ECE 569 Midterm

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1. Show that the following sets or functions are convex
   (a) $\mathcal{S} = \{x \in \mathbb{R}^n : \|x\|_1 + \|x\|_2 \leq 1\}$.

   Proof. Let $x, y \in \mathcal{S}$ be arbitrary and $\theta \in [0,1]$. Then

$$\begin{aligned}
\|\theta x + (1-\theta)y\|_1 + \|\theta x + (1-\theta)y\|_2 &\leq \|\theta x\|_1 + \|(1-\theta)y\|_1 + \|\theta x\|_2 + \|(1-\theta)y\|_2 \\
&= \theta\|x\|_1 + (1-\theta)\|y\|_1 + \theta\|x\|_2 + (1-\theta)\|y\|_2 \\
&= \theta\left(\|x\|_1 + \|x\|_2\right) + (1-\theta)\left(\|y\|_1 + \|y\|_2\right) \\
&\leq \theta(1) + (1-\theta)(1) = 1.
\end{aligned}$$

   We conclude that $\mathcal{S}$ is convex by definition. □

   (b) $\mathcal{S} = \left\{A \in \mathbb{S}^n : z^T A z \geq 1, z \in \mathcal{C}\right\}$, where $\mathcal{C} \subseteq \mathbb{R}^n$ (not necessarily convex).

   Proof. Let $A, B \in \mathcal{S}, z \in \mathcal{C}$ be arbitrary with $\theta \in [0,1]$. Then

$$\begin{aligned}
z^T\left(\theta A + (1-\theta)B\right)z &= z^T\theta A z + z^T(1-\theta)Bz \\
&= \theta z^T A z + (1-\theta)z^T B z \\
&\geq \theta(1) + (1-\theta)(1) = 1.
\end{aligned}$$

   We conclude that $\mathcal{S}$ is convex by definition. □

   (c) $\mathcal{S} = \mathcal{C}_1 - \mathcal{C}_2$ where $\mathcal{C}_1, \mathcal{C}_2$ are convex sets.

   Proof. Let $a, b \in \mathcal{S}, x_a, x_b \in \mathcal{C}_1, y_a, y_b \in \mathcal{C}_2$ be arbitrary and $\theta \in [0,1]$. Then

$$\begin{aligned}
\theta a + (1-\theta)b &= \theta(x_a - y_a) + (1-\theta)(x_b - y_b) \\
&= (\theta x_a + (1-\theta)x_b) - (\theta y_a + (1-\theta)y_b) \\
&= c_1 - c_2,
\end{aligned}$$

   where $c_1 = \theta x_a + (1-\theta)x_b \in \mathcal{C}_1$ and $c_2 = \theta y_a + (1-\theta)y_b \in \mathcal{C}_2$ by definition of convex sets. Then certainly $c_1 - c_2 \in \mathcal{S}$, thus $\mathcal{S}$ is convex. □

   (d) $f(x) = \sum_{i=1}^n \max 0, 1 - x_i$.

Proof. Let $x, y \in \operatorname{dom}(f)$ and $\theta \in [0,1]$. Then

$$\begin{aligned}
f(\theta x + (1-\theta)y) &= \sum_{i=1}^{n} \max\{0, 1 - (\theta x_i + (1-\theta)y_i)\} \\
&= \sum_{i=1}^{n} \frac{0 + (1 - (\theta x_i + (1-\theta)y_i)) + |(1 - (\theta x_i + (1-\theta)y_i)) - 0|}{2} \\
&= \sum_{i=1}^{n} \frac{1 - \theta x_i - (1-\theta)y_i + |1 - (\theta x_i - (1-\theta)y_i)|}{2} \\
&= \sum_{i=1}^{n} \frac{\theta + (1-\theta) - \theta x_i - (1-\theta)y_i + |\theta + (1-\theta) - \theta x_i - (1-\theta)y_i|}{2} \\
&\leq \sum_{i=1}^{n} \frac{\theta + (1-\theta) - \theta x_i - (1-\theta)y_i + |\theta - \theta x_i| + |(1-\theta) - (1-\theta)y_i|}{2} \\
&= \sum_{i=1}^{n} \frac{\theta - \theta x_i + \theta|1 - x_i| + (1-\theta) - (1-\theta)y_i + (1-\theta)|1 - y_i|}{2} \\
&= \theta \sum_{i=1}^{n} \frac{1 - x_i + |1 - x_i|}{2} + (1-\theta) \sum_{i=1}^{n} \frac{1 - y_i + |1 - y_i|}{2} \\
&= \theta \sum_{i=1}^{n} \max\{0, 1 - x_i\} + (1-\theta) \sum_{i=1}^{n} \max\{0, 1 - y_i\} \\
&= \theta f(x) + (1-\theta) f(y).
\end{aligned}$$

Since $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$, we conclude that f is convex. □

(e) $f(x,t) = -\log(t - \|x\|_2)$, where $\operatorname{dom}(f) = \left\{(x,t) \in \mathbb{R}^{n+1} : \|x\|_2 < t\right\}$.

