ECBS6001: Advanced Macroeconomics

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Lecture 15: Dynamic programming in continuous time

Outline

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- Today we start using our tools for dynamic programming.
  - no uncertainty
  - Markov chains
- We review some simple applications.

From last time

Solutions to examples

- Both CEU email servers had the following stationary distribution:

$$\pi_n^* = 1/N$$

  for $n = 0, 1, ..., N-1$ and 0 thereafter.
- It was easiest to see for server 2, because the Kolmogorov equation gave $\pi_n^* = \pi_{n-1}^*$ and $\pi_0^* = \pi_{N-1}^*$.

Solutions to the teamwork assignments

- For server 1,

$$\lambda\pi^*_{N-1} = (\lambda+\eta)\pi^*_N$$

so the stationary distribution is

$$\pi^*_n = \begin{cases} \frac{\lambda+\eta}{N(\lambda+\eta)+\lambda} & \text{if } n < N \\ \frac{\lambda}{N(\lambda+\eta)+\lambda} & \text{if } n = N \end{cases}$$

- Take $\eta \to \infty$ to get the result.

Discrete time

Dynamic programming with no uncertainty

- Suppose you have a vector of state variables x_t, and control variables c_t.
- The equation of motion for the state is

$$\Delta x_{t+1} = F(x_t, c_t).$$

- Per-period utility is

$$u(x_t, c_t)$$

- The sequential problem is

$$\max_{\{c_t\}} \sum_{t=1}^{\infty} \beta^t u(x_t, c_t) \text{ s.t. } \Delta x_{t+1} = x_t + F(x_t, c_t)$$

The recursive formulation

- The corresponding Bellman equation is

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V(x_{t+1})\}$$

- or substituting in the equation of motion

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

Solution

- The solution is a value function $V(x)$ that maps the state into the PDV of utility.
- Equivalently, the solution can be given as a policy function $c(x)$.
- What would change if time periods were days instead of years?

Continuous time

Moving to continuous time

- Let time periods be Δ apart.
- As before, we want to characterize the time series as Δ becomes smaller and smaller.
- We take the limit as $\Delta \to 0$.
  - We will have to rescale flows, but not stocks.

Differential equations

- Now dynamics are characterized by the differential equation:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{F(x_t, c_t, \Delta t)}{\Delta t} \equiv f(x_t, c_t).$$

Dynamic programming

- Back to our discrete-time Bellman:

$$V(x_t) = \max_{c_t} \{u(x_t, c_t) + \beta V[x_t + F(x_t, c_t)]\}$$

- Which of the objects here depend on the length of the time period?

Dynamic programming

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  - u, because utility is a *flow*: shorter time periods yield less utility
  - β, because shorter time periods are discounted less
  - F as we have seen above

Dynamic programming

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- Which of the objects here depend on the length of the time period?
  - u, because utility is a *flow*: shorter time periods yield less utility
  - β, because shorter time periods are discounted less
  - F as we have seen above
- Let us make this period dependence explicit:

$$V(x_t) = \max_{c_t} \left\{ u(x_t, c_t)\Delta + \frac{1}{1+\rho\Delta} V[x_t + f(x_t, c_t)\Delta] \right\}$$

- Now u is the *per-period* utility, ρ is the *per-period* discount rate, f is the *per-period* growth rate.

Infinitesimal periods

As you might expect, we take Δ to 0.
First multiply by $(1+\rho\Delta)$:

$$(1+\rho\Delta)V(x_t) = \max_{c_t}\{u(x_t,c_t)\Delta(1+\rho\Delta) + V[x_t + f(x_t,c_t)\Delta]\}$$

Then subtract $V(x_t)$:

$$\rho\Delta V(x_t) = \max_{c_t}\{u(x_t,c_t)\Delta(1+\rho\Delta) + V[x_t + f(x_t,c_t)\Delta] - V(x_t)\}$$

Then divide by Δ:

$$\rho V(x_t) = \max_{c_t}\left\{u(x_t,c_t)(1+\rho\Delta) + \frac{V[x_t + f(x_t,c_t)\Delta] - V(x_t)}{\Delta}\right\}$$

Now we're ready to take the limit.

The Hamilton-Jacobi-Bellman equation

$$\rho V(x_t) = \max_{c_t} \left[u(x_t, c_t) + V'(x_t) f(x_t, c_t) \right]$$

- What is different:
  - we have ρV, not V on the LHS
  - we have $\dot{V}$, not new value on RHS
- Intuition:
  - the per-period discount loss from my value should be compensated by
  - flow utility
  - and (expected) gains in future value

Example: eat-the-pie problem

- You have wealth W, accruing interest r per unit of time.
- You maximize

$$\int_{t=0}^{\infty} \exp(-\rho t) \ln c(t) dt.$$

1. Write down the Bellman equation.
2. Guess that $V(W) = a + b \ln W$ and solve for a and b.
3. What is the optimal consumption policy, $c(W)$?

Application to the Ramsey model

- Let

$$x_t = k_t,$$
$$u(x_t, c_t) = c_t^{1-\theta}/(1-\theta),$$
$$f(x_t, c_t) = f(k_t) - \delta k_t - c_t$$

- The Bellman equation is now

$$\rho V(k) = \max_c \left\{ \frac{c^{1-\theta}}{1-\theta} + V'(k)[f(k) - \delta k - c] \right\}$$

Deriving the Euler equation

- The FOC for optimal c:
$$c^{-\theta} = V'(k).$$
- Taking logs and differentiating wrt time
$$-\theta\hat{c} = \frac{V''(k)}{V'(k)}[f(k) - \delta k - c]$$
($\hat{x}$ denotes $\dot{x}/x$)
- Differentiating through the Bellman to express $V'(k)$:
$$\rho V'(k) = V''(k)[f(k) - \delta k - c] + V'(k)[f'(k) - \delta]$$
- Substituting in $V''(k)$:
$$\hat{c} = \frac{1}{\theta}[f'(k) - \delta - \rho] \equiv \frac{1}{\theta}[r(k) - \rho]$$

The consumption rule

$$\hat{c} = \frac{1}{\theta}[f'(k) - \delta - \rho] \equiv \frac{1}{\theta}[r(k) - \rho]$$

- Agents like to smooth consumption (especially with high θ).
- Consumption grows if $r > \rho$: the market return on my saving (late consumption) is higher than the private return from early consumption.
- Consumption growth is high if k is low (r is high).