MTH 511 HW 1

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1. **Exercise 3.22:** Show that $\|x\|_\infty \leq \|x\|_2$ for any $x \in \ell_2$, and that $\|x\|_2 \leq \|x\|_1$ for any $x \in \ell_1$. We split this proof into two parts: (a) $\|x\|_\infty \leq \|x\|_2$ and (b) $\|x\|_2 \leq \|x\|_1$.

   (a) $\|x\|_\infty \leq \|x\|_2$

   Proof. Fix $k \in \mathbb{N}$. Then clearly

$$|x_k|^2 \leq \sum_{i=1}^{\infty} |x_i|^2$$
$$|x_k| \leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$$
$$\|x\|_\infty = \sup_{j \in \mathbb{N}} |x_j| \leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} = \|x\|_2$$

   □

   (b) $\|x\|_2 \leq \|x\|_1$

   Proof. Recall that

$$\|x\|_1^2 = (|x_1| + |x_2| + \ldots + |x_n|)^2 = \left(\sum_{i=1}^{\infty} |x_i|\right)^2. \tag{1}$$

   By considering the multinomial expansion of (1), we see that

$$\left(\sum_{i=1}^{\infty} |x_i|\right)^2 \geq \sum_{i=1}^{\infty} |x_i|^2$$
$$\sum_{i=1}^{\infty} |x_i| \geq \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$$
$$\|x\|_1 \geq \|x\|_2 .$$

   □

2. **Exercise 3.23:** The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. (Note that c_0 is actually a linear subspace of ℓ_∞; thus c_0 is also a normed vector space under $\|\cdot\|_\infty$.) Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

   Proof. Recall the following:

   - By definition: $c_0 \subset \ell_\infty$.
   - By 3.22a: $\ell_2 \subset c_0$.
   - By 3.22b: $\ell_1 \subset \ell_2$.

   Note: recognize our proof of 3.22a relies on the fact that (x_n) converges, but is independent of what the sequence converges to. For this reason we are able to conclude that $\ell_2 \subset c_0$ by 3.22a.

   Chaining these results together, we conclude that

   $$\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty.$$

   □

3. **Exercise 3.25:** Using $\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$, state and prove lemma 3.7 and theorem 3.8 (also cover $p = 1, q = \infty$ for lemma 3.7).

   Lemma 3.7 (Holder's Inequality). *Let $1 < p < \infty$ and let q be defined by $1/p + 1/q = 1$. Given $x \in \ell_p$ and $y \in \ell_q$, we have $\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q$.*

   Theorem 3.8 (Minkowski's Inequality). *Let $1 < p < \infty$. If $x, y \in \ell_p$, then $x + y \in \ell_p$ and $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.*

4. **Exercise 3.36:** Given a metric space (M, d), prove a convergent sequence is Cauchy and a Cauchy sequence is bounded.

5. **Exercise 3.37:** A Cauchy sequence with a convergent subsequence converges.

6. **Exercise 3.39:** If every subsequence of (x_n) has a further subsequence that converges to x, then (x_n) converges to x.