MTH 511 HW 2

Brandyn Tucknott

Due 16 October 2025

1. Let V be a vector space and d a metric on V satisfying $d(x, y) = d(x - y, 0), d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y \in V$ and scalar α. Show that $\|x\| = d(x, 0)$ defines a norm on V. Give an example of a metric on a vector space $\mathbb{R}$ that fails to be associated with a norm in this way.

   Proof. First, we check the given norm satisfies all the properties of norms.

$$\begin{aligned}
\|x\| &= d(x, 0) = 0 \text{ iff } x = 0 \text{ since } d \text{ is a metric,} \\
\|\alpha x\| &= d(\alpha x, 0) = |\alpha| d(x, 0) = |\alpha| \, \|x\| \, , \\
\|x + y\| &= d(x + y, 0) = d(x, -y) \leq d(x, 0) + d(-y, 0) = \|x\| + \|y\| \, ,
\end{aligned}$$

   thus $\|x\|$ is a norm. Now consider the discrete metric. Generally speaking, the property $d(\alpha x, 0) \neq |\alpha| d(x, 0)$ does not hold, so a norm cannot be associated with this metric. $\square$

2. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

   Proof. We show this by contradiction. Let x a limit point of A and suppose $B_\varepsilon(x) \subset A$ contained finitely many points from A. Denote such points as $a_i \in_\varepsilon (x), i = 1, 2, \ldots, n$. Let $D = \min\{a_i\}$. Then for $\varepsilon < D$,

$$(B_\varepsilon(x) \backslash \{x\}) \cap A = \emptyset,$$

   and x is not a limit point. This is a contradiction, so we conclude our assumption that $B_\varepsilon(x)$ contained only finitely many points from A is incorrect. $\square$

3. Let A' be the set of limit points of a set A. Show that A' is closed, $\overline{A} = A' \cup A$, and $A' \cup A \subset\longleftrightarrow A$ is closed.

Proof. First we show that A' is closed. By theorem 4.9, we equivalently show for all $\varepsilon > 0, B_\varepsilon(x) \cap A' \neq \emptyset$, then $x \in A'$. If $x \in A'$ we are done, so suppose $x \in A'$. Then

$$B_\varepsilon(x) \cap A' = (B_\varepsilon(x) \backslash \{x\}) \cap A' \neq \emptyset.$$

But this is the definition of a limit point, so $x \in A'$. This contradicts our original assumption, so it must be that $x \in A'$, and thus A' is closed.

Next, we will show that $\overline{A} = A' \cup A$. Observe that for all $\varepsilon > 0$,

$$\begin{aligned} x \in A &\to x \in \overline{A}, \text{ and} \\ x \in A' &\to (B_\varepsilon(x)) \cap A \neq \emptyset \\ &\to B_\varepsilon(x) \cap A \neq \emptyset, x \in \overline{A} \end{aligned}$$

by proposition 4.10, and so $A' \cup A \subseteq \overline{A}$.

Similarly, for all $\varepsilon > 0$, we have that

$$x \in \overline{A} \to B_\varepsilon(x) \cup A \neq \emptyset.$$

Here we have two cases. If $x \in A$, then certainly $x \in A' \cup A$, and we are done. Suppose that $x \notin A$. Then by proposition 4.10, for all $\varepsilon > 0, B_\varepsilon(x) \cap A = (B_\varepsilon(x) \backslash \{x\}) \cap A \neq \emptyset$, which by definition of a limit points means $x \in A'$. Thus $\overline{A} \subseteq A' \cup A$.

Since $\overline{A} \subseteq A' \cup A$ and $A' \cup A \subseteq \overline{A}$, we conclude $\overline{A} = A' \cup A$.

It remains to show that $A' \subset A \longleftrightarrow A$ is closed. First we show $A' \subset A$ implies A is closed. Since $A' \subset A$, for all $\varepsilon > 0$,

$$(B_\varepsilon(x) \backslash \{x\}) \cap A \neq \emptyset \to B_\varepsilon(x) \cap A \neq \emptyset, x \in A.$$

Thus by theorem 4.9 A is closed. Now suppose A is closed. Contrapositively, we will equivalently show that given $x \in A'$ but $x \notin A$, then A is not closed. Since $x \in A'$, by definition for all $veps > 0$, we have that

$$\begin{aligned} &(B_\varepsilon(x) \backslash \{x\}) \cap A \neq \emptyset, \\ &\qquad B_\varepsilon(x) \cap A \neq \emptyset \text{ (equal since } x \notin A) , \end{aligned}$$

and since $x \notin A$ by theorem 4.9 we conclude A is not closed.

$\square$

4. Let E be a subset of a metric space M. Show that the complement of the interior of E is the closure of the complement of E.

