(\lambda+\eta)+\lambda} & \text{if } n < N \\ \frac{\lambda}{N(\lambda+\eta)+\lambda} & \text{if } n = N \end{cases}$$
- Take $\eta \to \infty$ to get the result.

Discrete time

Dynamic programming with no uncertainty

- Suppose you have a vector of state variables x_t, and control variables c_t.
- The equation of motion for the state is

$$\Delta x_{t+1} = F(x_t, c_t).$$

- Per-period utility is

$$u(x_t, c_t)$$

- The sequential problem is

$$\max_{\{c_t\}} \sum_{t=1}^{\infty} \beta^t u(x_t, c_t) \text{ s.t. } \Delta x_{t+1} = x_t + F(x_t, c_t)$$

The recursive formulation

- The corresponding Bellman equation is

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V(x_{t+1})\}$$

- or substituting in the equation of motion

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

Solution

- The solution is a value function $V(x)$ that maps the state into the PDV of utility.
- Equivalently, the solution can be given as a policy function $c(x)$.
- What would change if time periods were days instead of years?

Continuous time

Moving to continuous time

- Let time periods be Δ apart.
- As before, we want to characterize the time series as Δ becomes smaller and smaller.
- We take the limit as $\Delta \to 0$.
  - We will have to rescale flows, but not stocks.

Differential equations

- Now dynamics are characterized by the differential equation:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{F(x_t, c_t, \Delta t)}{\Delta t} \equiv f(x_t, c_t).$$

Dynamic programming

- Back to our discrete-time Bellman:

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

- Which of the objects here depend on the length of the time period?

Dynamic programming

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$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

- Which of the objects here depend on the length of the time period?
  - u, because utility is a *flow*: shorter time periods yield less utility
  - β, because shorter time periods are discounted less
  - F as we have seen above

Dynamic programming

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$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

- Which of the objects here depend on the length of the time period?
    - u, because utility is a *flow*: shorter time periods yield less utility
    - β, because shorter time periods are discounted less
    - F as we have seen above
- Let us make this period dependence explicit:

ϵ

$$V(x_t) = \max_{c_t} \left\{ u(x_t, c_t)\Delta + \frac{1}{1 + \rho\Delta} V[x_t + f(x_t, c_t)\Delta] \right\}$$

- Now u is the *per-period* utility, ρ is the *per-period* discount rate, f is the *per-period* growth rate.

Infinitesimal periods

As you might expect, we take Δ to 0.
First multiply by $(1 + \rho\Delta)$:

$$(1 + \rho\Delta)V(x_t) = \max_{c_t} \{u(x_t, c_t)\Delta(1 + \rho\Delta) + \mathbb{E}\, V[x_t + f(x_t, c_t)\Delta]\}$$

Then subtract $V(x_t)$:

$$\rho\Delta V(x_t) = \max_{c_t} \{u(x_t, c_t)\Delta(1 + \rho\Delta) + \mathbb{E}\, V[x_t + f(x_t, c_t)\Delta] - \mathbb{E}\, V(x_t)\}$$

Then divide by Δ:

$$\rho V(x_t) = \max_{c_t} \left\{ u(x_t, c_t)(1 + \rho\Delta) + \mathbb{E}\left(\frac{V[x_t + f(x_t, c_t)\Delta] - V(x_t)}{\Delta}\right) \right\}$$

Now we're ready to take the limit.

The Hamilton-Jacobi-Bellman equation

$$\rho V(x_t) = \max_{c_t} \left[u(x_t, c_t) + V'(x_t) f(x_t, c_t) \right]$$

$$= \frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial t}$$

- What is different:
  - we have ρV, not V on the LHS
  - we have $\dot{V}$, not new value on RHS

$$\langle \nabla V, \dot{x} \rangle$$

- Intuition:
  - the per-period discount loss from my value should be compensated by
  - flow utility
  - and (expected) gains in future value

Teamwork: eat-the-pie problem

- You have wealth W, accruing interest r per unit of time.
- You maximize
$$\int_{t=0}^{\infty} \exp(-\rho t) \ln c(t) dt.$$

1. Write down the Bellman equation.
2. Guess that $V(W) = a + b \ln W$ and solve for a and b.
3. What is the optimal consumption policy, $c(W)$?

