Volume of a N-Dimensional Ball

Brandyn Tucknott

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Theorem (Volume of an N-Ball) The volume of an n-dimensional ball with radius r can be expressed as follows:

$$(B^n(r)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right) \cdot \frac{n}{2}} r^n$$

Proof. Let ω_n be the surface characteristic for a $n-1$ dimensional sphere. Then we can express the volume of a n dimensional ball with radius r as

$$\begin{aligned}
\operatorname{Vol}(B^n(r)) &= \int_{B^n(r)} dr \\
&= \int_0^r \omega_n r^{n-1} dr \\
&= \omega_n \frac{r^n}{n}
\end{aligned}$$

Our new goal is to find an expression for the surface characteristic, which will complete our expression for the volume. Consider the multidimensional Gaussian, and observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\|\mathbf{x}\|_2^2} d\mathbf{x} &= \int_{-\infty}^{\infty} e^{-x_1^2 - \ldots - x_n^2} dx_1 \ldots dx_n \\
&= \underbrace{\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \ldots \int_{-\infty}^{\infty} e^{-x_n^2} dx_n}_{n \text{ times}} \\
&= \left(\sqrt{\pi}\right)^n = \pi^{n/2}
\end{aligned}$$

However, since it is a radial function, we can also evaluate the multidimensional Gaussian as

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\|\mathbf{x}\|_2^2} d\mathbf{x} &= \omega_n \int_0^{\infty} e^{-r^2} r^{n-1} dr \\
&= \frac{\omega_n}{2} \int_0^{\infty} e^{-r^2} r^{n-2} \cdot 2r dr \text{ set } t = r^2 \to dt = 2r \\
&= \frac{\omega_n}{2} \int_0^{\infty} e^{-t} t^{(n-2)/2} dt \\
&= \frac{\omega_n}{2} \int_0^{\infty} e^{-t} t^{n/2-1} dt \\
&= \frac{\omega_n}{2} \Gamma(n/2)
\end{aligned}$$

Setting the two results equal to each other, we can arive at a value for ω_n.

$$\begin{aligned}
\frac{\omega_n}{2} \Gamma(n/2) &= \pi^{n/2} \\
\omega_n &= \frac{2\pi^{n/2}}{\Gamma(n/2)}
\end{aligned}$$

Finally, we substitute this into our original volume expression, and conclude that

$$\begin{aligned}
\mathrm{Vol}(B^n(r)) &= \omega_n \frac{r^n}{n} \\
&= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{r^n}{n} \\
&= \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right) \frac{n}{2}} r^n
\end{aligned}$$

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