Proof. Recall that the interior A^o is the largest open set contained in A, and $\overline{A}$ is the smallest closed set containing A. Note that A^o is open, so $(A^o)^c$ is closed, and certainly the closure of a closed set is closed ($\overline{(A^o)^c} = (A^o)^c$). With some set algebra, we have that

$$\begin{aligned} A^o &\subseteq A \\ (A^o)^c &\supseteq A^c \\ \overline{(A^o)^c} &\supseteq \overline{A^c} \\ (A^o)^c &\supseteq \overline{A^c}. \end{aligned}$$

Similarly, note that $\left(\overline{A}\right)^c = M \backslash \overline{A} \to \overline{\left(\overline{A}\right)^c} = M \backslash A = (A^o)^c$.

$$\begin{aligned} \overline{A} &\supseteq A \\ \left(\overline{A}\right)^c &\subseteq A^c \\ \overline{\left(\overline{A}\right)^c} &\supseteq \overline{A^c} \\ (A^o)^c &\supseteq \overline{A^c}. \end{aligned}$$

We conclude that $(A^o)^c = \overline{A^c}$. $\square$

5. Show that a point $x \in A$ is an isolated point of A if and only if $(B_\varepsilon(x)\backslash\{x\}) \cap A = \emptyset$ for some $\varepsilon > 0$. Prove that a subset of $\mathbb{R}$ can have at most countably many isolated points, thus showing that every uncountable subset of $\mathbb{R}$ has a limit point.

   Proof. By definition of a limit point, for all $\varepsilon > 0, (B_\varepsilon(x)\backslash X) \cap A \neq \emptyset$, and if x is not a limit point, then this definition is negated:
$$\text{there exists } \varepsilon > 0, (B_\varepsilon(x)\backslash\{x\}) \cap A = \emptyset.$$

   For the other claim, suppose $A \subseteq \mathbb{R}$, and denote the isolated points of A as $I(A)$. For arbitrary $\varepsilon > 0$, consider the interval $(x - \varepsilon, x + \varepsilon)$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ with $a < b$ such that $a < x < b$. Choose a, b such that $(a, b) \subset (x - \varepsilon, x + \varepsilon)$. Define $\phi : I(A) \to \mathbb{Q}^2$ with $\phi(x) = (a, b)$. We wish to show that ϕ is injective. WLOG, let $x, y \in I(A)$ and $x < y$. We choose two rationals for each point: a_x, b_x, a_y, b_y. If $x < a_y$, then $a_x < x < a_y$, and $(a_x, b_x) \neq (a_y, b_y)$. If $x \geq a_y$, then $x \in B_\varepsilon(y)$, a contradiction since y is supposed to be isolated. Thus $(a_x, b_x) \neq (a_y, b_y)$, and ϕ is injective.

   Since ϕ is injective and $\mathbb{Q}$ is countable, there exist at most countable isolated points of A. $\square$

6. Verify each of the following formulas, where bdry(A) denotes the set of boundary points of A. (For my own ease, let A denote the boundary of A.)

   (a) $A = A^c$.

   Proof. By definition,

$$\begin{aligned} A^c &= \{x : B_\varepsilon(x) \cap A^c \neq \emptyset \text{ and } B_\varepsilon(x) \cap (A^c)^c \neq \emptyset\} \\ &= \{x : B_\varepsilon(x) \cap A^c \neq \emptyset \text{ and } B_\varepsilon(x) \cap A \neq \emptyset\} \\ &= A. \end{aligned}$$

   □

   (b) $\overline{A} = A \cup A^o$

   Proof. Let $U = A \cup A^o$. We will prove the given equality by showing $\overline{A} \subseteq U$ and $U \subseteq \overline{A}$. It is obvious that if $x \in A^o$ or $x \in A$, then $x \in \overline{A}$. Thus $U \subset \overline{A}$. Now suppose $x \in \overline{A}$. If $x \in A^o$ we are done, so suppose $x \notin A^o$. Negating the definition of an interior point gives us

$$\begin{gathered} B_\varepsilon(x) \cap A \not\subseteq A \longrightarrow \\ B_\varepsilon(x) \cap A^c \neq \emptyset \\ \text{and } B_\varepsilon(x) \cap A \neq \emptyset \text{ by definition of } x \in \overline{A}. \end{gathered}$$

   Then our collection is the set

$$\{x : B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^c \neq \emptyset\},$$

   for all $\varepsilon > 0$, which is by definition A. Thus $\overline{A} \subseteq U$, and we conclude $\overline{A} = U$. □

   (c) $M = A^o \cup A \cup (A^c)^o$

   Proof. Let $U = A^o \cup A \cup (A^c)^o$. Obviously $x \in U$ implies $x \in M$, so $U \subseteq M$.
   Suppose there existed $x \in M$ with $x \notin U$. Note that $\overline{A} = A \cup A^o$ and $A \cup (A^c)^o = \overline{A^c}$. Then

$$\begin{aligned} x &\notin U \\ x &\notin \overline{A} \cup \overline{A^c} \\ x &\notin M. \end{aligned}$$

   This is a contradiction, so clearly $x \in U$, and $M \subseteq U$. Thus $M = U$. □