Chapter 1 Prerequisites of Modern Macroeconomics

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In macroeconomics, we study the whole or aggregate performance of the economy. We consider the total output of products and services measured by the GDP, the aggregate of employment and unemployment, aggregate price changes the inflation, or breakdown of GDP such as consumption, investment, government purchases, exports, and imports as the main subjects of macroeconomics. We are interested in the phenomenon of economic growth and business cycle observed in the overall economy.

Macroeconomics can be traced to Keynes, a great macroeconomist in the history. He started macroeconomics as a separate field in economics after the Great Depression in the 1930s. He advocated the government should act strongly to increase government spending to save the economy during recessions. Keynesian economics is influential in present macroeconomics and government policy conducting.

Modern macroeconomics features dynamic, stochastic, general equilibrium models (short for DSGE models) after the rational expectation and neoclassical revolution lead by Robert Lucas, Thomas Sargent, Edward Prescott and so on. We will first present macroeconomic stylized facts that modern macroeconomics explores and explains, introduce some basic tools, and cover these basic elements through simple one-period and two-period general equilibrium models to familiarize you the essence of modern macroeconomics. These models shed light on the basic micro-foundation of later vastly-complicated infinite-horizon general equilibrium models. Whenever you feel astray in fully-fledged models of macroeconomics, you can always refresh your memory from the simplified models to get some intuition.

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We will cover the following materials in this chapter:

- some measurements of macroeconomics, Hodrick-Prescott filter;
- one-period model of consumption-leisure tradeoff, Lagrangian multiplier, general equilibrium, Walras law, welfare theorems;
- two-period model of consumption-saving tradeoff, intertemporal substitution, Ricardian equivalence;
- two-period model without government, risk-sharing
- two-period stochastic model, Markov chain

1 Some Measurements of Macroeconomy

Let's first check the time series of the US real GDP of 2009 dollar over the period. First look tells us there are two features outstanding:

- GDP increases over time;
- GDP fluctuates around the upward growth trend

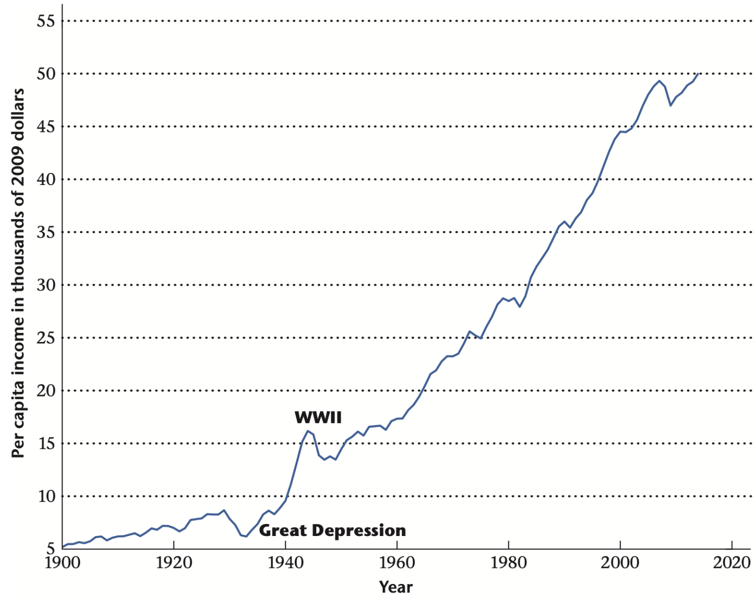
Actually, these two are important subjects of macroeconomics: economic growth and business cycle. All the subjects of macroeconomics can be classified into these two categories. It is easier to use the natural logarithm of GDP in the graph since the slope of the plot is equal to the growth rate of GDP.

$$slope = \frac{\ln y_t - \ln y_{t-1}}{t-(t-1)} = \ln \frac{y_t}{y_{t-1}} = \ln(1 + \frac{y_t - y_{t-1}}{y_t}) = \ln(1+g_t) \approx g_t$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{1+x} = 1 \Rightarrow \ln(1+x) \approx x$$

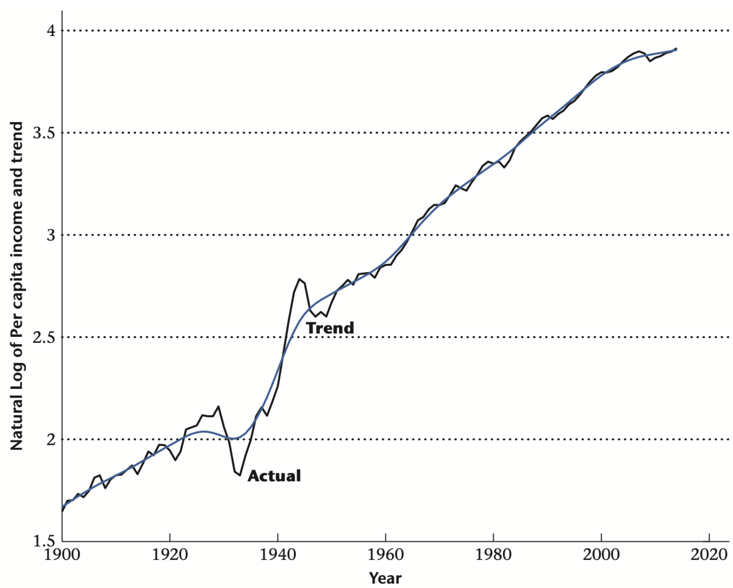
where the first equality comes from the L'Hospital's Rule. So when the growth rate g_t is small, it is equal to the difference between the natural logarithm of real GDP, slope of the logarithm graph.

Figure 1: US Real GDP, 2009 Dollar



The logarithm of real GDP is shown in Figure 2. Now the slope of the logarithm GDP is just the growth rate of GDP.

Figure 2: US Real GDP, 2009 Dollar



As GDP fluctuates around some trend, how can we get the trend x_t^s and cycle x_t^c from the original series x_t?

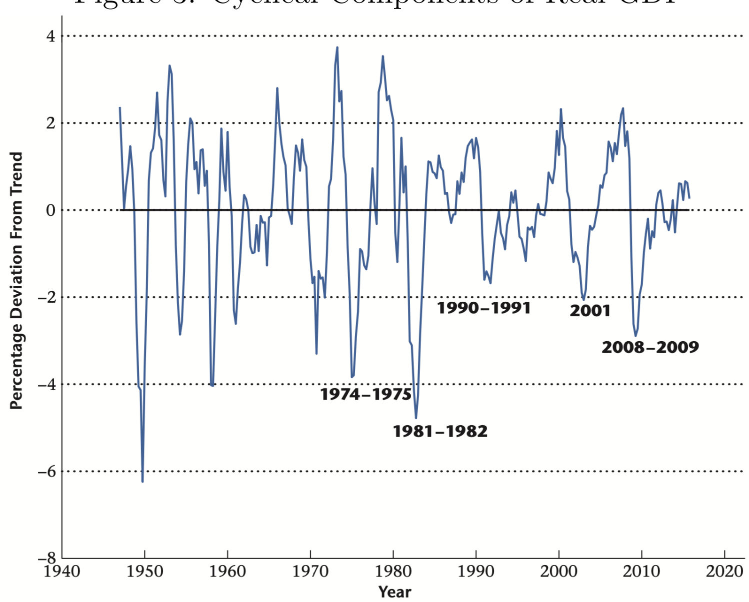
$$x_t = x_t^s + x_t^c$$

The basic idea is to use some criterion to get a "smooth" component, and then attribute the rest to the "cycle" component. In macroeconomics, we usually apply the Hodrick-Prescott filter to divide the macroeconomic time series to trend and cycle. Let λ be an exogenous constant chosen by the econometrician. The HP filter chooses a sequence of trend, x_t^s, to solve the minimization problem:

$$\min_{x_t^s} \quad \sum_{t=1}^{T}(x_t - x_t^s)^2 + \lambda \sum_{t=2}^{T-1} \left[(x_{t+1}^s - x_t^s) - (x_t^s - x_{t-1}^s)\right]^2$$

The basic idea behind this minimization problem is to minimize the squared deviation from the smoothed component, subject to a penalty for the smoothed component moving too much. If $\lambda = 0$, the solution would be $x_t^s = x_t$. As λ gets larger, you will not allow x_t^s move too much due to the penalty term. The higher λ is, the trend is smoother and the cycle component moves more fiercely. When λ goes to infinity, the smoothed series must be a line. λ is usually set to 1600 for quarterly data.

Figure 3: Cyclical Components of Real GDP



We can apply the HP filter to obtain the trend and cycle opponent, as shown in Figure 2. The cycle component is also plotted in Figure 3. If we compare the cyclical components of consumption and investment with GDP, we can see that consumption and investment both move steadily with GDP. We call this **procyclical**, measured by the correlation parameter of consumption and investment with GDP $corr(C_t^c, Y_t^c)$, $corr(I_t^c, Y_t^c)$. Note that consumption is less volatile than GDP, in that the cyclical component of consumption tend to be smaller than those in GDP. On the other hand, investment is more volatile than GDP. Actually, we can measure this by standard deviation of the cyclical component of the time series.

There are other empirical issues in macroeconomics which should be of interest to you. Read Chapter 1-3 of Williamson (2018) to get more details. These should be easy for you if you have already taken undergraduate macro course. For those who is not exposed to any macro course, you should read it carefully about the measurements. As this course is theory and method based, you should complete the empirics part by yourself. I will elaborate details about the empirics during the course if needed.

From the graph of US real GDP, we find two features of economic growth and business cycle. All macroeconomics subjects can be divided in to these two categories. Macroeconomists build various models to explain the behaviours of economic growth and business cycle. Here in this chapter, we use several simple model to explain the essence of modern macroeconomics.

2 One-Period Model: Consumption-Leisure Tradeoff

In modern macroeconomics, we model the real macro economy as a laboratory consisting of different and simplified markets. One simplest economy can be consisted of only two markets: goods market and labor market. Households[1] consume and buy goods in the goods market produced and sold by firms. Households provide and sell labor in the labor market utilized and bought by firms. In a competitive equilibrium, infinite many households and firms are price-takers, who have no power and have no effect on goods price and wage rate. The macro economy is in a competitive equilibrium when market prices are such that goods

[1] I will interchangeably use households and consumers in the notes.

Figure 4: Cyclical Components of Real Consumption

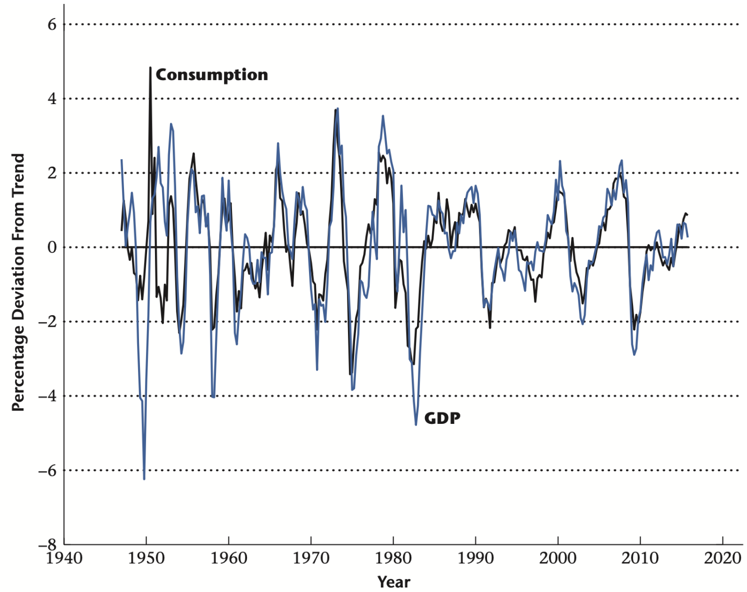
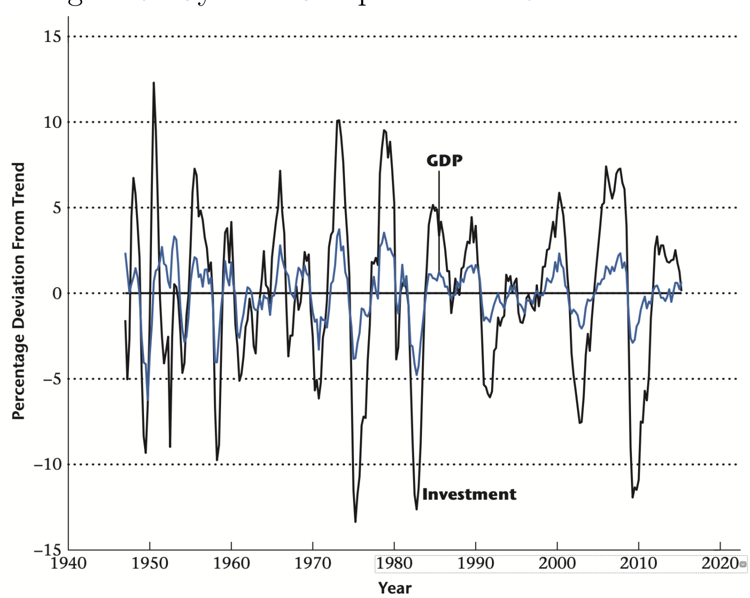


Figure 5: Cyclical Components of Real Investment



demand equals goods supply, and simultaneously labor demand equals labor supply.

