MTH 463 HW 8

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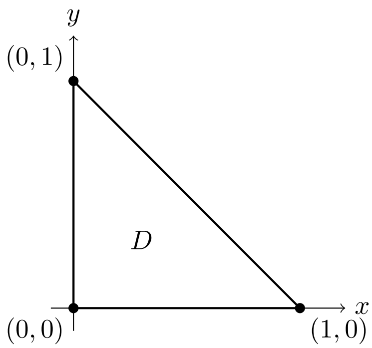
1. Denote by D the triangle denoted by the points $(0,0), (1,0),$ and $(0,1)$. Define

$$f(x,y) = 24xy\mathbb{1}_D(x,y)$$

(a) Show that $f(x,y)$ is a joint probability density for a pair of random variables X, Y.
Solution.



Since $f(x,y)$ is certainly nonnegative and integrable in the domain D, it remains to show that the integral in the domain D of $f(x,y)$ is 1.

$$\iint_D f(x,y)dA = \int_0^1 \int_0^{1-x} 24xy \cdot dydx = \int_0^1 12x(1-x)^2 dx = 1$$

(b) Find the marginal probability density functions of X and Y. Are X and Y independent?
Solution.

$$f_X(x) = \int_0^{1-x} f(x,y)dy = \int_0^{1-x} 24xydy = 12x(1-x)^2 \text{ and by symmetry } f_Y(y) = 12y(1-y)^2$$

Notice that

$$f_X(x) \cdot f_Y(y) = 12^2xy(1-x)^2(1-y)^2 \neq f(x,y), \text{ therefore } X, Y \text{ are not independent.}$$

(c) Find $E(X)$ and $\text{Var}(X)$.
Solution.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5}$$

$$E(X^2) = \int_0^1 x^2 \cdot 12x(1-x)^2 dx = \frac{1}{5}$$

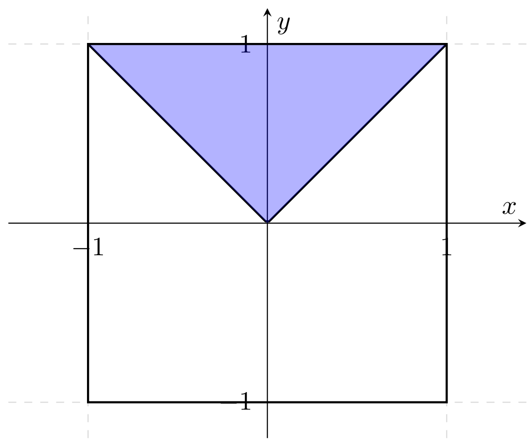
$$\text{Var}(X) = E(X^2) - E^2(X) = \frac{1}{25}$$

2. Let X and Y be independent uniformly distributed random variables distributed on the interval $[-1,1]$. That is,

$$f_{X,Y}(x,y) = \frac{1}{4}\mathbb{1}_D(x,y) \text{ where } D = [-1,1]\times[-1,1]$$

(a) Find $P(|X| < Y)$.
Solution.

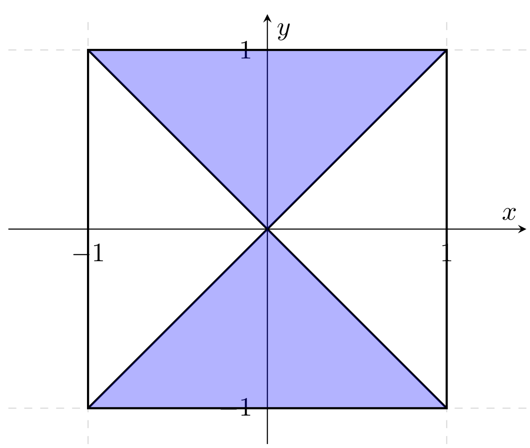
Region where $|x| < y$



$$P(|X| < Y) = \frac{1}{4} \cdot \text{ area of shaded region } = \frac{1}{4}$$

(b) Find $P(X^2 < Y^2)$.
Solution.

