ECBS6001: Advanced Macroeconomics

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Lecture 14: Continuous-time Markov chains

Outline

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- Continuous-time Markov chains.
  - transition rate matrix
  - steady-state distribution
  - Poisson process

Discrete time review

Markov chains

Example

- Take the following 2×2 transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$$

- The steady-state distribution is

$$\pi^* = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

Continuous time

Moving to continuous time

- Let us now take Δ to 0.
- The set of possible states is fixed, $S_1, ..., S_N$.
- What changes with Δ is the transition matrix: $P(\Delta)$.

Moving to continuous time

The transition "rate" matrix

- More generally, we know that $P(\Delta) - I$ is $O(\Delta)$.
- This means that
$$\lim_{\Delta \to 0} \frac{P(\Delta) - I}{\Delta} \equiv \Lambda$$
exists.
    - Because $P(\Delta)\mathbf{1} = \mathbf{1}$ (P is stochastic), $\Lambda\mathbf{1} = \mathbf{0}$.
    - The diagonal elements are negative,
    - the off-diagonals are positive.
- We call the matrix Λ the *transition rate matrix*.
    - In more formal math, Λ is called the *generator matrix* or the Q-matrix.
    - Together with an initial distribution, π_0, this fully characterizes the continuous-time Markov chain.

Examples

Example 1: Employment and unemployment

- The hazard rate of losing a job is δ.
  - The lifetime of a job is exponential with mean $1/\delta$.
  - Job loss is memoryless: you are just as likely to get fired on your 2nd day as on your 366th.
- The arrival rate of a new job for an unemployed is λ.
  - The spell of unemployment is exponential with mean $1/\lambda$.
  - Unemployment is memoryless: you are just as likely to find a job after 1 day of unemployment as after 365.

Example 1: continued

- State 0: employment.
- State 1: unemployment.

$$\Lambda = \begin{bmatrix} -\delta & \delta \\ \lambda & -\lambda \end{bmatrix}$$

Example 2: incoming emails

- Let $n(t)$ be the number of emails in your inbox at time t.
- We want to characterize the dynamics of n.
- Suppose emails arrive at random (think of spam).
  - You never erase email:
    $$\lambda_{i,j} = 0 \text{ if } j < i$$
  - No two emails arrive at the same time:
    $$\lambda_{i,i+s} = 0 \text{ for all } s \geq 2$$
  - Each new email arrives with a constant arrival rate:
    $$\lambda_{i,i+1} = \lambda$$
  - By construction, $\Lambda_i = \lambda$.
- This is called a Poisson process.

Example 2 (continued)

The transition rate matrix for the Poisson process:

$$\begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Forecasting with Markov chains

More generally

- More generally, the Kolmogorov equation is

$$\dot{\pi}(t) = \pi(t)\Lambda.$$

- Given an initial $\pi(0)$ and a transition rate matrix Λ, we can calculate the probability of each state in any future t.
  - Often there is no analytical solution for this ODE.
  - However, in dynamic programming it is sufficient to only look at the *immediate future*.
  - The transition rates will be sufficient to do recursive optimization.

The stationary distribution

- A stationary distribution π^* satisfies $\dot{\pi} = 0$, so
$$\pi^* \Lambda = 0.$$

Examples

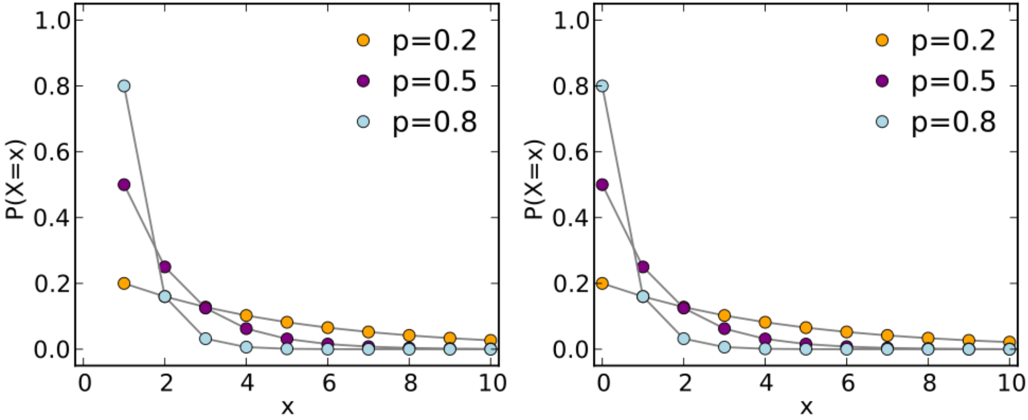
Exercises

Exercises

Take the 3 different email servers and

1. Write down the transition rate matrix.
2. Write down the Kolmogorov forward equation.
3. Solve for the steady-state distribution.

