Answer of Problem Set 1

1. Matlab basic operations: you should attach your Matlab codes and associated results in the answer.

   (a) Construct a vector from 0 to 100. The difference between two subsequent elements in the vector is 0.2. Count how many elements there are.

   (b) Construct an 5×5 identity matrix. Construct a 4×1 vector with all elements 1. Find the Kronecker product of $I_4 \otimes \mathbf{1}$.

   (c) Use for loop and while loop to find the sum of natural numbers from 1 to 100.

   (d) Write a function to find the sum of natural numbers from 1 to "N". (Basically, the argument of this function "sumN" is "n". Whenever you use this function, e.g. the sum from 1 to 20, just type "sumN(20)" and then you can get the sum.)

   (e) Use help in Matlab to learn the "find" function. What can this function do.

2. In this problem you will derive the HP filter and create your own code to implement it. Suppose you have a sequence of data, y_t, with $t = 1, \cdots, T$. Our objective is to find a trend $\{\tau\}_{t=1}^{T}$ to minimize the following objective function:

$$\min_{\tau_t} \quad \sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2, \quad \lambda \geq 0$$

   (a) Provide a verbal interpretation of this objective function and what exactly one is trying to get at when choosing a trend, $\{\tau\}_{t=1}^{T}$.

   The HP filter tries to find a time series trend that minimizes the difference of the time series and the trend component panelized by a volatile trend. That is, the HP filter balances the difference between the series and trend with the difference between the trend's change. If the trend follows closely with the time series, the difference between the trend's change will be high, and vice versa.

   (b) Prove that, if $\lambda = 0$, then the solution is $\tau_t = y_t, \forall t$. In the other words, the trend and the actual series would be identical.

   If $\lambda = 0$, the objective function is:

$$\min_{\tau_t} \quad \sum_{t=1}^{T}(y_t - \tau_t)^2$$

Minimization results $\tau_t = y_t$.

(c) Prove that, as $\lambda \to \infty$, then the solution is for the trend to be a linear time trend, i.e. for $\tau_t = \beta + \alpha t$ for some α and β.

If $\lambda \to \infty$, the second term of the objective will be infinity unless $\sum_{t=2}^{T-1}[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 = 0$. So

$$\tau_{t+1} - \tau_t = \tau_t - \tau_{t-1}$$

Let $\tau_t - \tau_{t-1} = \alpha$. Then $\tau_t = \alpha(t-1) + \tau_1 = \alpha t + (\tau_1 - \alpha) = \alpha t + \beta$.

(d) For the more general case in which $0 < \lambda < \infty$, derive analytical solutions for the trend. (Hint: notice that this a unconstrained optimization problem. When the objective is convex in τ_t. You should find the optimizer by making every derivative with respect to τ_t zero and then you can find the solution.)

Find the first order conditions for $t = 1, \cdots, T$.

$$\begin{aligned}
&-2(y_t - \tau_t) + 2\lambda(\tau_t - 2\tau_{t-1} + \tau_{t-2}) - 4\lambda(\tau_{t+1} - 2\tau_t + \tau_{t-1}) \\
&+ 2\lambda(\tau_{t+2} - 2\tau_{t+1} + \tau_t) = 0, \quad 3 \le t \le T-2
\end{aligned}$$

For $t = 1$, $t = 2$, $t = T-1$, and $t = T$,

$$\begin{aligned}
&-2(y_1 - \tau_1) + 2\lambda(\tau_3 - 2\tau_2 + \tau_1) = 0 \\
&-2(y_2 - \tau_2) + 2\lambda(\tau_3 - 2\tau_2 + \tau_1)(-2) + 2\lambda(\tau_4 - 2\tau_3 + \tau_2) = 0 \\
&2\lambda(\tau_{T-1} - 2\tau_{T-2} + \tau_{T-3}) - 2(y_{T-1} - \tau_{T-1}) + 2\lambda(\tau_T - 2\tau_{T-1} + \tau_{T-2})(-2) = 0 \\
&-2(y_T - \tau_T) + 2\lambda(\tau_T - 2\tau_{T-1} + \tau_{T-2}) = 0
\end{aligned}$$

Let $\Gamma = (\tau_1, \cdots, \tau_T)'$, $Y = (y_1, \cdots, y_T)'$, and

$$\Lambda = \begin{bmatrix}
1+\lambda & -2\lambda & \lambda & 0 & \cdots & \cdots & 0 \\
-2\lambda & 1+5\lambda & -4\lambda & \lambda & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \lambda & -4\lambda & 1+5\lambda & -2\lambda \\
0 & \cdots & \cdots & 0 & \lambda & -2\lambda & 1+\lambda
\end{bmatrix}$$

