MTH 312 HW 8

Brandyn Tucknott

11 March 2025

8.2.2. Let $C[0,1]$ be a collection of continuous functions on the closed interval [0, 1]. Decide which of the following are metrics on $C[0,1]$.

(a) $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$
Solution.
Since $|f(x) - g(x)| \geq 0$ with equality only when $f = g$ for all $x \in [0,1]$, this satisfies property (i). It is also obvious that

$$|f(x) - g(x)| = |g(x) - f(x)|$$

so property (ii) holds. Finally, observe that

$$|f(x) - h(x)| + |h(x) - g(x)| \geq |f(x) - h(x) + h(x) - g(x)| = |f(x) - g(x)|$$

so property (iii) holds. With this, we conclude that $d(f,g)$ is a metric.

(b) $d(f,g) = |f(1) - g(1)|$
Solution.
Consider when $f(x) = 0, g(x) = 1 - x$. Then

$$|f(1) - g(1)| = |0 - 0| = 0$$

but here $d(f,g) = 0$ with $f \neq g$, thus $d(f,g)$ is not a metric.

(c) $d(f,g) = \int_0^1 |f - g|$
Solution.
since $|f - g| \geq 0$, certainly $\int_0^1 |f - g| \geq 0$, with equality only when $f = g$. Thus property (i) holds. It is also clear that $\int_0^1 |f - g| = \int_0^1 |g - f|$, thus property (ii) holds. Now by triangle equality we have that

$$|f - g| \leq |f - h| + |h - g|$$
$$\int_0^1 \leq |f - g| \leq \int_0^1 |f - h| + \int_0^1 |h - g|$$

Thus property (iii) holds, and we conclude that $d(f,g)$ is a metric.

8.2.13. If E is a subset of a metric space (X, d), show that E is nowhere dense in X if and only if $\overline{E}^c$ is dense in X.

Proof. If $\overline{E}^c$ is dense in X, then

$$\begin{aligned} X &= \overline{\overline{E}^c} \\ \overline{\overline{E}^c}^c &= \emptyset \\ \left(\left(\overline{E}^c\right)^c\right)^\circ &= \emptyset \\ \overline{E}^\circ &= \emptyset \end{aligned}$$

Thus E is nowhere dense in X. Now suppose that E is nowhere dense in X. Then

$$\begin{aligned} \overline{E}^\circ &= \emptyset \\ \left(\left(\overline{E}^c\right)^c\right)^\circ &= \emptyset \\ \overline{\overline{E}^c}^c &= \emptyset \\ X &= \overline{\overline{E}^c} \end{aligned}$$

Thus $\overline{E}^c$ is dense in X. We conclude that

$$\overline{E}^c \text{ is dense in } X \iff E \text{ is nowhere dense in } X$$

$\square$

8.4.1. For $n \in \mathbb{N}$, let

$$n\# = n + (n-1) + (n-2) + \ldots + 2 + 1$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

Solution.
Observe that

$$n\# = n + (n-1)\#$$

for $n \geq 2$, and we can directly compute $1\# = 1$, but also that

$$\begin{aligned} 1\# &= 1 + 0\# \\ 0\# &= 1\# - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Similarly we can compute

$$\begin{aligned} 0\# &= 0 + (-1)\# \\ &= 0 + (-1) + (-2)\# \\ (-2)\# &= 1 \end{aligned}$$

Although there is nothing "natural" about the next definition, since

$$\begin{aligned} 1\# + (-1)\# &= 1 + 0 = 0 \\ 2\# + (-2)\# &= 3 + 1 = 4, \text{ we might guess that} \\ n\# + (-n)\# &= n^2 \end{aligned}$$

To evaluate the last value, we first evaluate the intermediate value $(-\frac{1}{2})\#$.

$$\begin{aligned} \frac{1}{2}\# &= \frac{1}{2} + (-\frac{1}{2})\# \\ \frac{1}{2}\# + (-\frac{1}{2})\# &= \frac{1}{2} + 2(-\frac{1}{2})\# \\ \frac{1}{4} &= \frac{1}{2} + 2(-\frac{1}{2})\# \\ (-\frac{1}{2})\# &= -\frac{1}{8} \end{aligned}$$

Using this, we evaluate $\frac{7}{2}\#$.

$$\begin{aligned} \frac{7}{2}\# &= \frac{7+5+3+1}{2} + (-\frac{1}{2})\# \\ &= 8 + -\frac{1}{8} \\ &= \frac{63}{8} \end{aligned}$$

(b) Now prove that $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$, and revisit Part (a).

Proof. We will use induction to prove the relation. As a base case, observe that when $n = 1$,

$$1\# = \frac{1}{2}(1)(1+1) = \frac{1}{2}(1)(2) = 1$$

This holds with the traditional definition of $n\#$, so we assume the definition holds up to $n-1$, and we aim to show this implies it is true for n.

$$\begin{aligned}
(n-1)\# &= \frac{(n-1)n}{2} \\
n + (n-1)\# &= n + \frac{n(n-1)}{2} \\
n\# &= \frac{2n + n^2 - n}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

$\square$

Using our new formula, we can now directly calculate the expressions.

$$\begin{aligned}
0\# &= \frac{0(0+1)}{2} = 0 \\
(-2)\# &= \frac{(-2)(-2+1)}{2} = 1 \\
\frac{7}{2}\# &= \frac{\frac{7}{2}\left(\frac{7}{2}+1\right)}{2} = \frac{63}{8}
\end{aligned}$$

These values align exactly with what was found in Part (a).