Translation

April 6, 2023

As is well known, in the type-free λ-calculus exactly the recursive functions are $definable$[1]. The concept used here of the definability of a function is also useful for the λ-calculus with types. We answer the question *here*[2] according to the then definable functions.

Types are 0 and with σ, τ also $(\sigma \to \tau)$. Terms (denoted by r^τ, s^τ, t^τ) formed from variables with types by application and λ-abstraction. Type indices that result from the context or that are immaterial, we often leave it out. Terms that differ only by bound renaming divorce, are identified. The relation $t \models t'$ (t is reducible to t') becomes inductively defined by

1. $(\lambda x.t)s \models t_x[s]$
2. When $t \models t'$ and $s \models s'$, so $ts \models t's'$
3. When $t \models t'$, so $\lambda x.t \models \lambda x.t'$
4. $t \models t$
5. When $t \models t'$ and $t' \models t''$, so $t \models t''$

A term is called in normal form if it has no subterms of the form $(\lambda x.t)s$. As is well known, for every term t there is a uniquely determined normal form t' with $t \models t'$. Two terms are equal if they have the same normal form. One can now introduce natural numbers as terms $\overline{n} \equiv \lambda\alpha.\alpha^n$ vom type $\nu \equiv (0 \to 0) \to (0 \to 0)$; where α^n is the n-th iterate of α, i.e. $\equiv \lambda x.\alpha(\alpha...(\alpha x)...)$. Every closed term of type ν in normal form is an $\overline{n}$. So defined every closed term t of the type $\nu \to (\nu \to ...(\nu \to \nu)...)$ is a number-theoretic one Function f determined by $t\overline{n_1}, ...\overline{n_k} = \overline{f(n_1, ..., n_k)}$. For example *will*[3]

1. $n + m$ define as $\lambda FG\alpha.(F\alpha) \circ (G\alpha)$
2. $n \cdot m$ define as $\lambda FG.F \circ G$
3. k (constant function) define as $\lambda F\alpha.\alpha^k$
4. $$d(n, m, i) = \begin{cases} n & i = 0 \\ m & i \neq 0 \end{cases}$$

   define as $\lambda FGH\alpha x.H(\lambda y.G\alpha x)(F\alpha x)$

[F, G, H variables of type $\nu, t^{\tau\to\tau} \circ s^{\tau\to\tau} \equiv \lambda x^\tau.t(sx)$]. The set of bare functions is apparently closed against insertions. So are all Polynomials definable, and more generally all functions resulting from polynomials Case distinction according to disappearance or non-disappearance of arguments are definable; in the 2-digit case, these are the functions

$$f(n,m) = \begin{cases} k & n = 0 \text{ and } m = 0 \\ P_1(m) & n = 0 \text{ and } m \neq 0 \\ P_2(m) & n \neq 0 \text{ and } m = 0 \\ P_3(n,m) & n \neq 0 \text{ and } m \neq 0 \end{cases}$$

with polynomials P_1, P_2, P_3. In the following it will be shown that this is all definable functions are.

So t defines a roughly 2-place function. Bring $tFG\alpha$ to normal form.

Each subterm (of the normal form of $tFG\alpha$) has one of the types 0, $0 \to 0$, or ν. Each subterm of type ν is identical to F or G. Possible subterms from Type $0 \to 0$ are:

1. α
2. With s also Fs, Gs.
3. With $s_1, ..., s_q$, formed according to (1), (2) also $\lambda y.s_1(...s_q(z)...)$, where z can also be identical to y.

In the following, s stands for subterms of the type $0 \to 0$. — For F, G set $\overline{n}$, $\overline{m}$, where initially $n, m \geq 1$ is assumed. By induction over s one shows: Each $s' \equiv s_{F,G}[\overline{n}, \overline{m}]$ is equal to $\alpha^{P(n,m)}$ or equal to a constant function $\lambda y.\alpha^{P(n,m)}z$ ($P(n,m)$ polynom). Proof: To (2)

$$\begin{aligned} (\lambda\beta.\beta^n)\alpha^{P(n,m)} &= \alpha^{P(n,m)\cdot n} \\ (\lambda\beta.\beta^n)(\lambda y.\alpha^{P(n,m)}z) &= \lambda y.\alpha^{P(n,m)}z \qquad\qquad (n \geq 1) \end{aligned}$$

To (3). Case 1: None of the s_i' is constant.

$$\lambda y.s_1'(...s_q'(z)...) = \lambda y.\alpha^{P_1(n,m)+\cdots+P_q(n,m)}z$$

Case 2: There is a first constant s_i', say $s_i' = \lambda\tilde{y}.\alpha^{P_i(n,m)}\tilde{z}$

$$\lambda y.s_1'(...s_q'(z)...) = \lambda y.\alpha^{P_1(n,m)+\cdots+P_q(n,m)}z$$

Since the normal form of $tFG\alpha$ does not contain a free variable of type 0, it is nach replacing F, G with $\overline{n}$, $\overline{m}$ ($n, m \geq 1$) equal to an $\alpha^{P(n,m)}$.

If about $n = 0, m \geq 1$, then one can replace all outermost subterms of the form Fs by $\lambda x.x$ and gets $\alpha^{P(m)}$ like before.