Basics of ML

Week - 4

1. Bayes' Theorem in Medical Diagnosis Problem

Bayes' Theorem is used to update the probability of a hypothesis given new evidence. The theorem is stated as:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Problem: Let's consider a patient is tested positive for a disease. The test is 99% sensitive and 95% specific. The prevalence of the disease in the population is 0.5%. If the patient tests positive, then we need to find the probability that they actually have the disease or not.

Given:

- Sensitivity $P(\text{Positive}|\text{Disease}) = 0.99$
- Specificity $P(\text{Negative}|\text{No Disease}) = 0.95$
- Prevalence $P(\text{Disease}) = 0.005$

We need $P(\text{Disease}|\text{Positive})$.

First, we need to calculate $P(\text{Positive})$:

$$P(\text{Positive}) = P(\text{Positive}|\text{Disease}) \cdot P(\text{Disease}) + P(\text{Positive}|\text{No Disease}) \cdot P(\text{No Disease})$$

Where:

$$P(\text{Positive}|\text{No Disease}) = 1 - \text{Specificity} = 1 - 0.95 = 0.05$$

$$P(\text{No Disease}) = 1 - P(\text{Disease}) = 0.995$$

So,

$$P(\text{Positive}) = (0.99 \cdot 0.005) + (0.05 \cdot 0.995)$$

$$P(\text{Positive}) = 0.00495 + 0.04975$$

$$P(\text{Positive}) = 0.0547$$

Now applying Bayes' Theorem:

$$P(\text{Disease}|\text{Positive}) = \frac{0.99 \cdot 0.005}{0.0547}$$

$$P(\text{Disease}|\text{Positive}) \approx \frac{0.00495}{0.0547}$$
$$P(\text{Disease}|\text{Positive}) \approx 0.0905$$

So, the probability that the patient actually has the disease given a positive test result is approximately 9.05%.

2. Finding Eigenvalues and Eigenvectors of a Given Matrix

Problem: Let's consider a given matrix A:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

we need to find the eigenvalues and corresponding eigenvectors.

Solution: **Eigenvalues**

$$\det(A - \lambda I) = 0$$

The determinant of $A - \lambda I$ is:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

Expanding along the first row:

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (2-\lambda)[(3-\lambda)(4-\lambda) - 1 \cdot 1] - 1[1 \cdot (4-\lambda) - 1 \cdot 1] + 1[1 \cdot 1 - 1 \cdot (3-\lambda)]$$

$$= (2-\lambda)[12 - 7\lambda + \lambda^2 - 1] - (4 - \lambda - 1) + (1 - 3 + \lambda)$$

$$= (2-\lambda)[\lambda^2 - 7\lambda + 11] - (3-\lambda) + (\lambda - 2)$$

$$= (2-\lambda)(\lambda^2 - 7\lambda + 11) - 5 + 2\lambda$$

$$= 2\lambda^2 - 14\lambda + 22 - \lambda^3 + 7\lambda^2 - 11\lambda - 5 + 2\lambda$$

$$= -\lambda^3 + 9\lambda^2 - 23\lambda + 17$$

Thus, the characteristic equation is:

$$-\lambda^3 + 9\lambda^2 - 23\lambda + 17 = 0$$

Solving for the roots of this cubic equation (eigenvalues):

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 6$$

Eigenvectors

$$\det(A - \lambda I)x = 0$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Solving $(A - I)x = 0$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvector is:

$$x = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solving $(A - 2I)x = 0$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvector is:

$$x = k \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 6$:

$$A - 6I = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Solving $(A-6I)x=0$:

$$\begin{pmatrix} -4 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvector is:

$$x = k \begin{pmatrix} 1 \\ \frac{5}{4} \\ \frac{11}{4} \end{pmatrix}$$

3. Calculating the Determinant and Inverse of a Matrix

Problem: Let's consider a matrix B:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

we need to calculate the determinant and, find its inverse.
Solution: **Determinant**:

$$\det(B) = 1 \cdot \det \begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 0 & 4 \\ 5 & 0 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 0 & 1 \\ 5 & 6 \end{pmatrix}$$

$$= 1 \cdot (1 \cdot 0 - 4 \cdot 6) - 2 \cdot (0 \cdot 0 - 4 \cdot 5) + 3 \cdot (0 \cdot 6 - 1 \cdot 5)$$
$$= -24 + 40 - 15$$
$$= 1$$

Inverse: Since $\det(B) \neq 0$, the inverse exists.
Using the formula for the inverse of a matrix:

$$B^{-1} = \frac{1}{\det(B)} \cdot \mathrm{adj}(B)$$

The adjugate of B (transpose of the cofactor matrix) is:

$$\mathrm{adj}(B) = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$$

So, the inverse of B is:

$$\begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix} = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$$

Therefore, the inverse of B is:

$$B^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

4. Properties and Applications

4.1 Properties of the Normal Distribution

- **Symmetry**: The Normal distribution is symmetric about its mean μ. This means that the left and right sides of the distribution are mirror images of each other.
- **Mean, Median, and Mode**: For a Normal distribution, the mean, median, and mode are all equal and located at the center of the distribution.
- **Standard Deviation (σ)**: The spread of the distribution is determined by its standard deviation. A larger standard deviation results in a wider and flatter curve, while a smaller standard deviation results in a narrower and taller curve.

4.2 Applications of the Normal Distribution

- **Statistical Inference**: Many statistical methods, including hypothesis testing and confidence intervals, are based on the assumption that the data follows a Normal distribution.
- **Natural and Social Sciences**: The Normal distribution is used to model a wide range of natural and social phenomena, such as heights, test scores, and measurement errors.
- **Finance and Economics**: Asset returns, risk assessments, and other financial metrics often assume Normal distribution for modeling purposes.

5. Calculating Probabilities Using the Normal Distribution

Problem: Let's consider the heights of adult males are normally distributed with a mean of 70 inches and a standard deviation of 3 inches. We need to calculate the probability that a randomly selected adult male is taller than 74 inches?

Solution: Standardization of the variable:

$$Z = \frac{X - \mu}{\sigma} = \frac{74 - 70}{3} \approx 1.33$$

$Z = 1.33$, $P(Z) = 0.9082$

The probability of being taller than 74 inches, we need the area to the right of $Z = 1.33$:

$$P(X > 74) = 1 - P(Z \leq 1.33) = 1 - 0.9082 = 0.0918$$

Thus, the probability that a randomly selected adult male is taller than 74 inches is approximately 0.0918.