And all the first order conditions are expressed as:

$$\Lambda\Gamma = Y$$

Then we can get the trend

$$\Gamma = \Lambda^{-1}Y$$

(e) Write a Matlab code to find the trend taking the actual data series y_t. To do this, express the first order conditions in (d) in matrix form as follows:

$$\begin{aligned}\Lambda\Gamma &= Y \\ \Gamma &= \Lambda^{-1}Y\end{aligned}$$

where Λ is a $T \times T$ matrix whose elements are functions of λ; Y is a $T \times 1$ vector of actual data. Γ is the vector of τ_t. Download quarterly and seasonly adjusted US real GDP data from 1947:Q1 to 2020:Q2 (FRED: https://fred.stlouisfed.org/). Take natural logs of the data and then use your code to compute the HP trend using $\lambda = 1600$. Show a plot of both the real GDP and the trend. Show a plot of the cyclical component. What is the standard deviation of the HP detrended GDP? How should you interpret this number?

See the matlab program 'hp.m'. The standard deviation of detrended GDP is 0.0169. It means the percentage change from the trend is averagely 1.69%.

3. Consider a more simplified one-period model without government. The consumer's problem is:

$$\begin{aligned}\max_{C,l} \quad & \ln C + \theta \ln l \\ s.t. \quad & C + wl = wh + \pi\end{aligned}$$

The firm is endowed with linear technology $y = zN^d$ instead of the Cobb-Douglous technology in the lecture notes.

$$\max_{N^d} \quad zN^d - wN^d$$

(a) Solve the consumer's problem. What is the marginal substitution rate of the leisure for consumption?

Formulate the Lagrangian function:

$$L(C, l, \lambda) = \ln C + \theta \ln l + \lambda(wh + \pi - C - wl)$$

The first order conditions are:

$$\begin{aligned}
L_C &= \frac{1}{C} - \lambda = 0 \\
L_l &= \frac{\theta}{l} - \lambda w = 0 \\
L_\lambda &= wh + \pi - C - wl = 0
\end{aligned}$$

The consumer considers w, π as given:

$$\begin{aligned}
C^* &= \frac{1}{1+\theta}(wh + \pi) \\
l^* &= \frac{\theta}{1+\theta}\frac{wh+\pi}{w}
\end{aligned}$$

$$MRS_{lC} = -\frac{dC}{dl} = \frac{U_l}{U_C} = w$$

(b) Solve the firm's problem.

$$\max_{N^d} \quad zN^d - wN^d$$

The profit of the firm is linear in N^d considering w as given. So

$$N^{d*} = \begin{cases} h & z > w \\ any & z = w \\ 0 & z < w \end{cases}$$

When $z = w$, the firm may choose any number between 0 and h, which has a profit of 0.

(c) Define the competitive equilibrium.

The competitive equilibrium is defined as an allocation $(C^*, l^*, N^{d*}, \pi^*)$ and a price system w^* such that:

- the allocation solves the the consumer's problem given the price system; the allocation solves the firm's problem given the price system
- all markets are cleared.

$$\begin{aligned}
C &= zN^d \\
h - l &= N^d
\end{aligned}$$

You can solve the equilibrium by clearing the labor market $h - l = N^d$. Under this market clearing condition, either $z > w$ and $z < w$ doesn't work. For example, if $z > w$, then $N^d = h$ and $h - l = h$ and we have $l = 0$. This is not equal to $l = \frac{\theta}{1+\theta}\frac{wh+(z-w)h}{w}$. You can also try to analyze when $z < w$, then $N^d = 0$ and $l = h$ and this is not equal to $l = \frac{\theta}{1+\theta}\frac{wh+0}{w}$. So at the CE, we should have $w^* = z$ and $\pi^* = 0$.

$$
\begin{aligned}
C^* &= \frac{1}{1+\theta}zh \\
l^* &= \frac{\theta}{1+\theta}h \\
N^{d*} &= \frac{1}{1+\theta}h
\end{aligned}
$$

(d) Define the social optimum of this problem. Write down the PPF. What is the marginal rate of transformation? Explain verbally why the PPF is not concave but linear or why the MRT is constant in the problem.

