MTH 464 HW 1

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1. A fair coin is tossed. If the coin results in a head, Die 1 with faces 1 through 6 is rolled and the up face is recorded. If the coin results in a tails, Die 2 with one face with value 1, 2 faces with value 2, and three faces with value 3 is rolled and the up face recorded. The dice are such that each face can be a top face with probability $\frac{1}{6}$. The process is repeated, with X_j denoting the value of the rolled die's top face in the j^{th} iteration.

   (a) Find $P(X_1 = 3)$
   Solution.

   $$P(X_1 = 3) = P(3 \text{ on Die } 1)P(\text{Die } 1) + P(3 \text{ on Die } 2)P(\text{Die } 2) = \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3}$$

   (b) Let Y denote the number of iterations needed until the first time the value of the rolled die's top face is 3. Find $P(Y = k)$.
   Solution.
   Note that X is exchangeable, i.e. $P(X_j = k) = P(X_1 = k)$. Using this observation allows us to compute the following:

   $$P(Y = k) = P(3 \text{ on } k^{th} \text{ roll}) + P(\text{no 3 on all } k-1 \text{ rolls}) =$$

   $$P(3 \text{ on the first roll}) + (1 - P(3 \text{ on the first roll}))^{k-1} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}$$

2. A bank classifies customers as having good or bad credit risks. Based on historical data, the bank observes that 1% of customers with good credit record and 10% of customers with bad credit record overdraft in their accounts in a given month. That is, with O denoting the event of overdraft in a month, and G, B denoting the events of good or bad credit record respectively, we have that $P(O|G) = 0.01, P(O|B) = 0.1$. A new customer, which the bank assigns a 70% chance of having a good credit risks, overdrafts in the first month. Find $P(G|O)$.
Solution.

$$P(O) = P(O|G)P(G) + P(O|B)P(B) = (0.01)(0.7) + (0.1)(0.3) = 0.007 + 0.03 = 0.037$$
$$P(G|O) = \frac{P(O|G)P(G)}{P(O)} = \frac{(0.01)(0.7)}{0.037} = 0.189$$

3. A random walker starts at 0 on the $x-$axis and at each time unit steps 1 unit to the left or right, each with probability $\frac{1}{2}$. Using the normal approximation to a binomial, estimate the probability that after 100 steps, the walker is more than 10 units away from his starting position. Express your answer as an integral of the pdf of a normal random variable over the appropriate interval and use the table of values provided to approximate its value.

<u>*Solution.*</u>

Define $X \sim \text{Bern}\left(\frac{1}{2}\right)$, and $Y = 2X - 1$. Observe that when $X = 0, Y = -1$, and when $X = 1, Y = 1$. Then our random walk is equivalent to $S = \sum_{j=1}^{100} Y_j$. We now calculate the mean and variance in order to use the DeMoivre-Laplace Central Limit Theorem.

$$E(Y) = E(2X - 1) = 2(\frac{1}{2}) - 1 = 0$$

$$\text{Var}(Y) = \text{Var}(2X - 1) = 4(\frac{1}{4}) = 1$$

Using these, we can find the mean and variance of our random walk where $n = 100$ steps:

$$\mu = nE(Y) = 100 \cdot 0 = 0$$

$$\sigma^2 = n\text{Var}(Y) = 100 \cdot 1 = 100$$

Then by DeMoivre-Laplace, $S \sim N(0, 100)$. We wish to find the probability $P(|S| > 10)$. Before we normalize the distribution, we need the z-score of S. This is easily calculated to be $Z = \frac{S-\mu}{\sigma} = \frac{S}{10}$

$$P(|S| > 10) = P\left(\left|\frac{S}{10}\right| > 1\right) = P(|Z| > 1) = 2P(Z > 1) = 2\left(1 - P(Z \leq 1)\right)$$

For continuity correction, we add $\frac{0.5}{10} = 0.05$ to the RHS of the inequality, giving us

$$P(|S| > 10) = 2P(1 - P(|Z| \leq 1.05))$$

This is easily evaluated to

$$2(1 - P(|Z| \leq 1.05)) = 2P(1 - 0.8531) = 0.2938$$

using a standard normal table, and we conclude that the probability of ending more than 10 units away after the random walk is 0.2938.

