Complex Analysis Chapter 1 Section 2

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Last Updated: 24 September 2025

1 Functions on the Complex Plane

1.1 Continuous Functions

Let f be a function on a set Ω of complex numbers. We say that f is **continuous** at a point $z_0 \in \Omega$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$. Equivalently, we can say for every sequence $\{z_1, z_2, \ldots\} \subset \Omega$ such that $\lim z_n = z_0$, then $\lim f(z_n) = f(z_0)$. The function f is continuous on Ω if it is continuous at every point in Ω. Sums and products of continuous functions are also continuous.

It is worth noting that the function f of the complex argument $z = x + iy$ is continuous if and only if it is continuous viewed as a function of the two real variables x, y.

By the triangle inequality, we see that if f is continuous, then the real-valued function defined by $z \to |f(z)|$ is continuous. We say that f attains a **maximum** at a point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in \Omega,$$

with the inequality reversed for the definition of a **minimum**.

Theorem 1.1. *A continuous function on a compact set Ω attains a maximum and minimum on Ω.* □

1.2 Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be open and f a complex-valued function on Ω. The function f is **holomorphic at the point** $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

converges. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, so that the quotient is well-defined. The limit of the quotient, when it exists, is denoted by $f'(z_0)$ and is called the **derivative of f at** z_0:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Take note that h is complex and can approach 0 from any direction.

The function f is **holomorphic on** Ω if it is holomorphic at every point of Ω. If C is a closed subset of $\mathbb{C}$, we say that f is **holomorphic on** C if f is holomorphic in some open set containing C. If f is holomorphic on $\mathbb{C}$, we say that f is **entire**.

Proposition 1.2. *If f and g are holomorphic in Ω, then:*

- $f + g$ *is holomorphic in* Ω *and* $(f + g)' = f' + g'$.
- fg *is holomorphic in* Ω *and* $(fg)' = f'g + fg'$.
- *If* $g(z_0) \neq 0$, *then* f/g *is holomorphic at* z_0 *and*

$$(f/g)' = \frac{gf' - fg'}{g^2}.$$

Moreover, if $f : \Omega \to U$ *and* $g : U \to \mathbb{C}$ *are holomorphic, then the chain rule holds;*

$$(g \circ f)'(z) = g'(f(z))f'(z) \text{ for all } z \in \Omega.$$

□

Complex-Valued Functions as Mappings