Documentation for the microTDFL

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Abstract

Abstract goes here.

1 Introduction

- Introductory remarks

2 Background

- Ohm's law
- Charge conservation
- EFIE
- Thin wire approximation
- Basically the same as section 2 in [1]

2.1 The EFIE

The electric field integral equation (EFIE) gives the electric field $\vec{E}(\vec{x}, t)$, at position $\vec{x}$ and time t, in terms of the charge and current densities and their time derivatives:

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} \Big[\rho(\vec{x}', t')\Big]_{\text{ret}} + \frac{\hat{R}}{cR} \left[\frac{\partial\rho(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial\vec{J}(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \right\}, \tag{1}$$

where ρ is the volume charge density, $\vec{J}$ is the current density, $\vec{R} = \vec{x} - \vec{x}'$, $R = \|\vec{R}\|$, $\hat{R} = \vec{R}/R$, and $t' = t - R/c$.

2.2 Thin-wire approximation

With the thin-wire approximation:

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int dl' \left\{ \frac{\hat{R}}{R^2} \Big[\lambda(\vec{x}', t')\Big]_{\text{ret}} + \frac{\hat{R}}{cR} \left[\frac{\partial\lambda(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial\vec{I}(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \right\} \tag{2}$$

3 Framework

3.1 Discretizing time and the channel

3.2 Storing the channel history

Since the charge and current densities in the EFIE are evaluated in retarded time, the electric field depends on a history of charges and currents, not just the present state of the channel. This history is stored as a vector of the form

$$\vec{h} = \begin{bmatrix} \vec{I}^{(m-1)} & \vec{I}^{(m-2)} & \cdots & \vec{I}^{(1)} & \vec{I}^{(0)} & \vec{Q}^{(0)} & 1 \end{bmatrix}, \tag{3}$$

where m is the number of time steps stored in the history vector, $\vec{I}^{(j)}$ is the vector of the $n-1$ currents at time step j and $\vec{Q}^{(j)}$ is the vector of the n charges at time step j. The 1 at the end of the vector allows an external electric field to be applied to the channel and can be varied to change the applied field. The currents are ordered from the most recent time step to the most ancient time step, and only the charges at the most ancient time step are stored. The charges at all other time steps can be computed from the currents. While it would be simpler to store the charges at every time step instead of computing them from the currents, this would nearly double the size of the vector; I opted for a smaller history vector and transition matrix at the cost of some additional complexity. The number of stored time steps m is determined by the longest light travel time between any two segments. The length of the history vector is $m(n-1)+n+1$.

In order to step the simulation forward in time, the currents at the next time step must be determined from the history of charges and currents. This is done by calculating the electric field two different ways at each current segment at the next time step. The first is through Ohm's Law; the electric field is R_l times the unknown future current. The other is computing the field from the EFIE, in terms of both the unknown currents and the history of charges and currents (and the applied field). Setting these equal to each other for each of the current segments yields an $(n-1)\times(n-1)$ linear system of equations; solving this system gives the currents at the next time step. Once the next currents are computed, the history vector has to be updated. The most ancient currents in the history vector are deleted, the others are shifted down, and the new currents are inserted at the top of the vector; the charges in the history vector are also updated to the new oldest time step.

This is the basic idea of how the simulation works. To carry out the simulation, a transition matrix can be computed that, when multiplied with the history vector, produces the updated history vector one time step forward. Thus instead of advancing through the simulation one step at a time, the matrix can be raised to any power to advance a large number of steps at once.

4 Computing the Electric Field

4.1 Applying the EFIE to the discretized channel

Consider the electric field at $\vec{x}_i$, a point on current segment i. From equation 2 we have

$$\vec{E}(\vec{x}_i,t) = \frac{1}{4\pi\epsilon_0}\int dl' \left\{ \frac{\hat{R}}{R^2}\Big[\lambda(\vec{x}',t')\Big]_{\mathrm{ret}} + \frac{\hat{R}}{cR}\left[\frac{\partial\lambda(\vec{x}',t')}{\partial t'}\right]_{\mathrm{ret}} - \frac{1}{c^2R}\left[\frac{\partial\vec{I}(\vec{x}',t')}{\partial t'}\right]_{\mathrm{ret}} \right\} \tag{4}$$