4. For $(x, y) \in \mathbb{R}^2$, define

$$f(x,y) = \begin{cases} \frac{1}{x}, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that f is a probability density function.
<u>*Solution.*</u>
We must verify that $\iint_D f(x,y)dA = 1$.

$$\iint_D f(x,y)dA = \int_0^1 \int_0^x \frac{1}{x} dydx = \int_0^1 1dx = 1$$

(b) Assume that f is the joint probability density function for random variables X, Y. Find the marginal densities $f_X(x)$ and $f_Y(y)$.
<u>*Solution.*</u>

$$f_X(x) = \int_0^x \frac{1}{x} dy = 1 \cdot \mathbb{1}_{(0,1)}(x)$$

$$f_Y(y) = \int_y^1 \frac{1}{x} dx = -\ln y \cdot \mathbb{1}_{(0,1)}(y)$$

(c) Find $E(X)$ and $E(Y)$.
<u>*Solution.*</u>

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x dx = \frac{1}{2}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 -y \ln y dy = \frac{1}{4}$$

(d) Are X, Y independent? Justify your answer.
<u>*Solution.*</u>
X, Y are not independent since $\frac{1}{x} \neq 1 \cdot -\ln y \longrightarrow f(x,y) \neq f_X(x) f_Y(y)$

(e) What is the conditional density of Y given $X = x$?
<u>*Solution.*</u>

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{x}}{1} = \frac{1}{x} \cdot \mathbb{1}_{(0,x)}(y)$$

5. Assume X is an exponential random variable with parameter 1 and that Y is an exponential random variable with parameter 2. Assume further that X, Y are independent.
   (a) Find $P(X > Y)$.
   Solution.
   Our domain D is such that $0 < y < x < \infty$, and we recognize that the joint pdf of X, Y is

   $f_{X,Y}(x, y) = f_X(x)f_Y(y) = e^{-x}2e^{-2y}$. To find $P(X > Y)$, we integrate the joint pdf over the domain

   $$\iint_D f_{X,Y}(x,y)dA = \int_0^\infty \int_0^x e^{-x}2e^{-2y}dydx = \int_0^\infty e^{-x}\left(1 - e^{-2x}\right)dx = \int_0^\infty e^{-x}dx - \int_0^\infty e^{-3x}dx = \frac{2}{3}$$

   We conclude that the probability $P(X > Y) = \frac{2}{3}$
   (b) Determine a such that $P(X > aY) = \frac{1}{2}$
   Solution.
   Define our domain D is such that $0 < ay < x < \infty$, and note the joint pdf of X, Y is

   $f_{X,Y}(x, y) = f_X(x)f_Y(y) = e^{-x}2e^{-2y}$. To find $P(X > aY)$, we integrate the joint pdf over the domain

   $$\iint_D f_{X,Y}(x,y)dA = \int_0^\infty \int_0^{\frac{x}{a}} e^{-x}2e^{-2y}dydx = \int_0^\infty e^{-x}\left(1 - e^{-\frac{2x}{a}}\right)dx = \int_0^\infty e^{-x}dx - \int_0^\infty e^{-\left(\frac{2}{a}+1\right)x}dx = 1 - \frac{1}{\frac{2}{a}+1}$$

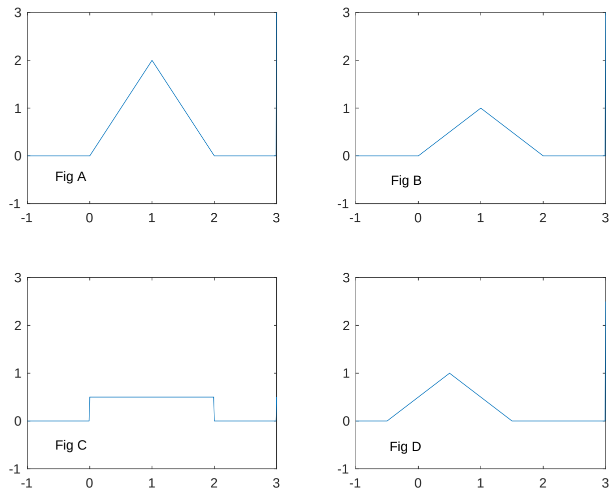
   We now set the equation to $\frac{1}{2}$ and solve for a.

   $$P(X > aY) = 1 - \frac{1}{\frac{2}{a}+1} = \frac{1}{2}$$

   $$\frac{1}{\frac{2}{a}+1} = \frac{1}{2} \longrightarrow a = 2$$

   We conclude that $P(X > aY) = \frac{1}{2}$ when $a = 2$.

