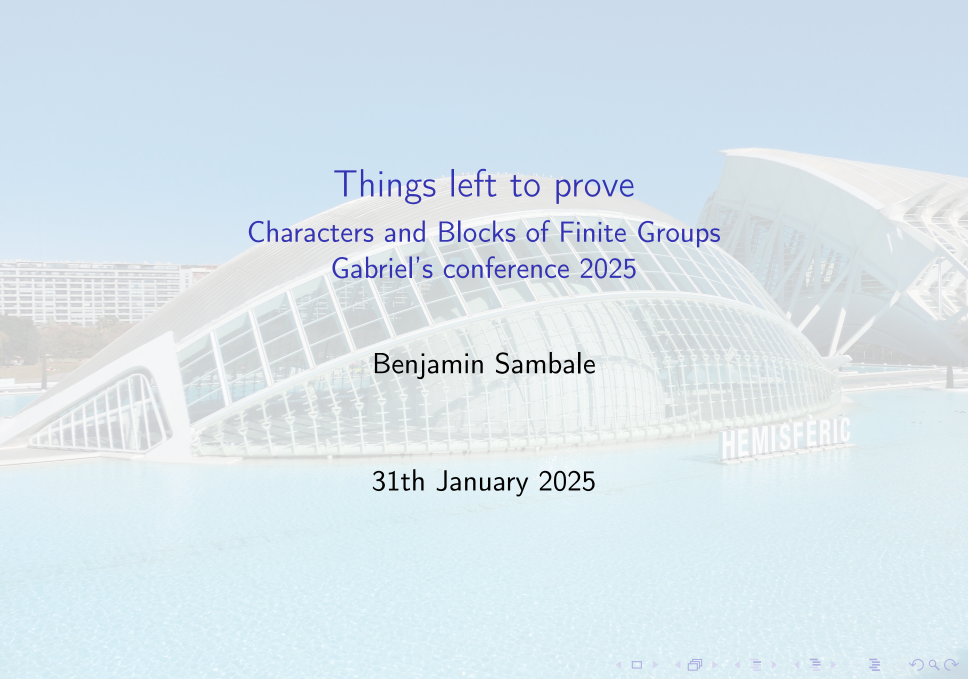

Things left to prove

Characters and Blocks of Finite Groups
Gabriel's conference 2025

Benjamin Sambale

31th January 2025

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- 2024: "All I would like to prove has been proved!", e-mail correspondence

Automorphisms

Let G be a finite group.

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Definition (Automorphism tower)

Let $\mathrm{Aut}^1(G) := \mathrm{Aut}(G)$, $\mathrm{Aut}^2(G) := \mathrm{Aut}(\mathrm{Aut}(G)), \ldots$.

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Theorem (Wielandt)

If $\mathrm{Z}(G) = 1$, *then* $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^{n+1}(G)$ *for some* n.

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Theorem (Hamkins)

The transfinite automorphism tower of any group is bounded by some cardinal number.

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Problem (Kourovka notebook 11.123)

- Is there a constant c such that $|\mathrm{Aut}^n(G)| < c$ for all n?
- Is $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^{n+1}(G)$ for some n?
- Can $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^m(G) \not\cong \mathrm{Aut}^{n+1}(G)$ with $n < m$ happen?

Automorphisms

Example

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- Center can stay non-trivial: $\mathrm{Aut}(D_8) \cong D_8$.
- The sequence can decrease arbitrarily long: $\mathrm{Aut}(C_{2\cdot 3^n}) \cong C_{2\cdot 3^{n-1}}$
- For $G = \texttt{SmallGroup}(32, 13)$ we have

n	1	2	3	4	5
$\lvert\mathrm{Aut}^n(G)\rvert$	2^7	2^{13}	2^{28}	2^{83}	?

Homomorphisms

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For every n, the number of $x \in G$ such that $x^n = 1$ is divisible by $\gcd(n, |G|)$.

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Theorem (Yoshida)

For every finite abelian group A,

$$|\mathrm{Hom}(A, G)| \equiv 0 \pmod{\gcd(|A|, |G|)}.$$

Homomorphisms

Conjecture (Asai–Yoshida)

For every finite group H,

$$|\mathrm{Hom}(H,G)| \equiv 0 \pmod{\gcd(|H/H'|,|G|)}.$$

Conjugacy classes

- Let $k(G)$ be the number of conjugacy classes of G.

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For every $\epsilon > 0$ there exists $\delta > 0$ such that

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Conjecture (Folklore?)

