MTH 312 HW 5

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7.2.3.

(a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0,$$

and in this case $\int_a^b f = \lim U(f, P_n) = \lim L(f, P_n)$

Proof. We wish to show the following two statements are equivalent:
(i) There exists a sequence of partitions satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

(ii) For all $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

If we can do this, then by Theorem 7.2.8 we are done.

(i) $\to$ (ii)
Assume (i) holds and $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|U(f, P_n) - L(f, P_n)| = U(f, P_n) - L(f, P_n) < \epsilon$$

for $n \geq N$, and $p_\epsilon = P_n$. Thus (ii) holds.

(ii) $\to$ (i)
Assume (ii) holds. Then for all $n \in \mathbb{N}$, there exists a partition P_n of $[a, b]$ such that $U(f, P_n) - L(f, P_n) < \frac{1}{n}$, and so

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

Now suppose there exists a sequence of partitions such that f is integrable on $[a, b]$. This gives the following inequalities:

$$L(f, P_n) \leq L(f), U(f) \leq U(f, P_n), L(f, P_n) \leq U(f, P_n) \longrightarrow$$

$$L(f, P_n) - U(f, P_n) \leq L(f) - U(f, P_n) = U(f) - U(f, P_n) = U(f, P_n) - L(f, P_n)$$

By the Squeeze Theorem $\lim U(f, P_n) = U(f) = \int_a^b f$ and $\lim L(f, P_n) = L(f) = \int_a^b f$. $\square$

(b) For each n, let P_n be a partition of $[0,1]$ into n equal subintervals. Find formulas for $U(f,P_n)$ and $L(f,P_n)$ if $f(x)=x$.

Solution.

For each $0 \le k \le n-1$ let $x_k = \frac{k}{n-1}$, and let $P_n = \{x_0, \dots x_{n-1}\}$. Since f is strictly increasing on $[0,1]$,

$$m_k = x_{k-1} = \frac{k-1}{n-1}, M_k = x_k = \frac{k}{n-1} \longrightarrow$$

$$U(f,P_n) = \sum_{k=1}^{n-1} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k}{(n-1)^2} = \frac{n}{2(n-1)}$$

$$L(f,P_n) = \sum_{k=1}^{n-1} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k-1}{(n-1)^2} = \frac{n}{2(n-1)} - \frac{1}{n-1}$$

(c) Use the sequential criterion for integrability to show directly that $f(x)=x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

Solution.

By Part (b), we have that

$$U(f,P_n) - L(f,P_n) = \frac{1}{n-1} \to 0$$

Then by Part (a), f is integrable on $[0,1]$ with

$$\int_0^1 f = \lim U(f,P_n) = \lim \frac{n}{2(n-1)} = \frac{1}{2}$$

7.2.6. A *Tagged Partition* $(P, \{c_k\})$ is one where in addition to a partition P, we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. Then define the corresponding Riemann sum

$$R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Riemann Original Integral Definition.
A bounded function f is integrable on $[a, b]$ with $\int_a^b f = A$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $P(f, [c_k])$ satisfying $\Delta_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \epsilon$$

Show that if f satisfies the Riemann definition above, then f is integrable in the sense of Definition 7.2.7.

Proof. Let $\epsilon > 0$, and $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta y_k < \delta$, it follows that

$$|R(f, P) - A| < \frac{\epsilon}{2}$$

Let $N \in \mathbb{N}$ satisfy $\frac{b-a}{N} < \delta$ for all k, and let $y_k = a + k\frac{b-a}{N}$. Let also Q_1 be the partition $\{y_0 \ldots y_n\}$ of $[a, b]$. Since $U(f)$ is the infimum of the set $\{U(f, Q) : Q \in \mathcal{P}\}$, there exists a partition Q_2 of $[a, b]$ such that

$$U(f) \leq U(f, Q_2) < U(f) + \frac{\epsilon}{4}$$

Now let $P = Q_1 \cup Q_2$ be the common refinement of Q_1, Q_2 and note that

$$\Delta x_k \leq \Delta y_k = \frac{b-a}{N} < \delta \longrightarrow$$

$$|R(f, P) - A| < \frac{\epsilon}{2} \tag{1}$$

Since $Q_2 \subseteq P$, by lemma 7.2.3 we have that

$$U(f) \leq U(f, P) \leq U(f, Q_2) < U(f) + \frac{\epsilon}{4} \tag{2}$$

Note that if M_k is the supremum of f over $[x_{k-1}, x_k]$, there exists some $c_k \in [x_{k-1}, x_k]$ such that

$$M_k - \frac{\epsilon}{4(b-a)} < f(c_k) \leq M_k$$

Furthermore

$$0 \leq U(f, P) - R(f, P) = \sum_{k=1}^{n} \Delta(M_k - f(c_k))x_k < \frac{\epsilon}{4(b-a)} \sum_{k=1}^{n} \Delta x_k = \frac{\epsilon}{4} \tag{3}$$

By equations (1), (2), (3):

$$|U(f) - A| \leq |U(f) - R(f, P)| + |R(f, P) - A| \leq |U(f) - U(f, P)| + |U(f, P) - R(f, P)| + \left| R(f, P) - A < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} \right| = \epsilon$$

Since epsilon was arbitrary, $U(f) = A$, and we similarly show $L(f) = A$, and so $U(f) = L(f)$ which satisfies Definition 7.2.7. $\square$

7.3.3. Let

$$f(x) = \begin{cases} 1, \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, \text{otherwise} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Proof. Let $P = \{x_0, \ldots, x_n\}$ be an arbitrary partition of $[0, 1]$. Notice that every subinterval $[x_{k-1}, x_k]$ contains at least one irrational y by the density of irrationals in $\mathbb{R}$. Since $f(y) = 0$ and f is strictly non-negative, it follows that $m_k = 0$ and thus $L(t, P) = 0$. Because P was an arbitrary partition, we know that $L(f) = 0$. It remains to be shown that f is integrable.

Let $c \in (0, 1)$ be given and let N be the smallest natural number such that $\frac{1}{N+1} < c$. Restricting f to $[c, 1]$, we get that

$$f(x) = \begin{cases} 1, \text{if } x = 1, \frac{1}{2}, \ldots, \frac{1}{N} \\ 0, \text{otherwise} \end{cases}$$

Let P_n be the evenly spaced partition of $[c, 1]$ satisfying $\Delta x_k \leq \frac{1}{n}$. If $n \geq N$ and each point $1, \ldots, \frac{1}{N}$ belongs to exactly one subinterval $[x_{k-1}, x_k]$, then $M_k = 1$ for exactly N indicies, and $M_k = 0$ for all the others. Then

$$U(f, P_n) = \sum_{k=1}^{n} M_k \Delta x_k \leq \frac{N}{n}$$

Since $L(f, P_n) = 0$, by the squeeze theorem we have that

$$\lim U(f, P_n) - L(f, P_n) = 0$$

By Exercise 7.2.3 f is integrable on $[c, 1]$, and by Theorem 7.3.2 on $[0, 1]$. We calculate $\int_0^1 f = 0$. $\square$