Problem Set 2

Due Date 2020.11.10

1. Consider the two-period model with consumption-saving tradeoff. In the consumer's problem, the consumer chooses c, c', b^d to maximize his lifetime utility.

$$
\begin{aligned}
\max_{c,b^d,c'} \quad & U(c,c') = u(c) + \beta u(c') && (1)\\
s.t. \quad & c + b^d = y - t && (2)\\
& c' = y' - t' + (1+r)b^d && (3)\\
& y, y', t, t', r, \quad given && (4)
\end{aligned}
$$

where $u(c) = \ln c$.

(a) Solve the consumer's problem.

(b) Show the comparative statics of the effect of the real interest rate r. Decompose the substitution effect and income effect of r.

2. Consider a 3-state Markov chain $\{\{e_i\}_{i=1}^3, \pi_0, P\}$ where $\pi_0 = (0.1, 0.2, 0.7)'$ and

$$
P = \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.4 & 0.5 & 0.1 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \tag{5}
$$

(a) As every number of P is positive, we know that this Markov chain has a unique stationary distribution. Calculate what the stationary distribution π_∞ is.

(b) Use matlab to prove the above result. (Hint: let $\pi_0 = (0.1, 0.2, 0.7)'$. Write a for loop to simulate $\pi_1, \cdots, \pi_T$ by using the equality $\pi'_{t+1} = \pi'_t P$ and then you can see that π_t is nearly the same as t goes to a large number.)

(c) Now try different initial distribution π_0, e.g., try $\pi_0 = (1, 0, 0)'$, $\pi_0 = (0, 0, 1)'$,$\pi_0 = (0.33, 0.33, 0.34)'$. Show that when t goes to a large number all the unconditional distributions go to the same one.

(d) We know that if every element of P is positive, there is a unique asymptotic stationary distribution. What if there is zero in P? Let's assume that

$$
P = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}
$$

Show that there are two independent eigenvector $(1,0,0)'$ and $(0,0,1)'$ associated with eigenvalue 1 for P.

(e) These two eigenvectors are both stationary for some initial unconditional distribution. Show that when $\pi_0 = (1,0,0)'$, $\pi_{t+1} = \pi_t = (1,0,0)'$, and when $\pi_0 = (0,0,1)'$, $\pi_{t+1} = \pi_t = (0,0,1)'$. Use Matlab to simulate π_t to show the above analytical results.

(f) Show that $\alpha*(1,0,0)'+(1-\alpha)*(0,1,1)'$ is also stationary for the initial distribution $\pi_0 = \alpha*(1,0,0)' + (1-\alpha)*(0,1,1)'$ where $\alpha \in [0,1]$.

(g) From the above two subquestions, what conclusions can you get?

3. (LS Excercise 2.1) Consider the Markov chain $\{\{e_i\}_{i=1}^2, \pi_0, P\}$, where $\pi_0 = (0.5, 0.5)'$ and

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} \tag{7}$$

and a random variable $y_t = \bar{y}x_t$ where $\bar{y} = (1,5)'$. Compute the likelihood (the probability of the data sample given the parameters) following three histories for $y_t, t = 0,1,2,3,4$. (Hint: calculate $Prob(y_0, y_1, y_2, y_3, y_4|\pi_0, P)$)

(a) 1,5,1,5,1.

(b) 1,1,1,1,1.

(c) 5,5,5,5,5.

4. Consider the optimal growth model with finite horizon. Different from the notes, we assume that capital depreciates at a rate of δ. And the utility function is $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$. So the social planner's problem is:

$$\max_{\{c_t\}_{t=0}^T, \{k_{t+1}\}_{t=0}^T} \quad \sum_{t=0}^{T} u(c_t) \tag{8}$$

$$s.t. \quad c_t + k_{t+1} \leq Ak_t^\alpha + (1-\delta)k_t, t = 0,1,\cdots,T \tag{9}$$

$$c_t, k_{t+1} \geq 0, t = 0,1,\cdots,T \tag{10}$$

$$k_0, given \tag{11}$$

(a) Solve the social planner's problem and get the characterization equations for the solution.

(b) Modify the Matlab program of the lecture's case and numerically solve the social planner's problem. Assume $A = 5, \alpha = 1/3, \beta = 0.99, \delta = 0.04, T = 100, k_0 = \frac{1}{3}\bar{k}$.

(c) Numerically (using Matlab) show that the sum of discounted utility when $k_{T+1} = 0$ is bigger than that when $k_{T+1} > 0$.

5. Consider the above optimal growth model but let's assume infinite horizon and other setups are the same.

(a) Show the social planner's problem.

(b) Show the Bellman equation for this model. Characterize the solution of this model. What are the Euler equation and transversality condition for the solution?

(c) Use phase diagram to show the dynamics of the solution.

(d) Assume the economy stays at the steady state first. Suddenly there is a time preference shock to the economy that people become more patient (Hint: β increases). Use phase diagram to show the transition dynamics of the solution. Verbally explain the economic intuition of the transition dynamics.

6. Consider a cake-eating problem. At the beginning of Period 0, there is cake with size a_0. In each period with left cake of size a_t, the consumer decides how much cake to eat c_t and the rest will be the cake of next period a_{t+1}. With consumption c_t, the consumer gets utility $u(c_t) = \ln(c_t)$. The consumer discounts his future utility by β.

   (a) If the economy lasts only T periods, then the consumer lives at the end of Period T and then the economy ends. The consumer's problem is to choose consumption c_t of every period and cake size left a_{t+1} to next period to maximize his lifetime utility. Show the consumer's problem.

   (b) Show that $c_0, c_1, c \dots, c_T, a_1, \cdots, a_T$ should be positive for the solution of the consumer's problem.

   (c) Apply the Kuhn-Tucker theorem to solve the consumer's problem. Interpret the economic meaning of the Lagrangian multiplier of each constraint.

   (d) If the Lagrangian multiplier associated with constraint $a_{T+1} \geq 0$ is μ_T, what does the condition $\mu_T a_{T+1} = 0$ mean economically.

   (e) Modify the matlab program of finite-horizon optimal growth model and solve numerically the cake-eating problem (Assume $a_0 = 2, \beta = 0.95$, T=100). Show a plot of all cake size $a_0, a_1, \cdots, a_T, a_{T+1}$. What change can you see if β increases?

   (f) Now assume infinite horizon from now on. Show the functional equation for the cake-eating problem. Characterize the solution of the model. What are the Euler equation and transversality condition for the solution?

   (g) Use phase diagram to show the dynamics of the solution. (Hint: you may notice that there is no steady state in this problem. There is only one phase in this model.)