

# MTH 311 Lab 6

Brandyn Tucknott

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**Theorem.** A set  $F \subset \mathbb{R}$  is a closed set  $\iff$  for every convergent sequence  $(a_n) \subset F$ ,  $\lim_{n \rightarrow \infty} a_n \in F$

1. Use the above theorem to prove that the following sets are not closed. In each case, this involves finding a suitable sequence  $(a_n)$ .

(a)  $A = \{x \in \mathbb{R} : 1 < x < 4\}$

*Proof.* Consider the sequence  $a_n = \frac{n+1}{n}$ . Then certainly for all  $n \in \mathbb{N}$ ,  $1 < \frac{n+1}{n} < 4$ . Additionally, the  $\lim_{n \rightarrow \infty} a_n = 1$ , which is not an element of  $A$ . We conclude that  $A$  is not closed.  $\square$

(b)  $B = \{\frac{1}{n} : n \in \mathbb{N}\}$

*Proof.* Consider the sequence  $a_n = \frac{1}{n}$ . Then for all  $n \in \mathbb{N}$ ,  $a_n \in B$  by definition of  $B$ , but the  $\lim_{n \rightarrow \infty} a_n = 0$ . Since 0 is not contained in  $B$ , we conclude that  $B$  is not closed.  $\square$

2. (a) Prove the forward direction of the theorem.

*Proof.* Given the set  $F \subset \mathbb{R}$  is a closed set, we want to show that every convergent sequence  $(a_n) \subset F$  has  $\lim_{n \rightarrow \infty} a_n \in F$ . Let  $(a_n)$  be an arbitrary convergent sequence in  $F$  and let  $x = \lim_{n \rightarrow \infty} a_n$ . We want to show that  $x \in F$ . We break this proof into two cases:

(i) There exists  $n \in \mathbb{N}$  such that  $a_n = x$ .

(ii) For all  $n \in \mathbb{N}$ ,  $a_n \neq x$ .

In case (i), we assume there exists an  $n \in \mathbb{N}$  such that  $a_n = x$ . Since  $(a_n) \subset F$ , by definition we have that  $x = a_n \in F$  for some  $n \in \mathbb{N}$ .

In case (ii), we assume that for all  $n \in \mathbb{N}$ ,  $a_n \neq x$ . Then  $x$  is definitionally a limit point of  $F$ . Since we are given  $F$  is closed, we conclude that  $x \in F$  by definition of a closed set (3.2.7).  $\square$

- (b) Prove the reverse direction of the theorem.

*Proof.* We wish to show that for all convergent sequences  $(a_n) \subset F$ , if  $x = \lim_{n \rightarrow \infty} a_n$  with  $x \in F$ , then  $F$  is a closed set. We do this with a proof by contrapositive, and instead we aim to show that if  $F$  is an open set, then for all convergent sequences  $(a_n) \subset F$ ,  $\lim_{n \rightarrow \infty} a_n \notin F$ .

Let  $F$  be an open set. Then by definition, there exists some limit point  $x \notin F$ . We aim to show there is a convergent sequence  $a_n \subset F$  that converges to  $x$ . Consider  $V_\epsilon(x) = (x - \frac{1}{\epsilon}, x + \frac{1}{\epsilon})$ . Clearly if  $\epsilon > 0$ , then  $V_\epsilon(x)$  is non-empty (we consider non-empty since  $x \notin F$ ), so by definition 3.2.4 we know that  $x$  is a limit point of  $F$ . Since  $x$  is a limit point of  $F$ , by theorem 3.2.5  $x = \lim_{n \rightarrow \infty} a_n$  for some sequence  $(a_n) \subset F$  where  $a_n \neq x$  for all  $n \in \mathbb{N}$ . With this we have shown there exists a sequence which converges to  $x \notin F$ , so we are done.  $\square$