ST 421 EC 1

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1. Determine whether the following statements are true or false. Justify your answer.
   (a) If A, B are independent, then A, B^c are also independent.
   <u>*Solution.*</u>

   $$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - (P(A)(1 - P(B))) = P(A)\left(1 - P(B)\right) = P(A)P(B^c)$$

   This statement is true.

   (b) For a discrete random variable Y, we have that $E(Y^2) \geq E^2(Y)$.
   <u>*Solution.*</u>

   $$\mathrm{Var}(Y) = E(Y^2) - E^2(Y) \geq 0 \longrightarrow$$
   $$E(Y^2) \geq E^2(Y)$$

   This statement is true.

   (c) If $Y \sim \text{Geometric}(p)$, then a random variable $Y^* = Y - 1$ has mean $\frac{1-p}{p}$.
   <u>*Solution.*</u>

   $$E(Y^*) = E(Y - 1) = E(Y) - E(1) = \frac{1}{p} - 1 = \frac{1-p}{p}$$

   This statement is true.

2. A quality control program at a plastic bottle production line involves inspecting the finished bottles for flaws such as microscopic holes. The probability the bottle has a flaw is 0.002. If a bottle has a flaw, the probability it will fail the inspection is 0.995. If a bottle does not have a flaw, the probability that it will pass the inspection is 0.990.

   Let events P denote passing inspection and F denote being flawed. Note we are given
   (i) $P(F) = 0.002$
   (ii) $P(P^c|F) = 0.995$
   (iii) $P(P|F^c) = 0.990$

   (a) If a bottle does not have a flaw, what is the probability that it will fail the inspection?
   Solution.

   $$P(P^c|F^c) = 1 - P(P|F^c) = 1 - 0.990 = 0.01$$

   (b) What is the probability a randomly selected bottle has a flaw and fails the inspection?
   Solution.

   $$P(P^c \cap F) = P(P^c|F)P(F) = 0.995 \cdot 0.002 \approx 0.002$$

   (c) If a bottle fails the inspection, what is the probability that it has a flaw?
   Solution.

   $$P(F|P^c) = \frac{P(P^c|F)P(F)}{P(P^c)} = \frac{P(P^c|F)(F)}{P(P^c|F) + P(P^c|F^c)} = \frac{0.995 \cdot 0.002}{0.995 + 0.01} \approx 0.002$$

3. Consider an experiment consisting in rolling a fair die twice. We can represent the possible outcomes by ordered pairs. Define the random variable Y to be the sum of the numbers observed in the rolls.

   (a) Write down all possible values the random variable Y can take.
   Solution.

   $$Y \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

   (b) Find $P(Y = 6)$.
   Solution.

   $$P(Y = 6) = |\{(1,5), (2,4), (3,3), (4,2), (5,1)\}| \cdot \frac{1}{36} = \frac{5}{36}$$

   (c) Suppose that somebody tells you that he performed the experiment and observed the value $Y = 4$. What is the probability that he obtained a 3 in the first roll?
   Solution.
   Let R_1 denote the first roll.

   $$P(R_1 = 3 | Y = 4) = \frac{|\{(3,1)\}|}{|\{(1,3), (2,2), (3,1)\}|} = \frac{1}{3}$$

4. Let Y be a random variable with pdf

$$f(y) = \begin{cases} 0.2, & -1 < y < 0 \\ 0.2 + cy, & 0 \leq y \leq 1 \end{cases}$$

(a) Find the cdf $F(y)$.
<u>*Solution.*</u>

$$F(y) = \int_{-\infty}^{y} f(t)dt = \begin{cases} 0.2(y+1), & -1 < y < 0 \\ \frac{cy^2}{2} + 0.2y + 0.2, & 0 \leq y \leq 1 \end{cases}$$

Recall that $\int_{-\infty}^{\infty} F(y)dy = 1$. Then

$$\int_{-\infty}^{\infty} F(y)dy = \int_{0}^{\infty} \frac{cy^2}{2} + 0.2y + 0.2 \cdot dy \tag{1}$$

This is true since

$$\int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{-\infty}^{y} f(t)dt = \int_{-1}^{0} f(t)dt + \int_{0}^{y} f(t)dt$$

We then continue to evaluate equation (1).

$$\int_{-\infty}^{\infty} F(y)dy = \left(\frac{cy^3}{6} + 0.2\frac{y^2}{2} + 0.2y\right)\Bigg|_0^1 = \left(\frac{c}{6} + 0.2\frac{1}{2} + 0.2\right) =$$

$$= \frac{c}{6} + 0.1 + 0.2 = \frac{c}{6} + 0.3 = 1 \longrightarrow$$

$$c = (1 - 0.3) \cdot 6 = 4.2$$

We can then plug $c = 4.2$ into our equation for $F(y)$, yielding

$$F(y) = \begin{cases} 0.2(y+1), & -1 < y < 0 \\ 2.1y^2 + 0.2y + 0.2, 0 \leq y \leq 1 \end{cases}$$

(b) Find $P(Y < \frac{1}{2} | Y > -\frac{1}{2})$.
<u>*Solution.*</u>

$$P\left(Y < \frac{1}{2} | Y > -\frac{1}{2}\right) = \frac{P(-\frac{1}{2} < Y < \frac{1}{2})}{P(Y > -\frac{1}{2})} = \frac{F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right)}{1 - F\left(-\frac{1}{2}\right)} = \frac{\frac{33}{40} - \frac{1}{10}}{\frac{9}{10}} \approx 0.806$$

