MTH 463 HW 5

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1. Assume X is a discrete random variable with the probability mass function

$$m(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{3} & x = 1 \\ \frac{1}{6} & x = 2 \end{cases}$$

Find $E(X), E(X^2),$ and $\text{Var}(X)$.

Solution.
We calculate the following to be

$$E(X) = \sum_{j=1}^{3} x_j \cdot P(x_j) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} = \frac{2}{3}$$

$$E(X^2) = \sum_{j=1}^{3} x_j \cdot P(x_j) = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{6} = 1$$

$$\text{Var}(X) = E(X^2) - E^2(X) = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

2. Let X be a Binomial random variable with parameters n, p. Show that

$$E\left(\frac{1}{1+X}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

Proof. First, note that since

$$k\binom{n}{k} = n\binom{n-1}{k-1} \longrightarrow$$

$$k\binom{n+1}{k} = (n+1)\binom{n}{k-1} \to \binom{n}{k-1} = \frac{k}{n+1}\binom{n+1}{k} \tag{1}$$

Also recognize that since

$$1 = 1^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}p^k(1-p)^{(n+1)-k} \longrightarrow$$

$$1 = \binom{n+1}{0}p^0(1-p)^{(n+1)-0} + \sum_{k=1}^{n+1}\binom{n+1}{k}p^k(1-p)^{(n+1)-k} = (1-p)^{n+1} + \sum_{k=1}^{n+1}\binom{n+1}{k}p^k(1-p)^{(n+1)-k} \longrightarrow$$

$$1-(1-p)^{n+1} = \sum_{k=1}^{n+1}\binom{n+1}{k}p^k(1-p)^{(n+1)-k} \tag{2}$$

These are important, and will be used later in the proof. To find $E\left(\frac{1}{1+X}\right)$, we simply compute it. Since $X \sim \text{binomial}(n, p)$,

$$E\left(\frac{1}{1+X}\right) = \sum_{k=0}^{n}\frac{1}{1+k}\binom{n}{k}p^k(1-p)^{n-k}) =$$

$$= \sum_{k=1}^{n+1}\frac{1}{1+(k-1)}\binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} =$$

$$\sum_{k=1}^{n+1}\frac{1}{k}\binom{n}{k-1}p^{k-1}(1-p)^{(n+1)-k} \tag{3}$$

By substituting equation (1) into equation (3), we get

$$E\left(\frac{1}{1+X}\right) = \sum_{k=1}^{n+1}\frac{1}{k}\left(\frac{k}{n+1}\binom{n+1}{k}\right)p^{k-1}(1-p)^{(n+1)-k} =$$

$$= \sum_{k=1}^{n+1}\frac{1}{n+1}\binom{n+1}{k}p^{k-1}(1-p)^{(n+1)-k} =$$

$$\frac{1}{p(n+1)}\sum_{k=1}^{n+1}\binom{n+1}{k}p^k(1-p)^{(n+1)-k} \tag{4}$$

Another substitution of (2) into (4) yields

$$E\left(\frac{1}{1-X}\right) = \frac{1}{p(n+1)} \cdot \left(1-(1-p)^{n+1}\right) = \frac{1-(1-p)^{n+1}}{p(n+1)}$$

This matches the claim, so we are done. $\square$

3. Let X be a Poisson random variable with parameter λ, that is for $k = 0, 1, 2, \ldots$

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Find $E\left(\frac{1}{1+X}\right)$.

Solution.

Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{5}$$

Similarly to Question (2), we directly calculate the expected value.

$$E\left(\frac{1}{1+X}\right) = \sum_{k=0}^{\infty} \frac{1}{1+k} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda^k}{k!} \cdot \frac{\lambda}{\lambda} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} =$$

$$= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} - \frac{e^{-\lambda}}{\lambda} \frac{\lambda^0}{0!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} - \frac{e^{-\lambda}}{\lambda} =$$

$$= \frac{e^{-\lambda}}{\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} - 1 \right) \tag{6}$$

We can substitute in equation (5) into equation (6) using e^{λ} instead of e^x. This gives us

$$E\left(\frac{1}{1+X}\right) = \frac{e^{-\lambda}}{\lambda} \left(e^{\lambda} - 1\right) = \frac{1 - e^{-\lambda}}{\lambda}$$

4. Let X be a continuous random variable with probability density function given by

$$f_X(x) = c(1-x^2)\mathbb{1}_{[-1,1]}(x) = \begin{cases} c(1-x^2) & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?
Solution.

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-1}^{1} c(1-x^2)dx = \int_{-1}^{1} cdx - \int_{-1}^{1} cx^2dx = 2c - \frac{c}{3}\cdot 2 = \frac{4}{3}c = 1$$

Solving for c gives $c = \frac{3}{4}$.