Application to the Ramsey model

- Let

$$x_t = k_t,$$
$$u(x_t, c_t) = c_t^{1-\theta}/(1-\theta),$$
$$f(x_t, c_t) = f(k_t) - \delta k_t - c_t$$

- The Bellman equation is now

$$\rho V(k) = \max_c \left\{ \frac{c^{1-\theta}}{1-\theta} + V'(k)[f(k) - \delta k - c] \right\}$$

Deriving the Euler equation

- The FOC for optimal c:

$$c^{-\theta} = V'(k).$$

- Taking logs and differentiating wrt time

$$-\theta \hat{c} = \frac{V''(k)}{V'(k)}[f(k) - \delta k - c]$$

($\hat{x}$ denotes $\dot{x}/x$)

- Differentiating through the Bellman to express $V'(k)$:

$$\rho V'(k) = V''(k)[f(k) - \delta k - c] + V'(k)[f'(k) - \delta]$$

- Substituting in $V''(k)$:

$$\hat{c} = \frac{1}{\theta}[f'(k) - \delta - \rho] \equiv \frac{1}{\theta}[r(k) - \rho]$$

The consumption rule

$$\hat{c} = \frac{1}{\theta}[f'(k) - \delta - \rho] \equiv \frac{1}{\theta}[r(k) - \rho]$$

- Agents like to smooth consumption (especially with high θ).
- Consumption grows if $r > \rho$: the market return on my saving (late consumption) is higher than the private return from early consumption.
- Consumption growth is high if k is low (r is high).

Uncertainty

Moments of a Markov chain

- Take a function $v()$ that assigns each state a value, $v_1, ..., v_N$.
- What is the expected value of v in a short period of time if we are in state s_i now?
- If $\Delta \approx 0$, the probability of moving from the state is ≈ 0, so the expected value is $\approx v_i$.
- It is more meaningful to talk about the "rate" of change, as in the case of differential equations.
- What is the *expected rate of change* in v?

$$\frac{E(dv)}{dt} = \lim_{\Delta \to 0} \frac{E(v_{t+\Delta}|t) - v_t}{\Delta}$$

Expectations

- Again, start from discrete time:

$$E[v(t+\Delta)|t] = (1 - \sum_{j \neq i} \Delta\lambda_{ij})v_i + \sum_{j \neq i} \Delta\lambda_{ij} v_j$$

- Subtract $v_i = v(t)$ from both sides and divide by Δ:

$$\frac{E[v(t+\Delta)|t] - v(t)}{\Delta} = \sum_{j \neq i} \lambda_{ij}(v_j - v_i)$$

- Taking $\Delta \to 0$ $\quad E(dv) = \sum \Delta \lambda_{ij} (v_j - v_i)$

$$E\left(\frac{dv}{dt}\right) \neq \quad \frac{E(dv)}{dt} = \sum_{j \neq i} \lambda_{ij}(v_j - v_i)$$

- Intuitively, the expected change is a weighted average of all potential changes.
- A change of $v_j - v_i$ arrives with arrival rate λ_{ij}, so has a weight λ_{ij}.

Dynamic programming

- We can derive the continuous-time HJB equation with uncertainty as

$$\rho V(x_i) = \max_c \left[u(x_i, c) + \frac{EdV(x)}{dt} \right]$$

- The only change is that we have *expected* change in V on the RHS.
  1. This holds if x is a jump process.
  2. x is continuous
  3. x is a mix of the two

Dynamic programming

- Suppose our state x follows a Markov chain as above.
- Arrival rates λ_{ij} may depend on past states (i) and on the policy variable c.
- Because x is a jump process,

$$\frac{EdV(x)}{dt} = \sum_{j \neq i} \lambda_{ij}[V(x_j) - V(x_i)].$$

- The HJB equation simplifies to

$$\rho V(x_i) = \max_c \left\{ u(x_i, c) + \sum_{j \neq i} \lambda_j(x_i, c)[V(x_j) - V(x_i)] \right\}$$

