MTH 511 HW 4

Brandyn Tucknott

5 November 2025

1. Prove theorem 5.5.

   Theorem 5.5. *Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Then the following are equivalent.*

   (i) f is a homeomorphism.

   (ii) $x_n \to_d x \leftrightarrow f(x_n) \to_\rho f(x)$.

   (iii) G is open in $M \leftrightarrow f(G)$ is open in N.

   (iv) E is closed in $M \leftrightarrow f(E)$ is closed in N.

   (v) $\hat{d}(x, y) = \rho(f(x), f(y))$ defined a metric on M equivalent to d.

   Proof. Our method will be to show that $i \to ii \to iii \to iv \to v \to i$.

   $(i) \to (ii)$: since f is a homeomorphism, we have that f, f^{-1} are continuous. Then by Theorem 5.1, we have that

$$\begin{aligned} x_n \to_d x &\longrightarrow f(x_n) \to_\rho f(x) \\ f(x_n) \to_\rho f(x) &\longrightarrow f^{-1}(f(x_n)) \to_d f^{-1}(f(x)) \text{ which is equivalent to} \\ f(x_n) \to_\rho x &\longrightarrow x_n \to_d x. \end{aligned}$$

   Thus $(i) \to (ii)$.

   $(ii) \to (iii)$: Property two says f, f^{-1} are continuous, so by Theorem 5.1 we have that

$$\begin{aligned} f(G) \text{ in } N \text{ open } &\longrightarrow G \text{ in } M \text{ open} \\ f^{-1}(f(G)) \text{ in } M \text{ open } &\longrightarrow f(G) \text{ in } N \text{ open } \text{ which is equivalent to} \\ G \text{ in } M \text{ open } &\longrightarrow f(G) \text{ in } N \text{ open }. \end{aligned}$$

   Thus $(ii) \to (iii)$.

   $(iii) \to (iv)$: This uses the exact same argument as $(ii) \to (iii)$, replacing the word 'open' for 'closed'.

   $(iv) \to (v)$: Certainly $\hat{d} = \rho$ is a metric, so it remains to be shown if it is an equivalent metric. Since we are given (iv), we know that

$$f^{-1}(f(E)) \text{ closed in } M \longrightarrow f^{-1}(E) \text{ closed in } N \text{ and} \tag{1}$$
$$f(E) \text{ closed in } N \longrightarrow E \text{ closed in } M. \tag{2}$$

   From Theorem 5.1 we know that f, f^{-1} are continuous, and thus f is a homeomorphism. This gives us $(i) \to (ii)$. We see then if $d(x_n, x) \to 0$, then $x_n \to_d x$ and by (ii) $x_n \to_\rho x$ which is equivalent to $x_n \to_{\hat{d}} x$. On the other hand, if $\hat{d}(x_n, x) \to 0$, then $x_n \to_{\hat{d}=\rho} x$ and by (ii) we have that $x_n \to_d x$. Thus $\hat{d}$ is equivalent to d.

$(v) \to (i)$: Property (v) tells us that

$$x_n \to_d x \longleftrightarrow x_n \to_{\hat{d}} x.$$

It is clear that

$$\begin{aligned} x_n \to_d x &\longrightarrow x_n \to_{\hat{d}} x \\ d(x_n, x) \to 0 &\longrightarrow \rho(x_n, x) \to 0 \\ x_n \to x &\longrightarrow f(x_n) \to f(x) \\ &\text{Thus } f \text{ is continuous.} \end{aligned}$$

Similarly, we have that

$$\begin{aligned} x_n \to_{\hat{d}} x &\longrightarrow x_n \to_d x \\ \rho(x_n, x) \to 0 &\longrightarrow d(x_n, x) \to 0 \\ f(x_n) \to x &\longrightarrow x_n \to x \\ &\text{Thus } f^{-1} \text{ is continuous.} \end{aligned}$$

Since f, f^{-1} is continuous with f one-to-one and onto, we conclude that f is a homeomorphism.

$\square$

2. Suppose we are given a point x and a sequence x_n in a metric space M, and suppose that $f(x_n) \to f(x)$ for every continuous real-valued function f on M. Prove that $x_n \to x$ in M.

Proof. Suppose $x_n \not\to x$. Then there exists some $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$d(x_{n_k}, x) \geq \varepsilon$$

for all k. Let $f : M \to \mathbb{R}$ be defined by

$$f(y) = \max(0, 1 - \frac{d(x,y)}{\varepsilon}).$$

For any $y, z \in M$, we have that

$$|f(y) - f(z)| \leq \frac{|d(x,y) - d(x,z)|}{\varepsilon} \leq \frac{d(y,z)}{\varepsilon},$$

so f is lipschitz and thus continuous (by exercise 5.19 in the book). Observe that

$$d(x_{n_k}, x) \geq \varepsilon \longrightarrow \frac{d(x_{n_k}, x)}{\varepsilon} \geq 1 \longrightarrow 1 - \frac{d(x_{n_k}, x)}{\varepsilon} \leq 0.$$

