MTH 483 HW 1

Brandyn Tucknott

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1. Let $z = 2 - 3i$ and $w = -1 + 2i$. Compute the following:
   (a) $z + 3\overline{w}$.
   Solution.
   $z + 3\overline{w} = 2 - 3i + 3(-1 - 2i) = -1 - 9i$.
   (b) z^3
   Solution.
   $z^3 = (2 - 3i)^3 = 2^3 + 3 \cdot 2^2(-3i) + 3 \cdot 2(-3i)^2 + (-3i)^3 = -46 - 9i$.
   (c) $w^2 + \overline{z} + i$.
   Solution.
   $w^2 + \overline{z} + i = (-1 + 2i)^2 + 2 + 3i + i = -1$
   (d) $w^2 + w$.
   Solution.
   $w^2 + w = (-3 - 4i) + (-1 + 2i) = -4$
2. Find the real and imaginary parts of each of the following:
   (a) $\frac{3+i}{3i}$.
   Solution.
   $\frac{3+i}{3i} = \frac{3}{3i} + \frac{i}{3i} = \frac{1}{3} + \frac{1}{i} = \frac{1}{3} - i$. Thus the real and imaginary components are $\frac{1}{3}$ and -1 respectively.
   (b) $(3 + 2i)^2 - (4 - i)^2$.
   Solution.
   $(3 + 2i)^2 - (4 - i)^2 = (5 + 12i) - (15 - 8i) = -10 + 20i$. Thus the real and imaginary components are -10 and 20 respectively.
   (c) i^n for any $n \in \mathbb{Z}$.
   Solution.
   We break this down into cases. Let $k \in \mathbb{Z}$. Then

$$
\begin{aligned}
n = 4k &\longrightarrow i^n = 1 \longrightarrow i^n = 1, i^n = 0 \\
n = 4k + 1 &\longrightarrow i \longrightarrow i^n = 0, i^n = 1 \\
n = 4k + 2 &\longrightarrow -1 \longrightarrow i^n = -1, i^n = 0 \\
n = 4k + 3 &\longrightarrow i^n = -i \longrightarrow i^n = 0, i^n = -1
\end{aligned}
$$

   (d) $\frac{7i}{2-i}$.
   Solution.
   $\frac{7i}{2-i} = \frac{-7+14i}{5}$. Thus the real and imaginary components are $-\frac{7}{5}$ and $\frac{14}{5}$ respectively.

3. Write in Polar Form:
   (a) $-1-\sqrt{3}i = -2e^{i\frac{\pi}{3}}$
   (b) $\frac{\sqrt{3}}{2} + \frac{i}{2} = e^{i\frac{\pi}{6}}$
   (c) $\frac{i}{10+10i} = \frac{1}{20} + i\frac{1}{20} = \frac{1}{10\sqrt{2}}e^{i\frac{\pi}{4}}$
   (d) $\frac{1+i}{1-i} = i = e^{i\frac{\pi}{2}}$
4. Write in Rectangular Form:
   (a) $\sqrt{2}e^{-i\frac{\pi}{2}} = -i\sqrt{2}$
   (b) $\frac{1}{2}\left(\cos(\frac{64\pi}{3}) + i\sin(\frac{64\pi}{3})\right) = \frac{1}{2}\left(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})\right) = \frac{1}{2}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -\frac{1}{4} - i\frac{\sqrt{3}}{4}$
   (c) $(1+i)^{30} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{30} = 2^{15}e^{\frac{30\pi}{4}} = 2^{15}e^{\frac{3\pi}{2}} = -2^{15}i$
   (d) $\frac{d}{d\phi}e^{\phi+i\phi} = ie^{\phi+i\phi} = ie^{\phi}\left(e^{i\phi}\right) = ie^{\phi}\left(\cos\phi + i\sin\phi\right) = -e^{\phi}\sin\phi + ie^{\phi}\cos\phi$
5. Given $x, y \in \mathbb{R}$, define the matrix $M(x,y) := \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. Show that

$$M(x,y) + M(a,b) = M(x+a, y+b) \text{ and } M(x,y)M(a,b) = M(xa-yb, xb+ya)$$

Proof.

$$\begin{aligned} M(x,y) + M(a,b) &= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= \begin{pmatrix} x+a & -(y+b) \\ y+b & x+a \end{pmatrix} \\ &= M(x+a, y+b) \end{aligned}$$

Similarly,

$$\begin{aligned} M(x,y)M(a,b) &= \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= \begin{pmatrix} xa-yb & -(xb+ya) \\ ay+xb & -by+ax \end{pmatrix} \\ &= M(xa-yb, xb+ya) \end{aligned}$$

□

6. Find $\cos(5t)$ and $\sin(5t)$ in terms of $\cos(t)$ and $\sin(t)$.

Proof. Using De Moivre's Identity and binomial theorem, we know that

$$\cos(5t) + i\sin(5t) = (\cos(t) + i\sin(t))^5 = \sum_{k=0}^{5}\binom{5}{k}\cos(t)^{5-k}\left(i\sin(t)\right)^k$$

Examining this, observe that $\cos(5t)$ is the real portion of the sum, or when k is even. Similarly, $\sin(5t)$ is determined when k is odd. Thus

$$\begin{aligned} \cos(5t) &= \binom{5}{0}\cos(t)^5 i^0\sin(t)^0 + \binom{5}{2}\cos(t)^3 i^2\sin(t)^2 + \binom{5}{4}\cos(t)i^4\sin(t)^4 \\ &= \cos(t)^5 - 10\cos(t)^3\sin(t)^2 + 5\cos(t)\sin(t)^4 \\ i\sin(5t) &= \binom{5}{1}\cos(t)^4 i\sin(t) + \binom{5}{3}\cos(t)^2 i^3\sin(t)^3 + \binom{5}{5}i^5\sin(t)^5 \\ &= i\left(5\cos(t)^4\sin(t) - 10\cos(t)^2\sin(t)^3 + \sin(t)^5\right) \end{aligned}$$

□

7. In this exercise, we derive the solution to the cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{1}$$

where $a, b, c \in \mathbb{R}$.

