CSM EECS 20N Lecture Notes

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0.1 Preface

These pieces of notes, to be updated irregularly, are meant to support upcoming syllabus changes in coming semesters of EECS 16A towards Signal Processing. The contents of these notes will very likely be a superset of any coming semester‘s syllabus that involves signal processing, and will therefore have extraneous parts out of some semesters‘ scopes. Specifically, these notes will be dedicated to the union of EECS 16A‘s past linear algebra content and EECS 20N‘s 2011 rendition‘s syllabus.

As the volume of this note will naturally exceed a 4-unit course worth of content, it will take a lot of time and effort to complete these notes. Therefore, the project is made open-source for interested parties to help maintaining with. It will be the primary author’s first time reading some of the involved contents in this note, so there may be a lack of upper-division-level insight in signal processing from this note.

The rest of preface will be written when it is finally not Christmas.

Contents

Part I

Linear Algebra is A Really Near Algebruh

Chapter 1

The Fundamentals of Linear Algebra

Hello

Chapter 2

Arithmetics of Vectors and Matrices

Hello

Chapter 3

Gaussian Elimination

Hello

Chapter 4

Linear Dependence

Hello

Chapter 5

Vector Spaces

Hello

Chapter 6

Eigen Show You The World: Eigenvalues, Eigenvectors, and Eigenpain

Hello

Chapter 7

Eigenspaces and Change of Basis

Hello

Chapter 8

Inner Products and Norms, Orthogonality

Hello

Chapter 9

Least Squares Algorithm: Where Machine Learning Starts

Hello

Part II

It is All About the Functions and Sets

Chapter 10

An Integral Review for Integration

Eventually, integrals will become a core aspect of this coursework. Therefore, to prepare students for upcoming mathematical prerequisite knowledge, we aim to review some core aspects of integrals here. If you have never learned integrals before, you probably should not be here; and, if you would still like to digest the contents here, you may want to consult other resources along with this one, as this resource will only discuss a summary of integral techniques.

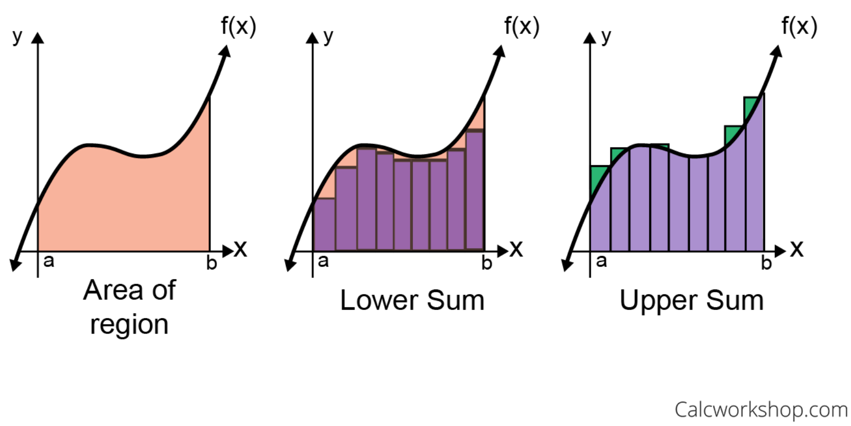
10.1 A Brief Review of An Integral

The most explicit interpretation of an integral is the area under curve: provided some curve that represents a single-variate function f, the area under its curve from $x = a$ to $x = b$ is an integral: $\int_a^b f(x)\,dx$.

Let us discuss this interpretation furthermore by discussing some methodologies within our scope for approximating the area under curve.

In the above figure, the "Lower Sum" and "Upper Sum" techniques (formally known as Reimann Sums) attempt to approximate the area under curve with the following algorithm:

1. Decide the width of the rectangles that will fill the area under curve (or, overfill).
2. Decide the height of the rectangle based on either the curve height at the left or right end of the rectangle.



Alternatively, we may also choose either the lower one or the higher one (respectively leading to the "Lower Sum" and "Higher Sum" techniques above).

3. Sum the areas of these rectangles.

We consider this as a discrete operation. Discrete here means that the mathematical operations is done on a non-continuous number line that does not contain every possible number. For example, by deciding the width of rectangle, we also decide to skip looking at the function values of any number that is not the rectangle width. However, as you may have noted, discrete operations do not provide a great approximation and may take a lot of computation as the number of rectangles increase.

What if there is, instead, a continuous method of doing so, such that we can consider rectangles of infinitesimal widths and sum up their areas? Such is the operation that we call integrals. In other words, integral operations are the discrete summation techniques we see in the above picture, except we now use infinitesimal-width rectangles such that we can most accurately portray the area under curve with the sum of those rectangles. This shows us that integrals are actually the analogy of summations in a continuous space.

While the summation methods using n rectangles to approximate the area under curve within $[a, b]$ may be simplified as a summation:

$$\sum_{i=0}^{n-1} \frac{b-a}{n} f(a + i \times \frac{b-a}{n})$$

The integral version of such method is instead written as:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{b-a}{n} f(a + i \times \frac{b-a}{n})$$

We will introduce its mathematical operations in the coming section.

10.2 Computing An Integral

To compute an integral $\int_a^b f(x)\, dx$, we first find the antiderivate of function f (which we would call F), then we would compute the value of integral as $F(b) - F(a)$.
The above very, very rough summary forgets to address several subtleties:

1. The definition of antiderivative and the computations required to find it.
2. The reason why we may claim the value of $\int_a^b f(x)\, dx$ to be the difference $F(b) - F(a)$.
3. What do we do when there exists a point $x' \in [a, b]$ at which the function f is not defined at.

