Complex Analysis Chapter 1 Section 3

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3 Integration along curves

A **parameterized curve** $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parameterized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ with $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a, b$, $z'(a), z'(b)$ are interpreted as one-sided limits:

$$z'(a) = \lim_{h \to 0, h > 0} \frac{z(a+h) - z(a)}{h} \text{ and } z'(b) = \lim_{h \to 0, h < 0} \frac{z(b+h) - z(b)}{h}.$$

These quantities are called the right-handed derivative at $z(a)$ and left handed derivative at $z(b)$. We say the parameterized curve is **piecewise-smooth** if z is continuous on $[a, b]$ and there exist points $a = a_0 < a_1 < \ldots < a_n = b$, where $z(t)$ is smooth on the intervals $[a_k, a_{k+1}]$. The right-handed derivative and left-handed derivative at a_k may differ for $k = 1, 2, \ldots, n-1$.

Two parameterizations

$$z : [a, b] \to \mathbb{C} \text{ and } \tilde{z} : [c, d] \to \mathbb{C}$$

are **equivalent** if there exists a continuously differentiable bijection $s \to t(s)$ from $[c, d] \to [a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says that orientation must be preserved: as s travels from c to d, $t(s)$ travels from a to b. The points $z(a)$ and $z(b)$ are called **end-points** of the curve and are independent on the parameterization. Since a curve γ carries an orientation, it is natural to say that γ begins at $z(a)$ and ends at $z(b)$. A smooth or piecewise-smooth curve is **closed** if $z(a) = z(b)$ for any of its parameterizations, and **simple** if it is not self-intersecting ($z(t) \neq z(s)$ unless $s = t$).

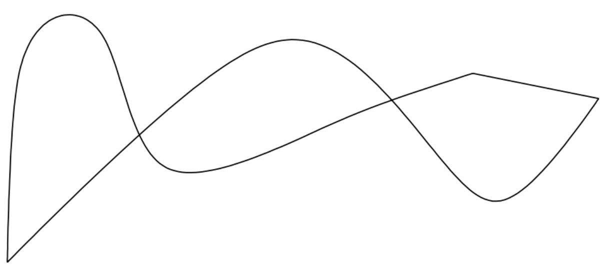

Figure 3. A closed piecewise-smooth curve

We will call any piecewise-smooth curves a **curve**, since these are our objects of primary concern. A basic example is a circle centered at z_0 with radius r, which is by definition

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The **positive orientation** (counterclockwise) is one given by the standard parameterization

$$z(t) = z_0 + re^{it} \text{ where } t \in [0, 2\pi],$$

while the **negative orientation** (clockwise) is the one given by

$$z(t) = z_0 + re^{-it} \text{ where } t \in [0, 2\pi].$$

In the following chapters, we denote by C the general positively oriented circle. Loosely speaking, a key theorem in complex analysis states that if a function is holomorphic is the interior of a closed curve γ, then

$$\int_\gamma f(z)dz = 0. \text{ (we explore this more next chapter)}$$

Given a smooth curve γ in $\mathbb{C}$ parameterized by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ, we define the integral of f along γ as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

For this definition to have meaning, we have to show that the right-hand integral is independent the choice of γ. Say that $\tilde{z}$ is an equivalent parameterization as above. Then the change of variables formula and chain rule imply that

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(z(t(s)))z'(t(s))t'(s)ds = \int_c^d f(z\tilde{(s)})\tilde{z}'(s)ds.$$

Thus the integral of f over γ is well-defined.

If γ is piecewise-smooth and $z(t)$ a piecewise-smooth parameterization, then

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt.$$

By definition, the length of a smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|\, dt.$$

If γ is piecewise smooth, the its length is the sum of its smooth parts.

Proposition 3.1. *Integration of continuous functions over curves satisfies the following properties:*

- *It is linear, that is, if $\alpha, \beta \in \mathbb{C}$, then*

$$\int_\gamma \alpha f + \beta g = \alpha \int_g ammaf + \beta \int_\gamma g$$

- *If γ^- is γ with the reverse orientation, then*

$$\int_\gamma f = -\int_{\gamma^-} f$$

- *One has the inequality*

$$\left|\int_\gamma f\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathit{length}(\gamma)$$

□