MTH 511 HW 3

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1. Prove that ℓ_2 is separable.

Proof. Let $\varepsilon^2 > 0$ be arbitrary, and consider a sequence $(r_1, r_2, \ldots, r_N, 0, 0, \ldots), N \in \mathbb{N}, r_k \in \mathbb{Q}$. Define also $S \subset \ell_2$ as the set of all possible sequences satisfying the above criterion. Consider first (y_n). Since $(y_n) \in \ell_2$, we know that $\sum_{i=1}^{\infty} y_i^2 < \frac{\varepsilon^2}{4}$, so certainly $\sum_{i=N+1}^{\infty} y_i^2 < \frac{\varepsilon^2}{4}$. By a similar frame of logic, we can show that

$$\sum_{i=1}^{\infty} y_i^2 < \frac{\varepsilon^2}{4}$$
$$\sum_{i=1}^{N} y_i^2 < \frac{\varepsilon^2}{4},$$

and also that

$$\sum_{i=1}^{\infty} x_i^2 < \frac{\varepsilon^2}{4}$$
$$\sum_{i=1}^{N} x_i^2 < \frac{\varepsilon^2}{4}.$$

It is easy to see then, that

$$\sum_{i=1}^{N}(y_i - x_i)^2 \le \sum_{i=1}^{N} y_i^2 + \sum_{i=1}^{N} x_i^2 < \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} < \frac{\varepsilon^2}{2}.$$

Then for all $(x_n) \in S$ and $(y_n) \in \ell_2$,

$$\begin{aligned}\sum_{i=1}^{\infty}(y_i - x_i)^2 &= \sum_{i=1}^{N}(y_i - x_i)^2 + \sum_{i=N+1}^{\infty}(y_i - x_i)^2 \\ &< \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{4} < \varepsilon^2.\end{aligned}$$

Thus $y \in B_\varepsilon(x) \cap S \to B_\varepsilon(x) \cap S \neq \emptyset$, and so $y \in \overline{S}$. So $\ell_2 \subseteq \overline{S}$, and since $\overline{S} \subseteq \ell_2$ is obvious, we conclude that $\overline{S} = \ell_2$. Since $|S| = \sum_{i=1}^{\infty} \mathbb{Q}^i$ is countable, S is countable and therefore dense, meaning ℓ_2 is separable. □

2. Show that ℓ_∞ is not separable.

Proof. Let $I \subseteq \mathbb{N}$ with a sequence (x_n) defined as

$$(x_k) = \begin{cases} 1, k \in I \\ 0, k \notin I \end{cases} .$$

Define $P = \{(x_n) : I \subseteq \mathbb{N}\}$. Then $|P| = |2^{\mathbb{N}}| = 2^{\aleph_0}$ is uncountable by Cantor's theorem. Now suppose $I_1 \neq I_2 \subseteq \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that

$$x_k(I_1) \neq x_k(I_2) \to d(x_k(I_1), x_k(I_2)) = 1.$$

Suppose also there exists a countable dense set $D \subseteq \ell_\infty$. Then for all $x \in P$, there exists $z \in D$ such that $z \in B_{1/2}(x)$. But open balls with radius $1/2$ must be disjoint, since for two centers $x \neq y, d(x, y) = 1$. Thus different $x \in P$ require distinct $z_x \in D$, giving an injection from $P \to D$. This is an injection from an uncountable set P to a countable set D, a contradiction. We conclude our assumption about the existence of a dense set $D \subseteq \ell_\infty$ is wrong, and in fact no such set exists. Then by definition, ℓ_∞ is not separable. $\square$

3. Prove that M has a countable open base if and only if M is separable.

Proof. First we show that M having countable open base implies M separable. Suppose M has countable open base $\mathcal{B}$, and let $x \in M, \varepsilon > 0$. For all $B \in \mathcal{B}$ with $B \neq \emptyset$, let $y_B \in B$. Define S to be

$$S = \{y_B : B \in \mathcal{B}, B \neq \emptyset\}.$$

Note that since $B_\varepsilon(x)$ is open, it is the union of some basis elements (definition of open base). So there exists $B \in \mathcal{B}$ such that $B \subseteq B_\varepsilon(x)$. Since $y_B \in B_\varepsilon(x)$ and $y_B \in S$, $y_B \in B_\varepsilon(x) \cap S$, so $B_\varepsilon(x) \cap S \neq \emptyset$ and $x \in \overline{S}$. We can define $\mathcal{B}' = \mathcal{B} \backslash \emptyset$ and a function $f : \mathcal{B}' \to S$ as $f : B \to y_B$ which is a map to its image, so f is surjective. Since S is countable and $M \subseteq \overline{S}, M = \overline{S}$, ($\overline{S} \subseteq M$ is obvious) S is countable and dense in M, thus M is separable.