This is one important feature of modern macroeconomics. General equilibrium that all markets are cleared simultaneously is embodied in most of macroeconomic models. Whenever we assess policy effects, we would not focus on some particular markets but the effects of all markets at the same time. For example, we assess the policy effect of consumption coupon during the covid-19 pandemic. Is it true that we only assess the effects of this policy on consumption in the goods market? What about the labor market's effects on goods market? It is not possible to get the big picture if we don't model the general equilibrium.

In a typical macroeconomic model, we need to specify following items:

1. The set of goods that consumers wish to consume;
2. Preference of consumers over alternative goods;
3. Technology of firms to produce goods;
4. Resources or endowments available to and various constraints faced by consumers and firms ;
5. Mechanism that consumers and firms interact in the economy

You may notice that this is quite similar to your microeconomics class in which consumers' and firms' problems, general equilibrium are the main contents. You are right. This is why we call it modern macroeconomics which is based on micro-foundation. This is a revolution from Keynesian macroeconomics which is lack of micro-foundation and was criticized by Robert Lucas in the 1970s, known as "Lucas Critique". You should answer the question what "Lucas Critique" is after you learn the advanced macroeconomic course.

Next, we explore the consumers' consumption-leisure optimization decision and firms' profit maximization decision in a toy model in Williamson (2018). We use the representative convention that there is only one consumer, one firm, and one government in the economy, or **you may consider the consumer and the firm as representative of infinite many consumers and firms who behave the same in the economy**. In default the consumer is simultaneously the owner of the firm. And this firm also hires the consumer who is also the owner of this firm. This seems a little strange to macro beginners. However, this structure enormously simplifies our analysis.

2.1 The representative consumer

The consumer is endowed with h hours and he can decide how many hours to work N^s and how many hours to rest as leisure l where $N^s + l = h$. The consumer consumes goods C and leisure l under the budget constraint $C \leq w(h-l) + \pi - T$. This budget constraint means that he can consume up to the point of his disposable income $w(h-l)+\pi-T$ where $w(h-l)$ is his wage income which is a product of wage rate w and working hours $h-l$, π is the profit of the firm which is redistributed to the owner of the firm, namely, the consumer, T is the lump-sum tax levied by the government. The consumption goods are numerair of this economy and the price of the goods is 1. The wage rate w is priced in terms of consumption goods. Then we have the budget constraint:

$$C \leq w(h-l) + \pi - T \tag{1}$$

The budget constraint means the consumer cannot consume more than his disposable income $w(h-l) + \pi - T$. If we move the item wl to the left hand side, we have the usually-seen budget constraint.

$$C + wl \leq wh + \pi - T \tag{2}$$

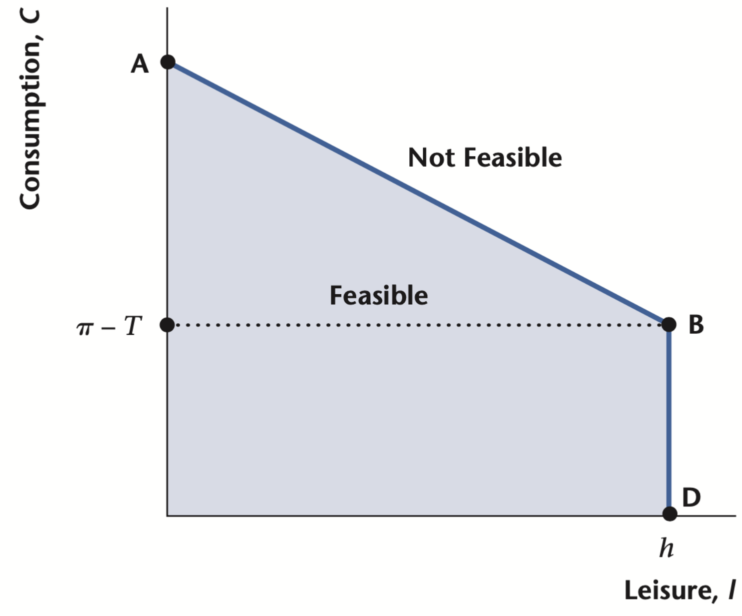
This budget constraint looks familiar to you if you have learned microeconomics. You can consider leisure l as another consumption goods with price w in terms of the price of C under the income $wh + \pi - T$. All the consumption bundles (C, l) under the budget constraint comprises of the feasible set in which the consumer can afford with his income, shown in Equation (3) and Figure 6.

$$\{(C, l) : C + wl \leq wh + \pi - T, C \geq 0, 0 \leq l \leq h\} \tag{3}$$

With this budget constraint, the consumer has following preference over (C, l) represented by the utility function:

$$U(C, l) = \ln C + \theta \ln l, \quad \theta > 0 \tag{4}$$

Figure 6: Budget Constraint



The utility function is 3-dimensional with U, C, and l. It's not easy to check the graph of 3-dimension function on the 2-dimension paper. Usually we show the utility function in 2-dimensional graph with C and l. The third dimension U is represented by the equivalence classes. This equivalence class is called indifference curve.

$$I_1 = [(C_1, l_1)]_{\sim} = \{(C, l) : U(C, l) = U(C_1, l_1)\} \tag{5}$$

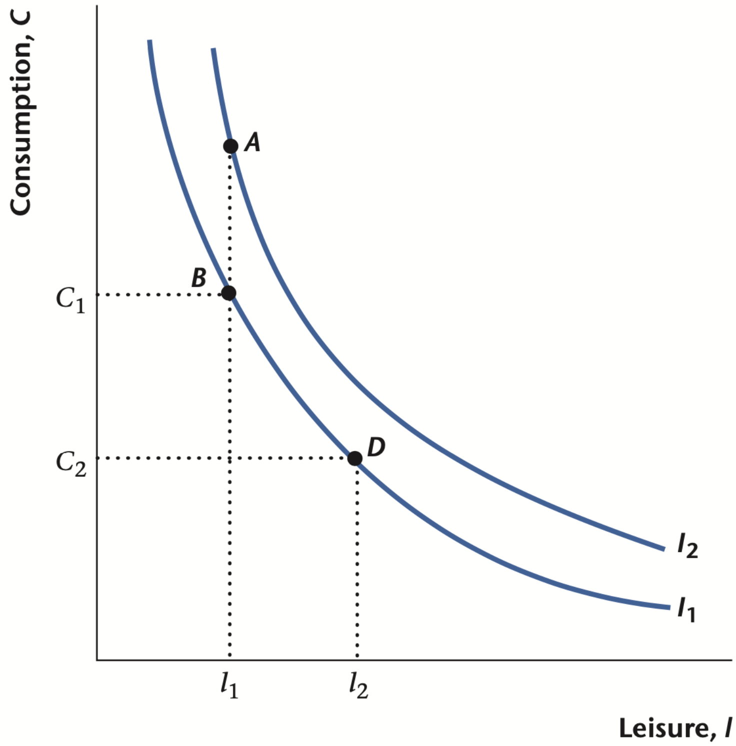
You can prove that the indifference curves do not cross.(Hint: just prove that $[(C_1, l_1)]_{\sim} \cap [(C_2, l_2)]_{\sim} = \emptyset$ if $U(C_1, l_1) \neq U(C_2, l_2)$). If the consumer is better off with higher C and l, the indifference curve with higher utility moves to the northeast direction.

There are several assumptions about consumer's preference:

1. More is preferred to less.
2. The consumer prefers diversity.
3. Both consumption goods and leisure are normal goods.

And you can see that the preference represented by the utility function (4) satisfies theses

Figure 7: Indifference Curve



three assumptions.

1. The utility function is increasing in its two arguments, which means that with more consumption, the consumer is better off and so is he with more leisure. This aligns with assumption 1 that more is preferred to less. From the utility function, we have:

$$\frac{\partial U}{\partial C} = U_C = \frac{1}{C} > 0$$
$$\frac{\partial U}{\partial l} = U_l = \frac{\theta}{l} > 0$$

The two positive partial derivatives imply two things: (1) first, the consumer is better off if we move the indifference curve to the northeast direction; (2) second, the indifference curve is downward sloping in the (C, l) space.

The first implication is trivial from the definition of indifference curve. Let's see the second implication. We know from the indifference curve I_1 of Equation (5) that:

$$U(C_1, l_1) = U(C, l), \forall (C, l) \in I_1$$

If we solve the total derivative to this function, we can see that the derivative is negative for all the points on the indifference curve.

$$\begin{aligned} 0 &= U_C dC + U_l dl \\ \frac{dC}{dl} &= -\frac{U_l}{U_C} < 0 \end{aligned} \tag{6}$$

We can also understand this result from economic intuition. Let's start with the consumption point (C_1, l_l). If the consumer increases his leisure, he is better off with more leisure and the consumption point will lie on a higher indifference curve. The consumption should be reduced if the consumer keeps his utility constant. That's why the indifference curve is downward sloping in the two-dimension (C, l) space. We call minus derivative of consumption goods with respect to leisure as the **marginal rate of substitution** of leisure for consumption goods when keeping the utility constant.

$$MRS_{l,C} = -\frac{dC}{dl} = \frac{U_l}{U_C} = \frac{\theta C}{l} \tag{7}$$

2. "The consumer prefers diversity" means diminishing marginal utility. The consumer always wants to balance consumption of different goods, which is why marginal utility becomes quite small when you have large quantity of some consumption goods, thus having more consumption goods accrue little to the consumer's utility. At this moment, consuming other goods of small quantity is of higher value to the consumer. This diminishing marginal utility is represented by the property of concavity of the utility function. We can see from the utility function (4) that the Hessian matrix is:

$$H = \begin{bmatrix} U_{CC} & U_{Cl} \\ U_{lC} & U_{ll} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C^2} & 0 \\ 0 & -\frac{\theta}{l^2} \end{bmatrix}$$

Since H is negative definite, the utility function is strictly concave in (C, l).

How about the indifference curve if the utility function is strictly concave? We can see that concavity of utility function means the difference curve is convex in the (C, l) space. We already know the first derivative of C with respect to l in Equation (6). We just need to differentiate it again.

$$\frac{d^2C}{dl^2} = \frac{U_l U_{Cl} - U_C U_{ll}}{U_C^2} > 0 \tag{8}$$

The indifference curve in the (C, l) space is downward sloping and convex. We can also understand this result from economic intuition. For example point B and D in Figure 7, leisure is higher at point D than point B and consumption is lower in point D than point B. Due to diminishing marginal utility, the marginal utility of leisure at point D is less than point B and the marginal utility of consumption at point D is higher than point B. When the utility is constant, to get one more unit of consumption goods, the leisure sacrificed in point D is higher than point B. This is because, as we increase the quantity of leisure and reduce the quantity of consumption, the consumer needs to sacrifice more and more leisure time to get another unit of consumption. The consumer needs to sacrifice more when you already have large quantity of leisure because of a preference for diversity. This is also called **diminishing marginal rate of substitution** of leisure for consumption goods.

3. "Both consumption goods and leisure are normal goods" means that when income increases both consumption goods and leisure will increase.

Let's write down the consumer's problem to maximize his utility under the budget constraint as microeconomics.

$$\max_{C,l} \quad U(C,l) = \ln C + \theta \ln l$$

$$s.t. \quad C + wl \leq wh + \pi - T$$

Notice that the constraint is an inequality constraint which can be solved by using Kuhn-Tucker conditions. Before that, we should be familiar with Lagrangian multiplier method which concentrates on equality constraints. We can notice that the maximizer of the consumer's problem should satisfy $C^* + wl^* = wh + \pi - T$.

Proposition 1. *If (C^*, l^*) is the maximizer of the consumer's problem, then $C^* + wl^* = wh + \pi - T$.*

Proof. We prove this proposition by contradiction. Suppose $C^* + wl^* = wh + \pi - T$ is not true, then $C^* + wl^* < wh + \pi - T$. There exists $\epsilon > 0$ such that

$$\begin{aligned} C^* + \epsilon + w(l^* + \epsilon) &< wh + \pi - T \\ \epsilon &< \frac{wh + \pi - T - C^* - wl^*}{1 + w} > 0 \end{aligned}$$

And $U(C^* + \epsilon, l^* + \epsilon) > U(C^*, l^*)$ because more is preferred to less. This contradicts to the condition that (C^*, l^*) is the maximizer of the consumer's problem. Then $C^* + wl^* = wh + \pi - T$ is true. Proof ends. $\square$

The inequality constraint becomes equality constraint for the optimizer (C^*, l^*). This is called that the inequality is binding, otherwise it is slack. Then the consumer's optimizing problem is:

$$\begin{aligned} &\max_{C,l} \quad U(C, l) = \ln C + \theta \ln l \\ &s.t. \quad g(C, l) = C + wl = wh + \pi - T = m \end{aligned}$$

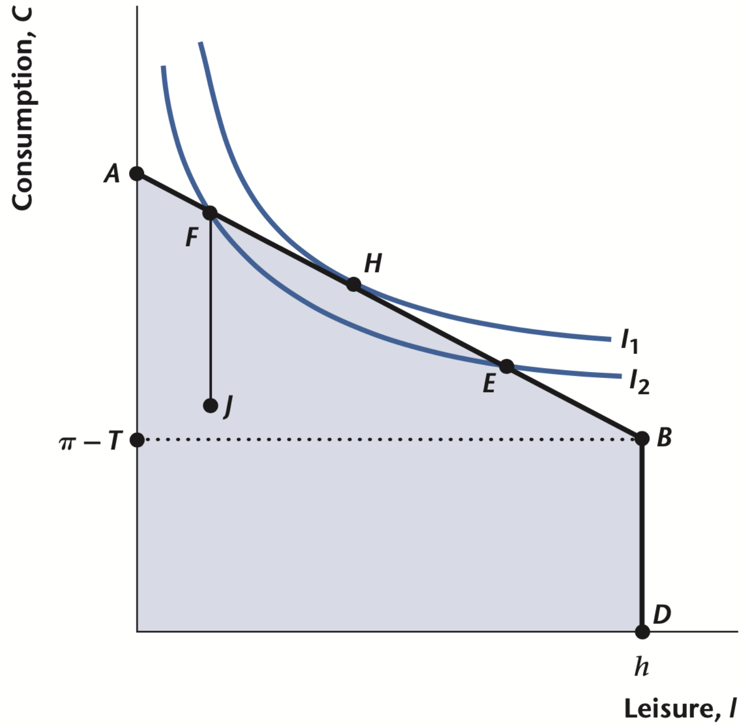
where we use $g(C, l) = m$ to stand for the constraint $C + wl = wh + \pi - T$.

