MTH 463 HW 3

Brandyn Tucknott

16 October 2024

1. Consider the following variation of the random walk. If you are at the origin 0, you stay there with probability p or take a step to 1 with probability 1 - p. In other words, a transition between states 0 and 1 occurs with probability 1 - p. Denote by P_n the probability that starting at 0, you are back at 0 after n steps. Show that for $n \geq 1$,

$$P_n = (2p-1)P_{n-1} + (1-p)$$

and $P_0 = 1$. Using this recurrence conclude that $P_n = \frac{1}{2} + \frac{1}{2}(2p-1)^n$.

Proof. First, we begin by showing the recursive definition for P_n.

$$P_n = p \cdot P(\text{at 0 after } n-1 \text{ moves}) + (1-p) \cdot P(\text{at 1 after } n-1 \text{ moves}) \longrightarrow$$

$$P_n = p \cdot P_{n-1} + (1-p) \cdot (1 - P_{n-1}) =\!\!\longrightarrow$$
$$P_n = P_{n-1}\left(p + -1 \cdot (1-p)\right) + (1-p) \longrightarrow$$
$$P_n = P_{n-1}(2p-1) + (1-p)$$

With this we have shown the recursive definition. We will now show that $P_n = \frac{1}{2} + \frac{1}{2} \cdot (2p-1)^n = \frac{1}{2} \cdot (1 + (2p-1)^n))$ using the recursive definition and induction.

As a base case, we need to show that P_1 has the same value for both formulas.

In the recurrence case:

$$P_1 = (2p-1)P_0 + (1-p) = (2p-1) + (1-p) = p$$

In the formulaic case:

$$P_1 = \frac{1}{2} + \frac{1}{2}(2p-1)^1 = \frac{1}{2} + p - \frac{1}{2} = p$$

Since both cases are equal, know there is at least one case where the formulaic definition holds. We aim to show that for a general case, if it is true for n, then it is true for $n+1$.

Assume that the relation holds for a general case n. We will now show that it holds for $n+1$.

$$P_{n+1} = (2p-1)P_n + (1-p) \longrightarrow$$

$$P_{n+1} = (2p-1) \cdot \left(\frac{1}{2}(1 + (2p-1)^n)\right) + (1-p) \longrightarrow$$

$$P_{n+1} = 2p \cdot \frac{1 + (2p-1)^n}{2} - \frac{1 + (2p-1)^n}{2} + (1-p) \longrightarrow$$

$$P_{n+1} = p \cdot (1 + (2p-1)^n) - \frac{1 + (2p-1)^n}{2} + (1-p) \longrightarrow$$

$$P_{n+1} = p + p \cdot (2p-1)^n - \frac{1}{2} - \frac{(2p-1)^n}{2} + 1 - p \longrightarrow$$

$$P_{n+1} = \frac{1}{2} + p \cdot (2p-1)^n - \frac{(2p-1)^n}{2} \longrightarrow$$

$$P_{n+1} = \frac{1}{2} + (2p-1)^n \cdot \left(p - \frac{1}{2}\right) \longrightarrow$$

$$P_{n+1} = \frac{1}{2} + (2p-1)^n \cdot \frac{1}{2}(2p-1) \longrightarrow$$

$$P_{n+1} = \frac{1}{2} + \frac{1}{2} \cdot (2p-1)^{n+1}$$

We conclude that given the recurrence relation $P_n = (2p-1)P_{n-1} + (1-p)$, the general formula $P_n = \frac{1}{2} + \frac{1}{2} \cdot (2p-1)^n$ is true. $\square$

2. An insurance company classifies drivers in three categories: good risks, average risks and bad risk drivers. Records indicate that the probability of good, average and bad risk drivers will be involved in an accident in a 1 year span are, respectively, 0.05, 0.15 and 0.30. If 20% of the population is a good risk, 50% is average risk and 30% is bad risk, what proportion of the people have accidents in a fixed year? If policy holder A had no accidents in a particular year, what is probability that this driver is a good or average risk?

<u>*Solution.*</u>
This problem asks two questions, and we will answer them one at a time. Let A be the event of an accident, G be an event of good risk, M denote the event of average risk (mean), and B be the event of bad risk.

