Midterm Exam

1. Consider a model with work-leisure tradeoff. The representative consumer has a preference of consumption and leisure given by $u(c,l) = \ln c + \theta \ln l$ where c is the consumption and l is the leisure constrained by the representative consumer's time endowment h. The rest time $n^s = h - l$ is the consumer's labor supply hired the representative firm owned by the consumer. The consumer buys the consumption goods c with his wage wn^s and the profit π redistributed from the firm. The representative firm has a linear technology that $y = zn^d$, where y is the production, z is the productivity, and n^d is the only factor hired by the firm. The firm has market power in the labor market so that the wage w the firm pays to the worker is a decreasing function of labor demand n^d. We assume that $w(n^d) = \bar{w} - \frac{1}{2}n^d$ where $\bar{w} < z < \bar{w} + h$.

   (a) Formulate the consumer's problem and the firm's problem.

   The representative consumer's problem is:

$$\max_{c,l} \quad \ln c + \theta \ln l$$
$$s.t. \quad c \leq (h-l)w + \pi$$

   The representative firm's problem is:

$$\max_{n^d} \quad \pi = zn^d - w(n^d)n^d$$

   (b) Solve the consumer's problem and the firm's problem. Show the marginal rate of substitution of leisure for consumption.

   For the consumer's problem, formulate the Lagrangian function:

$$L = \ln c + \theta \ln l + \lambda((h-l)w + \pi - c)$$

   The FOCs are:

$$L_c = \frac{1}{c} - \lambda = 0$$
$$L_l = \theta \frac{1}{l} - \lambda w = 0$$
$$L_\lambda = (h-l)w + \pi - c = 0$$

These give the two equilibrium conditions.

$$\frac{1}{c} = \theta\frac{1}{l}w$$
$$c = (h - l)w + \pi$$

where the first one shows the consumption-leisure tradeoff of the consumer and the second one is the budget constraint. We can solve c, l as

$$c = \frac{hw + \pi}{1 + \theta w^2}$$
$$l = \frac{\theta w^2 h + \theta w\pi}{1 + \theta w^2}$$

The marginal rate of substitution of leisure for consumption is:

$$MRS_{l,c} = \frac{\frac{1}{c}}{\theta\frac{1}{l}} = w$$

For the firm's problem, we can substitute the wage equation of the firm with market power and take the derivative w.r.t. n^d.

$$z - (\bar{w} - \frac{1}{2}n^d) + \frac{1}{2}n^d = 0$$

This gives us the equilibrium condition of labor demand.

$$n^d = z - \bar{w}$$

and

$$\pi = \frac{3}{2}(z - \bar{w})^2$$

(c) Show the market clearing conditions of goods market and labor market. Show that Walras law holds in this model.

In the equilibrium, the goods market and labor market clear.

$$y = c$$
$$n^s = n^d$$

The Walras law holds that either market clearing implies the clearing of the other market. Let's start with market clearing condition of goods market that $y = c$.

Then

$$c = y = zn^d = (h - l)w + \pi = n^s w + zn^d - wn^d$$
$$w(n^s - n^d) = 0 \Rightarrow n^s = n^d$$

If we start with labor market clearing condition that $n^s = n^d$. Then start with the consumer's budget constraint:

$$c = (h - l)w + \pi = n^s w + y - wn^d = y \Rightarrow c = y$$

(d) Define the competitive equilibrium. Solve the CE.

The competitive equilibrium is defined as an allocation $(c^*, l^*, n^{s*}, y^*, n^{d*}, \pi^*)$ and a price system w^* such that:

- Given the price system w^*, the allocation solves the consumer's problem.
- Given the price system w^*, the allocation solves the firm's problem.
- All markets clear.

Let's solve the CE with seven unkonwns and seven equations (one market clearing condition is redundant).

$$\begin{aligned}
c &= \frac{hw + \pi}{1 + \theta w^2} \\
l &= \frac{\theta w^2 h + \theta w \pi}{1 + \theta w^2} \\
n^s &= h - l \\
n^d &= z - \bar{w} \\
\pi &= \frac{3}{2}(z - \bar{w})^2 \\
y &= zn^d \\
n^s &= n^d
\end{aligned}$$

The CE is:

$$
\begin{aligned}
c^* &= \frac{hw^* + \pi^*}{1 + \theta w^{*2}} \\
l^* &= \frac{\theta w^{*2} h + \theta w^* \pi^*}{1 + \theta w^{*2}} \\
n^{*s} &= z - \bar{w} \\
n^{*d} &= z - \bar{w} \\
\pi^* &= \frac{3}{2}(z - \bar{w})^2 \\
y^* &= z(z - \bar{w}) \\
w^* &= \frac{-\frac{3}{2}\theta(z - \bar{w})^2 + \sqrt{\frac{9}{4}\theta^2(z - \bar{w})^4 + 4(z - \bar{w})(h - (z - \bar{w}))}}{2\theta(z - \bar{w})}
\end{aligned}
$$

(e) Define the Pareto optimum. Solve the PO.