The social planner solves the following problem:

$$
\begin{aligned}
\max_{C,l} \quad & \ln C + \theta \ln l \\
s.t. \quad & C = z(h-l)
\end{aligned}
$$

$$
L(C, l, \lambda) = \ln C + \theta \ln l + \lambda(z(h-l) - C)
$$

The first order conditions are:

$$
\begin{aligned}
L_C &= \frac{1}{C} - \lambda = 0 \\
L_l &= \frac{\theta}{l} - \lambda z = 0 \\
L_\lambda &= z(h-l) - C = 0
\end{aligned}
$$

We can solve the problem directly.

$$
\begin{aligned}
\bar{C} &= \frac{1}{1+\theta}zh \\
\bar{l} &= \frac{\theta}{1+\theta}h
\end{aligned}
$$

The social optimum is defined as an allocation $(\bar{c}, \bar{l})$ such that the allocation solves the social planner's problem.
The PPF is:

$$
PPF := \{(C, l) | C = z(h-l)\}
$$

We can see that the PPF is linear in l. This is determined by the linear technology of the firm's production function instead of the diminishing marginal product implied by the Cobb-Douglous production function.

$$MRT_{lC} = -\frac{dC}{dl} = z$$

(e) Compare the PO and CE. Show that CE is a PO in this problem. Check that $MRS = MRT$ in this case.

Compare the equilibrium conditions of CE and PO and we can find that they are the same. So CE is also a PO. In the CE, $w^* = z$. So $MRS_{lC} = MRT_{lC} = z$ for the CE.

(f) Apply the implicit function theorem to find the comparative statics of C^*, l^* with respect to z. Disentangle the substitution effect and income effect.

As we can get closed-form solution in this problem, we can directly get the derivatives.

$$\begin{aligned}\frac{dC^*}{dz} &= \frac{h}{1+\theta}\\ \frac{dl^*}{dz} &= 0\end{aligned}$$

Up to the technology increase, the equilibrium consumption increases but leisure doesn't change. We can decompose the aggregate effect into substitution effect and income effect. When technology increases, the equilibrium wage w^*, which is the price of leisure, increases. The substitution effect is the effect along the indifference curve where the original equilibrium point lies. We can find the substitution effect from the two conditions:

$$\begin{aligned}U^* =& U(C^*, l^*)\\ z =& \frac{U_l(C^*, l^*)}{U_C(C^*, l^*)}\end{aligned}$$

Apply the implicit function theorem:

$$\begin{aligned}0 =& U_C(C^*, l^*)dC^* + U_l(C^*, l^*)dl^*\\ U_{lC}dC^* + U_{ll}dl^* =& U_C dz + z(U_{CC}dC^* + U_{Cl}dl^*)\end{aligned}$$

Substitute all the partial derivatives:

$$\begin{aligned}\frac{dC^*}{dz}_{subs} =& \frac{\theta h}{(1+\theta)^2}\\ \frac{dl^*}{dz}_{subs} =& -\frac{\theta h}{(1+\theta)^2 z}\end{aligned}$$

$$
\begin{aligned}
\left.\frac{dC^*}{dz}\right|_{income} &= \frac{dC^*}{dz} - \left.\frac{dC^*}{dz}\right|_{subs} = \frac{h}{(1+\theta)^2} \\
\left.\frac{dl^*}{dz}\right|_{income} &= \frac{dl^*}{dz} - \left.\frac{dl^*}{dz}\right|_{subs} = \frac{\theta h}{(1+\theta)^2 z}
\end{aligned}
$$

We can see that the substitution effect for consumption is positive and that for leisure is negative. The income effect for consumption and leisure are both positive. Notice that the substitution effect and income effect of the technology to the leisure cancel out in our case, which is determined by the logarithm utility.

By the way, you can also try to change the utility function to the CRRA utility function $U(C,l) = \frac{C^{1-\gamma}}{1-\gamma} + \frac{l^{1-\gamma}}{1-\gamma}$. You can check that when $\gamma = 1$ the two effects cancel each other out.

4. Consider the economy with a frictional wage income tax t. Other setups are the same as the one-period model in our lecture notes. Basically, the representative consumer solves the problem:

$$
\begin{aligned}
\max_{C,l} \quad & \ln C + \theta \ln l \\
s.t. \quad & C + w(1-t)l = w(1-t)h + \pi - T
\end{aligned}
$$

The representative firm maximizes its profit:

$$
\max_{N^d} \quad zK^\alpha (N^d)^{1-\alpha} - wN^d
$$

The government runs a balanced budget.

$$
G = T + twN^d
$$

(a) Solve the consumer's problem. Describe verbally the tradeoff between consumption and leisure of the consumer by the equilibrium condition. What is the marginal substitution rate of the leisure for consumption?

