Complex Analysis Chapter 1 Section 3

Brandyn Tucknott

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3 Integration along curves

A **parameterized curve** $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parameterized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ with $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a, b$, $z'(a), z'(b)$ are interpreted as one-sided limits:

$$z'(a) = \lim_{h \to 0, h > 0} \frac{z(a+h) - z(a)}{h} \text{ and } z'(b) = \lim_{h \to 0, h < 0} \frac{z(b+h) - z(b)}{h}.$$

These quantities are called the right-handed derivative at $z(a)$ and left handed derivative at $z(b)$. We say the parameterized curve is **piecewise-smooth** if z is continuous on $[a, b]$ and there exist points $a = a_0 < a_1 < \ldots < a_n = b$, where $z(t)$ is smooth on the intervals $[a_k, a_{k+1}]$. The right-handed derivative and left-handed derivative at a_k may differ for $k = 1, 2, \ldots, n - 1$.

Two parameterizations

$$z : [a, b] \to \mathbb{C} \text{ and } \tilde{z} : [c, d] \to \mathbb{C}$$

are **equivalent** if there exists a continuously differentiable bijection $s \to t(s)$ from $[c, d] \to [a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says that orientation must be preserved: as s travels from c to d, $t(s)$ travels from a to b. The points $z(a)$ and $z(b)$ are called **end-points** of the curve and are independent on the parameterization. Since a curve γ carries an orientation, it is natural to say that γ begins at $z(a)$ and ends at $z(b)$. A smooth or piecewise-smooth curve is **closed** if $z(a) = z(b)$ for any of its parameterizations, and **simple** if it is not self-intersecting ($z(t) \neq z(s)$ unless $s = t$).

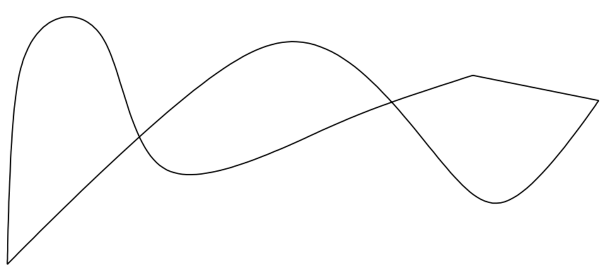

Figure 3. A closed piecewise-smooth curve

We will call any piecewise-smooth curves a **curve**, since these are our objects of primary concern. A basic example is a circle centered at z_0 with radius r, which is by definition

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The **positive orientation** (counterclockwise) is one given by the standard parameterization

$$z(t) = z_0 + re^{it} \text{ where } t \in [0, 2\pi],$$

while the **negative orientation** (clockwise) is the one given by

$$z(t) = z_0 + re^{-it} \text{ where } t \in [0, 2\pi].$$

In the following chapters, we denote by C the general positively oriented circle. Loosely speaking, a key theorem in complex analysis states that if a function is holomorphic is the interior of a closed curve γ, then

$$\int_\gamma f(z)dz = 0. \text{ (we explore this more next chapter)}$$

Given a smooth curve γ in $\mathbb{C}$ parameterized by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ, we define the integral of f along γ as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

For this definition to have meaning, we have to show that the right-hand integral is independent the choice of γ. Say that $\tilde{z}$ is an equivalent parameterization as above. Then the change of variables formula and chain rule imply that

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(z(t(s)))z'(t(s))t'(s)ds = \int_c^d f(z\tilde{(s)})\tilde{z}'(s)ds.$$

Thus the integral of f over γ is well-defined.

If γ is piecewise-smooth and $z(t)$ a piecewise-smooth parameterization, then

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt.$$

By definition, the length of a smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|\, dt.$$

If γ is piecewise smooth, the its length is the sum of its smooth parts.

Proposition 3.1. *Integration of continuous functions over curves satisfies the following properties:*

- *It is linear, that is, if $\alpha, \beta \in \mathbb{C}$, then*

$$\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$$

- *If γ^- is γ with the reverse orientation, then*

$$\int_\gamma f = -\int_{\gamma^-} f$$

- *One has the inequality*

$$\left| \int_\gamma f \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \textit{length}(\gamma)$$

Proof. The first property follows from linearity of the Riemann integral. The second property is a result of the integral being independent of our choice of γ. Let $\gamma : [a, b] \to \mathbb{C}$, and recall that $\gamma^- = \gamma(a+b-t)$. Note that if we let $s = a + b - t$, then $ds = -dt$.

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t) \\
&= \int_a^b f(\gamma^-(a+b-t))(-\gamma^{-\prime}(a+b-t))dt \\
&= \int_b^a f(\gamma^-(s))(-\gamma^{-\prime}(s)) - ds \\
&= \int_a^b f(\gamma^-(s))(-\gamma^{-\prime}(s))ds \\
&= -\int_{\gamma^-} f
\end{aligned}$$

For the third property, note that

$$\left|\int_\gamma f\right| \leq \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|dt \leq \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$$

as was to be shown. ∎

A **primitive** for f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in$.

Theorem 3.2. *If a continuous function f has a primitive in Ω, and γ is a curve in Omega that begins at w_1 and ends at w_2, then*

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

Proof. If γ is smooth, then by application of the chain rule and the fundamental theorem of calculus it is true. If $z : [a, b] \to \mathbb{C}$ is a parameterization of γ, then $z(a) = w_1$ and $z(b) = w_2$ and we have

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b F'(z(t))z'(t)dt \\
&= \int_a^b \frac{d}{dt}F(z(t))z'(t)dt \\
&= F(z(b)) - F(z(a)).
\end{aligned}$$

If γ is only piecewise-smooth, then we can obtain the telescopic sum

$$\begin{aligned}
\int_\gamma f(z)dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\
&= F(z(a_n)) - F(z(a_0)) \\
&= F(z(b)) - F(z(a))
\end{aligned}$$

∎

Corollary 3.3. *If γ is a closed curve in an open set Ω, and f is continuous and has a primitive in Ω, then*

$$\int_\gamma f(z)dz = 0.$$

Corollary 3.4. *If f is holomorphic in a region Ω and $f' = 0$, then f is constant.*

Proof. Fix a point $w_0 \in \Omega$. It suffices to show that $f(w) = f(w_0)$ for all $w \in \Omega$. Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w. Since f is clearly a primitive for f', we have

$$\int_\gamma f'(z)dz = f(w) - f(w_0).$$

By assumption, $f' = 0$ so the integral on the left is 0, and we conclude that $f(w) = f(w_0)$. $\square$

Remark on notation. When convenient, we follow the practice of using the notation $f(z) = O(g(z))$ to mean that there is a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for z in the neighborhood of the point in question. In addition, we say $f(z) = o(g(z))$ when $|f(z)/g(z)| \to 0$. We also write $f(z) \sim g(z)$ to mean that $f(z)/g(z) \to 1$.