MTH 463 HW 7

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1. A fair die is rolled 18,000 times. Use the DeMoivre-Laplace Central Limit Theorem to approximate to approximate the binomial distribution, estimate the probability that 6 comes up at least 3060 times.
   Solution.
   We approximate the binomial distribution as a normal distribution with $\mu = np, \sigma^2 = np(1-p)$ by the DeMoivre-Laplace Central Limit Theorem. We know we can use this since the number of trials n is sufficiently large. Note that

$$P(X = 6 \text{ occurs} \geq 3060 \text{ times}) = 1 - P(X = 6 \text{ occurs} \leq 3060 \text{ times})$$

   We now standardize the distribution with $Z = \frac{X-\mu}{\sigma}$, so to find the probability that $X \leq 3060$, we find the z-score for $x = 3060$, and match it to to a standardized normal distribution probability table.

   First, we calculate the mean and variance to be

$$\mu = np = 18000 \cdot \frac{1}{6} = 3000$$

$$\sigma^2 = \mu(1-p) = 3000 \cdot \frac{5}{6} = 2500 \to \sigma = 50$$

   We now compute a z-score,

$$z = \frac{x-\mu}{\sigma} = \frac{3060 - 3000}{50} = \frac{6}{5} = 1.2$$

   The table tells us that this is 0.8849, so we know that $P(X = 6 \text{ occurs} \leq 3060) = 0.8849$, and we conclude that

$$P(X = 6 \text{ occurs} \geq 3060 \text{ times}) = 1 - 0.8849 = 0.1151$$

2. An experiment consists of 1210 independent Bernoulli trials with probability of success $p = \frac{1}{11}$. Use the DeMoivre-Laplace Central Limit Theorem to estimate the probability of the event that

$$98 \leq \text{ Number of successes } \leq 116$$

Solution.
Since the number of trials is sufficiently large, we use the DeMoivre-Laplace Central Limit Theorem to estimate the probability that the number of successes is between 98 and 116 with

$$\mu = np = \frac{1210}{11} = 110$$

$$\sigma^2 = \mu(1-p) = 110 \cdot \frac{10}{11} = 100 \to \sigma = 10$$

Let X denote the number of successes and note that

$$P(98 \leq X \leq 116) = P(X \leq 116) - P(X \leq 98)$$

We find the z-score of 116 and 98 to find the probability that $X \leq 116, 98$ respectively.

$$z_{116} = \frac{116 - \mu}{\sigma} = \frac{116 - 110}{10} = \frac{6}{10} = 0.6 \longrightarrow P(X \leq 116) = 0.7257$$

$$z_{98} = \frac{98 - \mu}{\sigma} = \frac{98 - 110}{10} = \frac{-12}{10} = -1.2 \longrightarrow P(X \leq 98) = 0.1151$$

$$P(98 \leq X \leq 116) = P(X \leq 116) - P(X \leq 98) = 0.7257 - 0.1151 = 0.6106$$

3. Henry flips a fair coin 5 times every morning for 30 consecutive days. Let X denote the number of mornings over these 30 days in which all 5 flips were tails.

   (a) Compute to 4 decimal digits the probability that $P(X = 2)$.
   Solution.
   First, recognize that a success is when all 5 flips result in tails, so $p = \frac{1}{2^5} = \frac{1}{32}$. Then $X \sim \text{Bin}(n, p)$, and we directly calculate $P(X = 2)$ using the probability mass function of X.

$$P(X = 2) = \binom{n}{2} p^2 (1-p)^{n-2} = \binom{30}{2} \left(\frac{1}{32}\right)^2 \left(\frac{31}{32}\right)^{28} \approx 0.1746$$

   (b) Use the Poisson approximation to estimate $P(X = 2)$.
   Solution.
   Using the Poisson approximation, we let $\lambda = np = 30 \cdot \frac{1}{32} = \frac{15}{16}$. Then

$$P(X = 2) \approx e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\frac{15}{16}} \frac{\left(\frac{15}{16}\right)^2}{2!} \approx 0.1721$$

