MTH 463 HW 1

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1. A deck of only six cards $\{1, 2, 3, 4, 5, 6\}$ is shuffled so that each of the 6! orderings has equal probability of $\frac{1}{6!}$. Let A be the event that the card 1 is among the top three cards in the deck, and let B be the event that card 5 is the second card from the top.

   (a) Find $P(A)$ and $P(B)$.
   Solution.

$$P(A) = 3 \cdot \frac{5!}{6!} = \frac{1}{2}$$

$$P(B) = \frac{1}{6}$$

   (b) Find $P(A \cap B)$.
   Solution.

$$P(A \cap B) = \frac{2 \cdot 4!}{6!} = \frac{1}{15}$$

   (c) Find $P(A \cup B)$.
   Solution.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{6} - \frac{1}{15} = \frac{3}{5}$$

2. Consider events E, F, G. Find the expressions in terms of these sets of the following events. In your answers, use E^c, F^c, G^c if needed to denote the corresponding complementary events.

   (a) At least one of the events E, F G occurs.
   Solution.
   $P(E \cup F \cup G)$

   (b) Exactly one of E, F, G occurs.
   Solution.
   $P(E) + P(F) + P(G) - 2P(E \cap F) - 2P(E \cap G) - 2P(F \cap G) + 3P(E \cap F \cap G)$

   (c) At most one of the events E, F, G occurs.
   Solution.
   At most one is equivalent to saying the opposite of at least 2.
   $1 - (P(E \cap F) + P(E \cap G) + P(F \cap G) - 2P(E \cap F \cap G))$

   (d) None of E, F, G occur.
   Solution.
   $1 - P(E \cup F \cup G)$

   (e) Exactly two of E, F, G occur.
   Solution.
   $P(E \cap F) + P(E \cap G) + P(F \cap G) - 3P(E \cap F \cap G)$

3. Show that the probability that *exactly* one of two events A or B occurs is

$$P(A) + P(B) - 2P(A \cap B)$$

Proof. We know that the probability of A and/or B occurring is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If we can remove the case where A and B occur, we will be left with the probability of A xor B occurring. So

$$P(A \oplus B) = P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B).$$

$\square$

4. A fair die has three of its faces painted blue, two painted red, and the remaining painted green. The die is rolled 7 times. Let X denote the number of times that a blue face is rolled. Similarly let Y denote the number of times a red face is rolled, and G a green face. Find $P(X = 3, Y = 2, G = 2)$.

   Solution.

$$P(X = 3, Y = 2, G = 2) = \binom{7}{3, 2, 2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(\frac{1}{6}\right)^2 = \frac{7!}{3! \cdot 2! \cdot 2!} \cdot \frac{1}{8} \cdot \frac{1}{9} \cdot \frac{1}{36} = \frac{210}{2592} \approx 0.081$$

5. Consider the polynomial $P(x_1, ..., x_n) = (x_1 + ... + x_n)^r$ and let Q denote the n^{th}-order partial derivative of P with respect to $x_1, ..., x_n$, that is,

$$Q = \frac{\partial^n P}{\partial x_1 ... \partial x_n}$$

For $r > n$ find the number of different monomials in Q.
Solution.
This problem is equivalent to creating as many distinct sets of length r using $x_1, ...x_n$. This can be translated into a "stars and bars" problem, which has the distinct occurrences O of

$$O = \binom{n + r - 1}{r}. \tag{1}$$

Next, we can evaluate Q to be

$$Q = \frac{\partial^n P}{\partial x_1 ... \partial x_n} =$$
$$\frac{\partial^n}{\partial x_1 ... \partial x_n} (x_1 + ... + x_n)^r =$$
$$r \cdot (r - 1) \cdot ... \cdot (r - n + 1) (x_1 + ... + x_n)^{r-n} =$$
$$\frac{r!}{(r - n)!} (x_1 + ... + x_n)^{r-n}.$$

Our goal now is to see how many distinct monomials result from Q being expanded out, which we can count by (1) to be

$$\binom{n + (r - n) - 1}{r - n} =$$
$$\binom{r - 1}{r - n}.$$