If we put the indifference curve and budget constraint line in the same (C, l) space, we can see that the maximizer should be the point where the indifference curve is tangent to the budget constraint line shown in Figure 8. Since the indifference curve and budget constraint line are tangent at the point (C^*, l^*), the first derivative of the indifference curve $\frac{dC}{dl}^{IC}(C^*, l^*)$ and the budget constraint line $\frac{dC}{dl}^{BC}(C^*, l^*)$ at this point should be the same.

$$\frac{dC}{dl}^{IC}(C^*, l^*) = \frac{dC}{dl}^{BC}(C^*, l^*) \tag{9}$$

How can we get the two derivatives? You should refresh your memory about **implicit**

Figure 8: Consumer's Problem



function theorem.

$$
\begin{aligned}
\bar{U} = U(C, l) \Rightarrow 0 = U_C dC + U_l dl \Rightarrow \frac{dU}{dl} = -\frac{U_l}{U_C} \\
\bar{m} = g(C, l) \Rightarrow 0 = g_C dC + g_l dl \Rightarrow \frac{dg}{dl} = -\frac{g_l}{g_C}
\end{aligned}
$$

Substitute these two derivatives to the tangent point.

$$
-\frac{U_l(C^*, l^*)}{U_C(C^*, l^*)} = -\frac{g_l(C^*, l^*)}{g_C(C^*, l^*)} \tag{10}
$$

We can make a small transformation.

$$
\lambda^* = \frac{U_l(C^*, l^*)}{g_l(C^*, l^*)} = \frac{U_C(C^*, l^*)}{g_C(C^*, l^*)} \tag{11}
$$

And,

$$U_C(C^*, l^*) - \lambda^* g_C(C^*, l^*) = 0$$
$$U_l(C^*, l^*) - \lambda^* g_l(C^*, l^*) = 0$$

These two conditions along with the budget constraint corresponds to the ones that the first order partial derivatives of the following Lagrangian function at the point (C^*, l^*, λ^*) equal to zero.

$$L(C, l, \lambda) = U(C, l) - \lambda(g(C, l) - m) \tag{12}$$

with the partial derivatives zero.

$$\frac{\partial L}{\partial C}(C^*, l^*, \lambda^*) = U_C(C^*, l^*) - \lambda^* g_C(C^*, l^*) = 0 \tag{13}$$
$$\frac{\partial L}{\partial l}(C^*, l^*, \lambda^*) = U_l(C^*, l^*) - \lambda^* g_l(C^*, l^*) = 0 \tag{14}$$
$$\frac{\partial L}{\partial \lambda}(C^*, l^*, \lambda^*) = -(g(C^*, l^*) - m) = 0 \tag{15}$$

If (C^*, l^*) is the maximizer of the consumer's problem, the first order conditions of the Lagrangian should be satisfied. You can see that they are necessary conditions for the inner maximizer. In most of macroeconomic problems, these first order conditions are also sufficient under some conditions such as $U(C, l)$ is strictly concave and $g(U, l) = m$ is convex.

The utility value attained at the maximizer is a function of parameters of the consumer's problem θ and m. This is called **value function** $v(\theta, m)$, or **indirect utility function** in consumer's problem.

$$v(\theta, m) = U(C^*(\theta, m), l^*(\theta, m))$$

We are interested how the maximum value of the utility function changes when the deep parameters change. This is solved by the **envelope theorem**.

$$\frac{dv}{dx} = \frac{\partial L}{\partial x}(C^*(\theta, m), l^*(\theta, m), \lambda^*(\theta, m), \theta, m), \quad x = \theta, m \tag{16}$$

In particular, we are interested in how the maximum value of the utility function changes when we slack the budget constraint. According to the envelop theorem:

$$\frac{dv}{dm} = \frac{\partial L}{\partial m}(C^*(\theta, m), l^*(\theta, m), \lambda^*(\theta, m), \theta, m) = \lambda^*(\theta, m) \tag{17}$$

Proof.

$$\begin{aligned}
\frac{dv}{dm} &= U_C(C^*(\theta, m), l^*(\theta, m))C^*_m + U_l(C^*(\theta, m), l^*(\theta, m))l^*_m \\
&= \lambda^* g_C(C^*, l^*)C^*_m + \lambda^* g_l(C^*, l^*)l^*_m \\
&= \lambda^* \left[g_C(C^*, l^*)C^*_m + g_l(C^*, l^*)l^*_m\right] \\
&= \lambda^*(\theta, m)
\end{aligned}$$

where the first equality comes from the total derivative of the utility function $U(C^*, l^*)$ with respect to m evaluated at the optimum (C^*, l^*); the second comes from the Lagrangian first order conditions; the last equality comes from the total derivative of the budget constraint with respect to m.

$$g_C(C^*, l^*)C^*_m + g_l(C^*, l^*)l^*_m - 1 = 0$$

We also know that

$$\frac{\partial L}{\partial m}(C^*(\theta, m), l^*(\theta, m), \lambda^*(\theta, m), \theta, m) = \lambda^*(\theta, m)$$

Then

$$\frac{dv}{dm} = \frac{\partial L}{\partial m}(C^*(\theta, m), l^*(\theta, m), \lambda^*(\theta, m), \theta, m) = \lambda^*(\theta, m)$$

□

This conclusion is very important in economics especially in macroeconomics. The Lagrangian multiplier is equal to the marginal value of the value function with respect to disposable income. When the consumer has one more unit of disposable income, his maximum utility will increase $\lambda^* = U_C(C^*, l^*)$ units.

We may also understand the mathematical intuition of the envelope theorem from the Lagrangian function. The point (C^*, l^*, λ^*) is simultaneously the maximum of the Lagrangian function if you consider the three conditions (13, 14, 15) the neccesary conditions for optimizing the Lagrangian function without constraint.

$$\max_{C,l,\lambda} \quad L(C,l,\lambda) = L(C^*,l^*,\lambda^*) = U(C^*,l^*) - \lambda^*(g(C^*,l^*) - m) \tag{18}$$

If we change the parameters a bit, e.g. dm, for the Lagrangian function, there are two effects for the Lagrangian function: direct effect $\lambda^* dm$ by change of m; indirect effect $L_C(C^*,l^*,\lambda^*)C_m dm + L_l(C^*,l^*,\lambda^*)l_m dm + L_\lambda(C^*,l^*,\lambda^*)\lambda_m$ though change of C^*,l^*,λ^* by change of m. As the Lagrangian function is maximized at the point (C^*,l^*,λ^*), the first order conditions $L_C(C^*,l^*,\lambda^*) = 0, L_l(C^*,l^*,\lambda^*) = 0, L_\lambda(C^*,l^*,\lambda^*) = 0$. Then the indirect effect is zero. So if we change the parameters a bit, the total effect is equal to the direct effect caused by the parameter m because the maximum value of the Lagrangian function $L(C^*,l^*,\lambda^*)$ caused by change of C^*,l^*,λ^* should be zero. Then:

$$\begin{aligned}
\frac{dL}{dm}(C^*,l^*,\lambda^*) &= \frac{\partial L}{\partial C}(C^*,l^*,\lambda^*)\frac{dC^*}{dm} + \frac{\partial L}{\partial l}(C^*,l^*,\lambda^*)\frac{dl^*}{dm} + \frac{\partial L}{\partial \lambda}(C^*,l^*,\lambda^*)\frac{d\lambda^*}{dm} \\
&\quad + \frac{\partial L}{\partial m}(L^*,l^*,\lambda^*) \\
&= \frac{\partial L}{\partial m}(L^*,l^*,\lambda^*)
\end{aligned}$$

We also notice that at the optimum point (C^*,l^*,λ^*) of the Lagrangian function. We have $g(C^*,l^*) = m$. So,

$$L(C^*,l^*,\lambda^*) = U(C^*,l^*) = v(m,\theta) \tag{19}$$

Then

$$\frac{dL}{dm}(C^*,l^*,\lambda^*) = \frac{dU(C^*,l^*)}{dm} = \frac{dv(m,\theta)}{dm} \tag{20}$$

Then we have the envelope theorem result

$$\frac{dv(m,\theta)}{dm} = \frac{\partial L}{\partial m}(L^*,l^*,\lambda^*) = \lambda^* = U_C(C^*,l^*) > 0 \tag{21}$$

After we get some reference to Lagrangian multiplier method, we can go back to the consumer's problem.

$$\max_{C,l} \quad U(C,l) = \ln C + \theta \ln l$$
$$s.t. \quad g(C,l) = C + wl = wh + \pi - T = m$$

We can formulate the Lagrangen function as:

$$L(C,l,\lambda) = U(C,l) - \lambda(g(C,l) - m)$$

We can get the following FOCs (short for first order condition):

$$\frac{\partial L}{\partial C}(C^*, l^*, \lambda^*) = U_C(C^*, l^*) - \lambda^* g_C(C^*, l^*) = 0$$
$$\frac{\partial L}{\partial l}(C^*, l^*, \lambda^*) = U_l(C^*, l^*) - \lambda^* g_l(C^*, l^*) = 0$$
$$\frac{\partial L}{\partial \lambda}(C^*, l^*, \lambda^*) = -(g(C^*, l^*) - m) = 0$$

Simplify further and we have:

$$\frac{\partial L}{\partial C}(C^*, l^*, \lambda^*) = \frac{1}{C^*} - \lambda^* = 0$$
$$\frac{\partial L}{\partial l}(C^*, l^*, \lambda^*) = \frac{\theta}{l^*} - \lambda^* w = 0$$
$$\frac{\partial L}{\partial \lambda}(C^*, l^*, \lambda^*) = -(C^* + wl^* - m) = 0$$

We can combine the first two equations to:

$$MRS_{l,C}(C^*, l^*) = \frac{\theta C^*}{l^*} = w \tag{22}$$

The marginal rate of substitute of leisure for consumption goods at the optimal point should be equal to the given wage w.

We can solve the consumer's problem and get the equilibrium demand of consumption goods and leisure (C^*, l^*).

$$C^* = \frac{m}{1+\theta} = \frac{wh + \pi - T}{1+\theta} \tag{23}$$
$$l^* = \frac{\theta}{1+\theta}\frac{m}{w} = \frac{\theta}{1+\theta}\frac{wh + \pi - T}{w} \tag{24}$$

In economic models, we are interested in how models explain economic phenomenon. The variables explained by the model are called **endogenous variables**, and the variables that are given are called **exogenous variables**. As we see in the consumer's problem, the deep parameters θ, m are exogenous variables[2]; consumption goods C and $leisure$ are endogenous variables. We are interested in how the endogenous variables C, l behave when the exogenously-given variables θ, m change. In economics, we call this analysis **comparative statics**. Sometimes we call comparative statics policy experiments as if you do a policy experiment in the economy that resembles your model.

- The response of (C^*, l^*) with respect to a rise in disposable income m (income effect).

$$\frac{dC^*}{dm} = \frac{1}{1+\theta} > 0 \tag{25}$$
$$\frac{\partial l^*}{\partial m} = \frac{1}{1+\theta}\frac{1}{w} > 0 \tag{26}$$

  Holding wage constant, the consumer finds it optimal to consume more and enjoy more leisure (work less) if there is an increase in income (perhaps come from higher dividend income, or lower tax).

- The response of (C^*, l^*) with respect to a rise in wage w.

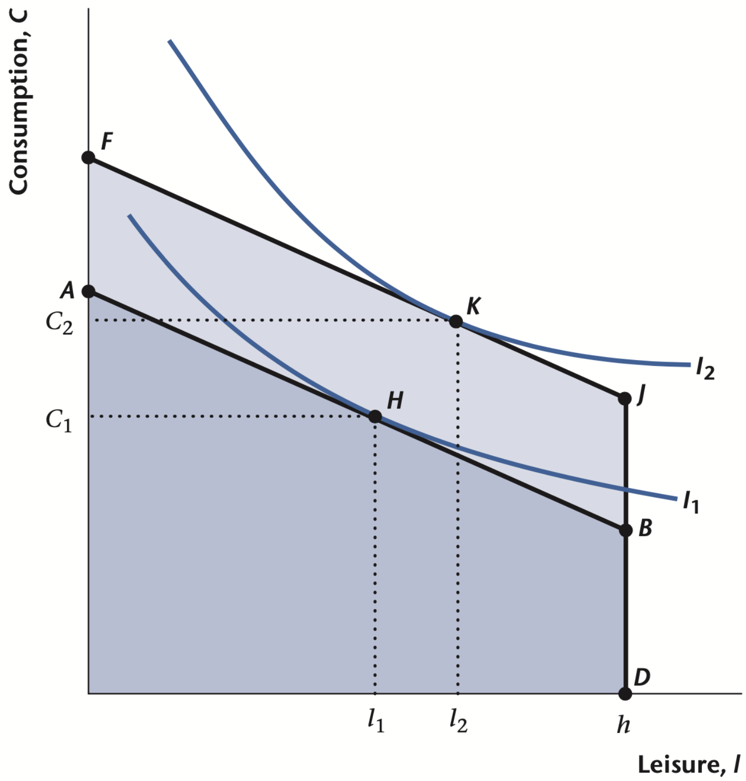
$$\frac{dC^*}{dw} = \frac{h}{1+\theta} > 0 \tag{27}$$
$$\frac{dl^*}{dw} = -\frac{\theta}{1+\theta}\frac{\pi - T}{w^2} < 0 \quad for \quad \pi - T > 0 \tag{28}$$

  Wage w is the price of leisure. If the price of leisure increases, there will be two effects of this change to the equilibrium consumption and leisure: **substitution effect** and **income effect**.

