MTH 464 HW 3

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1. Let $X_1, \dots, X_5 \sim \text{Unif}[0,1]$ be iid distributed. Let $X_{(1)}, \dots, X_{(5)}$ be its ordered values. The median of this sample of 5 random variables can be taken to be $X_{(3)}$. Find $P\left(\frac{1}{4} < X_{(3)} < \frac{3}{4}\right)$.

   <u>*Solution.*</u>
   Our approach will be to find the pdf for $X_{(3)}$, and then integrate over the appropriate bounds. To find the pdf, we use the formula

   $$f_{X_{(k)}} = \frac{n!}{(k-1)!(n-k)!} \left(F(x)\right)^{k-1} \left(1 - F(x)\right)^{n-k} f(x) \tag{1}$$

   Where f, F are the pdf and cdf for the distribution of X_k, which in this case will be $\text{Unif}[0,1]$. Using equation (1) evaluated at $n = 5, k = 3$, we get

   $$f_{X_{(3)}} = 30x^2(1-x)^2$$

   To find $P(\frac{1}{4} < X_{(3) < \frac{3}{4}})$ can be found using the cdf for $X_{(3)}$, found by integrating over its pdf.

   $$P(\frac{1}{4} < X_{(3)} < \frac{3}{4}) = \int_{\frac{1}{4}}^{\frac{3}{4}} 30x^2(1-x)^2 dx \approx 0.793$$

2. Let $X_{(1)}, \ldots, X_{(n)}$ be the ordered values of n iid random variables uniformly distributed on $[0,1]$. Define $X_{(0)} = 0$, $X_{n+1} = 1$. Show that for any $1 \leq k \leq n$

$$P\left(X_{(k+1)} - X_{(k)} > t\right) = (1-t)^n$$

Proof. Notice that $P\left(X_{(k+1)} - X_{(k)} > t\right) = P\left(X_{(k+1)} > t + X_{(k)}\right)$. Another way to view this is: for an arbitrary fixed $x_{(k)}$, the remaining points must fall outside of the interval $[x_{(k)}, X_{(k)} + t]$. This interval is of length t, and since $X_{(i)}$ is uniformly distributed along the interval $[0,1]$, the probability any $x_{(i \neq k)}$ falls outside of $[x_{(k)}, x_{(k)} + t] = 1 - t$. Since there are n independent points in which this must be true, we conclude

$$P\left(X_{(k+1)} - X_{(k)} > t\right) = (1-t)^n$$

$\square$

3. Let Z_1, Z_2 be independent standard normal random variables and define $X = Z_1, Y = Z_1 + Z_2$.

   (a) Find the joint density of (X, Y).
   <u>*Solution.*</u>
   Since Z_1, Z_2 are independent, we know the joint pdf to be

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} \tag{2}$$

   We compute the necessary Jacobian to be

$$J = \begin{pmatrix} \frac{\partial X}{\partial Z_1} & \frac{\partial X}{\partial Z_2} \\ \\ \frac{\partial Y}{\partial Z_1} & \frac{\partial Y}{\partial Z_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \longrightarrow \det J = 1$$

$$\left| \frac{1}{\det J} \right| = 1 \tag{3}$$

   Using a change of variables as well as Equations (2) and (3), we can find the pdf of X, Y to be

$$f_{X,Y}(x, y) = f_{Z_1,Z_2}(z_1 = x, z_2 = y - x) \cdot \left| \frac{1}{\det J} \right| = \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+(y-x)^2\right)} = \frac{1}{2\pi} e^{-\frac{1}{2}\left(2x^2-2xy+y^2\right)}$$

   (b) Find $\mathbb{E}(X), \mathbb{E}(Y)$.
   <u>*Solution.*</u>

$$\mathbb{E}(X) = \mathbb{E}(Z_1) = 0$$
$$\mathbb{E}(Y) = \mathbb{E}(Z_1 + Z_2) = \mathbb{E}(Z_1) + \mathbb{E}(Z_2) = 0$$

   (c) Find the Variance-Covariance matrix of the bivariate normal (X, Y).
   <u>*Solution.*</u>
   Observe that since Z_1, Z_2 are independent

$$\text{Var}(X) = \text{Var}(Z_1) = 1$$

$$\text{Var}(Y) = \text{Var}(Z_1 + Z_2) = 2$$

$$\begin{aligned} \text{Cov}(X, Y) = \text{Cov}(Y, X) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(Z_1(Z_1 + Z_2)) + \mathbb{E}(Z_1)\mathbb{E}(Z_1 + Z_2) = \\ = \mathbb{E}\left(Z_1^2 + Z_1 Z_2\right) + 0 = \mathbb{E}\left(Z_1^2\right) + \mathbb{E}(Z_1 Z_2) = 1 + \mathbb{E}(Z_1)\mathbb{E}(Z_2) = 1 \end{aligned}$$

   We can then construct the Variance-Covariance Matrix to be

$$\Sigma^2 = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \\ \sigma_{yx} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

4. Let $Z_1, Z_2 \sim N(0,1)$ be iid random variables. Show that

$$X = \frac{1}{\sqrt{2}}\left(Z_1 + Z_2\right), Y = \frac{1}{\sqrt{2}}\left(Z_1 - Z_2\right)$$

are also independent, identically distributed $N(0,1)$ random variables.