Let us address the above concerns one by one to validify the summarizing statement.

First of all, the antiderivaitve F of a function f is a function such that $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$. Provided so, the antiderivative of any function f is not unique. Let us proceed with a proof, or, an argument to prove or disprove a mathematical statement:

Explain 10.2.1. Antiderivative of Function $f(x)$ is Non-Unique

Suppose that the function f already has an antiderivative $F(x)$. Then, let C be a scalar, real constant. We may see that:

$$\begin{aligned} \mathrm{d}/\mathrm{d}x &= f(x) + \mathrm{d}/\mathrm{d}x\ C \\ &= f(x) + 0 = f(x) \end{aligned}$$

Usually, then, we denote the antiderivative of $f(x)$ as $F(x) + C$ for the above reason.

Meanwhile, the computation of integral from antiderivatives is secured by a famous theorem that we introduce below:

Theorem 10.2.1. Fundamental Theorems of Calculus

The first fundamental theorem of calculus. Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t)\,dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and

$$F'(x) = f(x)$$

for all x in (a, b) so F is an antiderivative of f.
This theorem formally defines what really is an antiderivative.
The second fundamental theorem of calculus. Let f be a real-valued function on a closed interval $[a, b]$ and F a continuous function on $[a, b]$ which is an antiderivative of f in (a, b):

$$F'(x) = f(x)$$

If f is Riemann integrable on $[a, b]$ (or, in other words, if the Reimann sum, approximation of area under curve converges as the rectangle width approaches infinitesimal), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

And when there exists a point $x' \in [a, b]$ at which the function f is not defined at, the integral is simply undefined. Usually, when we still want to find the area under curve, we can find a workaround by defining a function:

$$\hat{f}(x) = \begin{cases} f(x), & x \neq x' \\ \text{some other value}, & x = x' \end{cases}$$

and computing its integral instead.

10.3 Integral Tricks

There are several integral tricks that may become prevelant in the upcoming course contents. This section provides a survey of them.

10.3.1 Integral Identities

Let's review some integral identities:

Fundamentals.

1. $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|x|, & n = -1 \end{cases}$

2. $\int e^x\,dx = e^x + C$

3. $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

Trigonometric.

1. $\int \cos(x)\,dx = \sin(x) + C$
2. $\int \sin(x)\,dx = -\cos(x) + C$
3. $\int \sec^2(x)\,dx = \tan(x) + C$
4. $\int \csc^2(x)\,dx = -\cot(x) + C$
5. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$
6. $\int \csc(x)\tan(x)\,dx = -\csc(x) + C$

Inverse Trigonometric.

1. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) = -\cos^{-1}(x)$
2. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) = -\cot^{-1}(x)$
3. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) = -\csc^{-1}(x)$

10.3.2 Partial Fractions

The technique of Partial Fractions concern decomposing a rational function into several more rational functions that are easier to integrate. An example follows:

Explain 10.3.1. Example of Partial Fraction Trick

Compute the following expression:

$$\int \frac{1}{x^4-1}\,dx$$

Let us first attempt to decompose $\frac{1}{x^4-1}$ into a sum of several more rational functions. The first step of doing so is to find a way to factorize the denominator of our function:

$$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

Then, we look for some coefficients that allows for us to express:

$$\frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

How many coefficients we need can be explored upon with rules of thumbs that, for the sake of brevity, should not be discussed here.
Then, we manipulate such that:

$$\begin{aligned}
\frac{1}{x^4-1} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \\
&= \frac{1}{x^4-1} \cdot \left(A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)\right) \\
&= \frac{1}{x^4-1} \cdot \left((A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)\right)
\end{aligned}$$

Which allows us to define a system of linear equations to solve A, B, C, D for:

$$\begin{cases} A+B+C &= 0 \\ A-B+D &= 0 \\ A+B-C &= 0 \\ A-B-D &= 0 \end{cases}$$

The solution should come out to be $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$, $D = -\frac{1}{2}$. Therefore,

$$\begin{aligned} \int \frac{1}{x^4-1}\,dx &= \int (\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1})\,dx \\ &= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}(x) + C \end{aligned}$$

10.3.3 u-Substitution

Write later

10.3.4 Integration by Part

Write later

10.3.5 Trigonometric Substitutions

Write later

Chapter 11

Set!

Hello

Chapter 12

Functions

Hello

Part III

Introduction to Signals and Systems

Chapter 13

Introduction to Signals

Hello

Chapter 14

Introduction to Systems

Hello

Chapter 15

Mathematical Definitions for Functions and Signals

Hello

Chapter 16

Mathematical Properties of A System

Hello

Chapter 17

States

Hello

Chapter 18

Linear

Hello

Chapter 19

Linear Systems and Its States

Hello

Part IV

Signal Processing

Chapter 20

Linear Systems and Its States

Hello

Chapter 21

It’s Time to Get a Little Complex

Hello

Chapter 22

Frequency, Phase, Domain

Hello

Chapter 23

Fourier Expansion: Infinite Terms

Hello

Chapter 24

Linear Time-Invariant Systems

Hello

Chapter 25

Frequency Response and Fourier Series

Hello

Chapter 26

Introduction to Filtering

Hello

Chapter 27

Impulse Resopnse Filters

Hello

Part V

Sampling and Fourier Transform

Chapter 28

The Four Fourier Transforms, Part I

Hello

Chapter 29

The Four Fourier Transforms, Part II

Hello

Chapter 30

Fourier Transform vs. Fourier Series

Hello

Chapter 31

Sampling and Reconstruction

Hello

Chapter 32

The Nyquist-Shannon Sampling Theorem

Hello