MTH 312 HW 2

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6.3.2 Consider the sequence of functions $h_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
(a) Computer the pointwise limit of (h_n) then prove that the convergence is uniform on $\mathbb{R}$.
Proof. The pointwise limit of (h_n) is

$$\lim h_n = h = |x|$$

To show h_n convergences to h uniformly, let $\epsilon > 0$ be arbitrary. Choose $N > \frac{1}{\epsilon}$. Then for all $n \geq N$,

$$|h_n(x) - h(x)| = \left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| < \left|\sqrt{x^2} + \frac{1}{n} + |x|\right| = \left||x| + \frac{1}{n} - |x|\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{\frac{1}{\epsilon}}\right| < \epsilon$$

for all $x \in \mathbb{R}$. Thus we conclude that $h_n \to h$ uniformly.

$\square$

(b) Note that each h_n is differentiable. Show $g(x) = \lim h'_n(x)$ exists for all x, and explain how we can be certain that the convergence is not uniform on any neighborhood of zero.
Solution.
To show the limit exists, we simply compute it. Note that $h'_n(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$

$$g(x) = \lim h'_n(x) = \frac{x}{|x|} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

Note that since $g(x)$ is not defined at $x = 0$, the slopes of all $h'_n(x)$ would approach infinity as n increased. Logically then, the rate of increase cannot be bounded, and uniform convergence is impossible over any interval including $x = 0$.

6.3.5 Let $g_n(x) = \frac{nx+x^2}{2n}$, and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.
Solution.

$$\lim g_n(x) = g(x) = \frac{x}{2}, \longrightarrow g'(x) = \frac{1}{2}$$

(b) Compute $g_n'(x)$ and show the sequence (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

Proof. First, note that $g_n'(x) = \frac{1}{2} + \frac{x}{n}$. It is clear then, that $h(x) = \lim g_n'(x) = \frac{1}{2}$. We will now show that (g_n') converges uniformly, and from there conclude $h = f'$. Let $\epsilon > 0$, and choose $N > \frac{M}{\epsilon}$. Then

$$|g_n'(x) - h(x)| = \left|\frac{1}{2} + \frac{x}{n} - \frac{1}{2}\right| = \left|\frac{x}{n}\right| < \frac{M}{n} < \epsilon$$

Thus $g_n' \to h$ uniformly, and that there exists $x_0 = 0$ such that the sequence $g_n(x_0) = g(0) = 0$ converges. Then by Theorem 6.3.3, $g' = h = \frac{1}{2}$.

□

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = \frac{nx^2+1}{2n+x}$.

Proof. First, we find algebraically the limit as $n \to \infty$. This is directly computed as

$$\lim f_n(x) = f(x) = \frac{x^2}{2} \longrightarrow f'(x) = x$$

Next, we wish to show that (f_n') converges on every interval $[-M, M]$, and conclude $f'(x) = \lim f_n'(x)$. Note that $f_n'(x) = \frac{2nx}{2n+x} - \frac{nx^2+1}{(2n+x)^2}$, and thus $\lim f_n' = x$. Our approach will be to show $f_n' \to g$ uniformly, then conclude that $f' = g$.

With some algebra, we know that

$$|f_n'(x) - g(x)| = \left|\frac{2nx}{2n+x} - \frac{nx^2+1}{(2n+x)^2} - x\right| = \left|\frac{x^3 + 3nx^2 + 1}{4n^2x + 4nx + x^2}\right| \leq \left|\frac{M^3 + 3nM^2 + 1}{4n^2M + 4nM + M^2}\right|$$

This is independent of x, and clearly the denominator has the dominating term with respect to n. Thus as $n \to \infty, |f_n' - g| \to 0$, and we conclude that it converges uniformly. Since it converges uniformly, we again pick $x_0 = 0$ giving us $f_n'(x_0) = f_n'(0) = \frac{-1}{4n^2}$, which certainly converges to 0 as $n \to \infty$. By Theorem 6.3.3 we conclude that $f' = g = x$.

□