MTH 311 Lab 4

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1. Prove the following version of the Nested Interval Property:
   For each $n \in \mathbb{N}$, let $_n = [a_n, b_n]$, where $a_n < b_n$. Assume that the sequence $_n$ of any closed intervals is nested, i.e. $_n \supset_{n+1}$ for all $n \geq 1$. Prove that $\cap_{n=1}^{\infty}{}_n$ is a nonempty closed interval.

   Do this in the following steps:

   (a) Use the Monotone Convergence Theorem to prove that the sequences (a_n) and (b_n) converge.

   Proof. To use the Monotone Convergence Theorem, we must first show that the sequences (a_n) and (b_n) are both bounded and monotone.

   First, we show that they are both monotone. Consider the definition of $_n = [a_n, b_n],_n \supset I_{n+1}$. This leads us to the definition that $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ for all $n \geq 1$. Then by definition, $a_{n+1} \geq a_n$ for all $n \geq 1$, because if it were not, there would exist some $_k = [a_k, b_k]$ where $_{k-1} \supsetneq_k$. But this is a contradiction, so it must be that $a_{n+1} \geq a_n$ for all $n \geq 1$. By a similar argument, we can show that $b_{n+1} \leq b_n$ for all $n \geq 1$. This gives us that our sequences (a_n) and (b_n) are both monotone and increasing / decreasing respectively.

   It remains to show that (a_n) and (b_n) are both bounded. Since (a_n) is increasing, we wish to show it is bounded above, and similarly with (b_n) we wish to show it is bounded below. This is significantly easier given the condition that $a_n < b_n$ for all $n \geq 1$. This tells us that (a_n) is bounded above by the largest b_n, and that (b_n) is bounded below by the smallest a_n. We know these to be b_1 and a_1 respectively, so we concluded that both sequences are bounded.

   Since both (a_n) and b_n are monotone and bounded, we conclude that (a_n) and (b_n) converge by the Monotone Convergence Theorem. □

   (b) Let $a = \lim a_n$ and $b = \lim b_n$. Use the Order Limit Theorem to prove $a \leq b$.

   Proof. Since we have that $a_n < b_n$ for all $n \geq 1$ and the $\lim a_n = a$, $\lim b_n = b$, by the Order Limit Theorem we conclude that $a \leq b$. □

   (c) The proof of the Monotone Convergence Theorem shows that $a = \sup\{a_n : n \in \mathbb{N}\}$ and $b = \inf\{b_n : n \in \mathbb{N}\}$. Prove that $a_n \leq a \leq b_n \leq b$ for all $n \in \mathbb{N}$.

   Proof. By definition of $\sup(a_n)$ and $\inf(b_n)$ we have that $a_n \leq a$ and $b_n \leq b$ for all $n \in \mathbb{N}$. If we can show that $a \leq b_n$ for all $n \geq 1$, then we are done. We will do a proof by contradiction to show $a \leq b_n$. Assume that for some $k \in \mathbb{N}, b_k < a$. Then there exists some $a_{k+l}, l \in \mathbb{N}$ where $a_{k+l} > b_k$, breaking our initial condition of $a_n < b_n$ for all $n \in \mathbb{N}$ and leading to a contradiction. If this does not happen, then $a_n \leq b_k$ for all $n \in \mathbb{N}$, implying $b_k = \sup\{a_n : n \in \mathbb{N}\}$. This too is a contradiction, since $a = \sup\{a_n : n \in \mathbb{N}\}$ and $b_k < a$. Therefore $b_k \geq a$, and we conclude that $a_n \leq a \leq b_n \leq b$ for all $n \in \mathbb{N}$. □

   (d) Prove that $[a, b] \subset \cap_{n=1}^{\infty}{}_n$. In other words, prove that for every $x \in [a, b]$ and every $n \in \mathbb{N}, x \in_n$.

   Proof. Suppose that $x \in [a, b] = [\sup\{a_n : n \in \mathbb{N}\}, \inf\{b_n : n \in \mathbb{N}\}$. Then by definition $x \in [a_n, b_n] =_n$ for all $n \in \mathbb{N}$. From this, we can conclude that $x \in \cap_{n=1}^{\infty}{}_n$. □

(e) Prove that for every $x < a, x \notin \cap_{n=1}^{\infty}{}_n$. During your proof, use the fact that $a = \sup\{a_n : n \in \mathbb{N}\}$.

Proof. Suppose that $x < a$. Then there exists some $a_k > x$, since $a_n \leq a$ for all $n \in \mathbb{N}$ (more specifically, we let $a_k = x + \epsilon$ for some $\epsilon > 0$). Since there exists some $a_k > x$, the interval ${}_k = [a_k, b_k]$ with $x \notin_k$ exists, allowing us to conclude that $x \notin \cap_{n=1}^{\infty}{}_n$. □

(f) Use the result of Part (e) to prove $\cap_{n=1}^{\infty} \subset [a, b]$.

Proof. Suppose $x \in \cap_{n=1}^{\infty}{}_n$. Then by the contrapositive of Part (e), $x \geq a$. Suppose also that $x > b$. Then $x > b \geq b_n$ for all $n \in \mathbb{N}$, that is $x > b_n$ for all n. If this is true, then $x \notin_j$ for any $j \in \mathbb{N} \rightarrow x \notin \cap_{n=1}^{\infty}{}_n$, a contradiction. Thus $x > b$ is false, implying that $x \leq b$. Since $a \leq x \leq b$, we have that $x \in [a, b]$ and conclude $\cap_{n=1}^{\infty}{}_n \subset [a, b]$. □

The combination of Parts (d) and (f) then yields $\cap_{n=1}^{\infty} = [a, b]$, which is nonempty since $a \leq b$.