Proof. Let $h : \mathbb{R}_{>0} \to \mathbb{R}$ as $h(x) = -\log(x)$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}_{>0}$ as $g(x,t) = t - \|x\|_2$. It is sufficient to Show that h is convex (non-increasing) and g is concave. First we consider h. We compute the first and second derivatives of h to be

$$h'(x) = -\frac{\log(e)}{x}, h''(x) \qquad = \frac{\log(e)}{x^2}.$$

Since $h''(x) > 0$ for all $x \in \mathbb{R}_{>0}$, we conclude h is convex. Furthermore, since $h'(x) < 0$ on the same domain, we know h is non-increasing. Now consider g. We are done if we can show $-g$ is convex. Let $(x,t), (y,s) \in \operatorname{dom}(g) = \operatorname{dom}(f), \theta \in [0,1]$. Then

$$\begin{aligned}
-g(\theta x + (1-\theta)y, \theta t + (1-\theta)s) &= \|\theta x + (1-\theta)y\|_2 - (\theta t + (1-\theta)s) \\
&\leq \theta \|x\|_2 + (1-\theta)\|y\|_2 - \theta t - (1-\theta)s \\
&= -\theta(t - \|x\|_2) - (1-\theta)(s - \|y\|_2) \\
&= -\left(\theta g(x,t) + (1-\theta) g(y,s)\right).
\end{aligned}$$

Thus $-g$ is convex, or equivalently g is concave. Since h is convex (non-increasing) and g concave, $f = h \circ g$ is convex. □

2. Answer the following questions and provide justifications.

(a) Is the following optimization problem convex?

$$\min_{x\in\mathbb{R}^2} \frac{1}{2}(x_1^2 + x_2) \text{ s.t. } -10 \le x_2 \le 10, x_1 \ge 5.$$

Proof. First, we show that $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x) = 1/2(x_1^2 + x_2)$ is convex. We compute the gradient and hessian of f to be

$$\begin{aligned}\nabla f &= (x_1, 1/2)^T,\\ \nabla^2 f &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Now for arbitrary $x \in \mathbb{R}^2$, compute

$$\begin{aligned} x^T \nabla^2 f x &= (x_1, x_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (x_1, x_2)^T \\ &= (x_1, 0)(x_1, x_2)^T \\ &= x_1^2 \ge 0. \end{aligned}$$

Thus $\nabla^2 f \succeq 0$ and f is convex. Since f is convex and the optimization problem has convex constraints, we conclude that we have a convex optimization problem. □

(b) Let $A, B \in \mathbb{R}^{n\times n}, x_0 \in \mathbb{R}^n$. Is the following optimization problem convex?

$$\min_{x\in\mathbb{R}^n} x^T A^T A x \text{ s.t. } x^T B^T B x \le 50, \|x - x_0\| \le 10.$$

Proof. First, observe that

$$x^T A^T A x = (Ax)^T A x = \|Ax\|_2^2.$$

We can then rewrite the optimization problem as

$$\min_{x\in\mathbb{R}^n} \|Ax\|_2^2 \text{ s.t. } \|Bx\|_2^2 \le 50, \|x - x_0\| \le 10.$$

Since $\|Ax\|_2^2$ is convex and all constraints are also convex, we conclude the given problem is a convex optimization problem. □

(c) Let $A \in \mathbb{R}^{n\times n}, x_0 \in \mathbb{R}^n$. Is the following optimization problem convex?

$$\min_{x\in\mathbb{R}^n} x^T A^T A x \text{ s.t. } \|x - x_0\|_2 \ge 10.$$

Proof. Although our given function is convex (just as in question 2.b), here our given constraint is the set of all points outside of the circle of radius 10 centered at x_0 in $\mathbb{R}^2$. That set is clearly not convex, so our given optimization problem is not convex. □

(d) Consider the following optimization problem:

$$\min_{x_1,x_2} e^{x_1} + \frac{1}{2}(x_1 - x_2)^2 - x_1 - x_2 = f(x).$$

Verify that the above problem is a convex optimization problem.