The current in segment i is determined by the component of the electric field parallel to the segment. Letting $\hat{I}_i$ be the unit vector parallel to current segment i, the parallel component of the field is

$$\hat{I}_i\cdot\vec{E}(\vec{x}_i,t) = \frac{1}{4\pi\epsilon_0}\int dl' \left\{ \frac{\hat{I}_i\cdot\hat{R}}{R^2}\Big[\lambda(\vec{x}',t')\Big]_{\mathrm{ret}} + \frac{\hat{I}_i\cdot\hat{R}}{cR}\left[\frac{\partial\lambda(\vec{x}',t')}{\partial t'}\right]_{\mathrm{ret}} - \frac{1}{c^2R}\left[\hat{I}_i\cdot\frac{\partial\vec{I}(\vec{x}',t')}{\partial t'}\right]_{\mathrm{ret}} \right\}. \tag{5}$$

Breaking the integral over the entire channel into a sum of integrals over the charge segments and a sum of integrals over the current segments:

$$\begin{aligned}\hat{I}_i \cdot \vec{E}(\vec{x}_i, t) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} \left\{ \int dl' \frac{\hat{I}_i \cdot \hat{R}}{R^2} \Big[\lambda_j(\vec{x}', t')\Big]_{\text{ret}} + \int dl' \frac{\hat{I}_i \cdot \hat{R}}{cR} \left[\frac{\partial \lambda_j(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \right\} \\ - \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \left\{ \int dl' \frac{\hat{I}_i \cdot \hat{I}_j}{c^2 R} \left[\frac{\partial I_j(\vec{x}', t')}{\partial t'}\right]_{\text{ret}} \right\}.\end{aligned} \tag{6}$$

Also note that we have replaced $\vec{I}_j$ in the last term with $I_j \hat{I}_j$ (where $\hat{I}_j$ is a unit vector pointing along current segment j), allowing the time-independent part ($\hat{I}_j$) to be separated from the time-dependent part (I_j). We now suppose that the charge and current densities are spatially constant on any given segment, allowing us to pull them outside of the integrals. We also replace the charge density λ_j with the total charge on the segment Q_j divided by its length l_j:

$$\begin{aligned}\hat{I}_i \cdot \vec{E}(\vec{x}_i, t) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} \left\{ \Big[Q_j(t')\Big]_{\text{ret}} \int dl' \frac{\hat{I}_i \cdot \hat{R}}{l_j R^2} + \left[\frac{\partial Q_j(t')}{\partial t'}\right]_{\text{ret}} \int dl' \frac{\hat{I}_i \cdot \hat{R}}{l_j cR} \right\} \\ - \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \left\{ \left[\frac{\partial I_j(t')}{\partial t'}\right]_{\text{ret}} \int dl' \frac{\hat{I}_i \cdot \hat{I}_j}{c^2 R} \right\}.\end{aligned} \tag{7}$$

We now make the following definitions of the matrices **R2**, **R1L**, and **R1T**. These matrices depend on the geometry of the lightning channel but not on the charges or currents on the channel (past or present). They only need to be computed once for a particular channel geometry and then can be reused whenever the EFIE is used to compute the electric field on one of the current segments.

$$\mathbf{R2}_{ij} = \frac{1}{4\pi\epsilon_0} \int dl' \frac{\hat{I}_i \cdot \hat{R}}{l_j R^2} \tag{8}$$

$$\mathbf{R1L}_{ij} = \frac{1}{4\pi\epsilon_0} \int dl' \frac{\hat{I}_i \cdot \hat{R}}{l_j cR} \tag{9}$$

$$\mathbf{R1T}_{ij} = \frac{1}{4\pi\epsilon_0} \int dl' \frac{\hat{I}_i \cdot \hat{I}_j}{c^2 R} \tag{10}$$

We can now rewrite equation 7 using the geometric factor matrices **R2**, **R1L**, and **R1T**. We also rewrite the left side (the component of the electric field parallel to the channel) using Ohm's law:

$$I_i R_l = \sum_{j=1}^{n} \left\{ \mathbf{R2}_{ij} \Big[Q_j(t')\Big]_{\text{ret}} + \mathbf{R1L}_{ij} \left[\frac{\partial Q_j(t')}{\partial t'}\right]_{\text{ret}} \right\} - \sum_{j=1}^{n-1} \left\{ \mathbf{R1T}_{ij} \left[\frac{\partial I_j(t')}{\partial t'}\right]_{\text{ret}} \right\}. \tag{11}$$