Proof. First, we will show that X, Y have a standard normal distribution, and then we will show they are independent.

To show they are both standard normal, it is required that the expected value and variance X, Y is 0 and 1 respectively. Note that $\operatorname{Var}(Z_1) = \mathbb{E}\left(Z_1^2\right) - \mathbb{E}\left(Z_1\right)^2 = \mathbb{E}\left(Z_1^2\right) = 1$, similarly $\mathbb{E}\left(Z_2^2\right) = 1$.

$$\mathbb{E}(X) = \mathbb{E}\left(\frac{Z_1 + Z_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\mathbb{E}(Z_1) + \mathbb{E}(Z_2)\right) = 0$$
$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{Z_1 - Z_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\mathbb{E}(Z_1) - \mathbb{E}(Z_2)\right) = 0$$

$$\mathbb{E}\left(X^2\right) = \mathbb{E}\left(\frac{Z_1^2 + 2Z_1Z_2 + Z_2^2}{2}\right) = \frac{1}{2}\left(\mathbb{E}\left(Z_1^2\right) + 2\mathbb{E}(Z_1)\mathbb{E}(Z_2) + \mathbb{E}\left(Z_2^2\right)\right) = 1$$
$$\mathbb{E}\left(Y^2\right) = \mathbb{E}\left(\frac{Z_1^2 - 2Z_1Z_2 + Z_2^2}{2}\right) = \frac{1}{2}\left(\mathbb{E}\left(Z_1^2\right) - 2\mathbb{E}(Z_1)\mathbb{E}(Z_2) + \mathbb{E}\left(Z_2^2\right)\right) = 1$$

Therefore $\mu_X, \mu_Y = 0$ and $\sigma_X^2, \sigma_Y^2 = 1$, and we confirm that $X, Y \sim N(0,1)$.

Next, we claim they are independent. Note first the individual pdfs for X and Y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$$

Observe that $X + Y = \frac{2Z_1}{\sqrt{2}}, X - Y = \frac{2Z_2}{\sqrt{2}}$, and the Jacobian is

$$\left|\frac{1}{\det J}\right| = \left|\frac{1}{\det\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}}\right| = \left|\frac{1}{1}\right| = 1$$

$$f_{X,Y}(x,y) = f_{Z_1,Z_2}\left(z_1 = \frac{\sqrt{2}}{2}(x - +y), z_2 = \frac{\sqrt{2}}{2}(x-y)\right) \cdot \left|\frac{1}{\det J}\right| =$$
$$= \frac{1}{2\pi}e^{-\frac{1}{2}\left(\frac{\sqrt{2}}{2}(x+y)^2 + \frac{\sqrt{2}}{2}(x-y)^2\right)} \cdot 1 =$$
$$= \frac{1}{2\pi}e^{-\frac{1}{4}(x^2+2xy+y^2+x^2-2xy+y^2)} =$$
$$= \frac{1}{2\pi}e^{-\frac{1}{4}(2x^2+2y^2)} =$$
$$= \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} =$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} = f_X(x)f_Y(y)$$

Since $f_{X,Y} = f_X f_Y$, we conclude that $X, Y \sim \text{Unif}(0,1)$ are independent. $\square$

5. Let Z_1, Z_2 be independent standard normal random variables. Find an affine transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(X, Y) = T(Z_1, Z_2)$ is a bivariate normal random vector with the following properties:

$$\mathbb{E}(X) = 0, \mathbb{E}(Y) = 1, \mathrm{Var}(X) = 4, \mathrm{Var}(Y) = 1, \mathrm{Corr}(X, Y) = \frac{\sqrt{3}}{2}$$

Solution.
The affine transformation we seek is of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = M \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

We know $\mu_X = 0, \mu_Y = 1$, and rewrite the transformation as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = M \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We need to find M, where $\Sigma^2 = MM^T$. Since variance is given, we need to find covariance.

$$\rho = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} \longrightarrow \mathrm{Cov}(X, Y) = \sigma_X \sigma_Y \rho$$

Using this, we can derive the covariance to be

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X) = 2 \cdot 1 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

We can construct M as

$$M = \begin{pmatrix} \sigma_X & 0 \\ \sigma_Y \rho & \sigma_Y \sqrt{1 - \rho^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{\sqrt{3}}{2} & \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Finally, we define the affine transformation we seek as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2Z_1 \\ \frac{\sqrt{3}}{2} Z_1 + \frac{1}{2} Z_2 + 1 \end{pmatrix}$$