MTH 312 HW 7

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7.5.4. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show this conclusion does not follow if f is not continuous.

Proof. Let $F(x) = \int_a^x f$. Since $F(x) = 0$ everywhere on $[a, b]$, F must be differentiable on $[a, b]$ with $F'(x) = 0$ on for $x \in [a, b]$. Note also that f is continuous, and by the Fundamental Theorem of Calculus, we have that $F'(x) = f(x)$ for all $x \in [a, b]$. Thus $f(x) = 0$ for all $x \in [a, b]$.

For a counter example, consider $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Then we have that $\int_0^x f = 0$ for all $x \in [0, 1]$, but it is not the case that $f(x) = 0$ for all $x \in [0, 1]$. $\square$

7.6.2. Define

$$h(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin C \end{cases}$$

(a) Show h has discontinuities at each point of C and is continuous at every point in the complement of C. Thus h is not continuous on an uncountably infinite set.

Proof. Suppose that $x \notin C$. Since C is closed, the complement of C is open, and there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq C^c$. Thus h is zero on the interval $(x - \delta, x + \delta)$, and it follows that h is continuous at x. Now suppose $x \in C$. To show h is not continuous at x, it is sufficient to show:

$$\text{for any } \delta > 0, \text{ there exists } y \in (x - \delta, x + \delta) \text{ such that } y \notin C \tag{1}$$

However, if there exists δ which does not satisfy equation (1), then C contains a proper interval, which is a contradiction since it is totally disconnected (shown in Exercise 3.4.8). Thus h is not continuous at x. □

(b) Now prove that h is integrable on $[0, 1]$.
DISCLAIMER: Exercise 7.3.9 (d) makes this problem trivial, so I refrained from using it.

Proof. We that that since C has content zero, $D = C \cap [0, 1]$ also has content zero, and by Exercise 7.3.9 (a) h is integrable. Let P be a partition of $[0, 1]$. It follows that any subinterval $[x_{k-1}, x_k]$ of the partition P contains some $x \notin C$, thus $h(x) = 0$ and $L(h, P) = 0$. Since P was arbitrary, it follows that $\int_0^1 h = L(h) = 0$. □

7.6.3. Show that any countable set has measure zero.

Proof. Let $A \subseteq \mathbb{R}$ be a countable set, and $eps > 0$ be given. Choose $n \in \mathbb{N}$ such that $2^{-N} < \epsilon$. For each $n \in \mathbb{N}$, let

$$O_n = \left(a_n - \frac{\epsilon}{2^{N+n+1}}, a_n + \frac{\epsilon}{2^{N+n+1}}\right)$$

Then $A \subseteq \cup_{n=1}^{\infty} O_n$, and $|O_n| = 2^{-N-n}$. Then

$$\sum_{n=1}^{\infty} |O_n| = \sum_{n=1}^{\infty} 2^{-N-n} = 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \epsilon$$

Thus A has measure zero. $\square$