MTH 312 HW 1

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6.2.3 For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \text{ and } h_n(x) = \begin{cases} 1, & \text{if } x \geq \frac{1}{n} \\ nx, & \text{if } 0 \leq x \leq \frac{1}{n} \end{cases}$$

Answer the following questions about the sequences (g_n) and (h_n):
(a) Find the pointwise limit on $[0, \infty)$.
Solution.

$$\lim_{n\to\infty} g_n(x) = g(x) = \begin{cases} x, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & 1 < x < \infty \end{cases}$$

$$\lim_{n\to\infty} h_n(x) = h(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \end{cases}$$

(b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
Solution.
For (g_n), as $n \to \infty$, the rate at which different "sections" converge to their part in $g(x)$ is not equal. In particular, smaller x converges exponentially faster than other portions of the sequence, making uniform convergence impossible.

For (h_n), as $n \to \infty, nx$ becomes more and more "uncontained". Because of this, it cannot be that the sequence (h_n) is uniformly continuous.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.
Solution.
For both functions, consider instead the set $(1, 2)$.

For the sequence (g_n), we have the new pointwise limit $\lim_{n\to\infty} g_n(x) = g(x) = 0$. Then for arbitrary $\epsilon > 0$, choose $N > \frac{\ln \frac{1-\epsilon}{\epsilon}}{\ln 2} = \log_2 \frac{1-\epsilon}{\epsilon}$

$$|g_n(x) - g(x)| = \left| \frac{x}{1+x^n} - 0 \right| = \frac{x}{1+x^n} \leq \frac{1}{1+x^n} < \frac{1}{1+2^n} < \frac{1}{1+2^{\left(\log_2 \frac{1-\epsilon}{\epsilon}\right)}} = \frac{1}{1+\frac{1-\epsilon}{\epsilon}} = \epsilon$$

Similarly for the sequence (h_n), we have the new pointwise limit $\lim_{n\to\infty} h_n(x) = h(x) = 1$. Then for arbitrary $\epsilon > 0$, we have that

$$|h_n(x) - h(x)| = |1 - 1| = 0 < \epsilon, \text{so } N \in \mathbb{N} \text{ works (independent of } \epsilon \text{ in this case)}$$

6.2.9 Assume (f_n) and (g_n) are uniformly convergent sequences of functions.
(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

Proof. Note first that $(f_n + g_n) \to f + g$. Now let $\frac{\epsilon}{2} > 0$, and choose $N = \max(N_{f_n}, N_{g_n})$. Then for all $n \geq N$, we have that

$$|(f_n + g_n) - (f + g)| = |f_n - f + g_n - g| \leq |f_n - f| + |g_n - g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

We conclude that $(f_n + g_n)$ is uniformly convergent by definition of uniform convergence. □

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

Proof. Consider $(f_n) = \frac{1}{n}, (g_n) = x$. Then the pointwise limit on the interval $[0, \infty)$ of both functions are

$$f_n \to f = 0, \text{ and } g_n \to g = x$$

It is obvious that f_n, g_n are uniformly convergent to f, g respectively. However, on the domain $[0, \infty)$, $(f_n \cdot g_n) = \frac{x}{n}$ is not uniformly convergent (although the pointwise limit does exist).

□

(c) Prove that if there exists $M > 0$ such that $|f_n|, |g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

Proof. Let $A \subset \mathbb{R}$. Since $(f_n), (g_n)$ converge uniformly, there exists N_1, N_2 such that

$$\sup_{x \in A} |f_n(x) - f(x)| < \frac{\epsilon}{2M}, \text{ for } n \geq N_1$$

$$\sup_{x \in A} |g_n(x) - g(x)| < \frac{\epsilon}{2M}, \text{ for } n \geq N_2$$

(these definitions come from Baby Rudin 3rd edition, which I occasionally use as a secondary text)

Let $\epsilon > 0$ be arbitrary, and choose $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$|f_n g_n - fg| = |f_n g_n - f_n g + f_n g - fg| \leq |f_n g_n - f_n g| + |f_n g - fg| \leq |f_n|\,|g_n - g| + |g|\,|f_n - f| <$$

$$< |f_n| \frac{\epsilon}{2M} + |g| \frac{\epsilon}{2M} = \frac{\epsilon}{2M}(|f_n| + |g|) < \frac{\epsilon}{2M} \cdot 2M = \epsilon$$

Since this is true for any $x \in A$, definitionally $(f_n g_n)$ converges uniformly. □

6.2.10 Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Proof. Let $\epsilon > 0$. Define $O_n = \{x \in [a, b] : |f_n(x) - f(x)| < \epsilon\}$. Then O_n is open relative to $[a, b]$, with $\bigcup_{n \in \mathbb{N}} O_n = [a, b]$, hence O_n is an open cover for $[a, b]$. Since $[a, b]$ is compact, by the Heine-Borel Theorem, there is a finite subcover of $[a, b]$ such that

$$[a, b] = O_{n_1} \cup \ldots \cup O_{n_k}, \text{ where } n_1 < \ldots < n_k$$

Recall however, that (f_n) is monotone increasing, so

$$O_{n_1} \subset O_{n_2} \subset \ldots \subset O_{n_k} \longrightarrow [a, b] = O_{n_k}$$

Then by definition of O_{n_k},

$$[a, b] = \{x \in [a, b] : |f_{n_k}(x) - f(x)| < \epsilon\}$$

We conclude that for $\epsilon > 0$, choose $N = n_k$. For all $n \geq N$, we know that $|f_n(x) - f(x)| < \epsilon$ for arbitrary $x \in [a, b]$. Then by definition, $f_n \to f$ uniformly.

$\square$