  – Higher wage means the price of leisure is higher and then the consumer will substitute leisure for consumption goods. The substitution effect for consumption is positive and that for leisure is negative. In the graph, we can quantify the

[2] As you can see from the consumer's problem per se, the disposable income $m = wh + \pi - T$ is exogenously given. But m is endogenous variable in the whole model of the consumer and the firm. The wage w is determined by market-clearing of labor market.

Figure 9: Disposable Income Rises



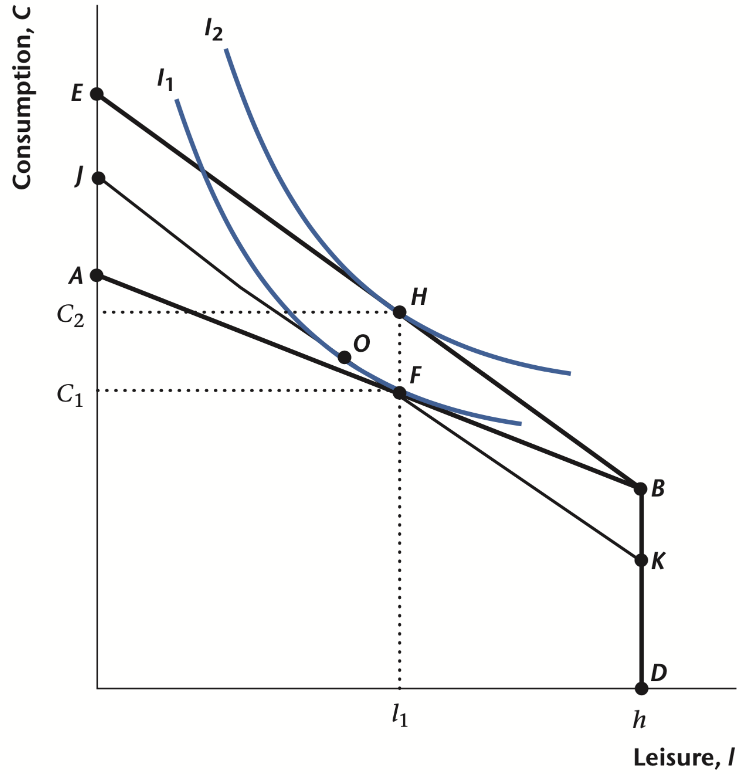
substitution effect by keeping the utility the same as the original consumption bundle with the new wage w by moving from the budget constraint from AB to JK. In mathematics, we can also quantify the substitution effect by applying the implicit function theorem. After wage increases, the new bundle still satisfies two conditions: the same utility as before; marginal rate of substitution equals to wage:

$$U^* = U(C^*, l^*)$$
$$w = \frac{U_l(C^*, l^*)}{U_C(C^*, l^*)}$$

Totally differentiate these two equations w.r.t. wage:

$$0 = U_C(C^*, l^*)dC^* + U_l(C^*, l^*)dl^*$$
$$U_{lC}dC^* + U_{ll}dl^* = U_C(C^*, l^*)dw + w(U_{CC}dC^* + U_{Cl}dl^*)$$

Figure 10: Wage Rises



The substitution effect is:

$$\frac{dC^*}{dw}_{subs} = \frac{-U_l}{U_{ll} + w\frac{U_l}{U_C}U_{CC}} = \frac{C^* l^*}{C^* + wl^*} = \frac{\theta(wh + \pi - T)}{(1+\theta)^2 w} > 0$$

$$\frac{dl^*}{dw}_{subs} = \frac{U_C}{U_{ll} + w\frac{U_l}{U_C}U_{CC}} = -\frac{l^{*2}}{\theta C^* + \theta wl^*} = -\frac{\theta(wh + \pi - T)}{(1+\theta)^2 w^2} < 0$$

– Higher wage also means that time endowment's value for the consumer is higher and he can earn more with the same working load. This results in higher income for the consumer. We know that both consumption and leisure are normal goods. So the income effects for consumption and leisure are both positive. In the graph, the income effects are represented by the move of the hypothetical budget constraint from JK to EB. In mathmatics, the income effect is the difference between the total effect and substitution effect.

$$\frac{dC^*}{dw}_{income} = \frac{dC^*}{dw} - \frac{dC^*}{dw}_{subs} = \frac{wh - (\pi - T)}{(1+\theta)^2 w}$$
$$\frac{dl^*}{dw}_{income} = \frac{dl^*}{dw} - \frac{dl^*}{dw}_{subs} = \frac{\theta(wh - (\pi - T))}{(1+\theta)^2 w^2}$$

Adding substitution effect and income effect resulting from higher wage, we know that the total effect for consumption is positive but the total effect for leisure is ambiguous. But in our utility function $U(C, l) = \ln C + \theta \ln l$, the substitution effect dominates for leisure and leisure decreases when wage increases. You should refer to Page 135-136 of Williamson (2018) to read carefully about the graph explanations of income and substitution effects.

After we find the equilibrium leisure demand, we can also find the labor supply function since the consumer allocates his time endowment h to two parts: leisure demand l and labor supply N^s. The consumer either works or enjoys leisure.

$$h = l + N^s \tag{29}$$

Then the equilibrium labor supply is:

$$N^s(w; \theta, \pi, T) = h - l^* = h - \frac{wh + \pi - T}{w} \tag{30}$$

It is easy to check that the labor supply curve is upward sloping as $\frac{\partial N^s}{\partial w} = -\frac{\partial l^*}{\partial w} > 0$.

2.2 The representative firm

The firm is endowed with exogenously-given capital K and employs labor N^d supplied by the consumer. The firm has a Cobb-Douglas function:

$$Y = zF(K, N^d) = zK^\alpha (N^d)^{1-\alpha}, 0 < \alpha < 1 \tag{31}$$

There are some properties for the technology represented by the Cobb-Douglas production function.

1. Constant resturn to scale.

$$\lambda Y = zF(\lambda K, \lambda N^d), \forall \lambda > 0 \tag{32}$$

2. More inputs mean more production. So the production function is monotonely increasing in both capital and labor, or **marginal product of capital** MP_K and **marginal product of labor** MP_N are positive.

$$\frac{\partial Y}{\partial K} = MP_K = z\alpha K^{\alpha-1}(N^d)^{1-\alpha} > 0 \tag{33}$$
$$\frac{\partial Y}{\partial N} = MP_N = z(1-\alpha)K^{\alpha}(N^d)^{-\alpha} > 0 \tag{34}$$

3. Diminishing marginal products. (Check it by yourself)

$$\frac{\partial MP_K}{\partial K} = Y_{KK} < 0, \frac{\partial MP_K}{\partial K} = Y_{NN} < 0 \tag{35}$$

4. Additional labor increase marginal product of capital and additional capital increases marginal product of labor.

$$\frac{\partial MP_K}{\partial L} = Y_{KN} = \frac{\partial MP_N}{\partial K} = Y_{NK} > 0 \tag{36}$$

The firm determines the optimal labor input choice N^d to maximize the firm's profit $\pi = zF(K, N^d) - wN^d$.

$$\max_{N^d} \quad zF(K, N^d) - wN^d \tag{37}$$

This is just a one-variable optimization problem. As the objective function is concave, the first order condition is sufficient for the maximizer of the firm's problem.

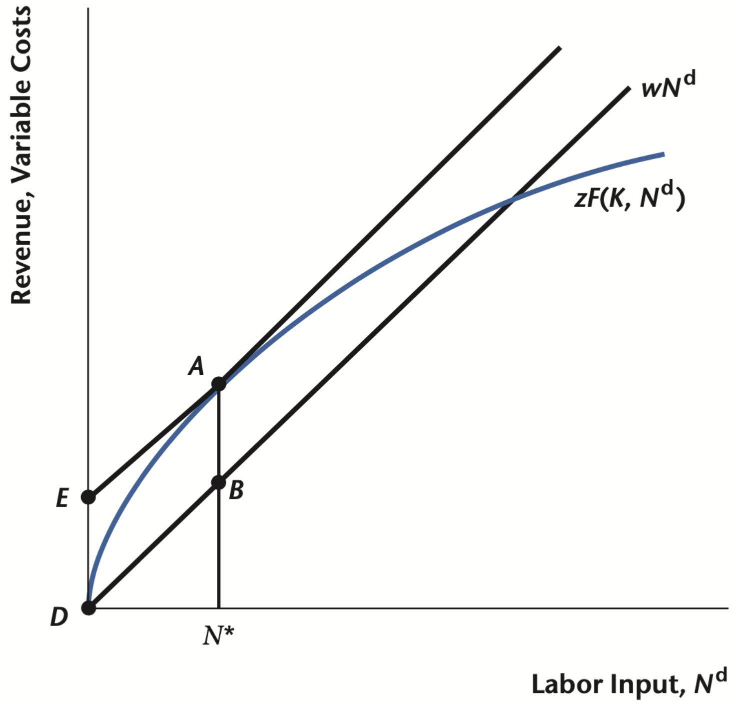
$$MP_N(N^{d*}) = z(1-\alpha)K^{\alpha}(N^{d*})^{-\alpha} = w \tag{38}$$

Then we can get the equilibrium labor demand function of the firm:

$$N^d(w; \alpha, z, K) = \left[\frac{(1-\alpha)zK^{\alpha}}{w}\right]^{\frac{1}{\alpha}} \tag{39}$$

It is easy to check that labor demand is downward sloping. Rising TFP z or capital K

Figure 11: Firm's Problem



results more labor demand.

2.3 Market Clearing

At the equilibrium, both goods market and labor market are cleared such that the demands of goods and labor equal to the supplies of goods and labor in each market. And we assume that the government sustains a balanced budget that government spending G equals to tax T.

$$Y = C + G \tag{40}$$

$$N^s = N^d \tag{41}$$

$$G = T \tag{42}$$

2.4 Competitive Equilibrium

After the consumer's problem, the firm's problem, and market-clearing conditions are specified, we can now define the competitive equilibrium.

Competitive Equilibrium A competitive equilibrium (CE) is an allocation $(C^*, l^*, N^{s*}, Y^*, N^{d*}, T^*)$ and a price system w^* given the exogenous variables (θ, G, z, K) such that

1. the consumer chooses (C, l, N^s) to maximize utility subject to his budget constraint, given (w, T, π);

$$\max_{C,l} U(C, l) = \ln C + \theta \ln l$$
$$s.t. C + wl \leq wh + \pi - T$$

2. the firm chooses N^d to maximize profit, with output $Y = zF(K, N^d)$ and maximized profit $\pi = Y - wN^d$ given (w, z, K)

$$\max_{N^d} \pi = zF(K, N^d) - wN^d$$

3. the goods market clears that goods demand equals to goods supply $C + G = Y$; the labor market clears that labor demand equals to labor supply $N^d = N^s$; the government runs a balanced budget $G = T$.

To find the competitive equilibrium, we have to solve the whole equation system to find the equilibrium allocation $(C^*, l^*, N^{s*}, Y^*, N^{d*}, T^*)$ and the price system w. In total, there are 7 unknowns. Let's see how many equations can be provided by the consumer's problem, firm's problem and market clearing.

In the consumer's problem, we have three equations:

$$w = \frac{\theta C}{l} \tag{43}$$
$$h = l + N^s \tag{44}$$
$$C = w(h - l) + Y - wN^d - T \tag{45}$$

In the firm's problem, we have two equations:

$$Y = zK^\alpha (N^d)^{1-\alpha} \tag{46}$$
$$w = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{47}$$

In the market clearing, we have three questions:

$$Y = C + G \tag{48}$$

$$N^s = N^d \tag{49}$$

$$G = T \tag{50}$$

In total, we have 8 equations for 7 unknowns. It's because we use two market clearing conditions of goods market and labor market, and one condition is redundant, which means that any two conditions imply the third one.

Walras Law : In a competitive equilibrium with N markets, market clearing in any $N-1$ markets implies market clearing in the remaining market.

We will show Walras Law by using our model here. Let's start with labor market clearing and balanced government constraint. Since the consumer maximizes his utility, his budget constraint is binding.

$$\begin{aligned} C &= w(h-l) + \pi - T && (51)\\ &= wN^s + Y - wN^d - G && (52)\\ &= Y - G && (53)\end{aligned}$$

where the second equality comes from $h - l = N^s$, $\pi = Y - wN^d$, and $G = T$; the third equality comes from $N^s = N^d$. Then we can deduce the goods market clearing condition:

$$Y = C + G$$

You may notice that this equality is just the national accounting identity without investment and net exports which our model abstracts from.

