MTH 511 HW 1

Brandyn Tucknott

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1. **Exercise 3.22:** Show that $\|x\|_\infty \leq \|x\|_2$ for any $x \in \ell_2$, and that $\|x\|_2 \leq \|x\|_1$ for any $x \in \ell_1$. We split this proof into two parts: (a) $\|x\|_\infty \leq \|x\|_2$ and (b) $\|x\|_2 \leq \|x\|_1$.

   (a) $\|x\|_\infty \leq \|x\|_2$

   Proof. Fix $k \in \mathbb{N}$. Then clearly

$$\begin{aligned} |x_k|^2 &\leq \sum_{i=1}^{\infty} |x_i|^2 \\ |x_k| &\leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \\ \|x\|_\infty = \sup_{j \in \mathbb{N}} |x_j| &\leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} = \|x\|_2 \end{aligned}$$

$\square$

   (b) $\|x\|_2 \leq \|x\|_1$

   Proof. Recall that

$$\|x\|_1^2 = (|x_1| + |x_2| + \ldots + |x_n|)^2 = \left(\sum_{i=1}^{\infty} |x_i|\right)^2. \tag{1}$$

   By considering the multinomial expansion of (1), we see that

$$\begin{aligned} \left(\sum_{i=1}^{\infty} |x_i|\right)^2 &\geq \sum_{i=1}^{\infty} |x_i|^2 \\ \sum_{i=1}^{\infty} |x_i| &\geq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \\ \|x\|_1 &\geq \|x\|_2 . \end{aligned}$$

$\square$

2. **Exercise 3.23:** The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. (Note that c_0 is actually a linear subspace of ℓ_∞; thus c_0 is also a normed vector space under $\|\cdot\|_\infty$.) Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

   Proof. Recall the following:

   - By definition: $c_0 \subset \ell_\infty$.
   - By 3.22a: $\ell_2 \subset c_0$.
   - By 3.22b: $\ell_1 \subset \ell_2$.

   Note: recognize our proof of 3.22a relies on the fact that (x_n) converges, but is independent of what the sequence converges to. For this reason we are able to conclude that $\ell_2 \subset c_0$ by 3.22a.

   Chaining these results together, we conclude that

$$\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty.$$

□

3. **Exercise 3.25:** Using $\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$, state and prove lemma 3.7 and theorem 3.8 (also cover $p = 1, q = \infty$ for lemma 3.7).

Lemma 3.7 (Holder's Inequality). *Let $1 < p < \infty$ and let q be defined by $1/p + 1/q = 1$. Given $f \in \ell_p$ and $g \in \ell_q$, we have $\sum_{i=1}^{\infty} |f(t)g(t)| \leq \|f\|_p \|g\|_q$.*

Proof. In the case where $p = 1, q = \infty$, note if $g_{max} = \max_{t\in[0,1]} |g(t)|$, then $|g(t)| \leq |g_{max}|$ and we have that

$$\begin{aligned}\int_0^1 |f(t)g(t)|dt &\leq \int_0^1 |f(t)g_{max}|dt \\ &= g_{max} \int_0^1 |f(t)|dt \\ &= \|g\|_\infty \|f\|_1 = \|g\|_q \|f\|_p .\end{aligned}$$

If $1 < p < \infty$, then by Young's Inequality we have the following:

$$\begin{aligned}\int_0^1 \left|\frac{f(t)g(t)}{\|f\|_p \|g\|_q}\right| dt &\leq \frac{1}{p}\int_0^1 \left|\frac{f(t)}{\|f\|_p}\right|^p dt + \frac{1}{q}\int_0^1 \left|\frac{g(t)}{\|g\|_q}\right|^q dt \\ &= \frac{1}{p}\frac{1}{\|f\|_p}^p \int_0^1 |f(t)|^p dt + \frac{1}{q}\frac{1}{\|g\|_q^q}\int_0^1 |g(t)|^q dt \\ &= \frac{1}{p}\frac{1}{\|f\|_p^p}\|f\|_p^p + \frac{1}{q}\frac{1}{\|g\|_q^q}\|g\|_q^q \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

So clearly $\int_0^1 |f(t)g(t)|dt \leq \|f\|_p \|g\|_q$. ∎

Before we can prove Theorem 3.8, we first need to prove the analogue of Lemma 3.5 for our use.

