MTH 464 Final

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1. The Strong Law of Large Numbers is a statement about the successive arithmetic averages of a sequence of iid random variables. Using the continuity of the exponential and logarithmic functions, find the almost sure limit of the successive geometric averages of iid non-negative random variables. That is assume that $\{X_j\}_{j=1}^{\infty}$ are iid non-negative random variables and assume that $\mathbb{E}\left(\ln(X)\right) = \rho$. Find

$$\lim_{n\to\infty}\left[\prod_{j=1}^{n} X_j\right]^{\frac{1}{n}}$$

Proof. Let $P_n = \prod_{j=1}^{n} X_j$. Observe that

$$\begin{aligned}
\ln\left(P_n\right)^{\frac{1}{n}} &= \ln\left(\prod_{j=1}^{n} X_j\right)^{\frac{1}{n}} \\
&= \frac{1}{n}\ln\left(\prod_{j=1}^{n} X_j\right) \\
&= \frac{1}{n}\sum_{j=1}^{n}\ln(X_j) \\
&= \mathbb{E}\left(\ln(X)\right) \\
&= \rho
\end{aligned}$$

Thus we can evaluate the limit as follows.

$$\begin{aligned}
\lim_{n\to\infty}\left[\prod_{j=1}^{n} X_j\right]^{\frac{1}{n}} &= \lim_{n\to\infty}\left[P_n\right]^{\frac{1}{n}} \\
&= \lim_{n\to\infty} e^{\ln(P_n)^{\frac{1}{n}}} \\
&= \lim_{n\to\infty} e^{\rho} \\
&= e^{\rho}
\end{aligned}$$

$\square$

2. Recall that if $Z = N(\mu, \sigma), X = e^Z$ is called a lognormal random variable with parameters μ, σ. In this problem, let Z be a standard normal. We know that the probability density function of X vanishes for $x \leq 0$, and for $x > 0$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{1}{2}(\log(x))^2\right)$$

(a) Evaluate $\mu_n = \mathbb{E}(X^n)$, the n^{th} moment of X.

Proof.

$$\begin{aligned}\mu_n &= \mathbb{E}(X^n) \\ &= \mathbb{E}\left(\left[e^Z\right]^n\right) \\ &= \mathbb{E}\left(e^{nZ}\right) = \text{ MGF(Z) at } t = n \\ &= e^{n\mu + n^2\sigma^2/2} \\ &= e^{n^2/2}\end{aligned}$$

since Z is standard normal. □

(b) Show that the power series

$$\sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}$$

is only defined at $t = 0$.

Proof. Recall the ratio test checks if a series $\sum a_n$ converges, and if L is defined to be

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

our series converges absolutely if $L < 1$. Conditional convergence is not needed for this problem, so we will ignore it. First we evaluate $\frac{a_{n+1}}{a_n}$.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\mu_{n+1}t^{n+1}/(n+1)!}{\mu_n t^n/n!} \\ &= \frac{\exp\left(\mu(n+1) + (n+1)^2\sigma^2/2\right) t^{n+1}/(n+1)!}{\exp\left(\mu n + \sigma^2 n^2/2\right)/n!} \\ &= \frac{t^{n+1}\exp\left(\mu + \sigma^2(2n+1)/2\right)\exp\left(\mu n + \sigma^2 n^2/2\right)/\left((n+1)n!\right)}{t^n \exp\left(\mu n + \sigma^2 n^2/2\right)/n!} \\ &= \frac{\exp\left(\mu + \sigma^2(2n+1)/2\right) t}{n+1} \\ &= \frac{e^{(2n+1)/2}}{n+1}\end{aligned}$$

since Z is standard normal. Thus we can evaluate the limit L to be

$$\begin{aligned}L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{te^{(2n+1)/2}}{n+1}\right|\end{aligned}$$

Since $\lim_{n\to\infty} \frac{e^n}{n}$ diverges, L clearly diverges unless $t = 0$, in which case $L = 0$, and $\sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}$ converges absolutely by the ratio test. □

(c) For $-1 \leq a \leq 1$ let

$$f_a(x) = f(x)\left[1 + a\sin(2\pi\log(x))\right]$$

and let $\mu_n^{(a)}$ denote the moments of this density. Show that $\mu_n^{(a)} = \mu_n$ for all $|a| \leq 1$.

Proof. Observe that

$$\begin{aligned}
\mu_n^{(a)} &= \int_0^\infty x^n f_a(x)dx \\
&= \int_0^\infty x^n f(x)dx + \int_0^\infty x^n f(x) a\sin\left(2\pi\log x\right)dx \\
&= \mu_n + a\int_0^\infty x^n f(x)\sin\left(2\pi\log x\right)dx
\end{aligned}$$

Thus we are done if we can show that $a\int_0^\infty x^n f(x)\sin\left(2\pi\log x\right) = 0$. First, observe that we can perform a change of variables

$$t = \ln x \leftrightarrow x = e^t, dt = \frac{1}{x}dx$$

Then we can rewrite the integral as

$$\begin{aligned}
a\int_0^\infty x^n f(x)\sin(2\pi\ln x) &= a\int_0^\infty x^n \frac{1}{\sqrt{2\pi}x}e^{-1/2(\ln x)^2}\sin(2\pi\ln x)dx \\
&= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{nt}\cdot e^{-t^2/2}\sin(2\pi t)dt \\
&= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{nt-t^2/2}\sin(2\pi t)dt
\end{aligned}$$