Dynamic programming

- More generally, suppose x follows a Markov chain, and y is a continuous state variable.
- The HJB equation can be written as

$$\rho V(x_i, y) =$$
$$\max_c \left\{ u(x_i, y, c) + \sum_{j \neq i} \lambda_j(x_i, y, c)[V(x_j, y) - V(x_i, y)] \right.$$
$$\left. + f(x_i, y, c) V_y(x_i, y) \right\}$$

Dynamic programming

- More generally, suppose x follows a Markov chain, and y is a continuous state variable.
- The HJB equation can be written as

$$\rho V(x_i, y) = \max_c \left\{ u(x_i, y, c) + \sum_{j \neq i} \lambda_j(x_i, y, c)[V(x_j, y) - V(x_i, y)] \right\} + f(x_i, y, c) V_y(x_i, y)$$

$V(x, y_{t+\Delta}) - V(x, y_t) \approx V_y \Delta$

  - This only applies if y does not change when x jumps.
    - y is an aggregate variable, x is individual
  - Otherwise the jump would have to be accounted for.
  - The reverse does not matter. Why?

$E\, V(x', y_{t+\Delta}) - V(x, y_t) \approx V_y \Delta \lambda \Delta \stackrel{=}{=} O(\Delta^2) = o(\Delta)$

Checklist

By now you should understand

1. forward Kolmogorov equation
2. stationary distribution
3. Poisson process
4. Poisson distribution
5. moments of a Markov chain
6. Hamilton–Jacobi–Bellman equation with
    - continuous deterministic states
    - jump processes

Applications

Applications

- We consider three simple applications
  1. A model of exogenous job loss and job finding.
  2. A model of endogenous job search.
  3. A model of a milk farm.

Exogenous job search

Exogenous job search

- There are only two states:
  - E_1: worker has a job
  - E_2: worker is unemployed
- Transition across states (job loss, job finding) is exogenous (for now).
  - No optimization involved.
  - This is just a simple way of calculating the NPV of a job.

Exogenous job search

- The hazard rate of losing a job is δ.
- The arrival rate of a new job for an unemployed is λ.

Exogenous job search

- The per-period value of holding a job is w.
- The per-period value of being unemployed is b.
- What is the overall (present discounted) value of a job?

$$V_J = \int_{t=0}^{\infty} \exp(-\rho t) u(S_t).$$

  - S_t is random and varies over time.
  - We may have to evaluate a complex integral.
  - Dynamic programming makes our lives easier.

The Bellman equation

- The Bellman equation characterizing the value of a job

$$\rho V_J = w + \delta(V_U - V_J).$$

- The Bellman equation characterizing the value of unemployment

$$\rho V_U = b + \lambda(V_J - V_U).$$

- Note that you can think of the value function as $V(s)$ with s taking only two values.
- This looks much simpler! A system of two linear equations with two unknowns.

Solution

- The solution is

$$
\begin{aligned}
\rho \, V_J &= \frac{1}{\cancel{\rho}}\left[\frac{\rho+\lambda}{\rho+\lambda+\delta}w + \frac{\delta}{\rho+\lambda+\delta}b\right] \\
\rho \, V_U &= \frac{1}{\cancel{\rho}}\left[\frac{\rho+\delta}{\rho+\lambda+\delta}b + \frac{\lambda}{\rho+\lambda+\delta}w\right]
\end{aligned}
$$

- The value of a job is the weighted average of the PDV of wages and the PDV of benefits.
- The weigths depend on all parameters.

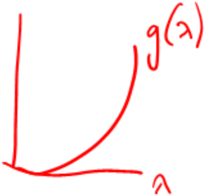
Comparative statics

- The value of a job is increasing in
  - wages
  - benefits
  - job finding rate
- Decreasing in
  - discount rate
  - firing rate
- Converges to the PDV of wages w/ρ as
  - δ goes to zero
  - λ goes to infinity

Endogenous job search

Endogenous job search

- We now endogenize job search. Everything else remains the same.
- To make sure that jobs arrive at rate λ, the unemployed has to pay a search cost $g(\lambda)$.
  - g is increasing, convex, twice differentiable, Inada
- Note that $g(\lambda)$ is a flow: the search effort at a given moment in time.
- λ is also a flow: the probability of success at a given moment in time.
- This still a memoryless process:
  - past search efforts have no effect
  - (just as time has no effect)



The Bellman equation

- The Bellman is now

$$\rho V_J = w + \delta(V_U - V_J)$$
$$\rho V_U = \max_{\lambda} \left[b - g(\lambda) + \lambda(V_J - V_U)\right]$$

- Note the maximization in the unemployed state.