Lemma 3.5. *Let $1 < p < \infty$ and $f, g \geq 0$. Then $(f + g)^p \leq 2^p(f^p + g^p)$. Consequently $f + g \in \ell_p$ whenever $f, g \in \ell_p$.* ∎

Proof. Let $1 < p < \infty$ and $f, g \geq 0$.

$$\begin{aligned}(f+g)^p &\leq (2\max(f,g))^p \\ &= 2^p (\max(f,g))^p \\ &= 2^p \max(f^p, g^p) \\ &\leq 2^p(f^p + g^p).\end{aligned}$$

It follows then that

$$\begin{aligned}\|f+g\|_p = \int_0^1 |f(t)+g(t)|^p dt &\leq 2^p\left(\int_0^1 |f(t)|^p dt + \int_0^1 |g(t)|^p dt\right) \\ &= 2^p\left(\|f\|_p^p + \|g\|^p\right) < \infty.\end{aligned}$$

Thus $f + g \in \ell_p$. ∎

Theorem 3.8 (Minkowski's Inequality). *Let $1 < p < \infty$. If $f, g \in \ell_p$, then $f + g \in \ell_p$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. Let $1 < p < \infty$ and $f, g \in \ell_p$. By Lemma 3.5 we have that $f + g \in \ell_p$. For the inequality, observe that

$$\begin{aligned} \|f+g\|_p^p &= \int_0^1 |f+g|^p dt \\ &\leq \int_0^1 |f|^p + |g|^p t \\ &= \int_0^1 |f|^p dt + \int_0^1 |g|^p dt \\ &= \|f\|_p^p + \|g\|_p^p . \end{aligned}$$

Thus we conclude that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

$\square$

4. **Exercise 3.36:** Given a metric space (M, d), prove a convergent sequence is Cauchy and a Cauchy sequence is bounded.

   Proof. First, we will show that a convergent sequence is bounded. Recall the definition for a convergent sequence:
   $$(x_n) \text{ converges to } x \leftrightarrow \text{ for all } \varepsilon > 0, (x_n) \text{ is eventually in } B_\varepsilon(x).$$
   Let $N \in \mathbb{N}$ be such that $\{x_n : n \geq N\} \subset B_\varepsilon(x)$ (we know this to be possible by the definition of (x_n) converging). Also let $S = \{x_n : 1 \leq n < N\}, m = \max(S)$. Then clearly $S \subset B_m(x)$, and we have that for all $n \geq 1$,
   $$\begin{cases} x_n \in B_m(x), & 1 \leq n < N \\ x_n \in B_\varepsilon(x), & n \geq N \end{cases}$$
   and from it we conclude that $x_n \in B_{\max(m,\varepsilon)}(x)$ and thus (x_n) is bounded.

   Since that $\{x_n : n \geq N\} \subset B_{\varepsilon/2}$, it follows that $\operatorname{diam}\left(B_{\varepsilon/2}\right) < \varepsilon$, and by definition (x_n) is Cauchy. Thus the convergent sequence (x_n) is Cauchy. To show a Cauchy sequence is bounded, it follows from the definition, there for all $\varepsilon > 0$ there exists $N \geq 1$ such that $\operatorname{diam}(x_n : n \geq N) < \varepsilon$. Then certainly $\{x_n : n \geq N\} \subset B_{\varepsilon/2}$. Since the set $\{x_n : 1 \leq n < N\}$ is a finite set of finite values, it is must be bounded. Since the leading terms and tail terms are bounded, the whole sequence must be bounded.

   $\square$

5. **Exercise 3.37:** A Cauchy sequence with a convergent subsequence converges.

Proof. Let (x_n) be Cauchy. Suppose for all subsequence $(a_n) \subset (x_n)$, (a_n) does not converge to a. Then by definition, there exists some $\varepsilon > 0$ where for all $N \geq 1$, there exists some $k \geq N$ such that $d(a_k, a) > \varepsilon/2$. Since $(a_n) \subset (x_n)$, it follows then that $\operatorname{diam}(\{x_n : n \geq k, n\}) > \varepsilon$. This is a contradiction to our definition of a Cauchy sequence, so it must be that there exists a convergent subsequence. $\square$

6. **Exercise 3.39:** If every subsequence of (x_n) has a further subsequence that converges to x, then (x_n) converges to x.

   Proof. We will do a proof by contrapositive. Given (x_n) does not converge to x, it is sufficient to show then there exists a subsequence where all further subsequences do not converge to x. Since (x_n) does not converge to x, by definition we have that there exists some $\varepsilon > 0$ where for all $N \in \mathbb{N}$, there is some $n_1 \in \mathbb{N}, d(x_{n_1}, x) \geq \varepsilon$.

   We will now inductively construct a subsequence $(a_n) \subset (x_n)$ for which (a_n) has no convergent subsequences. Choose $a_1 = x_{n_1}$; thus $d(a_1, x) \geq \varepsilon$. For the inductive step, assume we have chosen up to a_k, and wish to choose a_{k+1}. Choose $a_{k+1} \geq a_k + 1$, but still satisfying $d(a_{k+1}, x) \geq \varepsilon$. We have just constructed a subsequence $(a_n) \subset (x_n)$ such that for all $k \in \mathbb{N}, d(a_k, x) \geq \varepsilon$. It is clear then, that any subsequence $(b_n) \subset (a_n)$ will have the same property, and by definition will not converge to x. $\square$