We now show that M separable implies M has countable open base. Let M be separable. Then there exists $D \subseteq M$ which is countable and dense. Let $\mathcal{U} \subseteq M$ be open with $x \in \mathcal{U}$. By definition of open and dense, there exists some $\varepsilon/2 > 0$ such that $B_{\varepsilon/2}(x) \subset \mathcal{U}$, and there exists also $y \in D$ with $y \in B_{\varepsilon/2}(x)$. Then it must be that $x \in B_{\varepsilon/2}(y)$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ with $\varepsilon/2 < r << \varepsilon$. Thus

$$B_{\varepsilon/2}(y) \subset B_r(y) \subseteq \mathcal{U}.$$

This gives us the following:

- for all $x \in \mathcal{U}$, $x \in B_r(y) \to x \in \cup_{x\in\mathcal{U}} B_r(y)$
- for all $z \in \cup_{x\in\mathcal{U}B_r(y)}$, $z \in B_r(y)$ for some $x, B_r(y)$

This tell us that $\mathcal{U} \subseteq \cup_{x\in\mathcal{U}} B_r(y)$, and also that $\mathcal{U} \supseteq \cup_{x\in\mathcal{U}} B_r(y)$, so $\mathcal{U} = \cup_{x\in\mathcal{U}} B_r(y)$. We have shown any open set can be represented as a union of balls with rational radius, thus $\mathcal{B}$ is an open base. $\square$

4. Let $f : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

Proof. Assume for some $x \in M$, $f(x) \neq g(x)$. Then $\rho(f(x), g(x)) > 0$. Let $\varepsilon = \rho(f(x), g(x))$. By definition of continuity, for all $y \in M$, there exists some $\delta_f, \delta_g > 0$ such that

$$d(x, y) < \delta_f \to \rho(f(x), f(y)) < \varepsilon/2 \text{ and}$$

$$d(x, y) < \delta_g \to \rho(g(x), g(y)) < \varepsilon/2.$$

Let $\delta = \min\{\delta_f, \delta_g\}$. Since D is dense, there exists $y \in B_\delta(x)$ with $y \in D \cap B_\delta(x)$. Then

$$\begin{aligned}
\rho(f(x), g(x)) &\leq \rho(f(x), f(y)) + \rho(f(y), g(x)) \\
&= \rho(f(x), f(y)) + \rho(g(y), g(x)) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So $\rho(f(x), g(x)) < \varepsilon = \rho(f(x), g(x))$. This is a contradiction, and our assumption that $f(x) \neq g(x)$ is wrong. We conclude then, for all $x \in M, f(x) = g(x)$.

Here we show that $f(D)$ is dense in N given that f is onto. Let $y \in N$. Since f is onto, there exists $x \in M$ such that $y = f(x)$. By property of D being dense, and f continuous we know there exists some $z \in D$ such that $d(x, z) < \delta \to \rho(f(x), f(z)) < \varepsilon$. Thus $f(z) \in f(D) \cap B_\varepsilon(f(x)) \to B_\varepsilon(y) \cap f(D) \neq \emptyset$, and by definition of closure $y = f(x) \in \overline{f(D)}$. Since $y \in N$ gives us $y \in \overline{f(D)}$, we get $N \subseteq \overline{f(D)}$, and thus $N = \overline{f(D)}$. The image of a continuous function with countable domain is countable, so we conclude that $f(D)$ is dense in N. $\square$

5. Let $f : (M, d) \to (N, \rho)$ be continuous, and let A be a separable subset of M. Prove that $f(A)$ is separable.

Proof. It is sufficient to show that $f(A) = \overline{f(D^A)} \cap f(A)$ where $\overline{D^A} = \overline{D} \cap A$ for some countably dense set $D \subseteq M$. Let $x \in A$. Then $x \in A \cap \overline{D} = \overline{D^A}$ with $f(x) = y \in f(A)$. Since $x \in \overline{D^A}$, there exists some $z \in B^d_\delta(x) \cap D$, and by continuity

$$\begin{aligned} f(z) &\in B^\rho_\varepsilon(f(x)) \cap f(A) \\ f(z) &\in B^\rho_\varepsilon(y) \cap f(A) \\ f(z) &\in \overline{f(D)}. \end{aligned}$$

Since $f(z) \in \overline{f(D)}$ and $f(z) \in f(A)$, $f(z) \in \overline{f(D)} \cap f(A)$. So $\overline{f(D)} \cap f(A) \neq \emptyset$, and $y \in \overline{D^A}$. So for any $y \in f(A)$, we have that $y \in \overline{f(D^A)}$, thus $f(A) \subseteq \overline{f(D^A)}$. Since it is clear that $\overline{f(D^A)} \subseteq f(A)$, we have that $f(A) = \overline{f(D^A)}$, and $f(D^A)$ is dense in $f(A)$. Since $f(D^A)$ is the image of a continuous countable set, $f(A)$ is separable. □

6. Fix $y \in \ell_\infty$ and define $h : \ell_1 \to \ell_1$ by $h(x) = (x_n y_n)_{n=1}^{\infty}$. Show that h is continuous.

Proof. Let $\varepsilon > 0$ be arbitrary, and $y_s = \sup\{y_i\}$. Choose $\delta < \varepsilon/|y_s|$. Then for $x, z \in \ell_\infty$ given that $d(x,z) < \delta$, we have that

$$\begin{aligned}
d(x,z) &< \delta \\
\sum_{i=1}^{\infty} |x_i - z_i| &< \varepsilon/|y_s| \\
|y_s| \sum_{i=1}^{\infty} |x_i - z_i| &< \varepsilon \\
\sum_{i=1}^{\infty} |y_s||x_i - z_i| &< \varepsilon \\
\sum_{i=1}^{\infty} |y_i||x_i - z_i| &< \varepsilon (\text{ since certainly } \sum |y_i||x_i - z_i| < \sum |y_s||x_i - z_i|) \\
\sum_{i=1}^{\infty} |x_i y_i - z_i y_i| &< \varepsilon \\
\|h(x) - h(z)\|_1 &< \varepsilon.
\end{aligned}$$

$\square$