MTH 464 HW 2

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1. Let $U \sim \text{Unif}(0, 2\pi)$ and $T \sim \text{Exp}(1)$ be independent random variables. Show that $X = \sqrt{2T}\cos U, Y = \sqrt{2T}\sin U$ are independent standard normal random variables.

   Proof. Before any work is done, there are two important observations to make. The first is that

$$X^2 + Y^2 = 2T(\cos^2 U + \sin^2 U) \longrightarrow T = \frac{X^2 + Y^2}{2} \tag{1}$$

   The similarly we can find that

$$2T = \frac{X}{\cos U} = \frac{Y}{\sin U} \longrightarrow U = \arctan\left(\frac{Y}{X}\right) \tag{2}$$

   The second important observation is to recognize that since U, T are independent, the joint distribution of T, U can be written as

$$f_{T,U}(t, u) = \frac{1}{2\pi} e^{-t} \tag{3}$$

   From here, it is just a simple change of variables to show that $X, Y \sim N(0, 1)$. To make it simple, we compute the Jacobian beforehand:

$$J = \begin{pmatrix} \frac{\partial X}{\partial T} & \frac{\partial X}{\partial U} \\ \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial U} \end{pmatrix} = \begin{pmatrix} \frac{\cos u}{\sqrt{2t}} & -\sqrt{2t}\sin u \\ \frac{\sin u}{\sqrt{2t}} & \sqrt{2t}\cos u \end{pmatrix} \longrightarrow \det J = 1 \longrightarrow \frac{1}{|\det J|} = 1$$

   Now we directly use a change of variables to find the joint distribution for X, Y.

$$f_{X,Y}(x, y) = f_{T,U}(t(x, y), u(x, y)) \cdot \frac{1}{|\det J|} = f_{T,U}\left(t = \frac{x^2 + y^2}{2}, u = \arctan\left(\frac{Y}{X}\right)\right), \text{ by equations (1) and (2)}$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{x^2+y^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = f_X(x) \cdot f_Y(y)$$

   Where $f_X(x)$ is the marginal distribution of X, and $f_Y(y)$ is the marginal distribution of Y. It is obvious now to see that the joint is the product of two marginals, in particular $X, Y \sim N(0, 1)$ and also that the random variables X, Y are independent (since the product of the marginals is the joint). $\square$

2. Assume $X \sim \text{Exp}(\lambda), Y \sim \text{Exp}(\mu)$ be independent random variables.

   (a) Let $W = \min(X, Y)$. Show that $W \sim \text{Exp}(\lambda + \mu)$.

   Proof. Consider $P(X > w)$. Then by independence of X, Y, we can say that

$$P(W > w) = P(X > w \text{ and } Y > w) = P(X > w)P(Y > w) = e^{\lambda w} \cdot e^{-\mu w} = e^{-(\lambda+\mu)w}$$

   Note that

$$1 - e^{-(\lambda+\mu)w} = 1 - P(W > w) = P(W \le w)$$

   which is the cumulative distribution for an exponential distribution with parameter $\lambda + \mu$. We conclude then that $W \sim \text{Exp}(\lambda + \mu)$. □

   (b) Show that $P\left(W = X\right) = \frac{\lambda}{\lambda+\mu}$ and $P\left(W = Y\right) = \frac{\mu}{\lambda+\mu}$.

   Proof. In order to find $P(W = X)$, require that $X = \min(X, Y)$. That is,

$$P(W = X) = P(X \le Y) = \int_0^\infty \int_0^y \lambda e^{-\lambda x} \mu e^{-\mu y} dx dy = \frac{\lambda}{\lambda + \mu}$$

$$P(W = Y) = P(Y \le X) = \int_0^\infty \int_0^x \lambda e^{-\lambda x} \mu e^{-\mu y} dy dx = \frac{\mu}{\lambda + \mu}$$

□

3. Let $U \sim \text{Unif}[0,1]$ and $a \in (0,1)$.

   (a) Find the conditional distribution of U given that $U < a$, $P\left(U \leq u | U < a\right)$. Find also the corresponding conditional pdf.

   <u>*Solution.*</u>

$$F_{U|U<a} = P(U \leq u|U < a) = \frac{P(U < u \text{ and } U < a)}{P(U < a)} = \begin{cases} \frac{u}{a}, & 0 \leq 0 < u < a \\ 1, & \text{otherwise} \end{cases}$$

   We can find the pdf of the conditional by taking the derivative with respect to u.

$$f_{U|U<a} = \frac{d}{da} F_{U|U<a} = \begin{cases} \frac{1}{a}, & 0 < u < a \\ 0, & \text{otherwise} \end{cases}$$

   (b) Find the conditional distribution of U given that $U > a$.

