MTH 311 Lab 1

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1. For each $n \geq 1$, let $S_n = \sum_{k=1}^{n} \frac{1}{k}$.

   (a) Prove that $S_{2n} \geq \frac{1}{2} + S_n$ for all $n \geq 1$.

   Proof. First, we can rewrite the inequality as

   $$S_{2n} \geq S_n + \frac{1}{2} \longrightarrow$$

   $$S_{2n} - S_n \geq \frac{1}{2}.$$

   Substituting in the definition of S_n, we get

   $$\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \geq \frac{1}{2} \longrightarrow$$

   $$\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{1}{2}. \tag{1}$$

   From here, if we can show that (1) is true, then we are done. With this in mind, notice that the LHS of (1) is bounded by the value $\frac{1}{2n}$, namely that

   $$\sum_{k=n+1}^{2n} \frac{1}{k} \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}.$$

   We conclude that $S_{2n} \geq S_n + \frac{1}{2}$ for all $n \geq 1$. $\qquad \square$

   (b) Let $A = \{S_n : n \geq 1\}$. Is A bounded above?
   Solution.
   No, A is not bounded above because $\lim_{n \to \infty} S_n$ diverges since S_n is a harmonic series. Because it diverges, there is no value $x \in \mathbb{R}$ such that $x \geq S_n$ for all n.

2. Assume $0 < r < 1$, and let $S_n = \sum_{k=0}^{n} r^k$.

   (a) Show that
$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

   for all $n \geq 1$.

   Proof. Consider the expression $S_n - rS_n$. We can write this using the definition of S_n as

$$S_n - rS_n = \sum_{k=0}^{n} r^k - r \sum_{k=0}^{n} r^k \longrightarrow$$

$$S_n - rS_n = \sum_{k=0}^{n} r^k - \sum_{k=0}^{n} r^{k+1} \longrightarrow$$

$$S_n - rS_n = \sum_{k=0}^{n} r^k - \sum_{k=1}^{n+1} r^k \longrightarrow$$

$$S_n - rS_n = 1 - r^{n+1} \longrightarrow$$

$$S_n (1 - r) = 1 - r^{n+1} \longrightarrow$$

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

   We conclude that $S_n = \frac{1-r^{n+1}}{1-r}$ when $0 < r < 1$ and $S_n = \sum_{k=0}^{n} r^k$. $\square$

   (b) Let $A = \{S_n : n \geq 1\}$. Is A bounded above? Explain why or why not.
   Solution.
   Since S_n is a geometric series with a base $0 < r < 1$, we know that S_n converges as n tends to infinity. Because of this, we know that A is bounded above, in particular $sup(A) = \frac{1}{1-r}$.