(c) Find the mean and variance of Y.
<u>*Solution.*</u>

$$\mu = \int_{-\infty}^{\infty} y \cdot f(y)dy = 0.7$$

$$\sigma^2 = \int_{-\infty}^{\infty} y^2 \cdot f(y)dy - \mu^2 \approx 0.168$$

5. Suppose $Y \sim \text{Gamma}(\alpha, \beta)$. That is, Y has a pdf

$$f(y) = \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\beta\Gamma(\alpha)}\left(\frac{y}{\beta}\right)^{\alpha-1} \cdot e^{-\frac{y}{\beta}}, \text{ for } 0 < y < \infty \text{ and } \alpha, \beta > 0$$

(a) Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$.
Solution.
Recall that for a Gamma distribution, Y is defined on the interval $(0, \infty)$, and also that

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \tag{2}$$

Then we can verify by directly computing the integral.

$$\int_{-\infty}^{\infty} f(y)dy = \int_0^\infty f(y)dy = \int_0^\infty \frac{1}{\beta \cdot \Gamma(\alpha)} \cdot \left(\frac{y}{\beta}\right)^{\alpha-1} \cdot e^{-\frac{y}{\beta}}dy =$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(\frac{y}{\beta}\right)^{\alpha-1} \cdot e^{-\frac{y}{\beta}} \cdot \frac{1}{\beta}dy, \text{ Let } u = \frac{y}{\beta}, du = \frac{1}{\beta}dy \longrightarrow$$

$$\int_{-\infty}^{\infty} f(y)dy = \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}e^{-u}du$$

By equation (2), we know the integral with respect to u evaluates to $\Gamma(\alpha)$, so

$$\int_{-\infty}^{\infty} f(y)dy = \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha) = 1$$

(b) Without using moment generating functions, find the mean and variance of Y.
Solution.

$$\mu = \int_{-\infty}^{\infty} yf(y)dy = \int_0^\infty \frac{1}{\Gamma(\alpha)}\left(\frac{y}{\beta}\right)^{\alpha} e^{-\frac{y}{\beta}}dy, \text{ Let } u = \frac{y}{\beta}, du = \frac{1}{\beta}dy \longrightarrow$$

$$\mu = \frac{\beta}{\Gamma(\alpha)}\int_0^\infty u^\alpha e^{-u}du = \frac{\beta}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \frac{\alpha\beta\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta \text{ by equation (2)}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^\infty \frac{y^2}{\beta\Gamma(\alpha)}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-\frac{y}{\beta}}dy = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{\alpha+1}e^{-\frac{y}{\beta}}dy =$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \frac{\Gamma(\alpha+2)}{\left(\frac{1}{\beta}\right)^{\alpha+2}} = \frac{1}{\Gamma(\alpha)} \cdot \beta^2\Gamma(\alpha+2) = \alpha(\alpha+1)\beta^2$$

$$\sigma^2 = E(Y^2) - \mu^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

(c) Find the moment generating function of Y and use it to verify the results in Part (b).
Solution.

$$E\left(e^{tY}\right) = \int_{-\infty}^{\infty} e^{ty}f(y)dy = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{ty}y^{\alpha-1}e^{-\frac{y}{\beta}}dy = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{\alpha-1}e^{-y(\frac{1}{\beta}-t)}dy =$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \frac{\Gamma(\alpha)}{\left(\frac{1}{\beta}-t\right)^\alpha} = \left(\frac{1}{1-t\beta}\right)^\alpha$$

To verify the results of Part (b), we find the 1^{st} and 2^{nd} moments, and use them to calculate the mean and variance.

$$M_Y^{(1)}(0) = \alpha \left(\frac{1}{1-t\beta}\right)^{\alpha-1} \cdot \left.\frac{0-(-\beta)}{(1-t\beta)^2}\right|_{t=0} = \alpha\beta$$

So we confirm that $\mu = E(Y) = \alpha\beta$.

$$M_Y^{(2)}(0) = \frac{d}{dt}\alpha \left(\frac{1}{1-t\beta}\right)^{\alpha-1} \cdot \left.\frac{\beta}{(1-t\beta)^2}\right|_{t=0} =$$

$$= \left.\left(\alpha(\alpha-1)\left(\frac{1}{1-t\beta}\right)^{\alpha-2} \cdot \left(\frac{\beta}{(1-t\beta)^2}\right)^2 + \alpha\left(\frac{1}{1-t\beta}\right)^{\alpha-1} \cdot \frac{0-\beta(-2\beta)}{(1-t\beta)^4}\right)\right|_{t=0} =$$

$$= \alpha(\alpha-1)\beta^2 + 2\alpha\beta^2 = \alpha^2\beta^2 - \alpha\beta^2 + 2\alpha\beta^2 = \alpha^2\beta^2 + \alpha\beta^2$$

So we confirm that $\sigma^2 = E(Y^2) - \mu^2 = \alpha^2\beta^2 + \alpha\beta - \alpha^2\beta^2 = \alpha\beta^2$.