MTH 511 HW 6

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1. Prove lemma 8.8.

Lemma 0.1. *In a metric space M, the following are equivalent:*

(a) If $\mathcal{G}$ is a basis for any collection of open sets in M with $\cup\{G : G \in \mathcal{G}\}M$, then there are finitely many $G_1, G_2, \ldots, G_n \in \mathcal{G}$ with $\cup_{i=1}^{n} G_i M$.

(b) If $\mathcal{F}$ is any collection of closed sets in M such that $\cap F_i \neq \emptyset$ for all choices of finitely many sets $F_i, \ldots, F_n \in \mathcal{F}$, then $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$.

□

Proof. First, we will show that $(a) \implies (b)$. Suppose that (a) is true, and let $\mathcal{F}$ be a collection of closed sets such that every finite subcollection has nonempty intersection. For contradiction, assume that $\cap_{F\in\mathcal{F}} F = \emptyset$. Then for each $F \in \mathcal{F}$, let $G_F = M \setminus F$. We know this to be open, and since

$$\bigcap_{F\in\mathcal{F}} F = \emptyset \text{ if and only if } \bigcap_{F\in\mathcal{F}} G_F = M,$$

we know that $\{G_F : F \in \mathcal{F}\}$ covers M. By (a) there are finitely many $F_{F_1}, \ldots, G_{F_n}$ whose union is M, but then

$$\bigcap_{i=1}^{n} F_i = M \setminus \bigcup_{i=1}^{n} G_{F_i} = M \setminus M = \emptyset.$$

This is a contradiction, so we conclude that $\cap_{F\in\mathcal{F}} F \neq \emptyset$, and thus (b) is true.

Now we will show that $(b) \implies (a)$. Suppose that (b) is true, and let $\mathcal{G}$ be a collection of open sets with $\cup_{G\in\mathcal{G}} G = M$. For contradiction, assume that no finite subcollection of $\mathcal{G}$ covers M. Then for each $G \in \mathcal{G}$, let $F_G = M \setminus G$ and observe F_G is closed. For any $G_1, \ldots, G_n$, we have that

$$\bigcup_{i=1}^{n} G_i \neq M \implies \bigcap_{i=1}^{n} F_{G_i} = M \setminus \bigcup_{i=1}^{n} G_i \neq \emptyset.$$

Thus F_{G_i} forms a finite basis and by (b) we know that $\cap\{F_G : G \in \mathcal{G}\} \neq \emptyset$. However, this implies that $M \setminus \cup_{G\in\mathcal{G}} G \neq \emptyset$, a contradiction to $\cup_{G\in\mathcal{G}} G = M$. It must be then, that our assumption was false, and there exists a finite subcollection of $\mathcal{G}$ that covers M, and so (a) is true. □

2. Let $\mathcal{G}$ be an open cover for M. We say that $\varepsilon > 0$ is a *Lebesgue number* for $\mathcal{G}$ if each subset of diameter $< \varepsilon$ is contained in some $G \subset \mathcal{G}$. If M is compact, show that every open cover of M has a Lebesque number.

   Proof. Let $\mathcal{G}$ be an open cover for a compact metric space M. For each $x \in M$, choose $G_x \in \mathcal{G}$ such that $x \in G_x$. Because G_x is open, there exists $r_x > 0$ such that $B_{r_x}(x) \subset G_x$. Since the family $\left\{B_{r_x/2}(x) : x \in M\right\}$ is an open cover for M, by compactness it has a finite subcover, and there exist points $x_1, \ldots, x_n \in M$ with $M = \cup_{i=1}^n B_{r_{x_i}/2}(x_i)$. If we set $\varepsilon = \min\limits_{1 \le i \le n} \frac{r_{x_i}}{2}$. Certainly $\varepsilon > 0$, and it is sufficient to show that ε is a Lebesgue number for $\mathcal{G}$.

   Let $A \subset M$ be any set with diameter $< \varepsilon$. Then for any $a \in A$, since $B_{r_{x_i}/2}(x_i)$ covers M, there exists some $k \le n$ such that $a \in B_{r_{x_i}/2}(x_i)$ and so $d(a, x_k) < r_{x_k}/2$. For any other $b \in A$,

$$d(b, x_k) \le d(b, a) + d(a, x_k) < \varepsilon + \frac{r_{x_k}}{2} \le \frac{r_{x_k}}{2} + \frac{r_{x_k}}{2} = r_{x_k},$$

   and so $b \in B_{r_{x_k}}(x_k) \subset G_{x_k}$. Since every $b \in A$ lies inside G_{x_i}, we have that $A \subset G_{x_k}$. We conclude that every subset with diameter $< \varepsilon$ is contained by some $G \in \mathcal{G}$, and thus ε is a Lebesque number for $\mathcal{G}$. $\square$

3. Give an example of a continuous bounded map $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous. Can an unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous? Explain.

   Proof. One example of a continuous and bounded function but not uniformly continuous is $f(x) = \sin(x^2)$. For unbounded continuous functions which are uniformly continuous, we simply seek a Lipschitz continuous function, so this is certainly possible. For example, consider $f(x) = x$. It is certainly unbounded and 1-Lipschitz, and thus uniformly continuous. $\square$

4. Prove that $f : (M, d) \to (N, \rho)$ is uniformly continuous if and only if $\rho(f(x_n), f(y_n)) \to 0$ for any pair of sequences $(x_n), (y_n)$ in M satisfying $d(x_n, y_n) \to 0$.

