MTH 311 Lab 3

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10 October 2024

1. Use the definition of convergence of a sequence to prove

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

   (a) Begin by stating explicitly what is to be proved. For this statement, use the definition of convergence. The symbol ϵ should be involved.
   Solution.
   We want to show that $\left|\frac{n}{n^2+1} - 0\right| < \epsilon$

   (b) Before you use ϵ in your proof, first give ϵ a proper introduction
   Solution.
   Let $\epsilon > 0$ be an arbitrary real number.

   (c) $\frac{n}{n^2+1} < \frac{n}{n^2}$ for $n \geq 1$. When you apply the definition of convergence to this problem, this inequality will be useful.

   Proof. Notice that $\frac{n}{n^2+1} < \frac{n}{n^2}$ for $n \geq 1$, so it is sufficient to show that $\lim_{n\to\infty} \frac{n}{n^2} = 0$. Let $\epsilon > 0$ be an arbitrary real number. Choose $N \in \mathbb{N} > \frac{1}{\epsilon}$. Then for all $n \geq N$,

$$n > \frac{1}{\epsilon} \to \epsilon > \frac{1}{n} \to$$

$$\epsilon > \frac{n}{n^2} \to$$

$$\epsilon > \frac{n}{n^2} - 0 \to$$

$$\epsilon > \left|\frac{n}{n^2} - 0\right|$$

   since n is a natural number. With this we have shown that $\lim_{n\to\infty} \frac{n}{n^2} = 0$, so we conclude that the $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. $\square$

2. (a) The definition of convergence of a sequence (a_n) to a limit $a \in \mathbb{R}$ can be stated as follows:

$$\forall_{\epsilon>0}\exists_{N\in\mathbb{N}} \text{ such that } \forall_{n\geq N}, |a_n - a| < \epsilon$$

Find the negation of this statement.That is, state precisely what it means to say that a sequence does not converge to $a \in \mathbb{R}$.

<u>*Solution.*</u>
The negation of convergence to a is:

$$\exists_{\epsilon>0}\forall_{N\in\mathbb{N}} \text{ such that } \exists n \geq N, |a_n - a| > \epsilon$$

In English, the negation states that there exists some $\epsilon > 0 \in \mathbb{R}$ where for any $N \in \mathbb{N}$, there exists some $n \geq N$ such that $|a_n - a| > \epsilon$.

(b) Let $a_n = (-1)^n$ for all integers $n \geq 1$. Prove that the sequence (a_n) does not converge to 1; to do this, use your result from Part (a). (Actually, the sequence (a_n) does not converge to *anything*, but you do not need to show this.)

Proof. We wish to choose ϵ such that no matter the choice of N, there is always at least one n such that $|a_n - 1| > \epsilon$. Choose $\epsilon = 1$. Notice how no matter the choice of N, $|a_n - 1| = |(-1)^n - 1| = 0$ if n is even, and 2 if n is odd. Then for all $N \in \mathbb{N}$, we choose $n \geq N$ and n odd to satisfy

$$|(-1)^n - 1| > \epsilon \rightarrow$$

$$|-2| > 1 \longrightarrow 2 > 1$$

Since there exists $\epsilon > 0$ where for all natural numbers N, there is at least one $n \geq N$ to satisfying $|a_n - 1| > \epsilon$, we conclude that the sequence a_n does not converge to 1. $\square$