(a) Use the change of variables $x = y - \frac{a}{3}$ to transform the equation to the following reduced form:

$$y^3 + py + q = 0, \tag{2}$$

where $p = b - \frac{a^2}{3}, q = \frac{2a^3}{27} - \frac{ab}{3} + c$.

Proof.

$$\begin{aligned}
x^3 + ax^2 + bx + c &= (y - \frac{a}{3})^3 + a(y - \frac{a}{3})^2 + b(y - \frac{a}{3}) + c \\
&= y^3 - 3\frac{ay^2}{3} + 3\frac{a^2y}{9} - \frac{a^3}{27} + ay^2 - 2\frac{a^2y}{3} + \frac{a^2}{9} + by - \frac{ab}{3} + c \\
&= y^3 - \frac{a^2y}{3} + \frac{2a^3}{27} + by - \frac{ab}{3} + c \\
&= y^3 - \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) \\
&= y^3 + py + q = 0
\end{aligned}$$

□

(b) Let y be a solution of equation (2) written as $y = u + v$, and show that

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0$$

Proof.

$$\begin{aligned}
y^3 + py + q &= (u + v)^3 + p(u + v) + q \\
&= u^3 + 3u^2v + 3v^2u + v^3 + p(u + v) + q \\
&= u^3 + v^3 + 3uv(u + v) + p(u + v) + q \\
&= u^3 + v^3 + (3uv + p)(u + v) + q = 0
\end{aligned}$$

□

(c) Require that $3uv + p = 0$. Then directly we have $u^3v^3 = -\frac{p^3}{27}$ and by part (b) $u^3 + v^3 = -q$.

(d) Suppose R, W are numbers satisfying $R + W = -\beta$ and $RW = \gamma$. Show that R, W are solutions to the quadratic equation $X^2 + \beta X + \gamma = 0$.

Proof. Observe that

$$\begin{aligned}
(X - R)(X - W) &= X^2 - X(R + W) + RW \\
&= X^2 - X(-\beta) + \gamma \\
&= X^2 + \beta X + \gamma
\end{aligned}$$

Thus R, W are the two solutions to the given quadratic. □

(e) Use parts (c) and (d) to conclude that u^3 and v^3 are solutions to the quadratic equation $X^2+qX-\frac{p^3}{27}=0$.

Proof. By parts (c) and (d), we know that $u^3+v^3=-q$, and $v^3u^3=-\frac{p^3}{27}$. Thus u^3, v^3 are solutions to a quadratic equation of the form:

$$X^2-(v^3+u^3)X+v^3u^3=X^2+qX-\frac{p^3}{27}=0$$

We use the quadratic formula to derive values for u, v.

$$\begin{aligned} u^3 &= \frac{-q\pm\sqrt{q^2+\frac{4p^3}{27}}}{2} \\ &= -\frac{q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3} \end{aligned}$$

and thus

$$u=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} \text{ and } v=\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}$$

□

(f) Derive that

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}-\frac{a}{3}$$

is a solution of equation (1).

Proof. By parts (a), (b), and (e), since $x=y-\frac{a}{3}$ and $y=u+v$, we conclude that $x=u+v-\frac{a}{3}$. □

8. Consider Bombelli's equation $x^3 - 15x - 4 = 0$.

(a) Use Cardano's formula to derive the solution $x = u + v$, where

$$u = \sqrt[3]{2 + 11i} \text{ and } v = \sqrt[3]{2 - 11i}$$

Proof. To use the formula, first we find values for p, q.

$$\begin{aligned} p &= b - \frac{a^2}{3} \\ &= (-15) - \frac{0^2}{3} \\ &= -15 \\ q &= \frac{2a^3}{27} - \frac{ab}{3} + c \\ &= \frac{2 \cdot 0}{27} - \frac{0 \cdot (-15)}{3} + (-4) \\ &= -4 \end{aligned}$$

Next we use the values of p, q in Cardano's formula to find a solution for the given cubic equation.

$$\begin{aligned} x &= \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{a}{3} \\ &= \sqrt[3]{-\frac{-4}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}} + \sqrt[3]{-\frac{-4}{2} - \sqrt{\left(\frac{4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}} - \frac{0}{3} \\ &= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} \\ &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \end{aligned}$$

Since $x = u + v$, we conclude that $u = \sqrt[3]{2 + 11i}$ and $v = \sqrt[3]{2 - 11i}$. $\square$

(b) Notice that u, v have to be conjugate for $u + v$ to be real. Set $u = a + ib$ and $v = a - ib$. Show that $a = 2, b = 1$ works.

Proof. If $a = 2, b = 1$ with $u = a + ib, v = a - ib$, we know that $x = u + v = 2a = 4$, and we simply check if $x = 4$ satisfies the equation.

$$\begin{aligned} x^3 - 15x - 4\Big|_{x=4} &= 4^3 - 15(4) - 4 \\ &= 64 - 60 - 4 = 0 \end{aligned}$$

$\square$

(c) What is the real solution x of Bombelli's equation? What are the other two solutions of Bombelli's equation?

Proof. As determined in part (b) the real solution is $x = 4$, which we factor out to yield

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1)$$

Using the quadratic formula, we are able to find the remaining solutions.

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{12}}{2} \\ &= -2 \pm \sqrt{3} \end{aligned}$$

$\square$