4.2 Computing the geometric factors

Implicit in equations 8, 9, and 10 (specifically in $\hat{R}$ and R) is a point at which the electric field is being evaluated. The geometric factors $\mathbf{R2}_{ij}$, $\mathbf{R1L}_{ij}$, and $\mathbf{R1T}_{ij}$ represent the contributions of charge segment j and current segment j to the electric field at current segment i, but we didn't specify which point on the current segment. We could use the center of the segment, but for better results we average over multiple points along the segment. This is sort of like computing the average electric field on the segment, but not quite, since the retarded time is different at different points on the segment; we average the geometric factors over the entire segment but only use the retarded time for the center of the segment. We will demonstrate how equations 8, 9, and 10 are evaluated for a particular source segment j and a particular field point on current segment i in a channel with straight-line geometry; to compute an entry in one of these matrices, this result is averaged over a number of points along segment i.

We start with $\mathbf{R2}_{ij}$. Suppose charge segment j has endpoints at c and d (with $c < d$), and suppose we want to evaluate equation 8 at a point x on current segment i. First we need to evaluate the dot product $\hat{I}_i \cdot \hat{R}$. Both unit vectors point along the channel, so the dot product will be 1 or -1, depending on whether they point the same way. All the current segments are oriented the same way, so $\hat{I}_i$ always points in the positive direction. $\hat{R}$ points from the source point to the field point (from segment j to point x). If $x \geq d$ then $\hat{R}$ points in the positive direction, if $x \leq c$ then $\hat{R}$ points in the negative direction, and if $c < x < d$ (which is possible when segments i and j overlap) then parts of the segment are on either side of x and will contribute with opposite sign.

The only other part to deal with (other than multiplying by constants) is the integral of $1/R^2$. This integral diverges at $R = 0$, so we'll use the thin-wire approximation to replace R^2 with $R^2 + a^2$. Since R is just a distance along the channel, we can change dl' to dR, as long as we're careful about the limits of integration. The antiderivative is

$$\int \frac{dR}{R^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{R}{a}\right) + C. \tag{12}$$

If $c < x < d$, we break the integral into two parts; otherwise, we simply evaluate the antiderivative from one end of the channel to the other. Putting this all together (constants, dot product, and antiderivative), we get the following result:

$$\mathbf{R2}_{ij} = \begin{cases} \dfrac{-1}{4\pi\epsilon_0 l_j}\left[F(d-x) - F(c-x)\right], & x \leq c \\ \dfrac{1}{4\pi\epsilon_0 l_j}\left[-F(d-x) - F(c-x)\right], & c < x < d \\ \dfrac{1}{4\pi\epsilon_0 l_j}\left[F(d-x) - F(c-x)\right], & x \geq d \end{cases} \tag{13}$$

where $F(R)$ is the antiderivative given in equation 12. Note that in the $c < x < d$ case, there are only two terms because $F(0) = 0$. Again, this calculation is averaged for values of x along segment i to give the actual value of $\mathbf{R2}_{ij}$. We calculate $\mathbf{R1L}_{ij}$ in a similar way. This time the relevant antiderivative is

$$\int \frac{dR}{\sqrt{R^2 + a^2}} = \log\left(\sqrt{R^2 + a^2} + R\right) + C, \tag{14}$$

and the overall result (again, averaging along segment i) is:

$$\mathbf{R1L}_{ij} = \begin{cases} \dfrac{-1}{4\pi\epsilon_0 c l_j}\left[F(d-x) - F(c-x)\right], & x \leq c \\ \dfrac{1}{4\pi\epsilon_0 c l_j}\left[2F(0) - F(d-x) - F(c-x)\right], & c < x < d \\ \dfrac{1}{4\pi\epsilon_0 c l_j}\left[F(d-x) - F(c-x)\right], & x \geq d \end{cases} \tag{15}$$

where $F(R)$ is the antiderivative given in equation 14. Finally, we turn to $\mathbf{R1T}_{ij}$. Fortunately, the dot product $\hat{I}_i \cdot \hat{I}_j$ is always 1, so we can simply integrate over segment j (then average over segment i):

$$\mathbf{R1T}_{ij} = \frac{1}{4\pi\epsilon_0 c^2}\left[F(d-x) - F(c-x)\right] \tag{16}$$

where $F(R)$ is the antiderivative given in equation 14.