   (c) Try the normal approximation to estimate $P(X = 2)$. Compare your results and comment.
   Solution.
   Using the normal approximation, we let

$$\mu = np = \frac{15}{16}$$

$$\sigma^2 = \mu(1-p) = \frac{15}{16} \cdot \frac{31}{32} = \frac{465}{512} \to \sigma = \sqrt{\frac{465}{512}}$$

$$P(X = 2) \approx P(X \leq 2.5) - P(X \leq 1.5)$$

   Converting to a z-score, we have that

$$z_{2.5} = \frac{2.5 - \frac{15}{16}}{\sqrt{\frac{465}{512}}} \approx 1.64$$

$$z_{1.5} = \frac{1.5 - \frac{15}{16}}{\sqrt{\frac{465}{512}}} \approx 0.59$$

   This lets us evaluate our probability as

$$P(X = 2) \approx 0.9495 - 0.7224 = 0.2271$$

   We conclude that for n not sufficiently large enough, the normal approximation is not a good estimate, and that the Poisson may be a better estimate for smaller n.

4. Consider modeling losses from an accident using a uniform random variable X on the interval $[0, M]$ with $M > 0$. To reduce its risk, an individual purchases an insurance policy with deductible $D < M$. Denote Y by the random payment that the policy holder makes in the event of a loss, that is

$$Y = \begin{cases} X, & X \leq D \\ D, & D < X \leq M \end{cases}$$

(a) Compute the expected loss by the policy holder, $E(Y)$.
<u>*Solution.*</u>

$$E(Y) = \int_0^D x f_X(x)dx + \int_D^M D f_X(x)dx = \int_0^D x\frac{1}{M}dx + \int_D^M D\frac{1}{M}dx =$$

$$= \frac{1}{M}\frac{x^2}{2}\bigg|_0^D + \frac{D}{M}x\bigg|_D^M = \frac{D^2}{2M} + D - \frac{D^2}{M} = D - \frac{D^2}{2M}$$

(b) Compute the value $R = E(X)P(X \leq D)$. Show that $R \leq E(Y)$ with equality if and only if $D = M$.
<u>*Solution.*</u>

$$R = E(X)P(X \leq D) = \frac{M}{2}\int_0^D f_X(x)dx = \frac{M}{2M}x\bigg|_0^D = \frac{D}{2}$$

We now examine the inequality

$$R \leq E(Y) =$$

$$\frac{D}{2} \leq D - \frac{D^2}{2M} =$$

$$\frac{1}{2} \leq 1 - \frac{D}{2M}$$

$$\frac{D}{M} \leq 1$$

Obviously $D < M$ implies $\frac{D}{M} < 1$, and equality if and only if $D = M$.

5. Let Z be a standard normal random variable. For $\sigma > 0, \mu \in \mathbb{R}$, define $X = e^{\mu+\sigma Z}$.

   (a) Find the probability density function of X.
   Solution.
   We are trying to find $f_X(x)$ in terms of the random variable Z, and we do this using the Jacobian.

$$Z = \frac{\ln X - \mu}{\sigma} \longrightarrow \frac{dZ}{dX} = \frac{1}{\sigma X}$$

$$f_X(x) = f_Z(z) \cdot \left|\frac{dZ}{dX}\right| = f_Z(z) \cdot \frac{dZ}{dX} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{\sigma x} = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

   (b) Find $E(X)$.
   Solution.

$$E(X) = E\left(e^{\mu+\sigma Z}\right) = \int_{-\infty}^{\infty} e^{\mu+\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = \frac{e^{\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2-2\sigma z}{2}}\, dz = \frac{e^{\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2-2\sigma z+\sigma^2-\sigma^2}{2}}\, dz =$$

$$= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma)^2}{2}}\, dz, \text{ let } u = z - \sigma \to du = dz$$

$$= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\, du = \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^{\mu+\frac{\sigma^2}{2}}$$