(a) What proportion of the people have accidents in a fixed year?
To answer this, we consider the sum of the probabilities of an accident for each group weighted by their population size.

$$P(A) = P(A|G) \cdot (G \text{ pop. size}) + P(A|M) \cdot (M \text{ pop. size}) + P(A|B) \cdot (B \text{ pop. size}) \longrightarrow$$

$$P(A) = 0.05 \cdot 0.2 + 0.15 \cdot 0.5 + 0.30 \cdot 0.30 = 0.175$$

(b) If a policy holder A had no accidents in a particular year, what is the probability that this driver is a good or average risk?

$$P(G \cup M|\overline{A}) = P(G|\overline{A}) + P(M|\overline{A}) = \frac{P(\overline{A}|G)P(G)}{P(\overline{A}} + \frac{P(\overline{A}|M)P(M)}{P(\overline{A})} = \frac{0.95 \cdot 0.2}{0.825} + \frac{0.85 \cdot 0.5}{0.825} = 0.745$$

3. An urn has r red balls and b blue balls that are randomly removed one at a time. Let R_i denote the event that the i^{th} ball removed is red. Find

   (a) $P(R_i)$

   Solution.

   First, we look at $P(R_1)$ and $P(R_2)$.

$$P(R_1) = \frac{r}{r+b}$$

$$P(R_2) = P(r|R_1)P(R_1) + P(r|\overline{R_1})P(\overline{R_1}) = \frac{r-1}{r+b-1} \cdot \frac{r}{r+b} + \frac{r}{r+b-1} \cdot \frac{b}{r+b}$$

$$P(R_2) = \frac{r(r+b-1)}{(r+b)(r+b-1)} = \frac{r}{r+b}$$

   Since $P(R_2) = P(R_1)$, we can conclude that $P(R_i) = \frac{r}{r+b}$ (The full justification is an inductive proof, but that is not required for this problem).

   (b) $P(R_3|R_1)$

   Solution.

   Let AB denote $A \cap B$.

$$P(R_3|R_1) = P(R_3R_2|R_1) + P(R_3\overline{R_2}|R_1) = P(R_3|R_2R_1)P(R_2|R_1) + P(R_3|\overline{R_2}R_1)P(\overline{R_2}|R_1) \longrightarrow$$

$$P(R_3|R_1) = \frac{r-2}{r+b-2} \cdot \frac{r-1}{r+b-1} + \frac{r-1}{r+b-2} \cdot \frac{b}{r+b-1} = \frac{(r-1)(r+b-2)}{(r+b-1)(r+b-2)} = \frac{r-1}{r+b-1}$$

4. Suppose you are gambling against an infinitely rich adversary and at each stage of the game you either win or lose 1 token with probability p and $q = 1 - p$ respectively. Assume that you start with i tokens. Show that the probability you eventually go broke is

$$\begin{cases} 1, \text{ if } p \leq \frac{1}{2} \\ \left(\frac{q}{p}\right)^i, \text{ if } p > \frac{1}{2} \end{cases}$$

Proof. We prove this in two parts. First, we show that if $p \leq \frac{1}{2}$, the probability of loosing is 1, and second, we show that if $p > \frac{1}{2}$, the probability of loosing is $\left(\frac{q}{p}\right)^i$. Note also that the probability of winning P_i is

$$P_i = \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1}$$

where we start with i tokens and our opponent starts with $N - i$ tokens. Since our opponent starts with infinite tokens, the probability of us winning is

$$\lim_{N\to\infty} P_i.$$

(a) Assume that $p < \frac{1}{2}$. Then $q > p \longrightarrow \frac{q}{p} > 1 \longrightarrow$

$$\lim_{N\to\infty} P_i = \lim_{N\to\infty} \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} = 0$$

Since the probability of us winning is 0, the probability of us loosing is 1.
In the special case where $p = q = 0.5$, we have that the probability of winning is $\frac{i}{N}$, and when our opponent has infinite tokens, N tends towards infinity. So the probability of winning is

$$P_i = \lim_{n\to\infty} \frac{i}{N} = 0$$

So if our probability of winning is 0, the probability of loosing is 1.

(b) Now assume that $p > \frac{1}{2}$. Then we have that $p > q \longrightarrow \frac{q}{p} < 1 \longrightarrow$

$$\lim_{N\to\infty} P_i = \lim_{N\to\infty} \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} = \frac{\left(\frac{q}{p}\right)^i - 1}{-1} = 1 - \left(\frac{q}{p}\right)^i$$

Since the probability of winning is $1 - \left(\frac{q}{p}\right)^i$, we can conclude that the probability of loosing is $\left(\frac{q}{p}\right)^i$.

Now that we have shown both cases, we can conclude that if $p \leq \frac{1}{2}$, then the probability of us loosing is 1, and if $p > \frac{1}{2}$, then the probability of loosing is $\left(\frac{q}{p}\right)^i$.

∎