MTH 464 HW 5

Brandyn Tucknott

21 February 2025

1. Let $X, Y \sim \text{Exp}(\lambda)$ be iid.
   (a) Find the joint pdf of U, V.
   Solution.
   First, compute the jacobian:

$$J = \begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \longrightarrow \left| \frac{1}{\det J} \right| = \frac{1}{2}$$

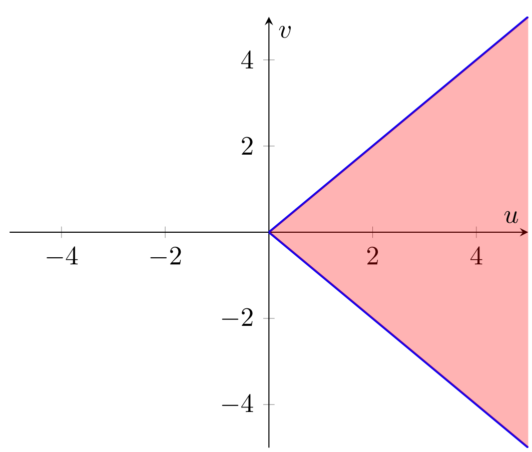
   Observe also that $x = \frac{u-v}{2}, y = \frac{u+v}{2}$. Then

$$f_{U,V}(u, v) = f_{X,Y}\left(x = \frac{u-v}{2}, y = \frac{u+v}{2}\right) \left| \frac{1}{\det J} \right| = \lambda e^{-\lambda \frac{u-v}{2}} \lambda e^{-\lambda \frac{u+v}{2}} \cdot \frac{1}{2} = \frac{\lambda^2}{2} e^{-\lambda u}$$

   (b) Show U, V are uncorrelated but not independent.
   Solution.
   To compute correlation, first compute the covariance.

$$\text{Cov}(U, V) = \text{Cov}(X + Y, Y - X) = \text{Cov}(X, Y) - \text{Cov}(X, X) + \text{Cov}(Y, Y) - \text{Cov}(Y, X) = \text{Var}(Y) - \text{Var}(X) = 0$$

   Since X, Y are iid. This allows us to conclude that $\text{Corr}(U, V) = 0$.
   To find wether or not U, V are independent, we will consider the region of U, V. Since it looks like



   Since the region U, V is not rectangular, and hence not a Cartesian product, we conclude U, V are not independent.

2. Let $\{U_j\}_{j=1}^{\infty} \sim \text{Unif}[0,1]$ be iid. For $0 \le x \le 1$, define $N(x) = \min\{n : \sum_{k=1}^{n} U_k > x\}$. Note that by convention, $\sum_{k=1}^{0} U_k = 0$, and also $P(N(x) \ge 1 = 1)$.

(a) Find $P(N(x) \ge 2)$.

<u>*Solution.*</u>

Observe that

$$P(N(x) \ge 2) = P\left(\sum_{k=1}^{1} U_k \le x\right) = P(U_1 \le x) = x$$

(b) Show by induction that $P(N(x) \ge n+1) = \frac{x^n}{n!}$.

Proof. We will first show by induction that $f_{\sum^n U_k}(x) = \frac{x^{n-1}}{(n-1)!}$ for $0 \le x \le n$ and $U_j \sim \text{Unif}[0, 1]$. For a base case, it is obvious that if $k = 1$, then $f_{U_1} = \frac{x^0}{0!} = 1$, which mateches with the pdf for a std uniform distribution. Now assume the relation holds up to $n-1$. We want to show it holds for n.

$$\begin{aligned} f_{\sum^n U_k} &= f_{\sum^{n-1} U_k + U_n} \\ &= \int_0^x f_{\sum^{n-1} U_k}(x-u) f_{U_k}(u) du \\ &= \int_0^x \frac{(x-u)^{n-2}}{(n-2)!} du \text{ let } v = x-u, dv = -du \\ &= -\int_x^0 \frac{v^{n-2}}{(n-2)!} dv = \int_0^x \frac{v^{n-2}}{(n-2)!} dv \\ &= \frac{x^{n-1}}{(n-1)!} \end{aligned}$$

Thus, we have shown that

$$f_{\sum^n U_k}(x) = \frac{x^{n-1}}{(n-1)!}, \text{ for } 0 \le x \le n \tag{1}$$

Now using equation (1) we directly compute

$$\begin{aligned} P\left(N(x) \ge n+1\right) &= P\left(\sum_{k=1}^{n} U_k \le x\right) \\ &= \int_0^x f_{\sum^n U_k}(x-u) f_U(u) du = \int_0^x f_{\sum^n U_k}(x-u) du = \int_0^x \frac{(x-u)^{n-1}}{(n-1)!} du, \text{ let } v = x-u \\ &= -\int_x^0 \frac{v^{n-1}}{(n-1)!} dv = \int_0^x \frac{v^{n-1}}{(n-1)!} dv = \frac{x^n}{n!} \end{aligned}$$

□

(c) Recall that if N is a positive integer valued random variable, then $E(N) = \sum_{k=1}^{\infty} P(N(x) \ge k)$. Conclude that $E(N) = e^x$.

Proof. We directly calculate calculate $E(N)$ using our result from Part (b).

$$\begin{aligned} E(N) &= \sum_{k=1}^{\infty} P(N(x) \ge k) \\ &= \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \\ &= \sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x \end{aligned}$$

□

3. Let $\{U_j\}_{j=1}^n \sim \text{Unif}[0,1]$ be iid and $U_{(j)}, j = 1, 2, \ldots, n$ be its order values. Recall that the pdf of $U_{(j)}$ is given by

$$f_{U_{(j)}}(u) = \frac{n!}{(j-1)!(n-j)!} u^{j-1}(1-u)^{n-j} \mathbb{1}_{[0,1]}(u)$$

Recall also that for all $a > 0, b > 0$, $\int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

(a) Find $\mathbb{E}\left(U_{(j)}\right)$ and $\text{Var}\left(U_{(j)}\right)$.

