MTH 311 Lab 7

Brandyn Tucknott

15 November 2024

1. Theorem 3.2.14 on page 92 states the following.
   (i) The union of a finite collection of closed sets in closed.

   (ii) The intersection of an arbitrary selection of closed sets is closed.

   (a) Prove that the union of a finite collection of compact sets is compact.

   Proof. By the characterization of compactness in $\mathbb{R}$, a set K is compact if and only if it is closed and bounded. We can equivalently show that the union of a finite collection of closed and bounded sets is also closed and bounded. By Theorem 3.2.14, we have that the union of a finite collection of closed sets is closed, and trivially we have that the union of a finite collection of bounded sets is bounded. We conclude then, that the union of a finite collection of compact sets is also compact. $\square$

   (b) Use an example to show that the union of an arbitrary selection of compact sets is not necessarily compact.

   Proof. Consider $S_n = [\frac{1}{n}, 1]$ for all $n \in \mathbb{N}$. Then $S = \bigcup_{n \in \mathbb{N}} S_n = (0, 1]$. Although S is bounded, it is not closed since the limit point 0 is not contained in S. Although each individual S_n is compact, since S is not both bounded and closed, S is not compact. $\square$

2. (a) Find a real number M such that $\left|x^2-1\right| \leq M\,|x-1|$ for all $x \in [0,2]$

   Solution.

   Since $|a \cdot b| = |a| \cdot |b|$ for $a, b \in \mathbb{R}$, we can solve for M.

$$\left|x^2-1\right| \leq M\,|x-1| \longrightarrow |x-1| \cdot |x+1| \leq M\,|x-1| \longrightarrow |x+1| \leq M \longrightarrow$$

$$M \geq x+1 \text{ since } x \in [0,2]$$

   To maximize this value, we choose $x = 2$ which yields $M = 3$.

   (b) Use the corresponding $\epsilon - \delta$ definition to prove that $\lim_{x\to 1} x^2 = 1$. In this example, for a given $\epsilon > 0$, the corresponding $\delta > 0$ is the minimum of two quantities.

   Proof. Given $\epsilon > 0$, choose $\delta = \min(1, \frac{\epsilon}{3})$. Suppose $0 < |x-1| < \delta$. Then we know that $|x-1| < 1$, and we observe that

$$|x+1| = |x-1+2| \leq |x-1| + |2| = |x-1| + 2 < 1 + 2 = 3$$

   From here, we check that $\left|x^2-1\right| < \epsilon$.

$$\left|x^2-1\right| = |x+1|\,|x-1| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

   Since for any given $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-1| < \delta$ then $\left|x^2-1\right| < \epsilon$, we conclude that $\lim_{x\to 1} x^2 = 1$. $\square$