# ST 421 HW 3

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**3.14** The maximum patent life for a new drug is 17 years. Subtracting the length of time required by the FDA for testing and approval of the drug provides the actual patent life for the drug—that is, the length of time the company has to recover research and development costs and make a profit. The distribution of the lengths of actual patent lives for new drugs is given below:

| Years, $y$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(y)$ | .03 | .05 | .07 | .1 | .14 | .2 | .18 | .12 | .07 | .03 | .01 |

**(a)** Find the mean patent life for a new drug.
*Solution.*

$$\mu = \sum_{i=3}^{13} i \cdot P(i) = 7.9$$

**(b)** What is the standard deviation of $Y =$ the length of life of a randomly selected new drug.
*Solution.*

$$\sigma = \sqrt{\mathrm{Var}\left([)\,Y\right]} = \sqrt{E[Y^2] - E^2[Y]} = \sqrt{\sum_{i=3}^{13} i^2 \cdot P(i) - 7.9^2} = \sqrt{67.14 - 7.9^2} = 2.175$$

**(c)** What is the probability the value of $Y$ falls between $\mu \pm 2\sigma$?
*Solution.*
$P(Y \in [\mu - 2\sigma, \mu + \sigma]) = P(Y \in [3.55, 12.25]) = .05 + .07 + .1 + .14 + .2 + .18 + .12 + .07 + .03 = 0.96$

**3.23** In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid \$15 for drawing a jack or queen and \$5 for a king or ace. A person who draws any other card is paid \$4. If a person plays this game, what is the expected gain?
*Solution.*

$$\mu = 4 \cdot \frac{9 \cdot 4}{52} + 5 \cdot \frac{4 \cdot 2}{52} + 15 \cdot \frac{4 \cdot 2}{52} = 5.846 \approx 5.85$$

So the expected gain from playing this game is \$5.85.

**3.30** Suppose $Y$ is a discrete random variable with mean $\mu$ and variance $\sigma^2$ and let $X = Y + 1$.

**(a)** Do you expect the mean of $X$ to be larger, smaller, or equal to $\mu = E[Y]$?
*Solution.*

$$E[X] = E[Y+1] = E[Y] + E[1] = E[Y] + 1$$

Therefore by linearity the expected value of $E[X] > E[Y]$.

**(b)** Use theorem 3.3 and 3.5 to express $E[X] = E[Y+1]$ in terms of $\mu = E[Y]$. Does this result agree with your answer to part (a)?
*Solution.*
Yes, and as mentioned in part (a), $E[X] = E[Y+1] = E[Y] + E[1] = E[Y] + 1 \rightarrow E[X] > E[Y]$.

**(c)** Recalling that variance is a measure of spread or dispersion, do you expect the variance of $X$ to be larger, smaller, or equal to $\sigma^2$?
*Solution.*

$$\text{Var}\left([\right) X] = E[X^2] - E^2[X] = E[Y^2 + 2Y + 1] - E^2[Y+1] = E[Y^2] + 2 \cdot E[Y] + 1 - E^2[Y] - 2 \cdot E[Y] - 1 = E[Y^2] - E^2[Y]$$

Compare this to

$$\text{Var}\left([\right) Y] = E[Y^2] - E^2[Y]$$

Since they are equal, we say that $\text{Var}\left([\right) X] = \text{Var}\left([\right) Y]$.

**(d)** Use definition 3.5 and the result in part (b) to show that

$$\text{Var}\left([\right) X] = E[X - E[X]] = E[(Y - \mu)^2] = \sigma^2$$

*Solution.*
As shown in part (c), these variances are equal without the need for theorems or part (b).

**3.40** The probability that a patient recovers from a stomach disease is 0.8. Suppose 20 people are known to have contracted this disease. What are the below probabilities? Let the event $R$ denote people recovering.

**(a)** Exactly 14 people recover?
<u>*Solution.*</u>

$$P(R = 14) = \binom{20}{14} \cdot 0.8^{14} \cdot 0.2^{6} = 0.109$$

**(b)** At least 10 people recover?
<u>*Solution.*</u>

$$P(R \geq 10) = 1 - P(R \leq 9) = 1 - \sum_{k=0}^{9} \binom{20}{k} \cdot 0.8^{k} \cdot 0.2^{20-k} = 0.999$$

**(c)** Between 14 and 18 people recover?
<u>*Solution.*</u>

$$P(14 \leq R \leq 18) = \sum_{k=14}^{18} \binom{20}{k} \cdot 0.8^{k} \cdot 0.2^{20-k} = 0.844$$

**(d)** At most 16 people recover?
<u>*Solution.*</u>

$$P(R \leq 16) = \sum_{k=0}^{16} \binom{20}{k} \cdot 0.8^{k} \cdot 0.2^{20-k} = 0.589$$

**3.70** An oil prospector will drill succession of holes in a given area to find a productive well. The probability he is successful on a given trial is 0.2.