Suppose we omit the national accounting identity equation to get 7 equations for 7 unknowns, and we can solve the whole CE to get the allocation $(C^*, l^*, N^{s*}, Y^*, N^{d*}, T^*)$ and the price system w^*. For most of the time, the equations are highly-nonlinear that we cannot have explicit solution. Like most macroeconomic problems, there is no explicit solution in our model either.

- Combine the FOCs of the consumer and the firm and using the market clearing condition of labor market:

$$\frac{\theta C}{h - N^d} = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{54}$$

- Obtain the national accounting identity and substitute the production function:

$$zK^\alpha (N^d)^{1-\alpha} = C + G \tag{55}$$

- Then we collapse the system of 7 equations 7 variables to the system of 2 equations 2 variables C, N^d.

$$\frac{\theta C}{h - N^d} = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{56}$$

$$zK^\alpha (N^d)^{1-\alpha} = C + G \tag{57}$$

- We can further substitute C into the first equation:

$$\frac{\theta(zK^\alpha (N^d)^{1-\alpha} - G)}{h - N^d} = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{58}$$

- Though this is only one equation with one unknown N^d, this equation is nonlinear in N^d and we can't find an explicit solution for N^d.

But we know that N^d is an implicit function of exogenously-given variables θ, z, K, G, α. Then we can apply the implicit function theorem to analyze the comparative statics.

$$F(N^d; \theta, z, K, G, \alpha) = \theta(zK^\alpha (N^d)^{1-\alpha} - G) - z(1-\alpha)K^\alpha (N^d)^{-\alpha}(h - N^d) = 0 \tag{59}$$

For example, if we want to know how the equilibrium labor change w.r.t. (short for with respect to) government spending G, $\frac{dN^d}{dG}$. Find the total derivative w.r.t. G.

$$F_{N^d} dN^d + F_G dG = 0 \Rightarrow \frac{dN^d}{dG} = -\frac{F_G}{F_{N^d}} \tag{60}$$

Compute the two partial derivatives:

$$
\begin{aligned}
F_G &= -\theta \\
F_{N^d} &= \theta z K^\alpha (1-\alpha)(N^d)^{-\alpha} + z(1-\alpha)K^\alpha \alpha (N^d)^{-\alpha-1}(h - N^d) + z(1-\alpha)K^\alpha (N^d)^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dN^d}{dG} &= -\frac{F_G}{F_{N^d}} \\
&= \frac{\theta}{\theta z K^\alpha (1-\alpha)(N^{d*})^{-\alpha} + z(1-\alpha)K^\alpha \alpha (N^{d*})^{-\alpha-1}(h - N^{d*}) + z(1-\alpha)K^\alpha (N^{d*})^{-\alpha}} > 0
\end{aligned}
$$

Even though we can't get the explicit solution, we can still apply the implicit function theorem to find the comparative statics through the characterization equations of the solution. We can do the same thing to the multiple-equation system to analyze the effects of rising G to more endogenous variables. For example, to the 2-equation system:

$$
\frac{\theta C}{h - N^d} = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{61}
$$

$$
zK^\alpha (N^d)^{1-\alpha} = C + G \tag{62}
$$

Or,

$$
\begin{aligned}
P(C, N^d, G) &= \theta C - z(1-\alpha)K^\alpha (N^d)^{-\alpha}(h - N^d) \\
Q(C, N^d, G) &= zK^\alpha (N^d)^{1-\alpha} - C - G
\end{aligned}
$$

We can find the total differentials of these two implicit functions P and G.

$$
\begin{aligned}
P_C dC + P_{N^d} dN^d + P_G dG &= 0 \\
Q_C dC + Q_{N^d} dN^d + Q_G dG &= 0
\end{aligned}
$$

Then

$$
\begin{aligned}
P_C \frac{dC}{dG} + P_{N^d} \frac{dN^d}{dG} &= -P_G \\
Q_C \frac{dC}{dG} + Q_{N^d} \frac{dN^d}{dG} &= -Q_G
\end{aligned}
$$

Or,

$$\begin{bmatrix} P_C & P_{N^d} \\ Q_C & Q_{N^d} \end{bmatrix} \begin{bmatrix} \frac{dC}{dG} \\ \frac{dN^d}{dG} \end{bmatrix} = \begin{bmatrix} -P_G \\ -Q_G \end{bmatrix}$$

From linear algebra[3], we have:

$$\begin{aligned}
\begin{bmatrix} \frac{dC}{dG} \\ \frac{dN^d}{dG} \end{bmatrix} &= \begin{bmatrix} P_C & P_{N^d} \\ Q_C & Q_{N^d} \end{bmatrix}^{-1} \begin{bmatrix} -P_G \\ -Q_G \end{bmatrix} \\
&= \frac{1}{P_C Q_{N^d} - P_{N^d} Q_C} \begin{bmatrix} Q_{N^d} & -P_{N^d} \\ -Q_C & P_C \end{bmatrix} \begin{bmatrix} -P_G \\ -Q_G \end{bmatrix} \\
&= \frac{1}{P_C Q_{N^d} - P_{N^d} Q_C} \begin{bmatrix} -Q_{N^d} P_G + P_{N^d} Q_G \\ Q_C P_G - P_C Q_G \end{bmatrix}
\end{aligned}$$

Next compute all the partial derivatives:

$$\begin{aligned}
P_C &= \theta \\
P_{N^d} &= z(1-\alpha)K^\alpha \alpha (N^d)^{-\alpha-1}(h - N^d) + z(1-\alpha)K^\alpha (N^d)^{-\alpha} \\
P_G &= 0 \\
Q_C &= -1 \\
Q_{N^d} &= zK^\alpha (1-\alpha)(N^d)^{-\alpha} \\
Q_G &= -1
\end{aligned}$$

Then substitute:

$$\begin{bmatrix} \frac{dC}{dG} \\ \frac{dN^d}{dG} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -[z(1-\alpha)K^\alpha \alpha (N^d)^{-\alpha-1}(h - N^d) + z(1-\alpha)K^\alpha (N^d)^{-\alpha}] \\ \theta \end{bmatrix}$$

where

$$\Delta = \theta z K^\alpha (1-\alpha)(N^d)^{-\alpha} + z(1-\alpha)K^\alpha \alpha (N^d)^{-\alpha-1}(h - N^d) + z(1-\alpha)K^\alpha (N^d)^{-\alpha}$$

[3] You can check that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This two equation approach is the same as the one that we collapse all the equations to one equation with only N^d. Actually, we know that the CE is characterized by the 7 equation system. If we want to analyze the comparative statics, it is OK to just apply the implicit function theorem to the 7-equation system and we can get the same result. It is complicated but not difficult. You should be careful with all the derivatives. Differentiate total derivatives from partial derivatives. Normally, with pencil and paper, we do not apply this theorem to multiple-equation system due to complication but collapse the system to one or two equations which is easier to deal with. But for most of the time, you may not collapse easily and nowadays you can use computer packages to do the differentials for you, such as Mathmatica or Matlab.

2.5 Production Possibility Frontier

In the CE, both goods market and labor market clear. If we express consumption goods and leisure in the CE in one diagram, we can get the production possibility function.

$$\begin{aligned} Y &= C + G \\ zF(K, h - l) &= C + G \end{aligned}$$

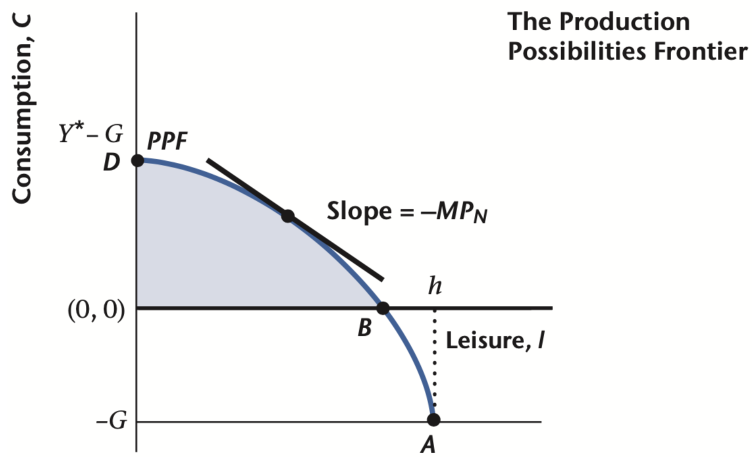
This is a relationship between C and l, given exogenous variables θ, α, z, K, G. The line with all the points (C, l) satisfies the relationship $zF(K, h - l) = C + G$ is the **production possibility Frontier**, which is shown in Figure 12. The PPF describes what the possibilities are for the economy as a whole in terms of the production of consumption goods and leisure. The PPF captures the important subject of this section: the tradeoff between leisure and consumption that the available production technology makes available for the representative consumer in the economy.

$$PPF; = \{(C, l) : C = zF(K, h - l) - G\} \tag{63}$$

We can check that the slope of the PPF is just minus marginal product of labor

$$Slope_{PPF} = \frac{dC}{dl} = Y_l = \frac{\partial Y}{\partial N}\frac{\partial N}{\partial l} = -Y_N = -MP_N \tag{64}$$

Figure 12: Production Possibilities Frontier



Note:

In microeconomics, we have another name for the negative of the slope of the PPF, which is the **marginal rate of transformation**. The marginal rate of transformation is the rate at which one good can be converted technologically into another. In our model, the marginal rate of transformation is the rate at which leisure can be converted in the economy into consumption through work. In other words, when the consumption bundle lies on the PPF, if he wants to have more leisure, he has to work less and consumes less due to less production. Denote the marginal rate of transformation of leisure for consumption as $MRT_{l,C}$. We have:

$$MRT_{l,C} = -slope_{PPF} = MP_N \tag{65}$$

The PPF is concave in l. Since the PPF is decreasing in l so concavity of PPF means the marginal rate of transformation of leisure for consumption goods is increasing in l. The concavity of PPF can be shown in:

$$\frac{d^2C}{dl^2} = -\frac{\partial Y^2}{\partial N^2}\frac{\partial N}{\partial l} = Y_{NN} < 0$$

We can understand this result from economic intuition. As we move from low leisure to high leisure point in the PPF, the consumer works less and his marginal product of labor increases and so consumption goods increases. Then the consumer needs to sacrifice more consumption goods to increase one unit of leisure in the high leisure point than the low leisure point in the PPF. This property is determined by diminishing marginal product of labor.

In the firm's problem, the firm hire the labor to the point that marginal product of labor equals to wage. And we see from PPF that in the CE the marginal rate of transformation also equals to wage. Then

$$MRT_{l,C} = MP_N = w \tag{66}$$

And in the consumer's problem, the equilibrium marginal rate of substitution of leisure for consumption equals to wage, so:

$$MRS_{l,C} = MRT_{l,C} = MP_N = w \tag{67}$$

In the CE of this model per se, the marginal rate of substitution of leisure for consumption is equal to the marginal rate of transformation, which is equal to the marginal product of labor. That is, because the consumer and the firm face the same market real wage in equilibrium, the rate at which consumer is willing to trade leisure for consumption is the same as the rate at which leisure can be converted into consumption goods using existing production technology.

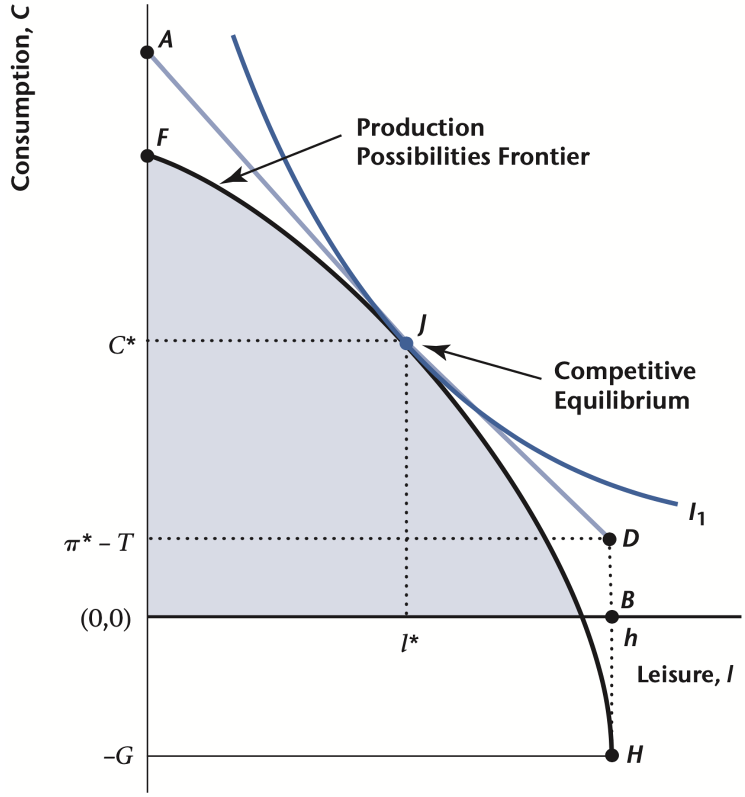
Think about this important question: what if the wage the consumer faces is not the same as the wage the firm faces? For example, there is a wage income tax for the consumer.