Proof. We begin by computing the gradient and hessian of f:

$$\begin{aligned}\nabla f &= (e^{x_1} + (x_1 - x_2) - 1, -(x_1 - x_2) - 1)^T \\ \nabla^2 f &= \begin{pmatrix} e^{x_1} + 1 & -1 \\ -1 & 1 \end{pmatrix}\end{aligned}$$

Now let $v \in \mathbb{R}^2$ be arbitrary. Then

$$\begin{aligned}
v^T \nabla^2 f(x) v &= (v_1, v_2) \begin{pmatrix} e^{x_1} + 1 & -1 \\ -1 & 1 \end{pmatrix} (v_1, v_2)^T \\
&= (v_1(e^{x_1} + 1) - v_2, v_2 - v_1)(v_1, v_2)^T \\
&= v_1(v_1(e^{x_1} + 1) - v_2) + v_2(v_2 - v_1) \\
&= v_1^2(e^{x_1} + 1) - 2v_1 v_2 + v_2^2 \\
&> v_1^2 - 2v_1 v_2 + v_2^2 \text{ since } e^{x_1} + 1 > 0 \\
&= (v_1 - v_2)^2 \geq 0.
\end{aligned}$$

We conclude that $\nabla^2 f \succ 0$ and by extension f is convex. Since f has no constraints and is convex, we can say that the given optimization problem is convex. $\square$

3. Let $b_\ell \in \mathbb{R}^n, \ell = 1, \dots, m$ be fixed vectors. Consider the following optimization problem.

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \sum_{\ell=1}^{m} (x - b_\ell)^T (x - b_\ell) = f(x).$$

(a) Verify that the above problem is a convex optimizatino problem.

Proof. Since $f(x) = 1/2 \sum_{\ell=1}^{m} (x - b_\ell)^T (x - b_\ell) = 1/2 \sum_{\ell=1}^{m} \|x - b_\ell\|_2^2$, we know f is convex (sum of positively weighted convex functions is convex). $\square$

(b) Find the optimal solution to the above problem in closed form.

Proof. Since we have an unconstrained convex optimization problem, the optimal solution x^* occurs at $\nabla f = 0$.

$$\begin{aligned}
\nabla f &= 0 \\
\frac{1}{2} \sum_{\ell=1}^{m} (2x^* - 2b_\ell^T) &= 0 \\
\sum_{\ell=1}^{m} x^* - b_\ell &= 0 \\
x^* m &= \sum_{\ell=1}^{m} \\
x^* &= \frac{1}{m} \sum_{\ell=1}^{m} b_\ell.
\end{aligned}$$

$\square$

(c) Consider a robustified version of the problem. We focus on the setting where each b_i is only an estimate of the true vector $\overline{b_\ell}$ such that

$$\overline{b_\ell} \in \mathcal{B}(b_\ell, r) := \left\{ \hat{b_\ell} : \left\| \hat{b_\ell} - b_\ell \right\|_2 \leq r \right\}$$

and the following reobust optimization:

$$\min_{x \in \mathbb{R}^n} \max_{\ell = 1, \dots, m, \hat{b_\ell} \in \mathcal{B}(b_\ell, r)} \frac{1}{2} \sum_{\ell=1}^{m} (\|x\|_2^2 - 2\hat{b}_\ell^T x). \tag{1}$$

Show that (1) is equivalent to the following convex optimization:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \sum_{\ell=1}^{m} \left(\|x - b_\ell\|_2^2 + 2r \|x\|_2 \right).$$

Proof. Call $f(x) = 1/2 \sum_{\ell=1}^{m} \|x\|_2^2 - 2\hat{b}_\ell^T x$ and $g(x) = 1/2 \sum_{\ell=1}^{m} \|x - b_\ell\|_2^2 + 2r \|x\|_2$. We begin by computing the gradients for each function, and observe that we are done if we can find a mapping F such that $F(x_f^*) = x_g^*$ where x_f^*, x_g^* is an optimal solution to f and g respectively. We begin by computing the gradient of f.