   <u>*Solution.*</u>

$$F_{U|U>a} = P(U \leq u|U > a) = \frac{P(U \leq u \text{ and } U > a)}{P(U > a)} = \begin{cases} 0, & u \leq a \\ \frac{\int_a^u 1 dt}{1-a}, & u > a \end{cases} = \begin{cases} \frac{u-a}{1-a}, & 1 \geq u > a \\ 0, & \text{otherwise} \end{cases}$$

   Similarly to Part (a), we can take the derivative of the conditional cumulative distribution to find the conditional pdf.

$$f_{U|U>a} = \frac{d}{du} F_{U|U>a} = \begin{cases} \frac{1}{1-a}, & 1 \geq u > a \\ 0, & \text{otherwise} \end{cases}$$

4. Assume that the number of years that a machine functions, denoted by T, is a random variable with hazard rate

$$\lambda(t) = \begin{cases} 0.2, & 0 < t < 2 \\ 0.2 + 0.3(t-2), & 2 \leq t < 5 \\ 1.1, & t > 5 \end{cases}$$

That is, $P(T > t) = e^{-\int_0^t \lambda(s) ds}$

(a) What is the probability that the machine will still be working six years after being purchased?
<u>*Solution.*</u>
To find the probability the machine still works after 6 years is equivalent to finding the probability that it does not break for at least 6 years.

$$P(T > 6) = e^{-\int_0^6 \lambda(s) ds} = e^{-\left(\int_0^2 0.2 ds + \int_2^5 0.2 + 0.3(s-2) ds + \int_5^6 1.1 ds\right)} \approx 0.0317$$

(b) If the machine is still working after six years of being purchased, what is the conditional probability that it will fail within the succeeding two years?
<u>*Solution.*</u>
Given $T > 6$, we want to find the $P(T \leq 8)$. Observe that $P(T \leq 8) = 1 - P(T > 8)$. Recall that $\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{P(T>t)}$. We can solve for the pdf to be $f(t) = P(T > t)\lambda(t)$.

$$P(T \leq 8 | T > 6) = \frac{P(T \leq 8 \text{ and } T > 6)}{P(T > 6)} = \frac{\int_6^8 f(t) dt}{0.0317} = \frac{\int_6^8 \lambda(t) \text{Exp}\left(-\int_0^t \lambda(s) ds\right) dt}{0.0317} =$$

$$= \frac{\int_6^8 1.1 \cdot e^{3.15 - 1.1t} dt}{0.0317} = \frac{e^{3.15} \int_6^8 1.1 e^{-1.1t} dt}{0.0317} = \frac{0.0282}{0.0317} \approx 0.8896$$

5. Let Z_1, Z_2 be standard normal random variables (iid) with mean 0 and variance 1. Show that $X = \frac{Z_1}{Z_2}$ is a Cauchy random variable with pdf given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

Proof. Define $Y = Z_2$, and observe that $X = \frac{Z_1}{Z_2} \rightarrow Z_1 = XZ_2 = XY$. To compute the pdf of X, our approach will be to first compute the joint of X, Y, and take the marginal of X. First, we compute the Jacobian

$$|J| = \left| \det \begin{pmatrix} \frac{\partial Z_1}{\partial X} & \frac{\partial Z_1}{\partial Y} \\ \frac{\partial Z_2}{\partial X} & \frac{\partial Z_2}{\partial Y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right| = |y|$$

Since Z_1, Z_2 are iid, we know their joint pdf is the product of their marginals, and since both are standard normal, we compute the joint pdf to be

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{z_1^2+z_2^2}{2}}$$

We are now able to solve for the joint pdf of X, Y

$$f_{X,Y}(x,y) = f_{Z_1,Z_2}\left(z_1 = xy, z_2 = y\right)|J| = \frac{1}{2\pi}|y|e^{-\frac{x^2y^2+y^2}{2}} = \frac{1}{2\pi}|y|e^{-\frac{y^2(x^2+1)}{2}}$$

All that remains is to find the marginal of X of the joint pdf X, Y, and confirm that it is identical to the pdf of Cauchy$(\theta = 0, \sigma = 1)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi}|y|e^{-\frac{y^2(x^2+1)}{2}}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} |y|e^{-\frac{y^2(x^2+1)}{2}}dy$$

Here we recognize that the integrand is even with respect to y, so we change the bounds (and in the process eliminate the absolute value) to be

$$f_X(x) = \frac{1}{2\pi} \cdot 2\int_0^{\infty} ye^{-\frac{y^2(x^2+1)}{2}}dy, \text{ let } u = \frac{y^2(x^2+1)}{2} \rightarrow du = y(x^2+1)dy$$

$$f_X(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+x^2}e^{-u}du = \frac{1}{\pi} \cdot \frac{1}{1+x^2}\int_0^{\infty} e^{-u}du = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

We conclude that $X \sim$ Cauchy$(\theta = 0, \sigma = 1)$. $\square$