MTH 511 HW 3

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1. Prove that ℓ_2 is separable.

Proof. Let $\varepsilon > 0$ be arbitrary, and consider a sequence $(r_1, r_2, \dots, r_N, 0, 0, \dots), N \in \mathbb{N}, r_k \in \mathbb{Q}$. Define also $S \subset \ell_2$ as the set of all possible sequences satisfying the above criterion. Consider first (y_n). Since $(y_n) \in \ell_2$, we know that $\sum_{i=1}^{\infty} y_i^2 < \frac{\varepsilon}{2}$, so certainly $\sum_{i=N+1}^{\infty} y_i^2 < \frac{\varepsilon}{2}$. By a similar frame of logic, we can show that

$$\sum_{i=1}^{\infty} y_i^2 < \frac{\varepsilon}{2}$$
$$\sum_{i=1}^{N} y_i^2 < \frac{\varepsilon}{2},$$

and also that

$$\sum_{i=1}^{\infty} x_i^2 < \frac{\varepsilon}{2}$$
$$\sum_{i=1}^{N} x_i^2 < \frac{\varepsilon}{2}.$$

It is easy to see then, that

$$\sum_{i=1}^{N}(y_i - x_i)^2 \leq \sum_{i=1}^{N} y_i^2 + \sum_{i=1}^{N} x_i^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \frac{veps}{1}.$$

Then for all $(x_n) \in S$ and $(y_n) \in \ell_2$,

$$\begin{aligned}\left(\sum_{i=1}^{\infty}(y_i - x_i)^2\right)^(1/2) &= \sum_{i=1}^{\infty}(y_i - x_i)^2 \\ &= \sum_{i=1}^{N}(y_i - x_i)^2 + \sum_{i=N+1}^{\infty}(y_i - x_i)^2\end{aligned}$$

$\square$

2. Show that ℓ_∞ is not separable.

3. Prove that M has a countable open base if and only if M is separable.

4. Let $f : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

5. Let $f : (M, d) \to (N, \rho)$ be continuous, and let A be a separable subset of M. Prove that $f(A)$ is separable.

6. Fix $y \in \ell_\infty$ and define $h : \ell_1 \to \ell_1$ by $h(x) = (x_n y_n)_{n=1}^{\infty}$. Show that h is continuous.