Region where $x^2 < y^2$



$$P(X^2 < Y^2) = \frac{1}{4} \cdot \text{ area of shaded region } = \frac{1}{2}$$

3. Let X, Y be independent random variables uniformly distributed on the interval $[-1,1]$. That is,

$$f_X(x) = f_Y(x) = \frac{1}{2}\mathbb{1}_{[-1,1]}(x)$$

Let $Z = X + Y$

(a) Find the probability density function of Z.
<u>*Solution.*</u>

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_{-\infty}^{\infty} \frac{1}{4} \cdot \mathbb{1}_{[-1,1]}(x)\mathbb{1}_{[-1,1]}(z-x)$$

Examing the indicator functions tells us that

$$-1 \le x \le 1, \text{ and}$$

$$-1 \le z - x \le 1 \longrightarrow z - 1 \le x \le z + 1$$

We can then rewrite out bounds of the integral as the minimum$(-1, z-1)$ and the maximum$(1, z+1)$.

$$f_Z(z) = \int_{\min(-1,z-1)}^{\max(1,z+1)} \frac{1}{4} dx = \begin{cases} \frac{1}{4}\int_{-1}^{z+1} dx = \frac{2+z}{4}, & z \le 0 \\ \frac{1}{4}\int_{z-1}^{1} dx = \frac{2-z}{4}, & z \ge 0 \end{cases}$$

(b) Check that $\int_{-\infty}^{\infty} f_Z(z)dz = 1$.
<u>*Solution.*</u>

$$\int_{-\infty}^{\infty} f_Z(z)dz = \int_{-2}^{2} f_Z(z)dz = \int_{-2}^{0} \frac{2+z}{4}dz + \int_{0}^{2} \frac{2-z}{4}dz = \frac{1}{2} + \frac{1}{2} = 1$$

(c) Find $E(Z)$ and $\mathrm{Var}(Z)$.
<u>*Solution.*</u>

$$E(Z) = \int_{-2}^{2} z f_Z(z)dz = \int_{-2}^{0} z \cdot \frac{2+z}{4}dz + \int_{0}^{2} z \cdot \frac{2-z}{4}dz = -\frac{1}{3} + \frac{1}{3} = 0$$

$$E\left(Z^2\right) = \int_{-2}^{2} z^2 \cdot f_Z(z)dz = \int_{-2}^{0} z^2 \cdot \frac{2+z}{4}dz + \int_{0}^{2} z^2 \cdot \frac{2-z}{4}dz = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\mathrm{Var}(Z) = E\left(Z^2\right) - E^2(Z) = \frac{2}{3}$$

4. Let $U \sim$ Uniform$[0,1]$ and $T \sim$ Exp(1) with U, T independent. Define $Z = U + T$.

(a) Find the probability density function of Z.
Solution.

$$f_Z(z) = \int_{-\infty}^{\infty} f_T(x) f_U(z-x) dx = \int_{-\infty}^{\infty} e^{-x} \cdot \mathbb{1}_{[0,\infty)}(x) \mathbb{1}_{[0,1]}(z-x) dx$$

Examining the indicator functions tells us that

$$0 \leq x < \infty \text{ and}$$

$$0 \leq z - x \leq 1 \longrightarrow z - 1 < x < z$$

We rewrite the bounds of the integral as maximum$(0, z-1)$ and minimum$(\infty, z) = z$.

$$f_Z(z) = \int_0^{\max(1,z)} e^{-x} dx = \begin{cases} \int_0^z e^{-x} dx = -e^{-z} + 1, & z \leq 1 \\ \int_{z-1}^z e^{-x} dx = -e^{-z} + e^{-(z-1)}, & z \geq 1 \end{cases}$$

(b) Check that $\int_{-\infty}^{\infty} f_Z(z) dz = 1$.
Solution.

$$\int_{-\infty}^{\infty} f_Z(z) dz = \int_0^1 -e^{-z} + 1 dz + \int_1^{\infty} -e^{-z} + e^{-(z-1)} dz = \left(e^{-1} - 1 + 1\right) + \left(-e^{-1} + 1\right) = 1$$

(c) Find $E(Z)$ and Var(Z).
Solution.