First-order condition

- The FOC for λ is
$$g'(\lambda^*) = V_J - V_U$$
- The "exogenous" Bellman still correctly calculates V_J and V_U once we substitute in the new benefits $b - g(\lambda^*)$ and the job finding rate λ^*.
- Find a λ^* that satisfies both the FOC and the Bellman.

Solution

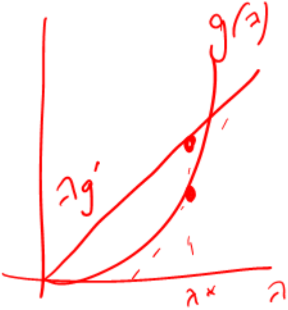
- Subtracting the two Bellman equations:

$$V_J - V_U = \frac{w - b + g(\lambda^*)}{\rho + \delta + \lambda^*}$$

- Substitute this into FOC to get an implicit solution for λ^*:

$$g'(\lambda^*) = \frac{w - b + g(\lambda^*)}{\rho + \delta + \lambda^*}$$

$$(\rho + \delta + \lambda^*)\, g'(\lambda^*) = w - b + g(\lambda^*)$$



Comparative statics

- Totally differentiating the implicit function...
- search intensity λ^* is
  - increasing in $w - b$
  - decreasing in $\rho + \delta$
  - decreasing with an upward shift of search costs

A milk farm

- Our last application considers a milk farm with n cows.
- The only state variable is n.
- The control variable is the feed c we give to each cow.

Flow profits

- Each cow gives $m(c)$ milk per period of time.
  - m is increasing and concave.
- Flow profits are
$$\pi = [pm(c) - wc]n,$$
  where
  - p is the price of milk
  - w is the price of feed

Dynamics

- New cows are born at rate λ to each existing cow (not modeled).
- Cows die at rate $\delta(c)$.
  - δ is decreasing and convex
- What is the dynamics of n?
- Over a Δ period of time, n becomes
  - $n+1$ with probability $\lambda n \Delta$
  - $n-1$ with probability $\delta(c) n \Delta$
  - n with probability $1-[\lambda+\delta(c)]\Delta$

Valuing the farm

- What is the value of the farm?
- The Bellman equation

$$
\begin{aligned}
\rho V(n) = \max_c \{ [pm(c) - wc]n + \\
\lambda n[V(n+1) - V(n)] + \\
\delta(c) n [V(n-1) - V(n)] \}
\end{aligned}
$$

Solution

- Guess that $V(n) = vn$.

$$\rho v n = \max_c \{[pm(c) - wc]n + \lambda n[v] + \delta(c)n[-v]\}$$

or

$$\rho v = \max_c \{pm(c) - wc + [\lambda - \delta(c)]v\}$$

- FOC for c

$$pm'(c) - w - v\delta'(c) = 0$$

- Find c and v so that both are satisfied.

Appendix

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$$\dot{W} = r W - c \quad + y$$

$$W_{t+\Delta} = W_t + r\Delta W_t - C_t\Delta + y_t\Delta$$

$$\frac{W_{t+\Delta} - W_t}{\Delta} = r W_t - C_t + y_t$$

$$\downarrow$$

$$\dot{W}_t$$

Empty slides for notes

$$\rho V(W_t) = \max_c \left\{ \ln(c_t) + V'(W_t)(rW_t - c_t) \right\}$$

$$\frac{1}{c_t} = V'(W_t) = \frac{b}{W_t} \qquad c_t = \frac{1}{b} W_t$$

$$\rho a + \rho b \ln W = -\ln b + \ln W_t \, 1 + \frac{b}{W_t}\left(rW_t - \frac{1}{b} W_t\right)$$

$$\frac{1}{b} = \rho$$

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$$a = \frac{1}{\rho}\left(\frac{v}{\rho} + \ln\rho - 1\right)$$

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