The Pareto optimum is defined as an allocation $(\hat{c}, \hat{l})$ such that the allocation solves the social planner's problem:

$$
\begin{aligned}
\max_{c,l} \quad & \ln c + \theta l \\
s.t. \quad & c \leq z(h - l)
\end{aligned}
$$

Formulate the Lagrangian function:

$$
L = \ln c + \theta l + (z(h - l) - c)
$$

$$
\begin{aligned}
L_c &= \frac{1}{c} - \lambda = 0 \\
L_l &= \theta \frac{1}{l} - \lambda z = 0 \\
L_\lambda &= z(h - l) - c = 0
\end{aligned}
$$

The allocation is:

$$
\begin{aligned}
\hat{c} &= \frac{zh}{1 + \theta z^2} \\
\hat{l} &= \frac{zh}{1 + \theta z^2}
\end{aligned}
$$

(f) What is the PPF in this model? Show the marginal rate of transformation of leisure for consumption. The PPF is:

$$PPF := \{(c, l) | c = z(h - l)\}$$

The marginal rate of transformation of leisure for consumption is minus slope of the PPF:

$$MRT_{l,c} = -\frac{dc}{dl} = z$$

(g) Compare the CE and PO. Explain why.

Clearly $c^* \neq \hat{c}$ and $l^* \neq \hat{l}$. So CE is not a PO in this model and PO is not a CE. Then the two fundamental theorems of welfare economics do not hold in this model. We can see that $MRS_{l,c} = w^* \neq z = MRT_{l,c}$. This is because the firm has market power in the labor market so the two theorems do not hold.

2. Comment on the following paragraph: "If the optimal growth model (Ramsey model) is meant to describe reality, it implies that we are living in an inherently unstable world. The saddle-path in the model is a knife-edge solution; even though the economy may just depart slightly from the saddle-path it will not return to equilibrium and will instead either explode or collapse."

   This comment is wrong. The phase diagram is drawn by the first order conditions of the optimal growth model, namely the Euler equation and the resource constraint. The FOCs are not alone the necessary and sufficient conditions for the solution. Both the FOCs and transversality condition are necessary and sufficient for the solution. So the paths depicted in the phase diagram are not necessary the solution path of the model but the path that satisfy the transversality condition simultaneously is the path of the solution, which is the saddle path. So for the optimal growth model, the solution of the saddle path is stable in terms of converging to the steady state. For any given k_t, c_t will be chosen on the saddle path which converges to the steady state. Any other chosen c_t doesn't satisfy the transversality condition, which is not the solution. Real world, if depicted by optimal growth model, is stable and not a knife-edge world.

3. Consider an optimal growth model with two sectors. A social planner seeks to maximize the utility of the representative consumer given by $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, where c_t is consumption at time t, l_t is leisure at time t. Sector 1 produces consumption goods using capital k_{1t} and labor n_{1t} according to the production function $c_t = f^1(k_{1t}, n_{1t})$. Sector 2 produces the capital goods according to the production function $k_{t+1} = f^2(k_{2t}, n_{2t})$. Total employment $n_t = n_{1t} + n_{2t}$,and leisure, l_t ,is constrained by the endowment of time, h, and satisfies $l_t + n_t = h$. Total capital is the sum of the amounts of capital used in each sector, that is, $k_t = k_{1t} + k_{2t}$, $k_0 > 0$ given.

   (a) Formulate the social planner's problem as a sequence problem.