$$\begin{aligned}
\nabla f(x_f^*) &= 0 \\
\frac{1}{2}\sum_{\ell=1}^{m} 2x_f^* - 2\hat{b}_\ell &= 0 \\
\sum_{\ell=1}^{m} x_f^* - \hat{b}_\ell &= 0 \\
mx_f^* - \sum_{\ell=1}^{m} \hat{b}_\ell &= 0 \\
x_f^* &= \frac{1}{m}\sum_{\ell=1}^{m} \hat{b}_\ell.
\end{aligned}$$

Now we compute the gradient of g.

$$\begin{aligned}
\nabla g(x_g^*) &= 0 \\
\frac{1}{2}\sum_{\ell=1}^{m} 2(x_g^* - b_\ell) + 2r\frac{x_g^*}{\left\|x_g^*\right\|_2} &= 0 \\
\sum_{\ell=1}^{m} x_g^* - b_\ell + r\frac{x_g^*}{\left\|x_g^*\right\|_2} &= 0 \\
x_g^* + \frac{rx_g^*}{\left\|x_g^*\right\|_2} &= \frac{1}{m}\sum_{\ell=1}^{m} b_\ell \\
x_g^*\left(1 + \frac{r}{\left\|x_g^*\right\|_2}\right) &= \frac{1}{m}\sum_{\ell=1}^{m} b_\ell
\end{aligned}$$

for $x \neq 0$. This tells us there is some scalar $\alpha \geq 0$ for which $\alpha x_g^* = 1/m \sum b_\ell$. Using this, we continue to evaluate the gradient. Let $s = 1/m \sum_{\ell=1}^{m} b_\ell$. Then

$$\begin{aligned}
x_g^* + \frac{rx_g^*}{\left\|x_g^*\right\|_2} &= \frac{1}{m}\sum_{\ell=1}^{m} b_\ell \\
\alpha\frac{1}{m}\sum_{\ell=1}^{m} b_\ell + \frac{r\alpha\frac{1}{m}\sum_{\ell=1}^{m} b_\ell}{\left\|\alpha\frac{1}{m}\sum_{\ell=1}^{m} b_\ell\right\|_2} &= \frac{1}{m}\sum_{\ell=1}^{m} b_\ell \\
\alpha s + \frac{r\alpha s}{\alpha\left\|s\right\|_2} &= s \\
\left(\alpha + \frac{r}{\left\|s\right\|_2}\right)s &= s,
\end{aligned}$$

so $\alpha + r/\left\|s\right\|_2 = 1$ and $\alpha = 1 - r/\left\|s\right\|_2$, allowing us to conclude that

$$\alpha x_g^* = s \longrightarrow x_g^* = \frac{s}{\alpha} = \frac{1}{\alpha m}\sum_{\ell=1}^{m} b_\ell.$$

Observe that since $\hat{b}_\ell \in B_r(b_\ell)$, there exists $u_\ell \in B_r(b_\ell)$ such that $\hat{b}_\ell - u_\ell = b_\ell$. Then we can write a function $F : \mathbb{R}^n \to \mathbb{R}^n_{\neq 0}$ defined as $F(x) = \frac{1}{\alpha}x - \frac{1}{m}\sum_{\ell=1}^{m} u_\ell$.

$$\begin{aligned}
F(x_f^*) &= \frac{1}{\alpha}x_f^* - \frac{1}{m}\sum_{\ell=1}^{m} u_\ell \\
&= \frac{1}{\alpha}\frac{1}{m}\sum_{\ell=1}^{m} \hat{b}_\ell - \frac{1}{m}\sum_{\ell=1}^{m} u_\ell \\
&= \frac{1}{\alpha}\frac{1}{m}\sum_{\ell=1}^{m} b_\ell + u_\ell - \frac{1}{m}\sum_{\ell=1}^{m} u_\ell \\
&= \frac{1}{\alpha}\frac{1}{m}\sum_{\ell=1}^{m} b_\ell \\
&= x_g^*.
\end{aligned}$$

Since there exists a mapping from x_f^* to x_g^*, the two given convex problems are equivalent (sidenote: u_ℓ is guaranteed fixed since it depends on b_ℓ [fixed by assumption] and $\hat{b}_\ell$ [chosen first; fixed while choosing x]). $\square$

4. Answer the following questions on constrained optimization problems.

(a) Consider the following optimization problem

$$\min_{x_1,x_2} 2x_1 + \frac{1}{2}(x_2-6)^2$$
$$\text{s.t. } x_1 + 2x_2 = 4.$$

Write down the KKT conditions for the equality constrained problem. You may let λ be the Lagrange multiplier. Find a KKT point $(x_1^*, x_2^*, \lambda^*)$ to the problem.

