MTH 312 HW 4

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6.6.5.
(a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, then prove the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.
Proof. By Taylor's Formula, we know that the coefficients for a series of a function centered at 0 can be computed as

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

We represent the Taylor series of f as

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $R > 0$ and $x \in [-R, R]$. By Lagrange's Remainder Theorem, for $N \in \mathbb{N}$, there exists some $c \in (-R, R)$ such that $|c| < |x|$ and the error function

$$E_N(x) = f(x) - \sum_{n=0}^{N} \frac{x^n}{n!}$$

satisfying

$$|E_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right| = \left| \frac{e^c}{(N+1)!} x^{N+1} \right| \leq \frac{e^c}{(N+1)!} R^{N+1}$$

Since factorial grows faster than exponential, the error approaches 0 as $n \to \infty$, and we conclude that on $[-R, R]$, $\sum a_n x^n \to f$ uniformly.

□

(b) Verify the formula $f'(x) = e^x$.
Solution.

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying the two series and collecting powers of x.
Solution.

$$e^x \cdot e^{-x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) =$$

$$\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) + \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) x + \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) \frac{x^2}{2} + \dots =$$

$$1 + (-1 + 1)\, x + \left(\frac{1}{2} + \frac{1}{2} - 1\right) x^2 + \left(-\frac{1}{6} + \frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) x^3 + \dots = 1$$

6.7.3.

(a) Find the second degree polynomial $p(x) = q_0 + q_1 x + q_2 x^2$ that interpolates the three points $(-1,1),(0,0),(1,1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1,1]$ on both axes.
<u>*Solution.*</u>
The second order polynomial which passes through all given points is obviously $P(x) = x^2$. The graph can be seen below after Part (b).

(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the point $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Add a sketch of this polynomial to the graph from Part (a).
<u>*Solution.*</u>
We are looking for a polynomial $a(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ which satisfies

$$a(-1) = 1, a(-0.5) = 0.5, a(0) = 0, a(0.5) = 0.5, a(1) = 1$$

Since $a(0) = 0$, we know that $a_0 = 0$, and we are left with 4 unknowns and 4 equations:

$$\begin{cases} a_4 - a_3 + a_2 - a_1 = 1 \\ \frac{1}{16}a_4 - \frac{1}{8}a^3 + \frac{1}{4}a^2 - \frac{1}{2}a_1 = \frac{1}{2} \\ \frac{1}{16}a_4 + \frac{1}{8}a^3 + \frac{1}{4}a^2 + \frac{1}{2}a_1 = \frac{1}{2} \\ a_4 + a_3 + a_2 + a_1 = 1 \end{cases} \longrightarrow \begin{pmatrix} -1 & 1 & -1 & 1 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Solving for the system above yields $a_1, a_3 = 0$, while $a_4 = -\frac{4}{3}, a_2 = \frac{7}{3}$. This gives us the final polynomial of $a(x) = -\frac{4}{3}x^4 + \frac{7}{3}x^2$.