ECE 569 Midterm

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1. Show that the following sets or functions are convex

   (a) $\mathcal{S} = \{x \in \mathbb{R}^n : \|x\|_1 + \|x\|_2 \leq 1\}$.

   Proof. Let $x, y \in \mathcal{S}$ be arbitrary and $\theta \in [0, 1]$. Then

   $$\begin{aligned} \|\theta x + (1-\theta)y\|_1 + \|\theta x + (1-\theta)y\|_2 &\leq \|\theta x\|_1 + \|(1-\theta)y\|_1 + \|\theta x\|_2 + \|(1-\theta)y\|_2 \\ &= \theta \|x\|_1 + (1-\theta)\|y\|_1 + \theta\|x\|_2 + (1-\theta)\|y\|_2 \\ &= \theta\left(\|x\|_1 + \|x\|_2\right) + (1-\theta)\left(\|y\|_1 + \|y\|_2\right) \\ &\leq \theta(1) + (1-\theta)(1) = 1. \end{aligned}$$

   We conclude that $\mathcal{S}$ is convex by definition. □

   (b) $\mathcal{S} = \left\{A \in \mathbb{S}^n : z^T A z \geq 1, z \in \mathcal{C}\right\}$, where $\mathcal{C} \subseteq \mathbb{R}^n$ (not necessarily convex).

   Proof. Let $A, B \in \mathcal{S}, z \in \mathcal{C}$ be arbitrary with $\theta \in [0, 1]$. Then

   $$\begin{aligned} z^T\left(\theta A + (1-\theta)B\right)z &= z^T\theta A z + z^T(1-\theta)Bz \\ &= \theta z^T A z + (1-\theta) z^T B z \\ &\geq \theta(1) + (1-\theta)(1) = 1. \end{aligned}$$

   We conclude that $\mathcal{S}$ is convex by definition. □

   (c) $\mathcal{S} = \mathcal{C}_1 - \mathcal{C}_2$ where $\mathcal{C}_1, \mathcal{C}_2$ are convex sets.

   Proof. Let $a, b \in \mathcal{S}, x_a, x_b \in \mathcal{C}_1, y_a, y_b \in \mathcal{C}_2$ be arbitrary and $\theta \in [0, 1]$. Then

   $$\begin{aligned} \theta a + (1-\theta)b &= \theta(x_a - y_a) + (1-\theta)(x_b - y_b) \\ &= (\theta x_a + (1-\theta)x_b) - (\theta y_a + (1-\theta)y_b) \\ &= c_1 - c_2, \end{aligned}$$

   where $c_1 = \theta x_a + (1-\theta)x_b \in \mathcal{C}_1$ and $c_2 = \theta y_a + (1-\theta)y_b \in \mathcal{C}_2$ by definition of convex sets. Then certainly $c_1 - c_2 \in \mathcal{S}$, thus $\mathcal{S}$ is convex. □

   (d) $f(x) = \sum_{i=1}^n \max 0, 1 - x_i$.

Proof. Let $x, y \in \operatorname{dom}(f)$ and $\theta \in [0,1]$. Then

$$\begin{aligned}
f(\theta x + (1-\theta)y) &= \sum_{i=1}^{n} \max\{0, 1 - (\theta x_i + (1-\theta)y_i)\} \\
&= \sum_{i=1}^{n} \frac{0 + (1 - (\theta x_i + (1-\theta)y_i)) + |(1 - (\theta x_i + (1-\theta)y_i)) - 0|}{2} \\
&= \sum_{i=1}^{n} \frac{1 - \theta x_i - (1-\theta)y_i + |1 - (\theta x_i - (1-\theta)y_i)|}{2} \\
&= \sum_{i=1}^{n} \frac{\theta + (1-\theta) - \theta x_i - (1-\theta)y_i + |\theta + (1-\theta) - \theta x_i - (1-\theta)y_i|}{2} \\
&\leq \sum_{i=1}^{n} \frac{\theta + (1-\theta) - \theta x_i - (1-\theta)y_i + |\theta - \theta x_i| + |(1-\theta) - (1-\theta)y_i|}{2} \\
&= \sum_{i=1}^{n} \frac{\theta - \theta x_i + \theta|1 - x_i| + (1-\theta) - (1-\theta)y_i + (1-\theta)|1 - y_i|}{2} \\
&= \theta \sum_{i=1}^{n} \frac{1 - x_i + |1 - x_i|}{2} + (1-\theta) \sum_{i=1}^{n} \frac{1 - y_i + |1 - y_i|}{2} \\
&= \theta \sum_{i=1}^{n} \max\{0, 1 - x_i\} + (1-\theta) \sum_{i=1}^{n} \max\{0, 1 - y_i\} \\
&= \theta f(x) + (1-\theta) f(y).
\end{aligned}$$

Since $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$, we conclude that f is convex. □

(e) $f(x,t) = -\log(t - \|x\|_2)$, where $\operatorname{dom}(f) = \left\{(x,t) \in \mathbb{R}^{n+1} : \|x\|_2 < t\right\}$.

Proof. Let $(x,t), (y,s) \in \operatorname{dom}(f)$ and recognize that $\theta(x,t) + (1-\theta)(y,s) = (\theta + (1-\theta)y, \theta t + (1-\theta)s)$. Then

$$\begin{aligned}
f(\theta x + (1-\theta)y, \theta t + (1-\theta)s) &= -\log(\theta t + (1-\theta)s - \|\theta x + (1-\theta)y\|_2) \\
&\leq -\log(\theta t + (1-\theta)s - \theta\|x\|_2 - (1-\theta)\|y\|_2) \\
&= -\log(\theta(t - \|x\|_2) + (1-\theta)(s - \|y\|_2))
\end{aligned}$$

□