Complex Analysis Chapter 1 Section 2

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2 Functions on the Complex Plane

2.1 Continuous Functions

Let f be a function on a set Ω of complex numbers. We say that f is **continuous** at a point $z_0 \in \Omega$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$. Equivalently, we can say for every sequence $\{z_1, z_2, \dots\} \subset \Omega$ such that $\lim z_n = z_0$, then $\lim f(z_n) = f(z_0)$. The function f is continuous on Ω if it is continuous at every point in Ω. Sums and products of continuous functions are also continuous.

It is worth noting that the function f of the complex argument $z = x + iy$ is continuous if and only if it is continuous viewed as a function of the two real variables x, y.

By the triangle inequality, we see that if f is continuous, then the real-valued function defined by $z \to |f(z)|$ is continuous. We say that f attains a **maximum** at a point $z_0 \in \Omega$ if

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in \Omega,$$

with the inequality reversed for the definition of a **minimum**.

Theorem 2.1. *A continuous function on a compact set Ω attains a maximum and minimum on Ω.* $\square$

2.2 Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be open and f a complex-valued function on Ω. The function f is **holomorphic at the point** $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

converges. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, so that the quotient is well-defined. The limit of the quotient, when it exists, is denoted by $f'(z_0)$ and is called the **derivative of f at z_0**:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Take note that h is complex and can approach 0 from any direction.

The function f is **holomorphic on** Ω if it is holomorphic at every point of Ω. If C is a closed subset of $\mathbb{C}$, we say that f is **holomorphic on** C if f is holomorphic in some open set containing C. If f is holomorphic on $\mathbb{C}$, we say that f is **entire**.

Proposition 2.2. *If f and g are holomorphic in Ω, then:*

- *$f+g$ is holomorphic in Ω and $(f+g)' = f' + g'$.*
- *fg is holomorphic in Ω and $(fg)' = f'g + fg'$.*
- *If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and*

$$(f/g)' = \frac{gf' - fg'}{g^2}.$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then the chain rule holds;

$$(g \circ f)'(z) = g'(f(z))f'(z) \text{ for all } z \in \Omega.$$

Complex-Valued Functions as Mappings

Recall that a fuction $F(x, y) = (u(x, y), v(x, y))$ is said to be differentiable at a point $P_0 = (x_0, y_0)$ if there exists a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \to 0 \text{ as } |H| \to 0, H \in \mathbb{R}^2$$

Equivalently we can write

$$F(P_0 + H) - F(P_0) = J(H) + |H| \, \Psi(H),$$

with $|\Psi(H)| \to 0$ as $H \to 0$. The linear transformation J is unique and is called the derivative of F at P_0. If F is differentiable, the partial derivatives of u and v exist and J is described with the standard basis in $\mathbb{R}^2$ by the Jacobian of F

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

With complex-differentiation, the derivative is a complex number $f'(z_0)$, and with the reals it is a matrix. However, there is a relation to be found involving the partials of u and v.

Given the following equations

$$\frac{f(z_0 + h) - f(z_0)}{h} \tag{1}$$

$$f(z_0 + h) - f(z_0) - ah = h\psi(h) \tag{2}$$

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \to 0 \text{ as } |H \to 0| \tag{3}$$

consider the limit when h is real. That is, $h = h_1 + ih_2$ with $h_2 = 0$. Then if we write $z = x + iy$, $z_0 = x_0 + iy_0$, and $f(z) = f(x, y)$, we find that

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0). \end{aligned}$$

Now taking h to be purely imaginary with $h = ih_2$, a similar argument shows that

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0). \end{aligned}$$

Therefore, if f is holomorphic, we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Writing $f = u + iv$ after seperating real and imaginary parts as well as using $1/i = -i$, we find that the partials of u and v exist, and they satisfy the following relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the **Cauchy-Riemann equations**. We can take this further and define two differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Proposition 2.3. *If f is holomorphic at z_0, then*

$$\frac{\partial f}{\partial \overline{z}} = 0 \textit{ and } f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if we write $F(x, y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

Proof. Taking real and imaginary parts, it is easy to see the Cauchy-Riemann equations are equivalent to $\partial f/\partial\overline{z} = 0$. By our earlier observation

$$\begin{aligned} f'(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) \\ &= \frac{\partial f}{\partial z}(z_0) \end{aligned}$$

and the Cauchy-Riemann equations give $\partial f/\partial z = 2\partial u/\partial z$. To prove that F is differentiable, it suffices to show that if $H = (h_1, h_2)$ and $h = h_1 + ih_2$, then the Cauchy-Riemann equations imply

$$J_F(x_0, y_0)(H) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) = f'(z_0)h,$$

where we have identified a complex number with the pair of real and imaginary parts. Another application of the Cauchy-Riemann equations give

$$\det J_F(x_0, y_0) = \frac{\partial u}{\partial}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left|2\frac{\partial u}{\partial z}\right|^2 = |f'(z_0)|^2.$$