6. Assume U, V are independent random variables uniformly distributed on the interval $[0, 1]$. Let $W = U + V$ and denote by $f_W(w)$ its density. Identify the correct graph for $f_W(w)$ and give a brief explanation for what is incorrect about the others.



<u>*Solution.*</u>
Figure B is the correct answer.

Figure A is wrong because the area of the function integrates to 2.
Figure C is wrong because it labels all possibilities as equally likely, which is not the case.
Figure D is wrong because it labels values below 0 as possible, which should not be the case.

7. Assume X is a random variable with differentiable pdf $f_X(x)$. Define the **mediam** of X as the value ν such that $P(X \leq \nu) = P(X \geq \nu) = \frac{1}{2}$. Show that the minimum of $g(a) = E\left(|X - a|\right)$ occurs at $a = \nu$.

Proof. We wish to show that $\frac{d}{da}g(a) = 0$

$$\frac{d}{da}g(a) = E\left(|X-a|\right) = \frac{d}{da}\int_{-\infty}^{\infty} |x-a|\, f_X(x)dx = \frac{d}{da}\left(\int_{-\infty}^{a} (a-x)f_X(x)dx + \int_{a}^{\infty} (x-a)f_X(x)dx\right) =$$

$$= \frac{d}{da}\left(\int_{-\infty}^{a} af_X(x)dx - \int_{-\infty}^{a} xf_X(x)dx + \int_{a}^{\infty} xf_X(x)dx - \int_{a}^{\infty} af_X(x)dx\right) =$$

$$= \frac{d}{da}\int_{-\infty}^{a} af_X(x)dx - \frac{d}{da}\int_{-\infty}^{a} xf_X(x)dx + \frac{d}{da}\int_{a}^{\infty} xf_X(x)dx - \frac{d}{da}\int_{a}^{\infty} af_X(x)dx =$$

$$= \frac{d}{da}a\int_{-\infty}^{a} f_X(x)dx - \frac{d}{da}\int_{-\infty}^{a} xf_X(x)dx + \frac{d}{da}\int_{a}^{\infty} xf_X(x)dx - \frac{d}{da}a\int_{a}^{\infty} f_X(x)dx =$$

$$= \left(\int_{-\infty}^{a} f_X(x)dx + a\frac{d}{da}\int_{-\infty}^{a} f_X(x)dx\right) - \frac{d}{da}\int_{-\infty}^{a} xf_X(x)dx + \frac{d}{da}\int_{a}^{\infty} xf_X(x)dx - \left(\int_{a}^{\infty} f_X(x)dx + a\frac{d}{da}\int_{a}^{\infty} f_X(x)dx\right) =$$

$$= F_X(a) + a\left(f_X(x)\right)\Big|_{-\infty}^{a} - \left(xf_X(x)\right)\Big|_{-\infty}^{a} + \left(xf_X(x)\right)\Big|_{a}^{\infty} - (1 - F(a)) - a\left(f_X(x)\right)\Big|_{a}^{\infty} =$$

$$= 2F_X(a) - 1 + af_X(a) - af_X(a) - af_X(a) + af_X(a) = 2F_X(a) - 1 = 0$$

Then $F_X(a) = \frac{1}{2}$, and $g(a)$ has a critical point at $a = \nu$. It remains to be shown that $a = \nu$ is minimal.

$$\frac{d}{da}g'(a) = \frac{d}{da}\left(2F_X(a) - 1\right) = 2f_X(a) > 0 \text{ for all } a \in \mathbb{R} \text{ by definition of a pdf}$$

Notice that for $\epsilon > 0$ and very small, $g'(\nu - \epsilon) = 2F_X(\nu - \epsilon) - 1 < 2F_X(\nu) - 1 = 0$. Thus $g'(\nu - \epsilon) < 0$. Observe also that $g'(\nu + \epsilon) = 2F_X(\nu + \epsilon) - 1 > 2F_X(\nu) - 1 = 0$. So $g'(\nu + \epsilon) > 0$.

Using both of these facts, we conclude that since $g'(\nu - \epsilon) < 0$ and $g'(\nu + \epsilon) > 0$ with $g''(\nu) > 0$, the lone critical point $a = \nu$ must be a minimum. □