MTH 312 HW 3

Brandyn Tucknott

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6.4.6 Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - ...$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Proof. First define $f_n = \frac{(-1)^n}{x+n}$, and rewrite f as

$$f(x) = \sum_{n=0}^{\infty} f_n = \sum_{0}^{\infty} \frac{(-1)^n}{x+n}$$

Recognize that the term-by-term differentiated series is

$$\sum_{n=0}^{\infty} f_n' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}$$

and also that each term in the differentiated series is bounded by

$$\left| \frac{(-1)^{n+1}}{(x+n)^2} \right| \leq \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges, by the Weierstrauss M-Test, we can deduce that $\sum f_n'$ converges uniformly on $(0, \infty)$. Now consider $x_0 = 1 \in (0, \infty)$. Then

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n}, \text{ which converges by the Alternating Series Test.}$$

Since $\sum f_n'$ converges uniformly and $f(x_0) = \sum f_n(x_0)$ converges for some $x_0 \in (0, \infty)$, the Term-by-term Differentiability Theorem tells us that $\sum f_n \to f$ and $\sum f_n' \to f_n'$ over $(0, \infty)$. Since f is differentiable, it is certainly defined and continuous over $(0, \infty)$.

$\square$

6.5.4 Assume $f(x) = \sum a_n x^n$ converges on $(-R, R)$.

(a) Show $F(x) = \sum \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

Proof. Let $x \in (-R, R)$. Then by Theorem 6.5.1, we know that $\sum f$ converges absolutely on $(-R, R)$, and so

$$\sum |a_n||x|^n \text{converges} \longrightarrow \sum |a_n||x|^{n+1} \text{converges}$$

Notice also that

$$\left| \frac{a_n}{n+1} x^{n+1} \right| = \frac{|a_n|}{n+1} |x|^{n+1} \leq |a_n||x|^{n+1}$$

so by the Comparison Test, we know that $\sum \frac{a_n}{n+1} x^{n+1}$ converges absolutely on $(-R, R)$. Since F converges on $(-R, R)$, it is certainly defined on $(-R, R)$, and by Theorem 6.5.7, $F' = f$.

$\square$

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g' = f$ on $(-R, R)$, find a power series representation for g.

<u>*Solution.*</u>

If you choose $g(x) = k + F(x)$, then $g' = f$. We can explicitly write out g as

$$g(x) = k + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

6.5.5

(a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $s \geq 1$.

Proof. Since $s > 0$ and $n \geq 1$, it is certainly true that $ns^{n-1} > 0$. Then choose $N \in \mathbb{N}$ such that $s \leq \frac{N}{N+1}$. Then for all $n \geq N$, we have that

$$s \leq \frac{n}{n+1} \to (n+1)s \leq n \to 0 < (n+1)s^n \leq ns^{n-1}$$

We conclude that the sequence (ns^{n-1}) is bounded below, eventually decreasing, and therefore bounded. $\square$

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Theorem 6.5.6 If $\sum a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum na_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

Proof. Since $t > |x| > 0$, by Part (a) we know that

$$n\left|\frac{x}{t}\right|^{n-1} \leq M$$

for some $M > 0 \in \mathbb{R}$ and all $n \in \mathbb{N}$. Note that $t \in (-R, R)$ by Theorem 6.5.1, since $\sum a_n t_n$ converges, it converges absolutely. Then

$$\sum a_n t^n \text{ abs. conv.} \longrightarrow \sum a_n t^{n-1} \text{ abs. conv.} \longrightarrow M \sum a_n t^{n-1} \text{ abs. conv.}$$

In other words, $\sum M|a_n|t^{n-1}$ converges, and make the following observation:

$$\left|na_n x^{n-1}\right| = n|a_n|x^{n-1} = n|a_n|\left|\frac{x}{t}\right|^{n-1} t^{n-1} \leq M|a_n|t^{n-1} \text{ for all } n \in \mathbb{N}.$$

By the Comparison Test, we conclude that $\sum na_n x^{n-1}$ is absolutely convergent on $(-R, R)$. $\square$