   Proof. First we will show the forward direction. Suppose f is uniformly continuous, and fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $d(x, y) < \delta \implies \rho(f(x), f(y)) < \varepsilon$ for all $x, y \in M$. If $d(x_n, y_n) \to 0$, then for large n we have that $d(x_n, y_n) < \delta$, and thus $\rho(f(x_n), f(y_n)) < \varepsilon$. Since ε was arbitrary, we conclude that $\rho(f(x_n), f(y_n)) \to 0$.

   For the backwards direction, suppose that $d(x_n, y_n) \to 0$ and $\rho(f(x_n), f(y_n)) \to 0$. We wish to show f is uniformly continuous. We do this by equivalently showing the contrapositive. That is, we wish to show if f is not uniformly continuous, then $d(x_n, y_n) \not\to 0$ or $\rho(f(x_n), f(y_n)) \not\to 0$. Suppose f is not uniformly continuous. Then there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, y \in M$ with $d(x, y) < \delta$ but $\rho(f(x), f(y)) \geq \varepsilon_0$. For each $n \geq 1$, choose x_n, y_n such that $d(x_n, y_n) < \frac{1}{n}$ but $\rho(f(x_n), f(y_n)) \geq \varepsilon_0$. Then $d(x_n, y_n) \to 0$, but $\rho(f(x_n), f(y_n))$ does not tend towards 0 since it is bounded below by ε_0. $\square$

5. Define $f : \ell_2 \to \ell_2$ by $f(x) = (x_n/n)_{n=1}^{\infty}$. Show that f is uniformly continuous.

Proof. For $x \in \ell_2$, write arbitrary $x = (x_n)_{n=1}^{\infty}$. Then

$$\|f(x)\|_{\ell_2}^2 = \sum_{n=1}^{\infty} \left| \frac{x_n}{x} \right|^2 = \sum_{n=1}^{n} \frac{|x_n|^2}{n^2} \leq \sum_{n=1}^{n} |x_n|^2 = \|\|\,\ell_2{}^2,$$

so $\|f(x)\|_{\ell_2} \leq \|x\|_{\ell_2}$, and for any $x, y \in \ell_2$ we have that

$$\|f(x) - f(y)\|_{\ell_2} = \|f(x-y)\|_{\ell_2} \leq \|x - y\|_{\ell_2}.$$

So f is 1-Lipschitz, and thus uniformly continuous. $\square$

6. Prove that a sequence of functions $f_n : X \to \mathbb{R}$ is uniformly convergent if and only if it is uniformly Cauchy. That is, prove that there exists some $f : X \to \mathbb{R}$ such that $f_n \rightrightarrows f$ on X if and only if for all $\varepsilon > 0$, there exists $N \geq 1$ such that $\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon$ whenever $m, n \geq N$.

Proof. Let f_n be a sequence of real-valued functions on X. First, we show the forwards direction. Assume $f_n \to f$ uniformly and fix $\varepsilon > 0$. Then there exists some $N \geq 1$ such that for all $n \geq N$, we have that $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon/2$. Then for any $m, n \geq N$ and $x \in X$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon$ and so f_n is uniformly Cauchy.

Now we show the backwards direction. Assume f_n is uniformly Cauchy, and fix $\varepsilon > 0$. Then there exists some N such that for all $m, n \geq N$, we have that $\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon$. Now fix $x \in X$. Then

$$|f_n(x) - f_m(x)| \leq \sup_{x \in X} |f_n - f_m| < \varepsilon,$$

and we have that the sequence $(f_n(x))_{n=1}^{\infty}$ is Cauchy. Then for $f(x) = \lim_{n \to \infty} f_n(x)$, we want to show $f_n \to f$ uniformly. Let $\varepsilon > 0$ and choose N such that for all $m, n \geq N$,

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$

We know this to be possible since f_n is uniformly Cauchy. Now fix $n \geq N, x \in X$. For all $m \geq N$,

$$|f_n(x) - f_m(x)| \leq \sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$

As $m \to \infty$, we have that $|f_n(x) - f_m(x)| \to |f_n(x) - f(x)|$ and so $|f_n(x) - f(x)| \leq \varepsilon$. Then

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon$$

for all $n \geq N$, and so $f_n \to f$ uniformly on X. $\square$

7. Let M be compact and $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto.

Proof. For contradiction, suppose f is not onto. Then $f(M) \subset M$ is compact, so there exists $a \in M \setminus f(M)$. Define $\delta := \inf_{x \in f(M)} d(a, x) > 0$ (positive since $a \notin f(M)$). Consider $a, f(a), f^2(a), f^3(a), \ldots$ for $m > n \geq n$ where $f^\alpha(a) = f(f^{\alpha-1}(a))$ with $f^0(a) = a$. Then

$$d(f^m(a), f^n(a)) = d(f^{m-n}(a), a).$$

Since $f^k(a) \in f(M)$ for all $k \geq 1$, $d(a, f^k(a)) \geq \delta$. Thus for all $m > n \geq 0$,

$$d(f^m(a), d^n(a)) \geq \delta.$$

We conclude that all distinct terms in the sequence $(f^n(a))_{n=0}^\infty$ are at least δ apart, and thus has no convergent subsequence. This is a contradiction to M being compact, so our assumption was false, and it must be that f is onto. Notice that if M is not compact, f need not be onto, and we can make no further conclusions. $\square$