A geometric distribution



The Poisson process

The Poisson process

- The possible states are $n = 0, 1, 2, ...$
- The Poisson process is characterized by an *arrival rate* λ (aka hazard rate).
- The transition rate matrix is

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- The Poisson process is used to characterize *rare, memoryless* events.

Examples

- Phone calls to emergency center.
- Particles emitted via radioactive decay.
- Views of the CEU website.

Characterizing the Poisson process

The two key characteristics of the Poisson process

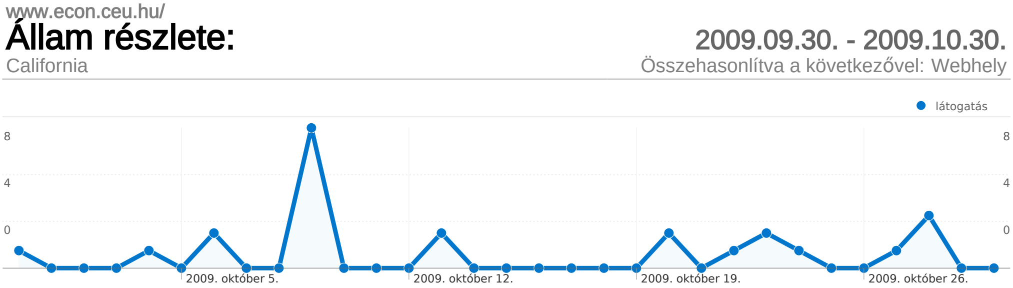
1. No two events happen at the same time ("rare events").
2. The future arrival of events is independent of past events ("memoryless").

Characterizing the Poisson process

The Poisson process may arise

- from a truly memoryless process
  - radioactive decay
- from the law of small numbers
  - view of the CEU website from California

Visits to econ.ceu.hu from California



Export shipments of shirts from the U.S.

Export shipments of shirts from the U.S. to Iceland

Counterexamples

- Emergency phone calls during a natural disaster.
- Arrival of guests at a restaurant.
- Your phone calls to your mother.

Properties of the Poisson process

- The waiting time between the $n-1$sth and nth arrival is T_n.
- T_n is random, exponentially distributed with parameter λ:

$$\Pr(T_n \leq t) = 1 - \exp(-\lambda t).$$

- Waiting times are independent.

Properties of the Poisson process

- Let $N = n(t+h) - n(t)$ denote the number of arrivals between t and $t+h$.
- N is a Poisson-distributed random variable with parameter λh.
- It takes on values $0, 1, 2, ...$ with pdf

$$\Pr(n = k) = \frac{\exp(-\lambda h)(\lambda h)^k}{k!}$$

Properties of Poisson processes (continued)

- Take two independent Poisson processes with arrival λ_1 and λ_2.
  - The sum is also a Poisson process with arrival $\lambda_1 + \lambda_2$.
  - The waiting time for the first arrival is exponential with parameter $\lambda_1 + \lambda_2$.

Properties of Poisson processes (continued)

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- Take a Poisson processes with arrival λ and a probability p.
- Kill each arrival with probability $1 - p$.
  - The new process is Poisson with arrival $p\lambda$.

Notation

- Let $J(t)$ denote the number of arrivals of a standard Poisson process between time 0 and time t.
- For any t, $J(t) \sim$ Poisson(t).
- Notation for a jump process of arrival λ and jumps of size x:

$$xJ(\lambda t)$$

- Graph.

Poisson representation of Markov chains

- Think of a Markov chain with N states.
- Starting in any given state, only $N-1$ things can happen (or nothing).
- Each $N-1$ jump has its own arrival rate.
- The first jump occurs with a Poisson arrival $\lambda_1 + ... + \lambda_{n-1}$ (see above).

Poisson representation of Markov chains (continued)

- Once there *is* a jump, which one is it?
- It could be any one of the $1, ..., n-1$.
- The probability of jump 1 is

$$\frac{\lambda_1}{\lambda_1 + ... + \lambda_{n-1}}.$$

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- This is a good old probability $\in [0, 1]$.
- This looks more like a discrete transition matrix.
- I find it useful to think about Markov chains as the sum of Poisson processes.

Checklist

By now you should understand

1. continuous-time Markov chain
2. arrival rate matrix
3. forward Kolmogorov equation
4. stationary distribution
5. Poisson process
6. Poisson distribution