

# MTH 312 HW 7

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4 March 2025

**7.5.4.** Show that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $\int_a^x f = 0$  for all  $x \in [a, b]$ , then  $f(x) = 0$  everywhere on  $[a, b]$ . Provide an example to show this conclusion does not follow if  $f$  is not continuous.

*Proof.* Let  $F(x) = \int_a^x f$ . Since  $F(x) = 0$  everywhere on  $[a, b]$ ,  $F$  must be differentiable on  $[a, b]$  with  $F'(x) = 0$  on for  $x \in [a, b]$ . Note also that  $f$  is continuous, and by the Fundamental Theorem of Calculus, we have that  $F'(x) = f(x)$  for all  $x \in [a, b]$ . Thus  $f(x) = 0$  for all  $x \in [a, b]$ .

For a counter example, consider  $f : [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Then we have that  $\int_0^x f = 0$  for all  $x \in [0, 1]$ , but it is not the case that  $f(x) = 0$  for all  $x \in [0, 1]$ . □

**7.6.2.** Define

$$h(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin C \end{cases}$$

- (a) Show  $h$  has discontinuities at each point of  $C$  and is continuous at every point in the complement of  $C$ . Thus  $h$  is not continuous on an uncountably infinite set.

*Proof.* Suppose that  $x \notin C$ . Since  $C$  is closed, the complement of  $C$  is open, and there exists some  $\delta > 0$  such that  $(x - \delta, x + \delta) \subseteq C^c$ . Thus  $h$  is zero on the interval  $(x - \delta, x + \delta)$ , and it follows that  $h$  is continuous at  $x$ . Now suppose  $x \in C$ . To show  $h$  is not continuous at  $x$ , it is sufficient to show:

$$\text{for any } \delta > 0, \text{ there exists } y \in (x - \delta, x + \delta) \text{ such that } y \notin C \quad (1)$$

However, if there exists  $\delta$  which does not satisfy equation (1), then  $C$  contains a proper interval, which is a contradiction since it is totally disconnected (shown in Exercise 3.4.8). Thus  $h$  is not continuous at  $x$ .  $\square$

- (b) Now prove that  $h$  is integrable on  $[0, 1]$ .

DISCLAIMER: Exercise 7.3.9 (d) makes this problem trivial, so I refrained from using it.

*Proof.* We note that since  $C$  has content zero,  $D = C \cap [0, 1]$  also has content zero, and by Exercise 7.3.9 (a)  $h$  is integrable. Let  $P$  be a partition of  $[0, 1]$ . It follows that any subinterval  $[x_{k-1}, x_k]$  of the partition  $P$  contains some  $x \notin C$ , thus  $h(x) = 0$  and  $L(h, P) = 0$ . Since  $P$  was arbitrary, it follows that  $\int_0^1 h = L(h) = 0$ .  $\square$

**7.6.3.** Show that any countable set has measure zero.

*Proof.* Let  $A \subseteq \mathbb{R}$  be a countable set, and  $\epsilon > 0$  be given. Choose  $n \in \mathbb{N}$  such that  $2^{-N} < \epsilon$ . For each  $n \in \mathbb{N}$ , let

$$O_n = \left( a_n - \frac{\epsilon}{2^{N+n+1}}, a_n + \frac{\epsilon}{2^{N+n+1}} \right)$$

Then  $A \subseteq \bigcup_{n=1}^{\infty} O_n$ , and  $|O_n| = 2^{-N-n}$ . Then

$$\sum_{n=1}^{\infty} |O_n| = \sum_{n=1}^{\infty} 2^{-N-n} = 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \epsilon$$

Thus  $A$  has measure zero. □