**(a)** What is the probability that the third hole drilled is the first to yield a productive well?
*Solution.*

$$P(3^{rd} \text{ hole first success}) = P(\text{fail}) \cdot P(\text{fail}) \cdot P(\text{success}) = 0.8 \cdot 0.8 \cdot 0.2 = 0.128$$

**(b)** If the prospector can afford to drill at most 10 wells, what is the probability he will fail to find a productive well?
*Solution.*

$$P(10 \text{ fails}) = 0.8^{10} = 0.107$$

**3.85** Find $E[Y(Y-1)]$ for a geometric random variable $Y$ by finding $\frac{d^2}{dq^2}\left(\sum_{y=1}^{\infty} q^y\right)$. Use this result to find the variance of $Y$.

<u>*Solution.*</u>

Note that $Q = \sum_{y=1}^{\infty} q^y = \frac{q}{1-q}$, so the double derivative of the series is equal to the double derivative of the sum value.

$$\frac{d^2}{dq^2}Q = \frac{d^2}{dq^2}\frac{q}{1-q} = \frac{2}{(1-q)^3}$$

If $q = 1-p$, then

$$E[Y(Y-1)] = p\sum_{y=1}^{\infty} y(y-1)q^{y-1} = pq\sum_{y=2}^{\infty} y(y-1)q^{y-2} = pq \cdot \frac{d^2}{dq^2}Q = pq \cdot \frac{2}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

Note that $E[Y] = \frac{1}{p}$. From here, we can calculate the variance to be

$$\text{Var}\left([\,]\, Y\right] = E[Y^2] - E^2[Y] = E[Y^2] - E[Y] - E^2[Y] + E[Y] = E[Y(Y-1)] - E^2[Y] + E[Y] =$$

$$\frac{2q}{p^2} - \frac{1}{p^2} + \frac{1}{p} = \frac{2q-1+p}{p^2} = \frac{1-p}{p^2}$$

**3.167** Let $Y$ be a random variable with mean 11 and variance 9. Using Chebyshev's theorem, find

**(a)** A lower bound for $P(6 < Y < 16)$.

*Solution.*

Let $k = \frac{5}{3}$, then $k\sigma = 5$, and $\mu \pm k\sigma = 6, 16$. Then by Chebyshev's inequality, we have that

$$P(\mu - k\sigma < Y < \mu + k\sigma) = P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{9}{25} = \frac{16}{25}$$

**(b)** The value of $C$ such that $P(|Y - 11| \geq C) \leq 0.09$.

*Solution.*

By Chebyshev's inequality, we know that

$$P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

This tells us that $C = k\sigma$ and $\frac{1}{k^2} = 0.09$. Then we solve for $k$, getting $k = \sqrt{\frac{1}{0.09}} = \sqrt{\frac{1}{\frac{9}{100}}} = \frac{10}{3}$. With the knowledge that $\sigma = 3$ and $k = \frac{10}{3}$, we calculate $C$ to be $C = k\sigma = \frac{10}{3} \cdot 3 = 10$.

**3.177** For a certain section of a pine forest, the number of diseased trees per acre $Y$ has a Poisson distribution with mean $\lambda = 10$. The diseased trees are sprayed with an insecticide at a cost of \$3 per tree, plus the fixed overhead cost for equipment rental at \$50. Letting $C$ denote the total spraying cost for a randomly selected acre, find the expected value and standard deviation for $C$. Within what interval would you expect $C$ to lie with probability at least 0.75.
*Solution.*
We know that the for a Poisson distribution, $E[X] = \text{Var}\left([\right) X] = \lambda$, so we know that

$$\mu = E[C] = E[3Y + 50] = 3E[Y] + 50 = 3\lambda + 50 = 80$$

$$\sigma = \sqrt{\text{Var}\left([\right) C]} = \sqrt{\text{Var}\left([\right) 3Y + 50]} = \sqrt{9\text{Var}\left([\right) Y] + 0} = \sqrt{9\lambda} = 3\sqrt{10}$$

By Chebyshev's inequality, we know that $P(|C - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$. Since we want our probability to be at least 0.75, we have

$$1 - \frac{1}{k^2} = 0.75 \longrightarrow k = 2$$

We now calculate $k\sigma = 2 \cdot 3\sqrt{10} = 6\sqrt{10}$, and we have the inequality

$$|C - \mu| < k\sigma \longrightarrow$$

$$-k\sigma < C - \mu < k\sigma \rightarrow$$

$$\mu - k\sigma < C < \mu + k\sigma$$

Subbing in our values for $\sigma, \mu, k$, we get

$$80 - 6\sqrt{10} < C < 80 + 6\sqrt{10}$$

$$61.026 < C < 98.974$$