We can refocus our attention on just the following integral:

$$\int_{-\infty}^\infty e^{nt-t^2/2}\sin(2\pi t)dt.$$

Let $g(t) = e^{nt-t^2/2}\sin(2\pi t)$. A hope would be that $g(t)$ is odd about some $t = k$, in which case the integral over the reals would evaluate to 0. With this new goal, we will show that given $n, g(t)$ is odd about $t = k = n$. That is,

$$g(n+t) = -g(n-t)$$

$$\begin{aligned}
-g(n-t) &= -\sin(2\pi(n-t))e^{n(n-t)-(n-t)^2/2} \\
&= \sin(2\pi(n+t))e^{n^2-nt-(n^2-2nt+t^2)/2} \\
&= \sin\left(2\pi(n+t)\right)e^{n^2-nt-n^2/2+nt-t^2/2} \\
&= \sin\left(2\pi(n+t)\right)e^{n^2+nt-n^2/2-nt-t^2/2} \\
&= \sin\left(2\pi(n+t)\right)e^{n(n+t)-(n^2+2nt+t^2)/2} \\
&= \sin\left(2\pi(n+t)\right)e^{n(n+t)-(n+t)^2/2} \\
&= g(n+t)
\end{aligned}$$

Thus given $n, g(t)$ is symmetric about $t = n$, and $\int_{-\infty}^\infty e^{nt-t^2/2}\sin(2\pi t)dt = 0$, allowing us to conclude that $\mu_n^{(a)} = \mu_n$. □

3. Let Y be a random variable that represents the value of a random number of donations to a foundation. A reasonable model is

$$Y = \sum_{j=1}^{N} R_j$$

where N denotes the number of donations, R_j the amount of the j^{th} donation. We assume $N \sim$ Geometric(p) and $\{R_j\}_{j=1}^{\infty}$ are iid lognormal random variables with parameter μ, σ for $j \geq 0$. That is,

$$R_j = \exp(\mu + \sigma Z_j) \text{ where } \{Z_j\}_{j=1}^{\infty} \sim N(0,1) \text{ are iid}$$

We further assume that N and $\{Z_j\}_{j=1}^{\infty}$ are independent.

(a) Find $\mathbb{E}(Y) = \mu_Y$ and $\mathrm{Var}(Y) = \sigma_Y^2$.

Proof. Note that if $R_j \sim$ lognormal(μ, σ^2), then $R_j = e^X$ where $X \sim N(\mu, \sigma)$. Note that

$$\begin{aligned}
\mathbb{E}(R_j) &= \mathbb{E}\left(e^X\right) \\
&= e^{\mu + \sigma^2/2} \\
\mathrm{Var}(R_j) &= \mathrm{Var}\left(e^X\right) \\
&= \mathbb{E}\left(e^{2X}\right) - \left(\mathbb{E}\left(e^X\right)\right)^2 \\
&= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}
\end{aligned}$$

We now directly calculate both expectation and variance of Y.

$$\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}\left(\sum_{j=1}^{N} R_j\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\sum_{j=1}^{N} R_j | N\right)\right) \\
&= \mathbb{E}\left(N\mathbb{E}(R_j)\right) \\
&= \mathbb{E}(N)\,\mathbb{E}(R_j) \\
&= \frac{1}{p} e^{\mu + \sigma^2/2} \\
\mathrm{Var}(Y) &= \mathrm{Var}\left(\sum_{j=1}^{N} R_j\right) \\
&= \mathbb{E}\left(\left[\sum_{j=1}^{N}\right]^2\right) - \left[\mathbb{E}\left(\sum_{j=1}^{N} R_j\right)\right]^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(\left[\sum_{j=1}^{N} R_j\right]^2 |N\right)\right) - \left[\mathbb{E}\left(\mathbb{E}\left(\sum_{j=1}^{N} R_j|N\right)\right)\right]^2 \\
&= \mathbb{E}\left(Ne^{2\mu + 2\sigma^2}\right) - \left[\frac{1}{p} e^{\mu + \sigma^2/2}\right]^2 \\
&= \mathbb{E}(N)\, e^{2\mu + 2\sigma^2} - \frac{1}{p^2} e^{2\mu + \sigma^2} \\
&= \frac{1}{p} e^{2\mu + \sigma^2}\left(e^{\sigma^2} - \frac{1}{p}\right)
\end{aligned}$$

$\square$

(b) Use the Chebyshev inequality to estimate $\mathbb{P}\left(|Y - \mu_Y| > \mu_Y\right)$.

Proof. Directly applying Chebyshev's inequality, we get

$$\begin{aligned}
\mathbb{P}\left(|Y - \mu_Y| > \mu_Y\right) &\leq \frac{\operatorname{Var}(Y)}{\mu_Y^2} \\
&= \frac{\frac{1}{p} e^{2\mu+\sigma^2}\left(e^{\sigma^2} - \frac{1}{p}\right)}{\frac{1}{p^2} e^{2\mu+\sigma^2}} \\
&= \frac{e^{\sigma^2} - \frac{1}{p}}{\frac{1}{p}} \\
&= p e^{\sigma^2} - 1
\end{aligned}$$

□