The social planner's problem is:

$$\max_{\{c_t\}_{t=0}^{\infty},\{l_t\}_{t=0}^{\infty},\{n_t\}_{t=0}^{\infty},\{n_{1t}\}_{t=0}^{\infty},\{n_{2t}\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty},\{k_{1t}\}_{t=0}^{\infty},\{k_{2t}\}_{t=0}^{\infty}} \quad \sum_{t=1}^{\infty} u(c_t, l_t)$$

$$\begin{aligned} s.t. \quad & c_t = f^1(k_{1t}, n_{1t}) \\ & k_{t+1} = f^2(k_{2t}, n_{2t}) \\ & n_{1t} + n_{2t} = n_t \\ & l_t + n_t = h \\ & k_{1t} + k_{2t} = k_t \\ & k_0, given \end{aligned}$$

(b) Formulate this problem as a dynamic programming problem. Clearly specify the state and control variables, and display the functional equation.

The state variable is k_t and the choice variables are $c_t, k_{t+1}, k_{1t}, k_{2t}, n_t, l_t, n_{1t}, n_{2t}$. The functional equation is:

$$v(k) = \max_{c,k',k_1,k_2,n,l,n_1,n_2} \quad u(c,l) + \beta v(k')$$

$$\begin{aligned} s.t. \quad & c = f^1(k_1, n_1) \\ & k' = f^2(k_2, n_2) \\ & n_1 + n_2 = n \\ & l + n = h \\ & k_1 + k_2 = k \\ & k, given \end{aligned}$$

(c) List the first order conditions of this problem.

Formulate the Lagrangian function as:

$$\begin{aligned} L = {} & u(c,l) + \beta v(k') + \lambda_1(f^1(k_1,n_1) - c) + \lambda_2(f^2(k_2,n_2) - k') + \lambda_3(n - n_1 - n_2) \\ & + \lambda_4(h - l - n) + \lambda_5(k - k_1 - k_2) \end{aligned}$$

The FOCs are:

$$
\begin{aligned}
L_c &= u_c - \lambda_1 = 0 \\
L_l &= u_l - \lambda_4 = 0 \\
L_{k'} &= -\lambda_2 + \beta v'(k') = 0 \\
L_{k_1} &= \lambda_1 f^1_{k_1} - \lambda_5 = 0 \\
L_{k_2} &= \lambda_2 f^2_{k_2} - \lambda_5 = 0 \\
L_n &= \lambda_3 - \lambda_4 = 0 \\
L_{n_1} &= \lambda_1 f^1_{n_1} - \lambda_3 = 0 \\
L_{n_2} &= \lambda_2 f^2_{n_2} - \lambda_3 = 0 \\
L_{\lambda_1} &= f^1(k_1, n_1) - c = 0 \\
L_{\lambda_2} &= f^2(k_2, n_2) - k' = 0 \\
L_{\lambda_3} &= n - n_1 - n_2 = 0 \\
L_{\lambda_4} &= h - l - n = 0 \\
L_{\lambda_5} &= k - k_1 - k_2 = 0
\end{aligned}
$$

The envelope theorem is:

$$
v'(k) = \frac{\partial L}{\partial k} = \lambda_5
$$

We know that $\lambda_5 = \lambda_1 f^1_{k_1} = u_c f^1_{k_1}$. Then

$$
\begin{aligned}
v'(k) &= u_c f^1_{k_1} \\
v'(k') &= u_c(c', l') f^1_{k_1}(k_1', n_1')
\end{aligned}
$$

With $\lambda_2 = \frac{\lambda_5}{f^2_{k_2}} = \frac{u_c f^1_{k_1}}{f^2_{k_2}}$, we have:

$$
\lambda_2 = \frac{u_c(c, l) f^1_{k_1(k_1, n_2)}}{f^2_{k_2}(k_2, n_2)} = \beta u_c(c', l') f^1_{k_1}(k_1', n_1')
$$

The solution $\{c_t\}_{t=0}^{\infty}, \{l_t\}_{t=0}^{\infty}, \{n_{1t}\}_{t=0}^{\infty}, \{n_{2t}\}_{t=0}^{\infty}, \{n_t\}_{t=0}^{\infty}, \{k_{t+1}\}_{t=0}^{\infty}, \{k_{1t}\}_{t=0}^{\infty}, \{k_{2t}\}_{t=0}^{\infty}$

is characterized by the following equations.

