Complex Analysis Chapter 1 Section 3

Brandyn Tucknott

Last Updated: 29 September 2025

3 Integration along curves

A **parameterized curve** $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parameterized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ with $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a, b$, $z'(a), z'(b)$ are interpreted as one-sided limits:

$$z'(a) = \lim_{h \to 0, h > 0} \frac{z(a+h) - z(a)}{h} \text{ and } z'(b) = \lim_{h \to 0, h < 0} \frac{z(b+h) - z(b)}{h}.$$

These quantities are called the right-handed derivative at $z(a)$ and left handed derivative at $z(b)$. We say the parameterized curve is **piecewise-smooth** if z is continuous on $[a, b]$ and there exist points $a = a_0 < a_1 < \ldots < a_n = b$, where $z(t)$ is smooth on the intervals $[a_k, a_{k+1}]$. The right-handed derivative and left-handed derivative at a_k may differ for $k = 1, 2, \ldots, n-1$.

Two parameterizations

$$z : [a, b] \to \mathbb{C} \text{ and } \tilde{z} : [c, d] \to \mathbb{C}$$

are **equivalent** if there exists a continuously differentiable bijection $s \to t(s)$ from $[c, d] \to [a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says that orientation must be preserved: as s travels from c to d, $t(s)$ travels from a to b. The points $z(a)$ and $z(b)$ are called **end-points** of the curve and are independent on the parameterization. Since a curve γ carries an orientation, it is natural to say that γ begins at $z(a)$ and ends at $z(b)$. A smooth or piecewise-smooth curve is **closed** if $z(a) = z(b)$ for any of its parameterizations, and **simple** if it is not self-intersecting ($z(t) \neq z(s)$ unless $s = t$).

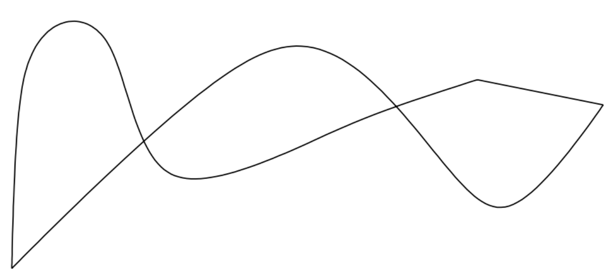

Figure 3. A closed piecewise-smooth curve