<u>*Solution.*</u>

$$\begin{aligned}
\mathbb{E}\left(U_{(j)}\right) &= \int_0^1 u \cdot \frac{n!}{(j-1)!(n-j)!} u^{j-1}(1-u)^{n-j} du \\
&= \frac{n!}{(j-1)!(n-j)!} \int_0^1 u^j (1-u)^{n-j} du \\
&= \frac{n!}{(j-1)!(n-j)!} \frac{\Gamma(j+1)\Gamma(n-j+1)}{\Gamma(n)} \\
&= \frac{n!}{(j-1)!(n-j)!} \frac{j!(n-j)!}{(n+1)!} \\
&= \frac{j}{n+1}
\end{aligned}$$

To compute the variance we need to quickly compute the $\mathbb{E}\left(U_{(j)}^2\right)$

$$\begin{aligned}
\mathbb{E}\left(U_{(j)}^2\right) &= \int_0^1 u^2 \cdot \frac{n!}{(j-1)!(n-j)!} u^{j-1}(1-u)^{n-j} du \\
&= \frac{n!}{(j-1)!(n-j)!} \int_0^1 u^{j+1} (1-u)^{n-j} du \\
&= \frac{n!}{(j-1)!(n-j)!} \frac{\Gamma(j+2)\Gamma(n-j+1)}{\Gamma(n+3)} \\
&= \frac{n!}{(j-1)!(n-j)!} \frac{(j+1)!(n-j)!}{(n+2)!} \\
&= \frac{(j+1)j}{(n+2)(n+1)}
\end{aligned}$$

Finally, we compute the variance to be

$$\begin{aligned}
\text{Var}\left(U_{(j)}\right) &= \mathbb{E}\left(U_{(j)}^2\right) - \left(\mathbb{E}\left(U_{(j)}\right)\right)^2 \\
&= \frac{j(j+1)}{(n+2)(n+1)} - \left(\frac{j}{n+1}\right)^2 \\
&= \frac{j(j+1)}{(n+2)(n+1)} - \frac{j^2}{(n+1)^2} \\
&= \frac{j(j+1)(n+1) - j^2(n+2)}{(n+1)^2(n+2)} \\
&= \frac{j(n-j+1)}{(n+1)^2(n+2)}
\end{aligned}$$

(b) Determine the value of j that minimizes $\text{Var}\left(U_{(j)}\right)$.

<u>*Solution.*</u>

Recognize first that the variance formula written in terms of j can be expanded to

$$\begin{aligned}\text{Var}\left(U_{(j)}\right) &= \frac{j(n-j+1)}{(n+1)^2(n+2)} \\ &= \frac{-j^2+jn+j}{(n+1)^2(n+2)}\end{aligned}$$

This is clearly a downward opening parabola, which means the minimum values will be at the endpoints, $j = 1, n$. They will be the same value because the variance is maximal at the vertex which can easily be shown to be equidistant from $j = 1, n$.

4. Suppose that conditioned on $Y = y$, X_1, X_2 are independent random variables with mean y. Show that $\operatorname{Cov}(X_1, X_2) = \operatorname{Var}(Y)$.

Proof. We are given that conditioned on $Y = y$, then X_1, X_2 are independent with mean y. Note that this means $\mathbb{E}(X_1|Y) = \mathbb{E}(X_2|Y) = Y$.

$$\begin{aligned}
\operatorname{Cov}(X_1, X_2) &= \mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\,\mathbb{E}(X_2) \\
&= \mathbb{E}(\mathbb{E}(X_1 X_2|Y)) - \mathbb{E}(\mathbb{E}(X_1|Y))\,\mathbb{E}(X_2|Y) \\
&= \mathbb{E}(\mathbb{E}(X_1|Y)\,\mathbb{E}(X_2|Y)) - \mathbb{E}(\mathbb{E}(X_1|Y))\,\mathbb{E}(\mathbb{E}(X_2|Y)) \\
&= \operatorname{Cov}(\mathbb{E}(X_1|Y), \mathbb{E}(X_2|Y)) \\
&= \operatorname{Cov}(Y, Y) = \operatorname{Var}(Y)
\end{aligned}$$

$\square$

5. Show that $\operatorname{Cov}(X, \mathbb{E}(Y|X)) = \operatorname{Cov}(X, Y)$.

Proof. Recall that

$$\mathbb{E}(Y|X) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \text{ and also that}$$
$$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$$

Then we can evaluate

$$\begin{aligned}
\mathbb{E}(X\mathbb{E}(Y|X)) &= \mathbb{E}(X)\,\mathbb{E}(Y|X) \\
&= \int_{-\infty}^{\infty} x f_X dx \int_{-\infty}^{\infty} y f_{Y|x} dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X \cdot f_{Y|X} dx dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y} dx dy \\
&= \mathbb{E}(XY)
\end{aligned}$$

We now directly evaluate the covariance to be

$$\begin{aligned}
\operatorname{Cov}(X, \mathbb{E}(Y|X)) &= \mathbb{E}(X\mathbb{E}(Y|X)) - \mathbb{E}(X)\,\mathbb{E}(\mathbb{E}(Y|X)) \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) \\
&= \operatorname{Cov}(X, Y)
\end{aligned}$$

□