Thus $\max(0, 1 - \frac{d(x_{n_k},x)}{\varepsilon}) = 0$, and $f(x_{n_k}) = 0$. Since $f(x_{n_k}) = 0$ and $f(x) = \max(0, 1 - \frac{d(x,x)}{\varepsilon}) = 1$, we have that $f(x_{n_k}) \not\to f(x)$, a contradiction. We conclude our assumption must be wrong, and it must be that $x_n \to x$. $\square$

3. Prove that a totally bounded metric space is separable.

Proof. Let M be a totally bounded metric space. Then for some points $x_1, \dots x_n \in M$, we have that

$$M = \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

for all $\varepsilon > 0$ and $n < \infty$. Define $D_n = \{x_1, \dots, x_n\}, D = \cup_{n=1}^{\infty} D_n$. Then D is a countable union of finite sets, and it remains to be shown that D is dense. Obviously $\overline{D} \subseteq M$, so we want to show that $M \subseteq \overline{D}$. Let $x \in M$. Then there exists some $x_k \in D_n$ with $k \leq n$ such that $x \in B_\varepsilon(x_k)$. Then clearly $x_k \in B_\varepsilon(x)$, and $B_\varepsilon(x) \setminus \{x\} \cap D \neq \emptyset$ and $x \in \overline{D}$ by definition of closure. We conclude that $\overline{D} = M$, so D is dense, and since it is countable we have that M is separable. $\square$

4. Let (M, d) and (N, ρ) be metric spaces with $f : M \to N$ a surjective function which satisfies

$$\rho(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in M$. Prove or find a counterexample to the following.

(a) If (N, ρ) is complete, then (M, d) is complete.

Proof. Let $M = (0, 1), N = \{0\}, f : M \to N$ be defined by $f(x) = 0$. Define also $d(x, y) = |x - y|$ and $\rho(0, 0) = 0$. Then certainly the given inequality holds with f onto. Every Cauchy sequence in N converges trivially to $0 \in N$, but the sequence $x_n = \frac{1}{n}$ converges to 0, which is not in M. We conclude the statement does not universally hold. $\square$

(b) (M, d) is complete, then (N, ρ) is complete.

Proof. Let $M = \mathbb{R}, N = (0, 1)$ with $d(x, y) = \rho(x, y) = |x - y|$. Define $f : M \to N$ as $f(x) = 1/2 \arctan(x) + 1/2$. Notice that since $\arctan(x)$ is invertible only when the domain is restricted to $(-\pi/2, \pi/2)$, the domain $\mathbb{R}$ makes it no longer one-to-one, but it retains its onto properties. Since $f'(x) = 1/(1 + x^2)$ is bounded by (0, 1), we have that

$$\begin{gathered}\left|\frac{f(x) - f(y)}{x - y}\right| \leq 1 \\ |f(x) - f(y)| \leq |x - y| \\ \rho(f(x), f(y)) \leq d(x, y).\end{gathered}$$

Again consider a sequence $x_n = 1/n$ in N. Certainly x_n is Cauchy, but it converges to $0 \notin N$, so the statement does not universally hold. $\square$

5. Fill in the details for the proof that ℓ_1 and ℓ_∞ are complete. (CURRENTLY INCOMPLETE)

   Proof. Here we show that ℓ_1 is complete. Let $f \in \ell_1$ be written as $f = (f(k))_{k=1}^{\infty}$, in which case $\|f\|_1 = \sum_{k=1}^{\infty} |f(k)|$. Let f_n be a sequence in ℓ_1, where $f_n = (f_n(k))_{k=1}^{\infty}$ and suppose that each f_n is Cauchy in ℓ_1. That is, suppose that for each $\varepsilon > 0$ there exists n_0 such that $\|f_n - f_m\|_1 < \varepsilon$ whenever $n, m \geq n_0$. The proof is broken into three steps, but since 2 and 3 are completed in the book, we will just do step 1 for both ℓ_1 and ℓ_∞.

   Step 1. (ℓ_1) $f(k) = \lim_{n\to\infty} f_n(k)$ exists in $\mathbb{R}$ for each k.
   We need to show $f \in \ell_1$ and that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

   □

6. Prove that c_0 is complete by showing that c_0 is closed in ℓ_∞.

Proof. Since a closed supspace of a complete metric space is complete, it is sufficient to show that $c_0 \subseteq \ell_\infty$ is closed. Let x_n be a sequence in c_0 converge to x in ℓ_∞. Let $\varepsilon > 0$ be arbitrary, and choose N large enough such that for all $n \geq N$,

$$\|x_n - x\|_\infty < \frac{\varepsilon}{2}.$$

For that index n, since x_n is in c_0, there exists K such that for all $k \geq K$,

$$\|(x_n)_k - (x_n)\|_\infty < \frac{\varepsilon}{2}.$$

Then for any $k \geq K$, we have that

$$\begin{aligned} |x_k| &\leq |x_k - (x_n)_k| + |(x_n)_k| \\ &\leq \|x_k - (x_n)_k\|_\infty + \|(x_n)_k\|_\infty \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus for all $k \geq K$, we have $|x_k| < \varepsilon$, and thus $x_k \to 0$ and x_k is in c_0. Since c_0 contains all its limit points in ℓ_∞, it is closed in ℓ_∞. $\square$