Proof. Let $f(x) = 2x_1 + 1/2(x_2-6)^2$, $h(x) = x_1 + 2x_2 - 4$, and λ a Lagrange multiplier. Then we require the following KKT conditions:

$$\begin{aligned} \nabla f(x) + \lambda \nabla h(x) &= 0, \\ h(x) &= 0. \end{aligned}$$

Observe that $h(x) = x_1 + 2x_2 - 4 = 0 \longrightarrow x_1 = 4 - 2x_2$. We can then compute the gradients of f and h to be

$$\nabla f = \begin{pmatrix} 2 \\ x_2 - 6 \end{pmatrix}, \nabla h = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

giving us a system of equations

$$\nabla f(x) + \lambda \nabla h(x) = 0 \longrightarrow \begin{cases} 2 + \lambda = 0 \\ x_2 - 6 + 2\lambda = 0 \end{cases}.$$

From this system, we obtain $\lambda = -2, x_2 = 10$, from which we compute $x_1 = 4 - 2x_2 = -16$. Thus a KKT point to the given problem is $(x_1^* = -16, x_2^* = 10, \lambda^* = -2)$. $\square$

(b) Consider the optimization problem

$$\min_{x_1,x_2} \frac{1}{2}(x_1-3)^2 + \frac{1}{2}(x_2+3)^2$$
$$\text{s.t. } x_1 + 2x_2 \geq 0$$
$$x_1^2 + x_2^2 \leq 1.$$

Write down the KKT conditions for the above problem. You may let μ_1, μ_2 be the dual variables corresponding to the first and second inequality, respectively. Find a KKT point $(x_1^*, x_2^*, \mu_1^*, \mu_2^*)$ to the problem.

Proof. For notational ease, let

$$\begin{aligned} f(x) &= \frac{1}{2}(x_1-3)^2 + \frac{1}{2}(x_2+3)^2 \\ g_1(x) &= -x_1 - 2x_2 \\ g_2(x) &= x_1^2 + x_2^2 - 1. \end{aligned}$$

The KKT conditions for this problem are as follows:

$$\begin{aligned} \mu_1, \mu_2 &\geq 0 \\ g_1(x), g_2(x) &\leq 0 \\ \mu_1 g_1(x), \mu_2 g_2(x) &= 0 \\ \nabla f(x) + \mu_1 \nabla g_1(x) + \mu_2 \nabla g_2(x) &= 0. \end{aligned}$$

Suppose we have that $g_1 = g_2 = 0$. The first equality $g_1 = 0$ allows us to derive a relationship between x_1 and x_2

$$g_1(x) = 0 \longrightarrow -x_1 - 2x_2 = 0 \longrightarrow x_1 = -2x_2,$$

and the second equation allows us to find values find them.

$$\begin{aligned} g_2(x) &= 0 \\ x_1^2 + x_2^2 - 1 &= 0 \\ (-2x_2)^2 + x_2^2 &= 1 \\ 5x_2^2 &= 1 \\ x_2 &= \pm\frac{\sqrt{5}}{5}. \end{aligned}$$

Let us consider $x_2 = -\sqrt{5}/5$. Then we know x_1, x_2 to be $\frac{2\sqrt{5}}{5}$ and $-\frac{\sqrt{5}}{5}$ respectively.
Next we compute the gradients for f, g_1, g_2 to be

$$\begin{aligned} \nabla f(x) &= \begin{pmatrix} x_1 - 3 \\ x_2 + 3 \end{pmatrix} \\ \nabla g_1(x) &= \begin{pmatrix} -1 \\ -2 \end{pmatrix} \\ \nabla g_2(x) &= \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}. \end{aligned}$$

Solving for the system of equations given by the last KKT condition

$$\nabla f(x) + \mu_1 \nabla g_1(x) + \mu_2 \nabla g_2(x) = 0 \longleftrightarrow \begin{cases} (x_1 - 3) - \mu_1 + 2\mu_2 x_1 = 0 \\ (x_2 + 3) - 2\mu_1 + 2\mu_2 x_2 = 0 \end{cases}$$

yields $\mu_1 = 3/5, \mu_2 = (9\sqrt{5}-5)/10$. This allows us to conclude a solution to the given KKT problem is $(x_1^* = 2\sqrt{5}/5, x_2^* = -\sqrt{5}/5, \mu_1^* = 3/5, \mu_2^* = (9\sqrt{5} - 5)/10)$. $\square$