4.3 Interpolating the charges and currents

5 The Structure of the Matrix

5.1 Solving for the next currents

The heart of the model is the system of linear equations that solves for the currents at the next time step. This system has the form

$$\begin{aligned} I_1 R_l &= c_{11} I_1 + c_{12} I_2 + c_{13} I_3 + \cdots + c_{1,n-1} I_{n-1} + d_1 \\ I_2 R_l &= c_{21} I_1 + c_{22} I_2 + c_{23} I_3 + \cdots + c_{2,n-1} I_{n-1} + d_2 \\ I_3 R_l &= c_{31} I_1 + c_{32} I_2 + c_{33} I_3 + \cdots + c_{3,n-1} I_{n-1} + d_3 \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned} \tag{17}$$

where the left side of each equation is the electric field as determined by Ohm's law and the right side is the field as computed from the EFIE. The currents $I_0, I_1, \ldots, I_{n-2}$ are the unknown currents at the next time step, each $c_{ij} I_j$ term is the contribution of current j to electric field at current segment i, and d_i is the contribution of earlier charges and currents (and the applied field) to the field at current segment i. In matrix form, we have

$$R_l \vec{I} = \left[c_{ij} \right] \cdot \vec{I} + \vec{d}. \tag{18}$$

Rearranging, we get

$$\left(\left[c_{ij} \right] - R_l \mathbb{I} \right) \vec{I} = -\vec{d}, \tag{19}$$

where $\mathbb{I}$ is the identity matrix. Defining $\mathbf{M}$ to be the matrix on the left side of equation 19, and defining $\vec{b} = -\vec{d}$ (to avoid negative signs), we simply have

$$\mathbf{M} \vec{I} = \vec{b}. \tag{20}$$

5.2 Updating the history matrix

We need to do three things to produce the next history vector: 1) shift the current history down to make room for the new time step, 2) compute the new currents and insert them in the history vector, and 3) update the charges stored in the vector. All three of these are done by multiplying a matrix by the old history vector (and the sum of these matrices is the overall transition matrix). Each of these matrices will be briefly discussed here, and the details of their implementation will be provided in the next section.

The first step is handled by the matrix $\mathbf{S}$. It shifts each time step down in the vector, except for the last time step, which is no longer stored. It leaves zeros at the top of the vector (where the new currents will be inserted), and preserves the bottom of the vector (the charges and the 1 for the applied field). That is, if

$$\vec{h}_{\text{old}} = \begin{bmatrix} \vec{I}^{(m-1)} & \vec{I}^{(m-2)} & \ldots & \vec{I}^{(1)} & \vec{I}^{(0)} & \vec{Q}^{(0)} & 1 \end{bmatrix}, \tag{21}$$

then multiplying the history vector by $\mathbf{S}$ gives

$$\mathbf{S} \vec{h}_{\text{old}} = \begin{bmatrix} \vec{0} & \vec{I}^{(m-1)} & \vec{I}^{(m-2)} & \ldots & \vec{I}^{(2)} & \vec{I}^{(1)} & \vec{Q}^{(0)} & 1 \end{bmatrix}. \tag{22}$$

The second step is a little more involved. From equation 20, we see that the new currents are $\vec{I} = \mathbf{M}^{-1} \vec{b}$. Since $\mathbf{M}$ does not depend on the charge/current history or in any other way evolve from step to step, we can compute it once and use its inverse in the transition matrix. The vector $\vec{b}$, however, does depend on the charge/current history, so we'll need a way of computing it from the history vector. In fact, we can compute a matrix $\mathbf{A}$ such that $\mathbf{A}\vec{h} = \vec{b}$. The new currents are then

$$\vec{I} = \mathbf{M}^{-1}\mathbf{A}\vec{h}. \tag{23}$$

The vector $\vec{I}$ is only contains the $n-1$ new currents, so we multiply it by another matrix $\mathbf{E}$ to extend it to the length of the history vector. That is, if we start with the history vector in equation 21, we now have

$$\mathbf{E}\mathbf{M}^{-1}\mathbf{A}\vec{h}_{\text{old}} = \begin{bmatrix}\vec{I}^{(m)} & \vec{0}\end{bmatrix}. \tag{24}$$

The third and final step is handled by the matrix $\mathbf{Q}$; it updates the charges from the most ancient time step in $\vec{h}_{\text{old}}$ to the most ancient time step in $\vec{h}_{\text{new}}$ (which is one time step later). The charges from $\vec{h}_{\text{old}}$ are carried along to $\vec{h}_{\text{new}}$ by $\mathbf{S}$, so all $\mathbf{Q}$ has to do is compute the change in the charges from one time step to the next (which is done by multiplying the net current into each segment by the time interval). Again starting with equation 21, the vector produced by $\mathbf{Q}$ is

$$\mathbf{Q}\vec{h}_{\text{old}} = \begin{bmatrix}\vec{0} & \vec{Q}^{(1)} - \vec{Q}^{(0)} & 0\end{bmatrix}. \tag{25}$$