$$
\begin{aligned}
&u_c(c_t, l_t) = \beta u_c(c_{t+1}, l_{t+1}) \frac{f^2_{k_2}(k_{2t}, n_{2t})}{f^1_{k_1}(k_{1t}, n_{1t})} f^1_{k_1}(k_{1t+1}, n_{1t+1}) \\
&u_l(c_t, l_t) = u_c(c_t, l_t) f^1_{n_1}(k_{1t}, n_{1t}) \\
&f^1_{n_1}(k_{1t}, n_{1t}) = \frac{f^1_{k_1}(k_{1t}, n_{1t})}{f^2_{k_2}(k_{2t}, n_{2t})} f^2_{n_2}(k_{2t}, n_{2t}) \\
&c_t = f^1(k_{1t}, n_{1t}) \\
&k_{t+1} = f^2(k_{2t}, n_{2t}) \\
&n_{1t} + n_{2t} = n_t \\
&l_t + n_t = h \\
&k_{1t} + k_{2t} = k_t \\
&k_0, given \\
&\lim_{t \to \infty} \beta^t v'(k_t) k_t = 0
\end{aligned}
$$

(d) State the Euler equation of the problem. Interpret the economic intuition behind the condition.

The Euler equation is:

$$u_c(c_t, l_t) f^1_{k_1}(k_{1t}, n_{1t}) = \beta u_c(c_{t+1}, l_{t+1}) f^1_{k_1}(k_{1t+1}, n_{1t+1}) f^2_{k_2}(k_{2t}, n_{2t})$$

Similarly, this equation still shows the intertemporal substitution of present consumption for future consumption to maximize life time sum of discounted utility. The LHS is the marginal cost of sacrificing one unit of capital in Sector 1. The RHS of the marginal benefit of sacrificing one unit of capital in Sector 1.

- Decreasing one unit of capital in Sector 1 means $f^1_{k_1}(k_{1t}, n_{1t})$ unit of consumption decrease because Sector 1 produces consumption goods. The marginal utility of consumption goods is $u_c(c_t, l_t)$. So the marginal cost of sacrificing one unit of capital in Sector 1 is $u_c(c_t, l_t) f^1_{k_1}(k_{1t}, n_{1t})$.
- Decreasing one unit of capital in Sector 1 means increasing one unit of capital in Sector 2, which increases the capital of next period by $f^2_{k_2}(k_{2t}, n_{2t})$. Other things constant, the increase of future capital increases future consumption goods by $f^1_{k_1}(k_{1t+1}, n_{1t+1}) f^2_{k_2}(k_{2t}, n_{2t})$. Then the marginal benefit is $\beta u_c(c_{t+1}, l_{t+1}) f^1_{k_1}(k_{1t+1}, n_{1t+1}) f^2_{k_2}(k_{2t}, n_{2t})$.

(e) State the transversality condition of this problem and interpret its economic meaning.

The transversality condition is:

$$\lim_{t \to \infty} \beta^t v'(k_t) k_t = 0$$

It means that the marginal utility of capital goes to zero asymptotically when time goes to infinity. We can not leave capital behind when time goes to farthest future.

(f) Characterize the solution of the social planner's problem.

The solution $\{c_t\}_{t=0}^{\infty}$, $\{l_t\}_{t=0}^{\infty}$, $\{n_{1t}\}_{t=0}^{\infty}$, $\{n_{2t}\}_{t=0}^{\infty}$, $\{n_t\}_{t=0}^{\infty}$, $\{k_{t+1}\}_{t=0}^{\infty}$, $\{k_{1t}\}_{t=0}^{\infty}$, $\{k_{2t}\}_{t=0}^{\infty}$ is characterized by the following equations.

$$u_c(c_t, l_t) = \beta u_c(c_{t+1}, l_{t+1}) \frac{f^2_{k_2}(k_{2t}, n_{2t})}{f^1_{k_1}(k_{1t}, n_{1t})} f^1_{k_1}(k_{1t+1}, n_{1t+1})$$

$$u_l(c_t, l_t) = u_c(c_t, l_t) f^1_{n_1}(k_{1t}, n_{1t})$$

$$f^1_{n_1}(k_{1t}, n_{1t}) = \frac{f^1_{k_1}(k_{1t}, n_{1t})}{f^2_{k_2}(k_{2t}, n_{2t})} f^2_{n_2}(k_{2t}, n_{2t})$$

$$c_t = f^1(k_{1t}, n_{1t})$$

$$k_{t+1} = f^2(k_{2t}, n_{2t})$$

$$n_{1t} + n_{2t} = n_t$$

$$l_t + n_t = h$$

$$k_{1t} + k_{2t} = k_t$$

$$k_0, given$$

$$\lim_{t\to\infty} \beta^t v'(k_t) k_t = 0$$