MTH 311 Lab 6

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Theorem. A set $F \subset \mathbb{R}$ is a closed set $\longleftrightarrow$ for every convergent sequence $(a_n) \subset F, \lim_{n\to\infty} a_n \in F$

1. Use the above theorem to prove that the following sets are not closed. In each case, this involves finding a suitable sequence (a_n).

   (a) $A = \{x \in \mathbb{R} : 1 < x < 4\}$

   Proof. Consider the sequence $a_n = \frac{n+1}{n}$. Then certainly for all $n \in \mathbb{N}, 1 < \frac{n+1}{n} < 4$. Additionally, the $\lim_{n\to\infty} a_n = 1$, which is not an element of A. We conclude that A is not closed. □

   (b) $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

   Proof. Consider the sequence $a_n = \frac{1}{n}$. Then for all $n \in \mathbb{N}, a_n \in B$ by definition of B, but the $\lim_{n\to\infty} a_n = 0$. Since 0 is not contained in B, we conclude that B is not closed. □

2. (a) Prove the forward direction of the theorem.

   Proof. Given the set $F \subset \mathbb{R}$ is a closed set, we want to show that every convergent sequence $(a_n) \subset F$ has $\lim_{n\to\infty} a_n \in F$. Let (a_n) be an arbitrary convergent sequence in F and let $x = \lim_{n\to\infty} a_n$. We want to show that $x \in F$. We break this proof into two cases:

   (i) There exists $n \in \mathbb{N}$ such that $a_n = x$.
   (ii) For all $n \in \mathbb{N}, a_n \neq x$.

   In case (i), we assume there exists an $n \in \mathbb{N}$ such that $a_n = x$. Since $(a_n) \subset F$, by definition we have that $x = a_n \in F$ for some $n \in \mathbb{N}$.

   In case (ii), we assume that for all $n \in \mathbb{N}, a_n \neq x$. Then x is definitionally a limit point of F. Since we are given F is closed, we conclude that $x \in F$ by definition of a closed set (3.2.7).

   □

   (b) Prove the reverse direction of the theorem.

   Proof. We wish to show that for all convergent sequences $(a_n) \subset F$, if $x = \lim_{n\to\infty} a_n$ with $x \in F$, then F is a closed set. We do this with a proof my contrapositive, and instead we aim to show that if F is an open set, then for all convergent sequences $(a_n) \subset F, \lim_{n\to\infty} a_n \notin F$.

   Let F be an open set. Then by definition, there exists some limit point $x \notin F$. We aim to show there is a convergent sequence $a_n \subset F$ that converges to x. Consider $V_\epsilon(x) = (x - \frac{1}{\epsilon}, x + \frac{1}{\epsilon})$. Clearly if $\epsilon > 0$, then $V_\epsilon(x)$ is non-empty (we consider non-empty since $x \notin F$), so by definition 3.2.4 we know that x is a limit point of F. Since x is a limit point of F, by theorem 3.2.5 $x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subset F$ where $a_n \neq x$ for all $n \in \mathbb{N}$. With this we have shown there exists a sequence which converges to $x \notin F$, so we are done. □