2.6 Pareto Optimality

In econmics, we are not only interested in what the competitive equilibrium is, but also whether the CE is optimal in the sense of Pareto optimality criteria. If there is no way to rearrange production or to reallocate goods so that someone is made better off without making someone else worse off. This kind of allocation is called **Pareto Optimum**.

In our model, there is only one representative agent the consumer and so we only need to make the consumer as well off as possible. We introduce **social planner** here. The social planner or benevolent totalitarian is virtual in economics. He knows completely about the

Figure 13: Competitive Equilibrium



structure of the economy: the preference, the technology, and he can dictates the allocation as if he is the God without worrying the price system. Namely, the social planner solves his problem:

$$\max_{C,l} U(C,l) = \ln C + \theta \ln l \tag{68}$$

$$s.t. \quad zK^{\alpha}(h-l)^{1-\alpha} = C + G. \tag{69}$$

The social planner's problem is to choose C and l, given the technology for converting l into C, to make the representative consumer as well off as possible. The constraint (71) is also called the **resource constraint**. That is, the social planner chooses a consumption bundle (C,l) on the PPF, and that is on the highest possible indifference curve for the consumer. So, we know that the optimal point should be the one at which the indifference curve is tangent to the PPF.

Pareto Optimum The Pareto optimum is defined as an allocation $(\bar{C}, \bar{l})$ that solve's the social planner's problem:

$$\max_{C,l} U(C,l) = \ln C + \theta \ln l \tag{70}$$

$$s.t. \quad zK^{\alpha}(h-l)^{1-\alpha} = C + G. \tag{71}$$

Basically, there are two conditions for the optimality consumption bundle:

- Tangent condition: $MRS_{l,C} = MRT_{l,C}$;
- Technology possibility condition: $(C, l) \in PPF$.

We can see that the CE satisfies these two conditions for optimality as shown in Figure 13. In the CE, the PPF is tangent to the indifference curve and the tangent point is on the PPF. So the CE is also Pareto optimal (short for PO). This is the celebrated **first fundamental theorem of welfare economics**.

First Fundamental Theorem of Welfare Economics The first fundamental theorem of welfare economics states that, under certain conditions, a competitive equilibrium is Pareto optimal.

The first theorem embeds in the spirit of Adam Smith who emphasizes on market's "invisible hand" can be optimal. Even though everybody is selfish and chase for his own interests, the markets can utilize this selfishness to do good to the people as a whole.

We can also see that the first theorem is true from that the solution characterization equations of PO are the same as that of CE.

Solve the social planner's problem by Lagrangian multiplier, and we can get:

$$\frac{\theta C}{l} = z(1-\alpha)K^{\alpha}(h-l)^{-\alpha} \tag{72}$$

$$C = zK^{\alpha}(h-l)^{1-\alpha} - G \tag{73}$$

The two equations (72) characterize the PO, which are the same as the two equations (61) collapsed from the 7-equation system of CE if we substitute $l = h - N^s$.

And we can have the second fundamental theorem of welfare economics here.

Second Fundamental Theorem of Welfare Economics The second fundamental theorem of welfare economics states that, under certain conditions, a Parato Optimum is a competitive equilibrium.

We can use our model to show that the second theorem is true. From the definition of CE, the CE is an allocation and a price system such that the consumer and the firm optimizes, and all markets clear. Some endogenous variables (C^*, l^*) are already given in the PO $(\hat{C}, \hat{l})$ which is the maximizer of the social planner's problem.

$$C^* = \hat{C}$$
$$l^* = \hat{l}$$

What remains is to construct other endogenous variables in the allocation of CE and price system wage.

$$w^* = MRS_{l,C}$$
$$T^* = G$$
$$N^{s*} = N^{d*} = h - \hat{l}$$
$$Y^* = zF(K, N^{s*})$$

So the CE with the allocation $(C^*, l^*, N^{s*}, N^{d*}, Y^*, T^*)$ and price system w^* is constructed by using the PO $(\hat{C}, \hat{l})$.

The second theorem is rather useful in large and complicated macroeconomic models in which there are up to more than 50 equations. The second theorem tells us the PO is a CE under some conditions. So we can directly use the PO to solve some variables in the allocation and use the allocation to construct the price system. It is much easier to solve the PO than to solve the CE by solving agent's problems one by one.

And also we can directly use the solution characterization equations in the PO to analyze the comparative statics in the CE. What is the effects of the government purchases on consumption goods and leisure?

The two theorems are true under some conditions. But under what conditions are they not true? Basically, if there are externality, distortional tax(not lump-sum tax), and monopoly in the economy, the two theorems don't hold. In the last subsection, there is a question that when there is labor wage income tax, what is the CE and PO? We can show that PO is not a CE in this case.

2.7 Summary of One Period Consumption-Leisure Model

In this simple one period model, we learn foundations of modern macroeconomics, which builds on microeconomics. All the demands and supplies come from agents' optimization problems. At the equilibrium, all markets are cleared and so the general equilibrium is emphasized in modern macroeconomics.

The key implication of the one period model is that the representative consumer trades off consumption and working load. If he chooses to work more, he can have more consumption goods but consume less leisure. At the equilibrium, the marginal rate of substitution of leisure for consumption should be equal to the marginal rate of transformation of leisure for consumption. If they are not equal, there is inefficiency and the consumer can change the allocation to improve welfare.

What if $MRS_{l,C} < MRT_{l,C}$? It means the worker works more than needed. He can trade consumption for leisure to make the two rates equal again.

3 Two-Period Model with Consumption-Saving Tradeoff

In the last section, we use one-period model to show the consumption-leisure tradeoff and general equilibrium. Intrinsically, the subjects of macroeconomics are dynamic in the sense that the variables of interest are changing through time. In this section, we will learn consumer's optimal choice in a simple dynamic world in which there are two periods.

3.1 Environment

We abstract from the firm's problem here but assume that the consumer is endowed with real income y and y' in the present period and future period. This simplification doesn't influence our main story the consumption saving tradeoff or intertemporal consumption tradeoff here. In economic models, we usually do this kind of simplification to illustrate the key mechanism and abstract other unnecessary complication.

Since the consumer is endowed with real income in the two periods, it is not necessary to assume that the consumer values leisure anymore and there is no consumption-leisure tradeoff here. The consumer has a preference represented by the following utility function:

$$U(c, c') = u(c) + \beta u(c') \tag{74}$$

Here the prime variable denotes the variable in the future period to simplify the notation. Just consider $c = c_t$ and $c' = c_{t+1}$. Likewise, we use this convention to other variables. The preference is separate on consumption of different periods so the marginal utility of present consumption is not influenced by change of future consumption. The consumer values less on the future consumption. It means that the consumer prefers more on present consumption if the quantities of consumption goods of the two periods are the same. So there is a **disount factor** $\beta < 1$ for the future consumption.

The preference of the consumer has similar properties as the one-period model: more is preferred than less; diversity is preferred; both present and future consumption goods are normal. Then the consumer is well off if the indifference curve moves to the northeast. And the indifference curve is decreasing and convex in the (c, c') space.

The government levies lump sum tax t from the consumer. The consumer distributes his present disposable income $y - t$ to two parts: consumption c or government bonds b^d with the interest rate r.

$$c + b^d \leq y - t \tag{75}$$

In the future period, the consumer has income endowment b', hands in lump-sum tax t' and can get bonds back with interest $(1+r)b$.

$$c' \leq y' - t' + (1+r)b^d \tag{76}$$

When $b^d > 0$, the consumer saves by buying government's bonds. You can consider it as lending to the government. When $b^d < 0$, the consumer borrows by selling government's bonds. Dividing the future budget constraint (76) by $(1+r)$ and adding the present budget constraint (75), we have the consumer's **lifetime budget constraint**.

$$c + \frac{c'}{1+r} \leq y - t + \frac{y' - t'}{1+r} \tag{77}$$

It states that the **present value** of lifetime consumption $c + \frac{c'}{1+r}$ cannot be larger than the present value of lifetime disposable income $y - t + \frac{y'-t'}{1+r}$. The present value here is in terms of the present consumption goods. That is, the relative price of the future consumption goods is $\frac{1}{1+r}$ in terms of the present consumption goods, or the relative price of the present consumption goods is $1+r$ in terms of the future consumption goods. This interpretation is quite important in modern macroeconomics as the real interest rate acts as the relative price of consumption goods of different periods. When r goes up, the relative price of present consumption goods goes up while the relative price of the future consumption goods goes down.

The government runs a balanced budget constraint in the two periods. As there is only one representative consumer, the aggregate tax, bonds, and government purchases equal to the representative one. We do not use aggregate variables T, B^s, G here.

$$t + b^s = g \tag{78}$$

$$t' = g' + (1+r)b^s \tag{79}$$

3.2 Pareto Optimality

In the one-period model, we introduces the first and second theorem of welfare economics. As there is no distortion, externality, and market power in the two-period model, the two welfare theorems hold here in this environment. Before the illustration of competitive equilibrium, we can check the Pareto optimality first here. The social planner's problem is:

$$\max_{c,c'} \quad u(c) + \beta u(c') \tag{80}$$

$$s.t. \quad c + g \leq y \tag{81}$$

$$c' + g' \leq y' \tag{82}$$

The benevolent totalitarian knows the preference of the consumer and the resource constraint. He dictates and distributes the allocation $(\bar{c}, \bar{c}')$ directly to maximize the utility of the consumer given y, y', g, g' without reference to any price system $1 + r$ and tax system t, t'.

We can construct the Lagrangian function to solve the problem[4]:

$$L(c, c', \lambda) = \ln c + \beta \ln c' + \lambda(y - c - g) + \lambda'(y' - c' - g')$$

We can get the FOCs as:

$$\frac{\partial L}{\partial c} = u'(c) - \lambda = 0$$
$$\frac{\partial L}{\partial c'} = \beta u'(c') - \frac{\lambda}{1+r} = 0$$
$$\frac{\partial L}{\partial \lambda} = y - c - g = 0$$
$$\frac{\partial L}{\partial \lambda'} = y' - c' - g' = 0$$

We can get the optimal consumption bundle as:

$$\bar{c} = y - g \tag{83}$$

$$\bar{c}' = y' - g' \tag{84}$$

Pareto Optimum The Pareto optimum of the two-period model is defined as an allocation $(\bar{c}, \bar{c}')$ such that the allocation solves the social planner's problem.

You can see that the allocation is quite simple here through solving the PO. We know that the second welfare theorem holds that we can construct the CE by the PO. As CE is

[4] Clearly you can show that equality should hold in the equilibrium.

an allocation and a price system such that the consumer solves his problem and all markets are cleared. We need to find the allocation and the price system.

Competitive Equilibrium The competitive equilibrium is an allocation $(c^*, c'^*, b^{d*}, b^{s*}, t^*, t'^*)$ and a price system $1 + r^*$ such that:

1. given the price system, the allocation solves the consumer's problem;

$$\begin{aligned} \max_{c,b,c'} \quad & U(c, c') = u(c) + \beta u(c') \\ s.t. \quad & c + b^d = y - t \\ & c' = y' - t' + (1 + r)b^d \end{aligned}$$

2. the goods market clears both in the present goods market and future goods market,

$$\begin{aligned} y &= c + g \\ y' &= c' + g' \end{aligned}$$

the bonds market clears,

$$b^s = b^d = b$$

and the government runs a balanced budget both in the present period (78) and in the future period (79).

$$\begin{aligned} g &= t + b^s \\ g' &= t' - (1 + r)b^s \end{aligned}$$

We can construct the allocation of CE from the optimal allocation of PO:

$$c^* = \bar{c}; \tag{85}$$

$$c^{'*} = \bar{c'}; \tag{86}$$

$$(t^*, t^{'*}, b^{s*}) = \{(t, t', b) : t + b^s = g; t' = g' + (1 + r^*)b^s\} \tag{87}$$

$$b^{d*} = b^{s*} \tag{88}$$

$$1 + r^* = MRS_{c,c'} = \frac{u'(c^*)}{\beta u(c^{'*})} \tag{89}$$

We can see that there are multiple CEs here as three variables $t^*, t^{'*}, b^{s*}$ should satisfy only two conditions. In all these CEs, present and future consumption $c^*, c^{'*}$ and the price system $1 + r^*$ are uniquely determined.

You can notice that $(t^*, t^{'*}, b^{s*})$ are not uniquely determined in this competitive equilibrium from Equation (87). It means that there are various combinations of $(t^*, t^{'*}, b^{s*})$ such that the competitive equilibrium holds. From economic intuition, it just states that it does not matter what approach the government collects its government income through tax or public bonds. The present consumption and future consumption $(c^*, c^{'*})$ do not change no matter when the government collects its tax, present or future, under the condition that the government's expenditures (g, g') do not change. If the government cuts tax, it has to increase public bonds to finance its government expenditures and in the next period it will increase future period tax. This is called **Ricardian Equivalence**, which was first discovered by the famous economist David Ricardo.

As there are multiple CEs by the Ricardian equivalence, we can construct the CE by assuming one number for $(t^*, t^{'*}, b^{s*})$ and then get the whole CE.

Suppose in the CE

$$t^* = x \in R$$

Then

$$b^{*s} = g - x$$
$$b^{*d} = b^{*d}$$
$$t^{'*} = g' + (1 + r^*)b^{s*}$$

We can see that in the equilibrium allocation and price system, the present consumption, future consumption, and real interest rate are uniquely determined by exogenously-given variables y, y', g, g', β. And other allocation variables t, t', b^s, b^d are not uniquely-determined in the model and they are actually determined by government's fiscal policy, which we will see in next subsection in detail.