□

Theorem 2.4. *Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f/\partial z$.*

Proof. Write

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h)$$

and

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h),$$

where $\psi_j(h) \to 0$ (for $j = 1, 2$) as $|h| \to 0$ and $h = h_1 + ih_2$. Using the Cauchy-Riemann equations, we find that

$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi(h),$$

where $\psi(h) = \psi_1(h) + \psi_2(h) \to 0$ as $|h| \to 0$. Therefore f is holomorphic and

$$f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

□

2.3 Power Series

A classic example of a power series is the complex **exponential** function, defined for all $z \in \mathbb{C}$ by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The above series converges absolutely for all $z \in \mathbb{C}$, which can be seen by observing

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!},$$

which allows us to compare $|e^z|$ with the power series $\sum_{n=0}^{\infty} |z|^n / n! = e^{|z|} < \infty$. Moreover, this shows that the power series of e^z converges uniformly in all discs in $\mathbb{C}$.

Theorem 2.5. *Given a power series* $\sum_{n=0}^{n} z^n$*, there exists* $0 \leq R \leq \infty$ *such that:*

- *If* $|z| < R$*, the series converges absolutely.*
- *If* $|z| > R$*, the series diverges.*

Moreover, if we use the convention that $1/0 = \infty$ *and* $1/\infty = 0$*, then* R *is given by Hadamard's formula*

$$\frac{1}{R} = \limsup |a_n|^{1/n}.$$

Remark. Notice this theorem says nothing about when $|z| = R$. This situation can either converge or diverge.

The number R is called the **radius of convergence** of the power series, and the region $|z| < R$ the **disc of convergence**. Moreover, the radius of convergence for the exponential function and geometric series are $R = \infty$ and $R = 1$ respectively.

More examples of power series in the complex plane are given below in the form of **trigonometric functions**. These are defined as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \text{ and } \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

A connection between these and the exponential function can be expressed as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

These are called **Euler formulas** for the sine and cosine functions.

Theorem 2.6. *The power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *defines a holomorphic function in its disc of convergence. The derivative of* f *is also a power series obtained by differentiating the term by term power series for* f*, that is,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f' *has the same radius of convergence as* f*.*

Proof. Since $\lim_{n\to\infty} n^{1/n} = 1$, by Hadamard's formula we know that

$$\frac{1}{R} = \lim_{n\to\infty} |n|^{1/n} = \lim_{a_n\to\infty} |na_n|^{1/n},$$

thus $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence.

To prove the first assertion, we must show that

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

gives the derivative of f.

Let R denote the radius of convergence of f, and let $|z_0| < r < R$. Denote

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \text{ and } E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

and write

$$f(z) = S_N(z) + E_N(z).$$

If h is chosen such that $|z_0 + h| < r$, we have

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right).$$

Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$, we see that

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where we use the fact that $|z_0| < r$ and $|z_0 + h| < r$. The expression on the right is the tail end of a convergent series, since g converges absolutely on $|z| < R$. Therefore, given $\varepsilon > 0$, we can find N_1 such that $N > N_1$ implies

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \varepsilon.$$

Also, since $\lim_{n\to\infty} S_N'(z_0) = g(z_0)$, we can find N_2 such that $N > N_2$ implies

$$|S_N'(z_0) - g(z_0)| < \varepsilon.$$

If we fix N so that $N > N_1, N_2$, we can find $\delta > 0$ so that $|h| < \delta$ implies

$$\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right| < \varepsilon,$$

since the derivative of a polynomrial is obtained by differentiating it term by term. Therefore

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right| < 3\varepsilon$$

whenever $|h| < \delta$, thereby concluding the proof of the theorem. □

Corollary 2.7. *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also obtained by term-wise differentiation.* □

So far, we have only dealt with power series centered about the origin, but we more generally express a power series centered at $z_0 \in \mathbb{C}$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The disc of convergence of f is now centered at z_0, but the radius is still given by Hadamard's formula. In fact, if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

f is obtained by simply translating g.

A function f defined on an open set Ω is said to be **analytic** (having a power series expansion) at a point $z_0 \in \Omega$ if there exists a power series $\sum a_n (z - z_0)^n$ centered at z_0 with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \text{ in a neighborhood of } z_0.$$

If f has a power series expansion point at every point in Ω, we say that f is **analytic on** Ω. By Theorem 2.6, an analytic function is also holomorphic. In the future we will prove the converse is true: every holomorphic function is analytic. For that reason we use the terms holomorphic and analytic interchangably.

2.4 Integration along curves

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