Adding together the matrices from these three steps gives the overall transition matrix,

$$\mathbf{G} = \mathbf{S} + \mathbf{E}\mathbf{M}^{-1}\mathbf{A} + \mathbf{Q} \tag{26}$$

When this matrix acts on the vector in equation 21, it produces the new history vector,

$$\vec{h}_{\text{new}} = \mathbf{G}\vec{h}_{\text{old}} = \begin{bmatrix}\vec{I}^{(m)} & \vec{I}^{(m-1)} & \dots & \vec{I}^{(2)} & \vec{I}^{(1)} & \vec{Q}^{(1)} & 1\end{bmatrix}. \tag{27}$$

6 Generating the Matrix

To generate the transition matrix, we compute the matrices $\mathbf{S}$, $\mathbf{E}$, $\mathbf{Q}$, $\mathbf{M}$, and $\mathbf{A}$ one at a time and then combine them as in equation 26.

6.1 Moving things around

The matrix $\mathbf{S}$ is responsible for shifting the currents down one time step in the history vector, while carrying along the charges and the last element in the vector. The currents are shifted down by a diagonal of $(m-1)(n-1)$ ones starting with $\mathbf{S}_{n,1}$, and the currents and applied field entry are carried forward by $n+1$ ones at the end of the main diagonal. As an example, if $n = 4$ and $m = 3$, then

$$\mathbf{S} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{28}$$

The matrix $\mathbf{E}$ is even simpler. All it does is extend an $(n-1)\times 1$ vector to an $m(n-1)+n+1\times 1$ vector by adding zeros; it is simply the $(n-1)\times(n-1)$ identity with $(m-1)(n-1)+n+1$ rows of zeros below it. Again using $n=4$ and $m=3$, it would be

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

6.2 Updating the charges

The job of $\mathbf{Q}$ is to update the charges stored in the history vector. For instance, to advance the simulation from step j to step $j+1$, the charges need to be updated from $Q^{(j)}$ to $Q^{(j+1)}$. This is done by adding the average net currents into each charge segment multiplied by the time interval ($\Delta Q = I\delta t$). For $n=4$ and $m=3$, the matrix is

$$\mathbf{Q} = \frac{\delta t}{2}\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&-1&0&0&0&0&0&0&0 \\ 0&0&0&1&-1&0&1&-1&0&0&0&0&0&0 \\ 0&0&0&0&1&-1&0&1&-1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}. \tag{30}$$

The n nonzero rows (four in the example) in $\mathbf{Q}$ represent the charges added to each of the n charge segments moving from step j to step $j+1$. The first $n-1$ nonzero columns represent the contributions from the $n-1$ currents at step $j+1$, and the last $n-1$ nonzero columns represent the contributions from the $n-1$ currents at step j. Consider the first three nonzero columns of the four nonzero rows in the example above. There is a -1 in the first row because the first current segment flows out of the first charge segment. The second row has a 1 for the first current segment (which flows into the second charge segment) and a -1 for the second current segment (which flows out of the second charge segment). The other two rows are similar. This pattern of 1s and -1s is repeated three columns over (and the entire matrix is divided by 2) in order to average the currents at steps j and $j+1$.

6.3 Solving for the new currents

7 Two-Stage Stepping

- Motivation for two-stage averaging scheme
- Details of implementation

8 Stability Analysis

- Overview of eigenvalue-based stability analysis
- Pretty plots of eigenvalues! (and comments thereupon)
- Effect of parameters on stability

9 Equilibrium Charge Distribution

- Plots of equilibrium charge distribution with and without external field
- Comparison to results in Jackson paper

10 Off-Channel Field

- Details of off-channel field calculation
- Effect of distance and orientation on measured field

11 Stepped Leaders

- Implementation of adding segments
- Plots of off-channel field, comparison to HAMMA data

12 Frequency Response of Channel to External Field

- Power spectral density of radiated field

Appendices

- Briefly describe functions in microTDFL.py and their inputs/outputs
- List of symbols/variable names/matrix names and their meanings?

References

[1] Carlson, Brant. "Mathematical development of the TDFL." Will properly cite later.

[2] Jackson, John David. Paper on charge density on straight wire.

[3] Cite Jackson's E&M book?