MTH 464 HW 6

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1. Assume (X_1, X_2) is a bivariate normal random variable, with Variance-Covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

and mean $\mathbb{E}(X_j) = \mu_j$ with $j = 1, 2$. Assume Z_1, Z_2 are iid standard normal random variables.

(a) Let

$$M = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}$$

Show that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = M \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

Proof. We just need to verify that X_1, X_2 have the same variance and covariance as shown in M. We explicitly calculate X_1, X_2 to be

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 Z_1 + \mu_1 \\ \sigma_2\rho Z_1 + \sigma_2\sqrt{1-\rho^2}Z_2 + \mu_2 \end{pmatrix}$$

Now we compute the variance and covariances.

$$\begin{aligned}
\mathrm{Var}(X_1) &= \mathrm{Var}(\sigma_1 Z_1 + \mu_1) = \sigma_1^2\mathrm{Var}(Z_1) + \mathrm{Var}(\mu_1) = \sigma_1^2 \\
\mathrm{Var}(X_2) &= \mathrm{Var}\left(\sigma_2\rho Z_1 + \sigma_2\sqrt{1-\rho^2}Z_2 + \mu_2\right) \\
&= \sigma_2^2\rho^2\mathrm{Var}(Z_1) + \sigma_2^2(1-\rho^2)\mathrm{Var}(Z_2) + \mathrm{Var}(\mu_2) = \sigma_2^2 \\
\mathrm{Cov}(X_1, X_2) &= \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) \\
&= \mathbb{E}(X_1X_2) - \mathbb{E}(\sigma_1 Z_1 + \mu_1)\mathbb{E}\left(\sigma_2\rho Z_1 + \sigma_2\sqrt{1-\rho^2}Z_2 + \mu_2\right) \\
&= \mathbb{E}\left((\sigma_1 Z_1 + \mu_1)\left(\sigma_2\rho Z_1 + \sigma_2\sqrt{1-\rho^2}Z_2 + \mu_2\right)\right) - \mu_1\mu_2 \\
&= \rho\sigma_1\sigma_2 + \mu_1\mu_2 - \mu_1\mu_2 \\
&= \rho\sigma_2\sigma_2
\end{aligned}$$

□

(b) Find M^{-1} such that

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = M^{-1}\begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix}$$

Proof. We can calculate M^{-1} to be

$$M^{-1} = \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ -\frac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_2\sqrt{1-\rho^2}} \end{pmatrix}$$

and the given condition necessarily follows. □

(c) Using Part (b), write X_2 as a linear combination of X_1 and Z_2 and note that X_1 and Z_2 are independent.

Proof. A true linear combination of the form $X_2 = aX_1 + bZ_2$ is not possible, but if we disregard the means μ_1, μ_2, we can derive the following equation.

$$X_2 - \mu_2 = \left(\frac{\rho\sigma_2}{\sigma_1}\right)(X_1 - \mu_1) + \left(\sigma_2\sqrt{1-\rho^2}\right) Z_2$$

□

(d) Show that $\operatorname{Cov}(X_2 - Y, X_1) = 0$. Conclude that the best mean square linear approximation to X_2 given X_1 is

$$Y = \rho\sigma_2\left(\frac{X_1 - \mu_1}{\sigma_1}\right) + \mu_2$$

Proof. First, we calculate the covariance.

$$\begin{aligned}
\operatorname{Cov}(X_2 - Y, X_1) &= \operatorname{Cov}(X_2, X_1) - \operatorname{Cov}(Y, X_1) \\
&= \rho\sigma_1\sigma_2 - \operatorname{Cov}\left(\rho\sigma_2\left(\frac{X_1 - \mu_1}{\sigma_1}\right) + \mu_2, X_1\right) \\
&= \rho\sigma_1\sigma_2 - \frac{\rho\sigma_2}{\sigma_1}\left(\operatorname{Cov}(X_1, X_1) - \operatorname{Cov}(\mu_1, X_1)\right) + \operatorname{Cov}(\mu_2, X_1) \\
&= \rho\sigma_1\sigma_2 - \frac{\rho\sigma_2}{\sigma_1}\left(\operatorname{Var}(X_1) - 0\right) + 0 \\
&= \rho\sigma_1\sigma_2 - \frac{\rho\sigma_2}{\sigma_1}\sigma_1^2 \\
&= \rho\sigma_1\sigma_2 - \rho\sigma_1\sigma_2 = 0
\end{aligned}$$

Our results tell us our error term $X_2 - Y$ is uncorrelated with X_1, which in turn implies our error is minimal. If it were not, there would be some non-zero correlation, which we could further minimize. □

(e) Find $\operatorname{Var}(Y), \operatorname{Var}(X_2 - Y)$

Proof. We directly compute the specified values.

