Problem 1: Dot Product of Random Vectors

Brandyn Tucknott

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(a) If $v, u \in \mathbb{R}^n$ for $n < \infty$ with

$$v_i = \begin{cases} 1, & \text{with probability } 1/2 \\ 0, & \text{with probability } 1/2 \end{cases}$$

$$u_i = \begin{cases} 1, & \text{with probability } 1/4 \\ 0, & \text{with probability } 3/4 \end{cases},$$

what is the expected value of $v \cdot u$ (with proof)?

Proof. We assume that the u_i, v_i are independent of each other since no reason has been given to lead us to suspect otherwise. Then by linearity of expectation and independence, we have that

$$\begin{aligned} \mathbb{E}(u \cdot v) &= \sum_{i=1}^{n} \mathbb{E}(u_i \cdot v_i) \\ &= \sum_{i=1}^{n} \mathbb{P}(u_i = 1, v_i = 1) \\ &= \sum_{i=1}^{n} \mathbb{P}(u_i = 1)\, \mathbb{P}(v_i = 1) \\ &= \sum i = 1^n \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{n}{8}. \end{aligned}$$

□

(b) What about for a more generalized case?

$$v_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases}$$

$$u_i = \begin{cases} 1, & \text{with probability } q \\ 0, & \text{with probability } 1 - q \end{cases},$$

what is the expected value of $v \cdot u$ (with proof)?

Proof. Again assuming independence, we use linearity to find that

$$\begin{aligned}
\mathbb{E}\left(u \cdot v\right) &= \sum_{i=1}^{n} \mathbb{E}\left(u_i \cdot v_i\right) \\
&= \sum_{i=1}^{n} \mathbb{P}\left(u_i = 1, v_i = 1\right) \\
&= \sum_{i=1}^{n} \mathbb{P}\left(u_i = 1\right) \mathbb{P}\left(v_i = 1\right) \\
&= \sum_{i=1}^{n} pq \\
&= npq.
\end{aligned}$$

$\square$