MTH 312 HW 6

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25 February 2025

7.4.3. Decide which of the following conjectures is true, and supply a short proof. For those that are not, give a counter example.

(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

Proof. Suppose $f : [0, 1] \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

Then $|f| = 1$ on [0, 1], so $|f|$ is integrable on [0, 1]. However, by theorem 7.4.2, $g(x) = \frac{1}{2}(f(x) + 1)$ should be integrable on $[0, 1]$. But g is Dirichlet's function, which we know to be un-integrable from $[0, 1]$. Since in this instance, the integrability of $|f|$ and f are different, we conclude this statement in false. $\square$

(b) Assume g is integrable and $g(x) \geq 0$ on [a, b]. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.

Proof. Consider $g : [0, 1] \to [0, 1]$ defined as

$$g(x) = \begin{cases} 1, & x = \frac{1}{n}, n \in \mathbb{N} \\ 0, \text{otherwise} & \end{cases}$$

In HW 5 we computed $\int_0^1 g = 0$. There are clearly an infinite number of points of the form $\frac{1}{n} > 0 \in [0, 1]$, but $\int_0^1 g = 0$. We conclude this statement is false. $\square$

(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Proof. If there exists y_0 s.t. $g(y_0) > 0$, then there exists $\delta > 0$ s.t. if we define $I = [a, b] \cap [y_0 - \delta, y_0 + \delta]$, then $x \in I \longrightarrow g(x) > 0$. In particular, for $\epsilon = \frac{g(y_0)}{2} > 0$,

$$g(x) \in [g(y_0) - \epsilon, g(y_0) + \epsilon] \longrightarrow g(x) > g(y_0) - \epsilon = \frac{g(y_0)}{2} = \epsilon > 0$$

Let $c = \inf I, d = \sup I$. By theorem 7.4.1,

$$\int_a^b g = \int_a^c g + \int_c^d g + \int_d^b g$$

Since $g \geq 0$, by theorem 7.4.2 we have that $\int_a^c g, \int_d^b g \geq 0$, and also $\int_c^d g \geq \epsilon(d - c) > 0$

We conclude this statement is true. $\square$

7.4.8. For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} \frac{1}{2^n}, & \text{if } \frac{1}{2^n} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{1}{2^n} \end{cases}$$

Set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H$.

Proof. Let $N \in \mathbb{N}$, let $H_N : [0,1] \to \mathbb{R}$ be the N^{th} partial sum of H. Then

$$H_N(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2^N}\right] \\ \frac{2^k - 1}{2^N}, & \text{if } x \in \left(\frac{1}{2^{N-k+1}}, \frac{1}{2^{N-k}}\right] \end{cases}$$

for $k \in [1, N]$. Observe that each H_N is piecewise constant, thus by theorem 7.4.1 it is integrable. To explicitly compute the integral on $[0,1]$,

$$\begin{aligned} \int_0^1 H_N &= \sum_{k=1}^{N} \int_{2^{-(N-k+1)}}^{2^{N-k}} H_N \\ &= \sum_{k=1}^{N} \left(\frac{2^k - 1}{2^N}\right)\left(\frac{1}{2^{N-k}} - \frac{1}{2^{N-k+1}}\right) \\ &= \frac{2}{3} - \frac{1}{6 \cdot 4^{N-1}} - \frac{1}{4^N} + \frac{1}{2^{N+1}} \end{aligned}$$

Thus

$$\int_0^1 H_N = \lim_{N \to \infty} \int_0^1 H_N = \frac{2}{3}$$

$\square$

7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some h on $[a, b]$ then g is continuous on $[a, b]$.

<u>*Solution.*</u>

This statement is false, and is apparent when we consider the function

$$h(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Here, h is differentiable, but h' is not continuous.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

<u>*Solution.*</u>

Since g is continuous on $[a, b]$, g is integrable on $[a, b]$, so we define $h(x) = \int_a^x g$, and by the Fundamental Theorem of Calculus, we have that $h' = g$. We conclude this statement is true.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

<u>*Solution.*</u>

If we consider

$$h(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then $\int_a^x h = 0$, everywhere and thus differentiable, but h is not continuous at $c = 0$. We conclude this statement is false.