$$\begin{aligned}
\operatorname{Var}(Y) &= \operatorname{Var}\left(\frac{\rho\sigma_2}{\sigma_1}(X_1 - \mu_1) + \mu_2\right) \\
&= \frac{\rho^2\sigma_2^2}{\sigma_1^2}\left(\operatorname{Var}(X_1) - \operatorname{Var}(\mu_1)\right) + \operatorname{Var}(\mu_2) \\
&= \frac{\rho^2\sigma_2^2}{\sigma_1^2}\left(\sigma_1^2 - 0\right) + 0 = \rho^2\sigma_2^2 \\
\operatorname{Var}(X_2 - Y) &= \operatorname{Var}\left(X_2 - \frac{\rho\sigma_2}{\sigma_1}(X_1 - \mu_1) - \mu_2\right) \\
&= \operatorname{Var}(X_2) - \frac{\rho^2\sigma_2^2}{\sigma_1^2}\left(\operatorname{Var}(X_1) - \operatorname{Var}(\mu_1)\right) - \operatorname{Var}(\mu_2) \\
&= \sigma_2^2 - \frac{\rho^2\sigma_2^2}{\sigma_1^2}\left(\sigma_1^2 - 0\right) - 0 \\
&= \sigma_2^2(1 - \rho^2)
\end{aligned}$$

□

2. Let $Y = aX + b$ where a, b are constants and X is a random variable with moment generating function $M_X(t)$. Express the moment generating function $M_Y(t)$ of Y in terms of M_X.

Proof.

$$\begin{aligned} M_Y(t) &= \mathbb{E}\left(e^{Yt}\right) \\ &= \mathbb{E}\left(e^{t(aX+b)}\right) \\ &= \mathbb{E}\left(e^{taX+tb}\right) \\ &= e^{tb}\mathbb{E}\left(e^{taX}\right) \\ &= e^{tb}M_X(ta) \end{aligned}$$

$\square$

3. Let X have a moment generating function $M_X(t)$. Define the cumulant generating function $\Psi_X(t) = \ln M_X(t)$. Show that

$$\left.\frac{d^2\Psi}{dt^2}\right|_{t=0} = \mathrm{Var}\left(X\right)$$

Proof. First, we calculate the second derivative with respect to t of $\Psi_X(T)$.

$$\begin{aligned}
\frac{d^2\Psi}{dt^2} &= \frac{d}{dt}\left[\frac{d}{dt}\ln M_X(t)\right] \\
&= \frac{d}{dt}\left[\frac{1}{M_X(t)}M_X'(t)\right] \\
&= \frac{M_X(t)M_X''(t) - M_X'(t)M_X'(t)}{M_X^2(t)}
\end{aligned}$$

We now evaluate this at $t = 0$, keeping in mind that $M_X(0) = 1, M_X'(0) = \mathbb{E}\left(X\right), M_X''(0) = \mathbb{E}\left(X^2\right)$:

$$\begin{aligned}
\left.\frac{d^2\Psi}{dt^2}\right|_{t=0} &= \left.\frac{M_X(t)M_X''(t) - M_X'(t)M_X'(t)}{\left(M_X(t)\right)^2}\right|_{t=0} \\
&= \frac{M_X(0)M_X''(0) - \left(M_X'(0)\right)^2}{\left(M_X(0)\right)^2} \\
&= \frac{1 \cdot \mathbb{E}\left(X^2\right) - \left(\mathbb{E}\left(X\right)\right)^2}{1^2} \\
&= \mathbb{E}\left(X^2\right) - \left(\mathbb{E}\left(X\right)\right)^2 \\
&= \mathrm{Var}\left(X\right)
\end{aligned}$$

□

4. Recall that a non-negative Y is called a *lognormal random variable* with parameters μ and σ^2 if $X = \ln(Y)$ is a normal random variable with mean μ and variance σ^2. Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

Proof. Using the moment generating function for $M_X(t)$, observe that

$$\mathbb{E}(Y) = \mathbb{E}\left(e^X\right) = M_X'(t) = e^{\mu + \frac{\sigma^2}{2}}$$

Similarly, note that

$$\mathbb{E}\left(Y^2\right) = \mathbb{E}\left(e^{2X}\right) = M_X''(t)\Big|_{t=0} = e^{2(\mu+\sigma^2)}$$

which we use to compute the variance

$$\text{Var}(Y) = \mathbb{E}\left(Y^2\right) - \left(\mathbb{E}(Y)\right)^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$$

$\square$

5. Show that for random variables X, Y,

$$\mathbb{E}\left((X-W)^2\right) = \mathbb{E}\left(X^2\right) - \mathbb{E}\left(W^2\right)$$

where $W = \mathbb{E}\left(X|Y\right)$.

Proof.

$$\mathbb{E}\left((X-W)^2\right) = \mathbb{E}\left(X^2 - 2XW + W^2\right) = \mathbb{E}\left(X^2\right) - 2\mathbb{E}\left(XW\right) + \mathbb{E}\left(W^2\right)$$

From here, recognize that we will be done if we can show

$$-2\mathbb{E}\left(XW\right) + \mathbb{E}\left(W^2\right) = -\mathbb{E}\left(W^2\right) \text{ or equivalently } \mathbb{E}\left(XW\right) = \mathbb{E}\left(W^2\right)$$

We directly calculate $\mathbb{E}\left(XW\right)$ to be

$$\begin{aligned}
\mathbb{E}\left(XW\right) &= \mathbb{E}\left(X\mathbb{E}\left(X|Y\right)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(X\mathbb{E}\left(X|Y\right)|Y\right)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(X|Y\right)\mathbb{E}\left(X|Y\right)\right) \\
&= \mathbb{E}\left(W^2\right)
\end{aligned}$$

□