3.3 Competitive Equilibrium

Competitive Equilibrium The competitive equilibrium is an allocation $(c^*, c^{'*}, b^{d*}, b^{s*}, t^*, t^{'*})$ and a price system $1 + r^*$ such that:

1. given the price system, the allocation solves the consumer's problem;

$$\max_{c,b,c'} \quad U(c, c') = u(c) + \beta u(c')$$
$$s.t. \quad c + b^d = y - t$$
$$c' = y' - t' + (1 + r)b^d$$

2. the goods market clears both in the present goods market and future goods market,

$$y = c + g$$
$$y' = c' + g'$$

the bonds market clears,

$$b^s = b^d = b$$

and the government runs a balanced budget both in the present period (78) and in the future period (79).

$$g = t + b^s$$
$$g' = t' - (1+r)b^s$$

We know from Walras Law that one market clearing condition is redundant (prove it by yourself, you may try that the goods markets clearing conditions can be deduced by assuming the bonds market clears and the government runs a balanced budget). For now, we just delete the market clearing condition of the goods market $c + g = y, c' + g' = y'$.

For the consumer's problem,

$$\begin{aligned} \max_{c,b^d,c'} \quad & U(c,c') = u(c) + \beta u(c') \\ s.t. \quad & c + b^d = y - t \\ & c' = y' - t' + (1+r)b^d \\ & with (y, y', t, t') given \end{aligned}$$

Construct the Lagrangian function of the consumer's problem:

$$L(c, b^d, c', \lambda, \lambda') = u(c) + \beta u(c') + \lambda(y - t - c - b^d) + \lambda'[y' - t' + (1+r)b^d - c']$$

Get the FOCs:

$$L_c = u'(c) - \lambda = 0;$$
$$L_{c'} = u'(c') - \lambda' = 0;$$
$$L_b = -\lambda + \lambda'(1+r) = 0;$$
$$L_\lambda = y - t - c - b^d = 0;$$
$$L_{\lambda'} = y' - t' + (1+r)b^d - c' = 0;$$

Transform:

$$
\begin{aligned}
u'(c) &= \beta(1+r)u'(c') \\
c &= y - t - b^d \\
c' &= y' - t' + (1+r)b^d
\end{aligned}
$$

Let's drop b^d for the moment.

$$
u'(c) = \beta(1+r)u'(c') \tag{90}
$$

$$
c + \frac{c}{1+r} = y - t + \frac{y'-t'}{1+r} \tag{91}
$$

Equation (90) is the celebrated neoclassical intertemporal consumption equation, also known as **Euler equation**. It states how the consumer dynamically choose consumption goods across different periods under the lifetime constraint to maximize his lifetime utility. The optimal choice of $(c^*, c^{'*})$ is the point that the marginal cost of one unit sacrifice of present consumption $u'(c)$ is equal to the marginal benefit of this sacrifice $\beta(1+r)u'(c')$. It is a common source for modern macroeconomics and can be dated to Irving Fisher and Milton Friedman, the cornerstone of permanent income theory of consumption.

Let's assume logarithm utility for the consumer $u(c) = \ln c$, and then the solution is:

$$
\begin{aligned}
c^* &= \frac{1}{1+\beta}(y - t + \frac{y'-t'}{1+r}) \\
c^{'*} &= \frac{\beta}{1+\beta}(1+r)(y - t + \frac{y'-t'}{1+r}) \\
b^{d*} &= y - t - \frac{1}{1+\beta}(y - t + \frac{y'-t'}{1+r})
\end{aligned}
$$

- You can see that $(c^*, c^{'*})$ is a function of the lifetime wealth $y - t + \frac{y'-t'}{1+r}$. The optimal present consumption is not only a function of present disposable income $y - t$ but also a function of future disposable income $y' - t'$. The **marginal propensity of consumption**, the consumption increase of one unit income increase, is:

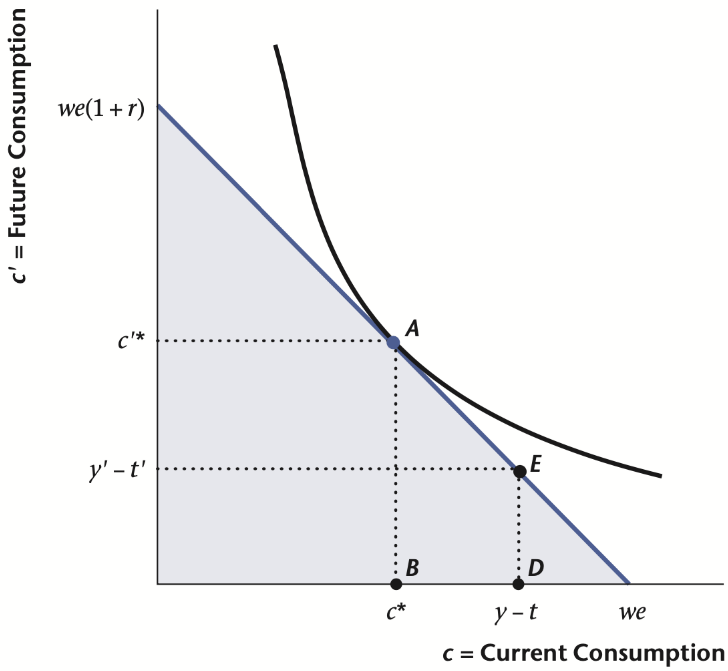
  $$
  \frac{dc^*}{dy} = \frac{1}{1+\beta} < 1
  $$

  It states that the present consumption increase will be less than the transitory income increase. The consumer smooths this present income increase and saves the remaining for future consumption. That is the reason why you can see that the cycle component of consumption is less volatile than that of GDP in the data, as shown in Figure 4.

- The marginal rate of substitution of present consumption goods for future present goods is equal to minus slope of the indifference curve, which is equal to minus slope of the budget constraint at the optimal point $(c^*, c^{'*})$.

$$MRS_{c,c'}(c^*, c^{'*}) = 1 + r.$$

Figure 14: Consumer's Problem



You can check the comparative statics by implicit function theorems. Please read carefully Chapter 9 of Williamson (2018) about the verbal explanation of the graph approach of the two-period model and related comparative statics.

Collect all the conditions of the CE excluding the redundant market clearing condition implied by Walras law:

$$u'(c) = \beta(1+r)u'(c') \tag{92}$$

$$c = y - t - b^d \tag{93}$$

$$c' = y' - t' + (1+r)b^d \tag{94}$$

$$g = t + b^s \tag{95}$$

$$g' = t' - (1+r)b^s \tag{96}$$

$$b^d = b^s \tag{97}$$

We can see that in the CE there are six conditions for seven endogenous variables including six variables of allocation (c, c', t, t', b^d, b^s) and one variable of price system r. Then normally there are infinite-many solutions for this system. Basically, we know that Ricardian equivalence holds here. As the CE we construct from PO, we can assume the tax t^* in the CE is an arbitrary number x:

$$t^* = x \tag{98}$$

Then

$$c^* = \frac{1}{1+\beta}(y - g + \frac{y' - g'}{1+r}) \tag{99}$$

$$c'^* = \frac{\beta}{1+\beta}(1+r)(y - g + \frac{y' - g'}{1+r}) \tag{100}$$

$$t'^* = g' + (1+r) \tag{101}$$

$$b^{s*} = g - x \tag{102}$$

$$b^{d*} = y - x - c^* \tag{103}$$

We can get the equilibrium price system $1 + r^*$ by clearing the bonds market:

$$b^{d*} = b^{s*} \Rightarrow 1 + r^* = \frac{y' - g'}{\beta(y - g)} \tag{104}$$

Substitute this $1 + r^*$ into the above endogenous variables:

$$c^* = y - g \tag{105}$$

$$c^{'*} = y' - g' \tag{106}$$

$$t^{'*} = g' + \frac{y' - g'}{\beta(y - g)}(g - x) \tag{107}$$

$$b^{s*} = g - x \tag{108}$$

$$b^{d*} = g - x \tag{109}$$

Competitive Equilibrium The competitive equilibrium is an allocation $(c^*, c^{'*}, b^{d*}, b^{s*}, t^*, t^{'*})$ given by Equation (99, 100, 103, 102, 98, 101) and a price system $1 + r^*$ given by Equation (104) such that:

1. given the price system $1 + r^*$, the allocation solves the consumer's problem;

$$\max_{c,b,c'} \quad U(c, c') = u(c) + \beta u(c')$$
$$s.t. \quad c + b = y - t$$
$$c' = y' - t' + (1 + r)b$$

2. the goods market clears both in the present goods market and future goods market,

$$y = c + g$$
$$y' = c' + g'$$

the bonds market clears,

$$b^s = b^d = b$$

and the government runs a balanced budget both in the present period and in the future period.

$$g = t + b$$
$$g' = t' - (1 + r)b$$

- Consumption smoothing. Notice that consumer's present disposable income is $y - t$ and his optimal present consumption choice is $y - g$. So when $t < g$, the consumer saves for future consumption and vice versa. This phenomenon is called **consumption smoothing**. The consumer optimizes intertemporally by moving his present income for future consumption if $t < g$ and vice versa.

- Ricardian equivalence. From the government's choices, we know that

$$(t^*, t^{'*}, b^*) = \{(t, t', b) : g = t + b; g' = t' - (1 + r)b\}$$

Any combinations of (t, t', b) that satisfy the above two equations satisfy the requirements of the CE. As there are three variables for two equations, there are infinitely many solutions normally. Intuitively, it states that once government spendings g, g' do not change, the consumer's choices c, c' do not change no matter what means the government collects funding for expenditures, through tax or public bonds.

If the government cuts taxes in the present period, the consumer will not increase his present consumption but buy more public bonds b or save more. In the future period, the consumer uses this saving to hand in more tax t' to the government. This is the celebrated Ricardian equivalence in neoclassical macroeconomics. This is clearly in contrast with Keynesian macroeconomics which states that tax cutting will boost disposable income and thus present consumption. The difference lies in the accounting of dynamic optimization in neoclassical macroeconomics. The consumer optimizes intertemporally instead of just considers one period.

We can also understand the Ricardian equivalence from the consumer's problem:

$$\begin{aligned}
\max_{c,b,c'} \quad & U(c, c') = u(c) + \beta u(c') \\
s.t. \quad & c + b^d = y - t \\
& c' = y' - t' + (1 + r)b^d
\end{aligned}$$

The lifetime consumer's budget constraint is:

$$c + \frac{c'}{1+r} = y - t + \frac{y' - t'}{1+r}$$

According to the government's budget $g = t + b^s, g' = t' - (1+r)b^s$, the government's lifetime budget is:

$$t + \frac{t'}{1+r} = g + \frac{g'}{1+r}$$

Substitute this into the consumer's budget, then the consumer's problem is:

$$\max_{c,b,c'} \quad U(c,c') = u(c) + \beta u(c')$$
$$s.t. \quad c + \frac{c'}{1+r} = y - g + \frac{y' - g'}{1+r}$$

Then you can see that the government's fiscal policy doesn't influence the consumer's choices of present consumption and future consumption.

4 Two-Period Model without Government

Imagine there is no government in the economy, the representative consumer has endowment income y, y' in the present and future period. He decides present consumption c, future consumption c', and buy bonds b[5] to maximize his lifetime utility.

The Consumer's Problem

$$\max_{c,b,c'} \quad U(c,c') = u(c) + \beta u(c')$$
$$s.t. \quad c + b^d = y$$
$$c' = y' + (1+r)b^d$$
$$y, y', r, \quad given$$

[5]You can consider the representative consumer as one of the infinitely many consumers and so the representative consumer has no power on prices. Ex ante some consumers issue bonds and some buy bonds.

We can form a Lagrangian:

$$L = u(c) + \beta u(c') + \lambda(y - c - b^d) + \lambda'(y - (1+r)b^d - c')$$

The FOCs are:

$$\begin{aligned}
L_c &= u'(c) - \lambda = 0 \\
L_{c'} &= \beta u'(c') - \lambda' = 0 \\
L_{b^d} &= -\lambda + \lambda(1+r) = 0
\end{aligned}$$

Combine the three conditions:

$$u'(c) = \beta(1+r)u'(c')$$

This is still the Euler condition. You should be able to explain the economic intuition verbally by yourself. Assume logarithm utility function $u(c) = \ln c$:

$$\begin{aligned}
c &= \frac{1}{1+\beta}(y + \frac{y'}{1+r}) \\
c' &= \frac{\beta}{1+\beta}(1+r)(y + \frac{y'}{1+r}) \\
b^d &= y - \frac{1}{1+\beta}(y + \frac{y'}{1+r})
\end{aligned}$$

Market Clearing In the CE, two markets are cleared.

- Goods markets

$$\begin{aligned}
c &= y \\
c' &= y'
\end{aligned}$$

- Bonds market. As there is only representative consumer and no government, everybody behaves the same. There is no reason why ex post some consumers lend and some consumers borrow. In the equilibrium, nobody lends and borrows.

$$b^d = b^s = 0$$

From Walras Law, one market clearing condition is redundant. For example, we just use the bonds market clearing condition and the goods market clearing conditions can be deduced from the consumer's budgets. From the bonds market clearing condition, we can get the equilibrium real interest rate.

$$b^d = b^s = 0 \Rightarrow 1 + r = \frac{y'}{\beta y} \tag{110}$$

Competitive Equilibrium The competitive equilibrium of the two-period model without government is an allocation $(c^* = y, c^{'*} = y', b^{d*} = 0, b^{s*} = 0)$ and a price system $1 + r^* = \frac{y'}{\beta y}$ such that :

1. given the price system, the allocation solves the consumer's problem;
2. all markets clear.

