MTH 463 HW 6

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1. A random variable X is produced through the following experiment. First, a fair die is rolled to get an outcome Y taking value in the set $\{1, 2, 3, 4, 5, 6\}$. Then, if $Y = k$, X is chosen uniformly from the interval $[0, k]$. Find the cumulative distribution function $F_X(x)$ and the probability density function $f_X(x)$ for $3 < x < 4$.
   Solution.
   First, note that the probability that Y takes on a particular k value $P(Y = k) = \frac{1}{6}$. Next, we find the cumulative distribution function $F_X(x)$. Note that since our random variable X is bounded by $[0, k]$,

   $$P(X \leq x) = P(X \in [0, k] \leq x \in (3, 4))P(Y = k) = 1 \cdot P(Y = k) \text{ for } k = 1, 2, 3$$

   Our next goal will be to find $F_X(x)$ for $k = 4, 5, 6$. We do this by recognizing that the interval from $[0, x]$ contains values less than or equal to x, so

   $$P(X \leq x) = \frac{x - 0}{k - 0} P(Y = k) \text{ for } k = 4, 5, 6$$

   We can then write

   $$P(X \leq x) = \begin{cases} 1 \cdot P(Y = k), & k = 1, 2, 3 \\ \frac{x}{k} P(Y = k), & k = 4, 5, 6 \end{cases}$$

   We then derive the cumulative distribution function to be

   $$F_X(x) = \sum_{k=1}^{6} P(X \leq x) = \frac{1}{6}\left(1 + 1 + 1 + \frac{x}{4} + \frac{x}{5} + \frac{x}{6}\right) = \frac{1}{2} + \frac{37x}{360}$$

   Since we have $F_X(x)$, we can easily derive $f_X(x)$ to be

   $$f_X(x) = \frac{dF_X}{dx} = \frac{37}{360}$$

2. (a) A fire station is to be located along a road of length A, with $A < \infty$. It is assumed that fires will occur at a point uniformly chosen on $(0, A)$. Find the location to place the fire station so that it minimizes the distance to the fire. That is, find $a \in (0, A)$ such that $E(|X - a|)$ is minimized with $X \sim \text{Uniform}(0, A)$.

Solution.

Our goal is to find a formula in terms of a for the expected value $|X - a|$, then minimize said formula.

$$E(|X-a|) = \int_0^A |x-a| \frac{1}{A} dx = \frac{1}{A}\left(\int_0^a (a-x)dx + \int_a^A (x-a)dx\right) = \frac{1}{A}\left(\left(ax - \frac{x^2}{2}\right)\Big|_0^a + \left(\frac{x^2}{2} - ax\right)\Big|_a^A\right) =$$

$$= \frac{1}{A}\left(\left(a^2 - \frac{a^2}{2}\right) + \left(\frac{A^2}{2} - aA - \frac{a^2}{2} + a^2\right)\right) = \frac{a^2}{A} - a + \frac{A}{2}$$

We now optimize the expected value by taking the derivative, setting it to 0, and solving for a.

$$\frac{dE(|X-a|)}{da} = \frac{d}{da}\left(\frac{a^2}{A} - a + \frac{A}{2}\right) = 0 \longrightarrow$$

$$\frac{2a}{A} - 1 = 0 \longrightarrow a = \frac{A}{2}$$

(b) Now suppose the road is of infinite length. If the distance to a fire from point 0 is an exponential random variable with parameter $\lambda > 0$, where should the fire station be located?

Solution.

Our approach is the same as in Part (a).

$$E(|X-a|) = \int_0^\infty |x-a|\,\lambda e^{-\lambda x} dx = \int_0^a (a-x)\lambda e^{-\lambda x} dx + \int_a^\infty (x-a)\lambda e^{-\lambda x} dx =$$

$$= \lambda\left(\int_0^a (a-x)e^{-\lambda x} dx + \int_a^\infty (x-a)e^{-\lambda x} dx\right)$$

Simple integration by parts gives you

$$E(|X-a|) = \lambda\left(\frac{a}{\lambda} + \frac{2}{\lambda^2}e^{-\lambda a} - \frac{1}{\lambda^2}\right) = a + \frac{2}{\lambda}e^{-\lambda a} - \frac{1}{\lambda}$$

We then solve for $\frac{dE}{da} = 0$, giving us

$$\frac{dE(|X-a|)}{da} = \frac{d}{da}\left(a + \frac{2}{\lambda}e^{-\lambda a} - \frac{1}{\lambda}\right) = 1 - 2e^{-\lambda a} = 0 \longrightarrow$$

$$2e^{-\lambda a} = 1 \longrightarrow a = -\frac{\ln\left(\frac{1}{2}\right)}{\lambda} \longrightarrow$$

$$a = \frac{\ln(2)}{\lambda}$$

3. Assume that Y is uniformly distributed on $[0,5]$. What is the probability that the roots of the equation

$$4x^2 + 4xY + Y + 2 = 0$$

are both real?
<u>*Solution.*</u>
We first find an explicit formula for the roots using the quadratic equation. This gives us

$$x = \frac{-4xy \pm \sqrt{(4y)^2 - 4 \cdot 4 \cdot (y+2)}}{2 \cdot 4}$$

Recognize that the roots are both real when the quantity under the square root is non-negative.

$$(4y)^2 - 4 \cdot 4 \cdot (y+2) \geq 0 =$$

$$16y^2 - 16(y+2) \geq 0 =$$

$$y^2 - y - 2 = (y-2)(y+1) \geq 0 \tag{1}$$

This lets us redefine our problem, and if we can find and interval on which $y \geq 0$, all roots of the original polynomial will be real. Notice this parabola is concave up, so the all points $y \in (-\infty, -1] \cup [2, \infty)$ are greater than or equal to 0, while all points $y \in (-1, 2)$ are less than 0.

Since We are interested in the section where $y \geq 0$, we examine $y \in (-\infty, -1] \cup [2, \infty)$. Since we know y is only defined on the interval $[0, 5]$, we can simplify the interval to

$$y \in (-\infty, -1] \cup [2, \infty) \equiv y \in [2, 5]$$

We are given that Y is uniformly distributed across the interval, so we simply take the ratios to get our final probability.

$$P(4x^2 + 4xY + Y + 2 = 0 \text{ has real roots}) = \frac{5-2}{5-0} = \frac{3}{5}$$

4. Assume that T is an exponential with random variable with parameter λ, that is, it's cumulative distribution function is

$$F(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Define the random variable U taking values on the interval $[0, 1]$ by

$$U = F(t)$$

Show that U is uniformly distributed on $[0, 1]$.
<u>*Solution.*</u>
Consider $P(U \leq u)$. This can be written out as

$$P(U \leq u) = P(F(T) \leq u) = P(T \leq F^{-1}(u)) = F(F^{-1}(u)) = u$$

This is by definition the CDF of a uniform distribution, so we know that U is uniformly distributed. If we now consider the interval on which U is defined, it will just be the interval on which $F(t)$ is defined, which would be $[0, 1]$ by definition of a CDF. From this, we conclude that $U \sim \text{Uniform}(0, 1)$.