Formulate the Lagrangian function:

$$
L(C,l,\lambda) = \ln C + \theta \ln l + \lambda(w(1-t)h + \pi - T - C - w(1-t)l)
$$

The first order conditions are:

$$
\begin{aligned}
L_C &= \frac{1}{C} - \lambda = 0 \\
L_l &= \frac{\theta}{l} - \lambda(1-t)w = 0 \\
L_\lambda &= w(1-t)h + \pi - T - C - w(1-t)l = 0
\end{aligned}
$$

We can solve the problem directly.

$$\begin{aligned}\frac{\theta C}{l} &= (1-t)w \\ C + w(1-t)l &= w(1-t)h + \pi - T\end{aligned}$$

The consumer's equilibrium condition shows the key tradeoff that the marginal rate of substitution of leisure for consumption $\frac{\theta C}{l}$ should be equal to the market price of leisure in terms of consumption goods $(1-t)w$. The difference is that the effective wage rate the consumer receives is now after-price wage $(1-t)w$.

$$MRS_{lC} = (1-t)w$$

(b) Solve the firm's problem. The firm's problem is:

$$\max_{N^d} \quad \pi = zK^{\alpha}(N^d)^{1-\alpha} - wN^d$$

The first order condition is:

$$w = z(1-\alpha)K^{\alpha}(N^d)^{-\alpha}$$

(c) Define the competitive equilibrium.

The CE is defined as an allocation $(C^*, l^*, \pi^*, T^*, N^{d*})$ and a price system w^* such that:

- the allocation solves the consumer's problem given the price system; the allocation solves the firm's problem;
- all markets are cleared that

$$\begin{aligned}zK^{\alpha}(N^d)^{1-\alpha} &= C + G \\ h - l &= N^d\end{aligned}$$

Collect all the equilibrium conditions (one market-clearing condition is omitted):

$$\begin{aligned}\frac{\theta C}{l} &= (1-t)w \\ C + w(1-t)l &= w(1-t)h + \pi - T \\ w &= z(1-\alpha)K^{\alpha}(N^d)^{-\alpha} \\ \pi &= zK^{\alpha}(N^d)^{1-\alpha} - wN^d \\ h - l &= N^d \\ G &= T + twN^d\end{aligned}$$

Substitute T, π, N^d, w into other equations, the system can be collapsed as:

$$\frac{\theta C}{l} = (1-t)z(1-\alpha)K^\alpha(h-l)^{-\alpha}$$
$$C + G = zK^\alpha(h-l)^{1-\alpha}$$

(d) Define the Pareto optimum of this problem. Write down the PPF. What is the marginal rate of transformation?

The PO of this problem is defined as an allocation $\bar{C}, \bar{l}$ such that the allocation solve's the social planner's problem.

$$\max_{C,l} \quad \ln C + \theta \ln l$$
$$s.t. \quad zK^\alpha(h-l)^{1-\alpha} = C + G$$

Formulate the Lagrangian function:

$$L(C, l, \lambda) = \ln C + \theta \ln l + \lambda(zK^\alpha(h-l)^{1-\alpha} - C - G)$$

Then we can get the equilibrium conditions:

$$\frac{\theta C}{l} = z(1-\alpha)K^\alpha(h-l)^{-\alpha}$$
$$C + G = zK^\alpha(h-l)^{1-\alpha}$$

The PO is characterized by the above two conditions.
MRT is the minus slope of the PPF.

$$\{(C, l)|C + G = zK^\alpha(h-l)^{1-\alpha}\}$$

$$MRT_{lC} = -\frac{dC}{dl} = z(1-\alpha)K^\alpha(h-l)^{-\alpha} = w$$

(e) Compare the PO and CE. Show that CE is not a PO in this problem. You can see that why the first theorem of welfare economics does not hold when there is frictional tax? Check that $MRS \neq MRT$ in this case.

Compare the characterization conditions of PO and CE, we can find that they are not the same in the frictional wage income tax. Check the MRT of this problem here. MRT is the minus slope of the PPF.

$$\{(C, l)|C + G = zK^\alpha(h-l)^{1-\alpha}\}$$

$$MRT_{lC} = -\frac{dC}{dl} = z(1-\alpha)K^{\alpha}(h-l)^{-\alpha} = w \neq MRS_{l,C} = (1-t)w$$

And we can find that $MRT_{lC} = \frac{MRS_{lC}}{1-t} \neq MRS_{lC}$ in this problem.