MTH 464 HW 4

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1. Let X be a random variable with $\mu = \mathbb{E}\left(X\right)$ and $\sigma^2 = \mathrm{Var}\left(X\right)$. Determine a, b such that $Y = a + bX$ satisfies $\mathbb{E}\left(Y\right) = 0, \mathrm{Var}\left(Y\right) = 4\sigma^2$, and $\mathrm{Corr}\left(X, Y\right) = -1$.

$$\mathbb{E}\left(Y\right) = \mathbb{E}\left(a + bX\right) = \mathbb{E}\left(a\right) + \mathbb{E}\left(bX\right) = a + b\mathbb{E}\left(X\right) = a + b\mu = 0 \tag{1}$$

$$\mathrm{Var}\left(Y\right) = \mathrm{Var}\left(a + bX\right) = \mathrm{Var}\left(a\right) + \mathrm{Var}\left(bX\right) = b^2\mathrm{Var}\left(X\right) = b^2\sigma^2 = 4\sigma^2 \tag{2}$$

$$\mathrm{Corr}\left(X, Y\right) = \frac{\mathrm{Cov}\left(X, Y\right)}{\sigma_x\sigma_y} = \frac{\mathbb{E}\left(XY\right) - \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)}{\sigma_x\sigma_y} = \frac{\mathbb{E}\left(X(a + bX)\right)}{\sigma_x\sigma_y} = -1 \tag{3}$$

Solving for equation (1) gives $a = -b\mu$. Using this, we can further reduce equation (3) into

$$\mathrm{Corr}\left(X, Y\right) = \frac{\mathbb{E}\left(aX + bX^2\right)}{\sigma_x\sigma_y} = \frac{a\mathbb{E}\left(X\right) + b\mathbb{E}\left(X^2\right)}{\sigma_x\sigma_y} = \frac{a\mu + b(\sigma^2 + \mu^2)}{\sigma \cdot 2\sigma} = \frac{-b\mu^2 + b\sigma^2 + b\mu^2}{2\sigma^2} =$$

$$= \frac{b}{2} = -1 \longrightarrow b = -2$$

Since $a = -b\mu$, we can sub-in $b = -2$ and conclude that $a = 2\mu, b = -2$.

2. Assume that the joint density of X, Y is given by

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-(y+\frac{x}{y})} \mathbb{1}_{[0,\infty)\times[0,\infty)}(x,y)$$

(a) Check that f is a pdf.

<u>*Solution.*</u>

To verify that f is a pdf, we need to check that it integrates to 1 over its domain, and also that $0 \leq f_{X,Y}(x,y) \leq 1$ for all $x, y \in \mathbb{R}$.

Examining f reveals it is negative only when $y < 0$, and since $y \in [0, \infty)$, we conclude that $f \geq 0$.

To check that it integrates to 1 over its domain, we compute

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = \int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-(y+\frac{x}{y})} dx dy = \int_0^{\infty} e^{-y} dy = 1$$

(b) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

<u>*Solution.*</u>

$$\mathbb{E}(X) = \int_0^{\infty} \int_0^{\infty} x f_{X,Y}(x,y) dx dy = \int_0^{\infty} \int_0^{\infty} \frac{x}{y} e^{-y-\frac{x}{y}} dx dy = \int_0^{\infty} y e^{-y} dy = 1$$

$$\mathbb{E}(Y) = \int_0^{\infty} \int_0^{\infty} y f_{X,Y}(x,y) dx dy = \int_0^{\infty} \int_0^{\infty} e^{-y-\frac{x}{y}} dx dy = \int_0^{\infty} y e^{-y} dy = 1$$

(c) Show that the $\operatorname{Cov}(X, Y) = 1$.

Proof. Note that

$$\operatorname{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X) Y$$

Since we calculated $\mathbb{E}(X), \mathbb{E}(Y)$ in Part (b), all that remains is to calculate $\mathbb{E}(XY)$ and put them together.

$$\mathbb{E}(XY) = \int_0^{\infty} \int_0^{\infty} x e^{-y-\frac{x}{y}} dx dy = \int_0^{\infty} y^2 e^{-y} dy = 2$$

We compute $\operatorname{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 2 - 1 \cdot 1 = 1$, and we are done. $\square$

3. Let $Z \sim N(0,1)$ be a standard normal random variable. Let I be independent of Z such that $P(I = -1) = P(I = 1) = \frac{1}{2}$. Define $Y = Z \times I$.

   (a) Show that Y is a standard normal random variable.

   Proof. Recall that for a standard normal distribution, $\Phi(a) = 1 - \Phi(-a)$. Then

$$P(Y \leq a) = P(ZI \leq a) = \frac{1}{2}P(Z \leq a) + \frac{1}{2}P(Z \geq -a) =$$

$$= \frac{1}{2}\Phi(a) + \frac{1}{2}(1 - \Phi(-a)) = \frac{1}{2}\Phi(a) + \frac{1}{2}\Phi(a) = \Phi(a)$$

   Since Y has the same cdf as a standard normal, we conclude that Y is standard normal. □

   (b) Show that $\operatorname{Cov}(Y, Z) = 0$.

   Proof.

$$\operatorname{Cov}(Y,Z) = \mathbb{E}(YZ) - \mathbb{E}(Y)\,\mathbb{E}(Z) = \mathbb{E}(ZI \cdot Z) - 0 = \mathbb{E}(Z \cdot Z) = \mathbb{E}\left(Z^2\right) = 0$$

   □

   (c) Show that Z, Y are not independent. This provides an example of uncorrelated variables which are not independent.

   Proof. Consider the case where $z = y = a$. If Z, Y were independent, given both $Z, Y \sim N(0,1)$ we would expect

$$P(Z \leq a \wedge Y \leq a) = P(Z \leq a)P(Y \leq a) = (\Phi(a))^2$$

   We now directly calculate the joint probability to be

$$P(Z \leq a \wedge Y \leq a) - P(Z \leq a \wedge ZI \leq a) =$$

$$= P(I = 1)P(Z \leq a \wedge ZI \leq a | I = 1) + P(I = -1)P(Z \leq a \wedge ZI \leq a | I = -1) =$$

$$= \frac{1}{2}P(Z \leq a \wedge Z \leq a) + \frac{1}{2}P(Z \leq a \wedge Z \geq -a) = \frac{1}{2}\Phi(a) + \frac{1}{2}(\Phi(a) - \Phi(-a)) =$$

$$= \Phi(a) - \frac{\Phi(-a)}{2} \neq (\Phi(a))^2$$

   We conclude that since $P(Z \leq a \wedge Y \leq a) \neq P(Z \leq a)P(Y \leq a)$, Z, Y are not independent. □