Proof

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$$\nu = (0 \to 0) \to (0 \to 0)$$

Let's consider t - two-digit function and N, M - two numerals and f as general number function, then their types are:

$$\begin{aligned} t &: \nu \to \nu \to \nu \\ N, M &: \nu \\ f &: 0 \to 0 \end{aligned}$$

Then lets consider normal form of $t' = tNMf : 0 \to 0$

Import to notice that t is closed.

We will proof that any subterm of t' has one of types from set $S = \{0, 0 \to 0, \nu\}$.

Declare function for our comfort in usage T which return type of expression in some context:

$$\forall e - \text{term}, \Gamma - \text{context} : T(\Gamma, e) = \tau, \text{ if } \Gamma \vdash e : \tau$$

For this let introduce Lemma which will helps us in it:

Lemma 0.1. *Consider some expression in normal form e in context Γ with such requirements:*

1. *Lets consider e has type τ then τ must be in set S*
2. *For each free variable $x \in FV(e)$ is true that $T(\Gamma, x) \in S$*

Then each subterm of e also has type which is in set S

Proof. We will proof this by induction on each step considering expression constructed by one of the possible ways.

Consider all possible deconstructions of expression e

1. $e = c_i$ where $c_i \in FV(e)$
2. $e = \lambda x.B$ then $\tau = t_1 \to t_2$
3. $e = PQ$ then $\exists \alpha. P : \alpha \to \tau, Q : \alpha$

For case 1 the lemma is explicitly satisfied since e doesn't have any subterm, except itself so lemma is correct

For case 2 consider few options

Option 1:

$$\begin{cases} \tau & = 0 \to 0 \\ t_1 & = 0 \\ t_2 & = 0 \end{cases}$$

Then $FV(B) = FB(e) \cup \{x\}$ and because of B exists in context $\Gamma' = \Gamma, x : 0$ and has type 0 then by induction case is clear that each B's subterm has correct type, but because if x s is subterm B, then x is subterm of e.

Other option:

$$\begin{cases} \tau & = \nu \\ t_1 & = 0 \to 0 \\ t_2 & = 0 \to 0 \end{cases}$$

Same proof as for option 1.

For case 3 let's recall that e in normal form that means P - is not lambda (otherwise it would be reduced) and two options

1. $P \in FV(e)$ than the same variants for type τ as for case 2 with induction step for Q
2. $P = P_2Q_1$

We can repeat choosing option 2 finite number of times because we are in normal form.

Then consider

$$e = PA_1A_2...A_k, \text{where } P \in FV(e)$$

Again few options:

$$\begin{cases} e & : 0 \\ P & : \nu \\ A_1 & : 0 \to 0 \\ A_2 & : 0 \end{cases}$$

$$\begin{cases} e & : 0 \\ P & : 0 \to 0 \\ A_1 & : 0 \end{cases}$$

$$\begin{cases} e & : 0 \to 0 \\ P & : \nu \\ A_1 & : 0 \to 0 \end{cases}$$

Again induction

$\square$