$$E(Z) = \int_0^1 z \cdot \left(-e^{-z} + 1\right) dz + \int_1^{\infty} z \cdot \left(-e^{-z} + e^{-(z-1)}\right) dz = \left(2e^{-1} - \frac{1}{2}\right) + \left(-2e^{-1} + 2\right) = \frac{3}{2}$$

$$E\left(Z^2\right) = \int_0^1 z^2 \cdot (-e^{-z} + 1) dz + \int_1^{\infty} z^2 \cdot (-e^{-z} + e^{-(z-1)}) dz = \left(5e^{-1} - \frac{5}{3}\right) + \left(-5e^{-1} + 5\right) = \frac{10}{3}$$

$$\text{Var}(Z) = E\left(Z^2\right) - E^2(Z) = \frac{10}{3} - \left(\frac{3}{2}\right)^2 = \frac{13}{12}$$

5. Particles are subject to collisions which cause them to split into two parts, each part taking a fraction of the parent mass. Suppose that this fraction is uniformly distributed on the interval $[0,1]$. Following a single particle through several splittings, we obtain a fraction of the original particle

$$Z_n = X_1 \cdot \ldots \cdot X_n \text{ where } \{X_j\}_{j=1}^{\infty} \text{ are iid uniformly distributed on } [0,1] \text{ random variables}$$

(a) Show that the density for a random variable $T_1 = -\ln(X_1)$ is an exponential random variable with parameter $\lambda = 1$.

Proof. Since $T_1 = -\ln(X_1)$, we can rearrange the variables into

$$X_1 = e^{-T_1}, \frac{dX_1}{dT_1} = -e^{-T_1}$$

We can find the pdf of T_1 by writing it in terms of X_1 and correcting it with the Jacobian.

$$f_{T_1}(t) = f_{X_1}(x) \cdot \left|\frac{dX_1}{dT_1}\right| = \mathbb{1}_{[0,1]}(x) \cdot \left|-e^{-t}\right| = e^{-t}\mathbb{1}_{[0,\infty)}(t)$$

□

(b) We know that the sum of independent exponential random variables with parameter λ has a Gamma distribution. That is, for $n \geq 1$, set

$$Y_n = \sum_{j=1}^{n} T_j \text{ where } \{T_j\}_{j=1}^{\infty} \text{ are iid } T_1 \sim \text{Exp}(\lambda), \text{ then}$$

$$f_{Y_n}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!}\lambda e^{-\lambda t}\mathbb{1}_{[0,\infty)}(t)$$

Show that

$$f_{Z_n}(z) = \frac{1}{(n-1)!}(-\ln(z))^{n-1}\mathbb{1}_{[0,1]}(z)$$

Proof. Before any proof, note that some simple computations will yield

$$Y_n = T_1 + \ldots + T_n = -\ln X_1 - \ldots - \ln X_n = \ln X_1^{-1} + \ldots + \ln X_n^{-1} = \ln(X_1 \cdot \cdots \cdot \ldots \cdot \cdots X_n)^{-1} =$$

$$= \ln \frac{1}{Z_n} = -\ln Z_n, \longrightarrow Y_n = -\ln Z_n, \left|\frac{dY_n}{dZ_n}\right| = \frac{1}{Z_n}, \text{ and } t = -\ln z$$

Again, to derive a pdf for Z_n, we will use the pdf of Y_n and use the Jacobian to correct.

$$f_{Z_n}(z) = f_{Y_n}(t) \cdot \left|\frac{dY_n}{dZ_n}\right| = \frac{t^{n-1}}{(n-1)!}e^{-t} \cdot \frac{1}{z}\mathbb{1}_{[0,\infty)}(t), \text{ and plugging in } t = -\ln z \text{ gives}$$

$$f_{Z_n}(z) = \frac{(-\ln z)^{n-1}}{(n-1)!} \cdot e^{\ln z} \cdot \frac{1}{z}\mathbb{1}_{[0,1]}(z) = \frac{(-\ln z)^{n-1}}{(n-1)!}\mathbb{1}_{[0,1]}(z)$$

□