Is there a constant c such that $k(G) > c \log |G|$ for all G?

Constituents

Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G.

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Problem (Knutson–Murray)

Let P be a p-group and $\chi \in \mathrm{Irr}(P)$. Is there a generalized character ψ such that $\chi\psi$ is the regular character of P?

Constituents

Call $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ fully ramified if θ^G is a multiple of an irreducible character.

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Let $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ such that all constituents of θ^G have the same degree. Then G is solvable.

Conjecture holds whenever θ^G has at most two constituents (Higgs).

Field of values

For $\chi \in \mathrm{Irr}(G)$ define the abelian number fields

$$\mathbb{Q}(\chi) := \mathbb{Q}\big(\chi(g) : g \in G\big) \subseteq \mathbb{Q}_{|G|} \subseteq \mathbb{C},$$
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By the Kronecker–Weber theorem, every abelian number field lies in some $\mathbb{Q}_n$.

Theorem (Fein–Gordon)

For every abelian number field F there exist a group G and $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = F$.

Field of values

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Example (quadratic fields)

Let $d \mid 210$ or $d \in \{-231, -11, 13, 17\}$. Then there exists G with $\mathbb{Q}(G) \cong \mathbb{Q}(\sqrt{d})$.

Field of values

Theorem (Robinson–Thompson)

For $n > 24$ we have

$$\mathbb{Q}(A_n) = \mathbb{Q}\Big(\sqrt{(-1)^{\frac{p-1}{2}}p} : p \text{ odd prime } \leq n,\ p \neq n-2\Big).$$

In particular, $\mathrm{Gal}(\mathbb{Q}(A_n)|\mathbb{Q})$ *is an elementary abelian* 2*-group.*

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- Is there a solvable group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{-5})$?

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Problem (Hung–Tiep)

Let $z = \zeta_1 + \ldots + \zeta_n \in \mathbb{C}$ be a sum of roots of unity. Let $c(z)$ be the smallest m such that $z \in \mathbb{Q}_m$. Is $|\mathbb{Q}_{c(z)} : \mathbb{Q}(z)| \leq n$?

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- Let $\mathbb{Z}_G$ be the ring of integers of $\mathbb{Q}(G)$.

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Example

Conjecture holds for nilpotent groups. For $G = C_{15} \rtimes D_{16}$,

$$\mathbb{Z}_G/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}.$$

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Theorem (Navarro–S.)

If G is π-separable with solvable Hall π-subgroups, then

$$l(G) = \sum_P k^0(\mathrm{N}_G(P)/P)$$

where P runs through the nilpotent π-subgroups of G up to conjugation.

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For $\pi = \{p\}$, this is Alperin's weight conjecture for p-solvable groups.

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Define the weight of a group P by

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Conjecture (Navarro–S.)

For every π-separable group G,

$$l(G) = \sum_P \mu(P) k^0(\mathrm{N}_G(P)/P)$$

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- If P is solvable, but not nilpotent, then $\mu(P) = 0$ (Carter subgroups).
- On the other hand,

P	A_5	A_6	A_7	A_8	$\mathrm{PSL}(3,4)$	A_5^2	$S_n (3 \leq n \leq 11)$
$\mu(P)$	1	-2	1	-1	6	-1	0

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A subset $J \subseteq \operatorname{Irr}(G)$ is a union of p-blocks if and only if

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Conjecture (Harada)

It is enough to fix $g = 1$ in Osima's theorem.

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Conjecture (ordinary basic set, Geck)

There exist $\chi_1, \ldots, \chi_l \in \mathrm{Irr}(B)$ such that $\chi_1^0, \ldots, \chi_l^0$ is a $\mathbb{Z}$-basis for the ring of generalized Brauer characters of B.

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This holds in a strong sense for "most" groups of Lie type (Brunat–Dudas–Taylor).

Number of characters

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- Let d be the defect of B. Then $p^d[\chi, \chi]^0 \in \mathbb{N}$ for all $\chi \in \mathrm{Irr}(B)$.

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- The case $\chi = 1_G$ was conjectured by Murai and implies Frobenius conjecture: G is p-nilpotent iff $|G^0| = |G|_{p'}$ (known via CFSG).

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Example

Conjecture holds for symmetric groups and "ATLAS groups". For the principal 2-block B of the Monster, $\gamma(B) \approx 39.5$.

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- *25th anniversary of my wedding. I have promised my wife not to think on mathematics for a couple of days. Now you are making my promise easy to break. . .*