In this model, the present and future consumption both equal to their endowments. That's because there is no saving technology in this economy so that the representative consumer cannot smooth consumption intertemporally by saving. Nobody borrows and lends in this economy and then the consumer cannot hedge aggregate risk in the economy. That is, he cannot move some present goods for future consumption if his present income increases. Without saving technology, the aggregate risk cannot be hedged or risk-shared.

We can see from the equilibrium real interest rate $1 + r = \frac{y'}{\beta y}$ that the real interest rate is positively related to future income y' and negatively related to present income y. We can understand this through the demand and supply of bonds. If the present income increases, the consumer wants to smooth consumption intertemporally and the demand for bonds increases but bonds supply is fixed at zero. So the price of bonds will increase and the real interest rate decreases (You can consider the bond's price is $q = \frac{1}{1+r}$).

Compare the above two models, the consumer can smooth consumption in the economy with government, and thus can have a higher sum of discounted utility than the economy without government. The government provides a saving technology, which enhances the welfare of the consumer.

5 Stochastic Two-Period Model

Previous models are all certain. Here in this subsection, we will introduce uncertainty to the simple two-period model to show the stochastic feature of modern macroeconomics. As we have already introduced dynamic and general equilibrium, by adding stochastic feature this model is the simplest DSGE model of macroeconomics. Before the stochastic two-period model, we first introduce one important stochastic process in macroeconomics: Markov chain.

5.1 Markov Chains

A stochastic process is a sequence of random vectors $\{x_t\}$. The sequence is ordered by a time index in discrete time models. A stochastic process $\{x_t\}$ is said to have the **Markov property** if for all $k \geq 1$ and all t,

$$Prob(x_{t+1}|x_t, x_{t-1}, \cdots, x_{t-k}) = Prob(x_{t+1}|x_t) \tag{111}$$

A stochastic process with Markov property is a **Markov Chain**. A time-invariant Markov chain is defined by a triple of objects, an n-dimensional state space consisting of vectors $e_i, i = 1, \cdots, n$, where e_i is an $n \times 1$ unit vector where ith entry is 1 and all other entries are zero; an $n \times n$ transition matrix P, which records the probabilities of moving from one value of the state to another in one period; and an $n \times 1$ vector π_0 whose ith element is the unconditional probability of being in state i at time 0: $\pi_{0i} = Prob(x_0 = e_i)$. The elements of P are

$$P_{ij} = Prob(x_{t+1} = e_j|x_t = e_i). \tag{112}$$

The transition matrix P and the unconditional distribution vector π_0 must satisfy the following assumption:

- For $i = 1, \cdots, n$, the matrix P satisfies

$$\sum_{j=1}^{n} P_{ij} = 1.$$

- The vector π_0 satisfies

$$\sum_{i=1}^{n} \pi_{0i} = 1.$$

A matrix P that satisfies (113) is called a **stochastic matrix**. The stochastic matrix defines the probabilities of moving from one value of the state to another in one period. The probability of moving from one value of the state to another in two periods is determined by P^2:

$$\begin{aligned}
&Prob(x_{t+2} = e_j | x_t = e_i) \\
=&\sum_{h=1}^{n} Prob(x_{t+2} = e_j | x_{t+1} = e_h, x_t = e_i) Prob(x_{t+1} = e_h | x_t = e_i) \\
=&\sum_{h=1}^{n} Prob(x_{t+2} = e_j | x_{t+1} = e_h) Prob(x_{t+1} = e_h | x_t = e_i) \\
=&\sum_{h=1}^{n} P_{hj} P_{ih} \\
=&(P^2)_{ij}
\end{aligned}$$

where the first equality comes from the total probability equation ($P(A) = \sum_i P(A|B_i) * P(B_i)$), the second equality comes from the Markov property, and the last equality comes from the proper of matrix multiplication. $(P^2)_{ij}$ is the ijth element of the matrix P^2. Likewise, we can know that the probability of moving from one value of the state to another in k periods is determined by P^k:

$$Prob(x_{t+k} = e_j | x_t = e_i) = (P^k)_{ij} \tag{113}$$

The unconditional probability of $x_t = e_j$ are determined by:

$$Prob(x_1 = e_j) = \sum_{i=1}^{n} Prob(x_1 = e_j | x_0 = e_i) Prob(x_0 = e_i) = \sum_{i=1}^{n} P_{ij} \pi_i$$

Then the unconditional probability distribution of x_1 is:

$$\pi_1' = \pi_0' P$$

Likewise, we know that

$$\pi_k' = \pi_0' P^k$$

So the unconditional probability distribution evolves according to

$$\pi_{t+1}' = \pi_t' P.$$

An unconditional distribution is called **stationary** or **invariant** if it satisfies

$$\pi_{t+1} = \pi_t$$

that is, the unconditional distribution remains the same through time.

$$\pi' = \pi' P$$

or

$$(I - P')\pi = 0.$$

which shows that the stationary unconditional distribution π is an eigenvector associated with a unit eigenvalue of P'.

For an arbitrary initial distribution π_0, if the limit of π_t is independent of π_0, we say that the Markov chain is **asymptotically stationary** with a unique invariant distribution. We call a solution π_∞ a stationary distribution or an invariant distribution of P. Then π_∞ must the the unique eigenvector associated with unit eigenvalue of P'.

$$(I - P')\pi_\infty = 0$$

Theorem 2. *Let P be a stochastic matrix with $P_{ij} > 0, \forall i, j$. Then P has a unique stationary distribution, and the process is asymptotically stationary.*

Theorem 3. *Let P be a stochastic matrix with $P_{ij}^n > 0, \forall i, j$ and ≥ 1. Then P has a unique stationary distribution, and the process is asymptotically stationary.*

The above two theorems states under what conditions the stochastic matrix has a unique stationary distribution.

One interesting question is how we forecast the state of next period given the information of this period.

$$
\begin{aligned}
E[x_{t+1}|x_t = e_i] &= \sum_{j=1}^{n} Prob(x_{t+1} = e_j|x_t = e_i)e_j \\
&= \sum_{j=1}^{n} P_{ij}e_j = P'_{i,\cdot}
\end{aligned}
$$

where $P_{i,\cdot}$ is the ith row of the stochastic matrix P. Let $y = (y(1), y(2), \cdots, y(n))'$ be an $n \times 1$ vector of real numbers and define $y_t = y'x_t$, so that $y_t = y(i)$ if $x_t = e_i$. This is a more relevant question in macroeconomics as we model the state as y_t instead of the unit vectors x_t.

$$
\begin{aligned}
E[y_{t+1}|y_t = y(i)] &= E(y'x_{t+1}|y'x_t = y(i)) \\
&= y'E(x_{t+1}|x_t = e_i) = y'P'_{i,\cdot} = (Py)_i
\end{aligned}
$$

where $(Py)_i$ is the ith element of the column vector Py. Or you can get the result through the definition of expectation as $E(x_{t+1}|x_t = x_i)$.

$$
\begin{aligned}
E[y_{t+1}|y_t = y(i)] &= \sum_{j=1}^{n} Prob(y_{t+1} = y(j)||y_t = y(i))y(j) \\
&= \sum_{j=1}^{n} Prob(x_{t+1} = e_j|x_t = e_i)y(j) \\
&= \sum_{j=1}^{n} P_{ij}y(j) = (Py)_i
\end{aligned}
$$

Likewise the kth step expectation of y_t is:

$$E(y_{t+k}|y_t = y(i)) = \sum_{j=1}^{n} Prob(y_{t+k} = y(j)|y_t = y(i))y(j) \tag{114}$$

$$= \sum_{j=1}^{n} Prob(x_{t+k} = e_j|x_t = e_i)y(j) \tag{115}$$

$$= \sum_{j=1}^{n} (P^k)_{ij} y_j = (P^k y)_i \tag{116}$$

where the last equality comes from $Prob(x_{t+k} = e_j|x_t = e_i) = (P^k)_{ij}$. Stacking all n rows together, we express this as:

$$E(y_{t+k}|y_t) = P^k y$$

5.2 Consumer's Problem with Stochastic Income

Basically, the model is similar to the two-period model without the government. The representative consumer is endowed with present and future income y and y'. The difference lies in that the two income flows are stochastic.

Let's assume that y, y' both take two possible values $y(1), y(2)$ and the distribution of y is as follows:

$$y = \begin{cases} y(1) & Prob(y = y(1)) = \pi_1 \\ y(2) & Prob(y = y(2)) = \pi_2 \end{cases}$$

where $\pi_1 + \pi_2 = 1$. We also assume that before the present period, the value of present income y is already drawn from the distribution of y. That is, the consumer knows the value of y, $y = y(1)$ or $y = y(2)$ in the present period.

The future income y' is not known in the present period and will be drawn from the distribution of y' in the future period. The transition of y to y' is governed by a Markov chain $((y(1), y(2))', \pi, P)$ and the stochastic matrix P is given by:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{117}$$

where P_{ij} denotes the probability of the future income $y(j)$ when the present income is $y(i)$ and $P_{i1} + P_{i2} = 1, i = 1, 2$. So (y, π, P) is a Markov Chain.

The consumer will choose present consumption c, bonds holding b^d, future consumption $c'(1)$ when future income is $y'(1)$, and future consumption $c'(2)$ when future income is $y'(2)$ given the present income $y(i), i = 1, 2$ to maximize his lifetime expected utility.

$$
\begin{aligned}
\max_{c,b^d,c'(1),c'(2)} \quad & \ln c + P_{i1}\beta \ln c'(1) + P_{i2}\beta \ln c'(2) \\
s.t. \quad & c + b^d = y(i) \\
& c'(1) = y'(1) + (1+r)b^d \\
& c'(2) = y'(2) + (1+r)b^d
\end{aligned}
$$

Form the Lagrangian function as:

$$
\begin{aligned}
L_i(c, b^d, c'(1), c'(2)) = \ln c + P_{i1}\beta \ln c'(1) + P_{i2}\beta \ln c'(2) + \lambda(y(i) - c - b^d) \\
+ \lambda'(1)[y'(1) + (1+r)b^d - c'(1)] + \lambda'(2)[y'(2) + (1+r)b^d - c'(2)]
\end{aligned}
$$

The FOCs are:

$$
\begin{aligned}
\frac{\partial L_i}{\partial c} &= u'(c) - \lambda = 0 \\
\frac{\partial L_i}{\partial b^d} &= -\lambda + \lambda'(1)(1+r) + \lambda'(2)(1+r) = 0 \\
\frac{\partial L_i}{\partial c'(1)} &= P_{i1}\beta u'(c'(1)) - \lambda'(1) = 0 \\
\frac{\partial L_i}{\partial c'(2)} &= P_{i2}\beta u'(c'(2)) - \lambda'(2) = 0
\end{aligned}
$$

Combine these equations:

$$
u'(c) = P_{i1}\beta u'(c'(1))(1+r) + P_{i1}\beta u'(c'(1))(1+r) \tag{118}
$$

This is the Euler condition in the stochastic model. It states similar intuition as the deterministic model that the consumer is trading off present consumption with future consumption. The marginal cost of sacrificed present consumption should be equal to the expected marginal benefit in the future period. Notice that

$$P_{i1}\beta\ln c'(1) + P_{i2}\beta\ln c'(2) = E(\ln c'|y(i))$$

This is the expected utility from stochastic future consumption of the consumer, which depends on the realization of the present income $y(i)$ since P_{i1}, P_{i2} depends on the realization of $y(i)$. This is known as the Markov property. As $y(i)$ stands for the information or the state of what the consumer knows in the present period, we usually denotes the conditional expected utility as

$$E_0(\ln c') = E(\ln c'|y(i)) = P_{i1}\beta\ln c'(1) + P_{i2}\beta\ln c'(2)$$

This E_0 denotes the expectation of $\ln c'$ conditional on information of Period 0 or present period. Likewise, we can use this kind of notation to reorganize the whole consumer's problem.

$$\begin{aligned}
\max_{c,b^d,c'} \quad & E_0(\ln c + \beta\ln c') \\
s.t. \quad & c + b^d = y \\
& E_0 c' = E_0 y' + (1+r)b^d
\end{aligned}$$

Form the Lagrangian function as:

$$L(c,b^d,c') = E_0\big[\ln c + \beta\ln c' + \lambda(y - c - b^d) + \lambda'[E_0 y' + (1+r)b^d - E_0 c']\big]$$

The FOCs are:

$$\begin{aligned}
\frac{\partial L}{\partial c} &= u'(c) - \lambda = 0 \\
\frac{\partial L}{\partial b^d} &= -\lambda + E_0\lambda'(1+r) = 0 \\
\frac{\partial L}{\partial c'} &= E_0[\beta u'(c') - \lambda'] = 0
\end{aligned}$$

Combine these

$$u'(c) = E_0\beta u'(c')(1+r)$$

This is just the Euler equation (118) if we use the conditional expectation notation. If we use time t to denote the present period and use $t+1$ to denote future period, then the Euler equation is:

$$u'(c_t) = E_t \beta u'(c(t+1))(1+r)$$

This is a more familiar Euler equation in macroeconomics. Basically, we will not explicitly states the uncertainty structure of this economy such as the initial distribution π of y and the stochastic matrix P. You should know that c_{t+1} is stochastic and depends on time t information: the realization of the current state y.