(b) Find the cumulative distribution of X.
Solution.

$$F_X(x) = \int_{-\infty}^{x} c(1-t^2)dt = c\left(t - \frac{t^3}{3}\right)\Bigg|_{-1}^{x} = c\left(\left(x - \frac{x^3}{3}\right) - \left(-1 + \frac{1}{3}\right)\right) = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right)$$

(c) Find $E(X)$ and $\mathrm{Var}(X)$.
Solution.

$$E(X) = \int_{-\infty}^{\infty} x{\cdot}P(x)dx = \int_{-1}^{1} x{\cdot}c(1-x^2)dx = c\int_{-1}^{1} x-x^3dx = c{\cdot}0 = 0 \text{ since } x-x^3 \text{ is an odd function}$$

$$\mathrm{Var}(X) = c\int_{-\infty}^{\infty} x^2(1-x^2)dx - E^2(X) = c\int_{-1}^{1} x^2 - x^4dx - 0 = c\cdot 2\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 =$$

$$= 2c\cdot\frac{2}{15} = \frac{4}{15}\cdot\frac{3}{4} = \frac{3}{15}$$

(d) Find $P(|X| < \frac{1}{2})$.
Solution.

$$P\left(|X| < \frac{1}{2}\right) = P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = P\left(X < \frac{1}{2}\right) - P\left(X < -\frac{1}{2}\right) = F_X\left(\frac{1}{2}\right) - F_X\left(-\frac{1}{2}\right) =$$

$$= \frac{3}{4}\left(\frac{1}{2} - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 + \frac{2}{3}\right) - \frac{3}{4}\left(-\frac{1}{2} + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 + \frac{2}{3}\right) = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{24} + \frac{2}{3} + \frac{1}{2} - \frac{1}{24} - \frac{2}{3}\right) = \frac{3}{4}\cdot\frac{11}{12} = \frac{11}{16}$$

5. A real valued random variable X is said to have a Standard Cauchy Distribution if it has pdf

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

(a) Find $F_X(x)$, the cumulative distribution of X. Check that your answer satisfies $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

<u>*Solution.*</u>

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{-\infty}^{x} \frac{1}{\pi} \cdot \frac{1}{1+t^2} dt = \frac{1}{\pi} \cdot \arctan t \Big|_{-\infty}^{x} =$$

$$= \frac{1}{\pi}\left(\arctan(x) - \arctan(-\infty)\right) = \frac{\arctan(x)}{\pi} - (-\frac{\pi}{2\pi}) = \frac{\arctan x}{\pi} + \frac{1}{2}$$

We now do a sanity check and see if our answer satisfies the limits as x approaches positive and negative infinity.

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} \frac{\arctan x}{\pi} + \frac{1}{2} = \frac{\arctan(-\infty)}{\pi} + \frac{1}{2} = -\frac{\pi}{2}\frac{1}{\pi} + \frac{1}{2} = 0$$

$$\lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} \frac{\arctan x}{\pi} + \frac{1}{2} = \frac{\arctan(\infty)}{\pi} + \frac{1}{2} = \frac{\pi}{2}\frac{1}{\pi} + \frac{1}{2} = 1$$

(b) Show that $Y = \frac{1}{X}$ is also a Standard Cauchy random variable.

Proof. Since $Y = \frac{1}{X}, y = \frac{1}{x} \to x = \frac{1}{y}$. We wish to show that the pdf for Y is

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

We can accomplish this by writing the cdf for Y in terms of X and taking the derivative with respect to y.

$$F_Y(y) = P(Y \le y) = P(\frac{1}{X} \le \frac{1}{x}) = P(X \ge x) = 1 - P(X \le x) = 1 - F_X(x)$$

Note that since we are dealing with continuous random variables, $P(X > x)$ and $P(X \ge x)$ are indistinguishable since $\mathbb{R}$ is uncountably infinite $\to P(X = x) = 0$. Then we compute

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - F_X(x)\right) = \frac{d}{dy}\left(1 - \left(\frac{\arctan(x)}{\pi} + \frac{1}{2}\right)\right) =$$

$$\frac{d}{dy}\left(1 - \frac{\arctan x}{\pi} - \frac{1}{2}\right) = \frac{d}{dy}\left(\frac{1}{2} - \frac{\arctan x}{\pi}\right) = -\frac{d}{dy}\left(\frac{\arctan x}{\pi}\right)$$

Substituting in $x = \frac{1}{y}$, we end up with

$$f_Y(y) = -\frac{d}{dy}\left(\frac{\arctan\left(\frac{1}{y}\right)}{\pi}\right) = \frac{-1}{\pi} \cdot \left(\frac{1}{1+\left(\frac{1}{y}\right)^2} \cdot \frac{-1}{y^2}\right) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

This is exactly what we were hoping to show, so we conclude that if X is a Standard Cauchy random variable, then $Y = \frac{1}{X}$ is as well. □