On redundant Sylow subgroups

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Abstract

A Sylow p-subgroup P of a finite group G is called *redundant* if every p-element of G lies in a Sylow subgroup different from P. Generalizing a recent theorem of Maróti–Martínez–Moretó, we show that for every non-cyclic p-group P there exists a solvable group G such that P is redundant in G. Moreover, we answer several open questions raised by Maróti–Martínez–Moretó.

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1 Introduction

By Sylow's theorem, every p-element of a finite group G lies in some Sylow p-subgroup of G. In the past, group theorists were interested in groups with *trivial-intersection* Sylow subgroups, i. e. every non-trivial p-element lies in a *unique* Sylow subgroup. In the present paper we are interested in opposite situation: groups whose p-elements all lie in at least two Sylow subgroups for a fixed prime p. Mikko Korhonen [9] has asked 10 years ago whether such groups actually exist. A positive answer was given by Jack Schmidt [9] using a group G with elementary abelian Sylow p-subgroups. Most recently, Maróti–Martínez–Moretó [11, Theorem A] have shown that for a given p-group P of exponent p there exists a solvable group G with $P \in \mathrm{Syl}_p(G)$ such that every element of P lies in a Sylow subgroup different from P. They called such a Sylow subgroup *redundant* in G, and so do we (by Sylow's theorem, either all or no Sylow p-subgroup is redundant and in the former case every p-element lies in at least two Sylow subgroups).

In general, it is easy to see that redundant Sylow subgroups must be non-cyclic. Maróti–Martínez–Moretó have speculated on p. 483 that the restriction on the exponent of P in their theorem might be superfluous. In this paper, we show in Theorem 1 that this is indeed the case. In contrast to the proof of [11, Theorem A] (which depends a deep theorem of Turull and the solvable case of Thompson's theorem), our proof is elementary. Using a refined method in Theorem 2, we also provide examples where the number of Sylow p-subgroups $\nu_p(G)$ of G only depends on p. In particular, we show that $\nu_2(G) = 27$ is the smallest possible value for a group G with a redundant Sylow 2-subgroup. We also the determine the minimum of $\nu_p(G)$ for p-solvable groups in Theorem 7. This is a contribution to [11, Question 8.5].

Let G_p be the set of p-elements of G. In [11, p. 846], the authors state that there are very few groups G such that $|G_p| < \nu_p(G)$ and only examples with elementary abelian Sylow p-subgroups are known. Our

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construction yields such examples for every non-cyclic p-group P. This leads to a negative answer to [11, Question 8.7]. On the other hand, we provide a positive answer to [11, Question 8.8] in Theorem 9.

2 Results

Theorem 1. *For every non-cyclic p-group P and every prime $q \neq p$ there exists an elementary abelian q-group N such that P acts on N and $G := N \rtimes P$ has the following properties:*

(i) P is redundant in G.

(ii) $|G_p| < \frac{1}{q^{p-1}}|G|$.

(iii) G_p is covered by $\frac{1}{q^{p-1}}\nu_p(G)$ Sylow p-subgroups.

Proof.

(i) Let V be the regular $\mathbb{F}_qP$-module with basis $B := \{v_x : x \in P\}$. Then P acts trivial on $Z := \langle \prod_{x\in P} v_x \rangle$. Let $N := V/Z \cong C_q^{|P|-1}$ and $G := N \rtimes P$. Since P acts transitively on B, it follows that $\mathrm{C}_N(P) = 1$ and $\mathrm{N}_G(P) = P$. Let $x \in P$. Since P is not cyclic,

$$w := \prod_{c\in\langle x\rangle} v_c Z \in \mathrm{C}_N(x) \setminus \{1\}.$$

Hence, $x = wxw^{-1} \in wPw^{-1} \in \mathrm{Syl}_p(G) \setminus \{P\}$. This shows that P is redundant in G.

(ii) Let $R \subseteq P$ be a set of representatives for the conjugacy classes of P. By construction, every p-element of G is conjugate to a unique element $x \in R$. Let $g \in \mathrm{C}_G(x)$ and write $g = ny$ with $n \in N$ and $y \in P$. Then $xy \equiv xg \equiv gx \equiv yx \pmod N$ and therefore $[x, y] \in P \cap N = 1$. This shows that $y \in \mathrm{C}_P(x)$, $n \in \mathrm{C}_N(x)$ and $\mathrm{C}_G(x) = \mathrm{C}_N(x)\mathrm{C}_P(x)$. Every right coset C of $\langle x\rangle$ in P determines an element $w_C := \prod_{c\in C} v_c \in \mathrm{C}_V(x)$. It is easy to check that the elements $\{w_C : C \in \langle x\rangle\backslash P\}$ form a basis of $\mathrm{C}_V(x)$. This yields

$$|\mathrm{C}_N(x)| = |\mathrm{C}_V(x)/Z| = q^{|P:\langle x\rangle|-1} \geq q^{p-1}.$$

Hence,

$$|G_p| = \sum_{x\in R} |G : \mathrm{C}_G(x)| = \sum_{x\in R} |P : \mathrm{C}_P(x)||N : \mathrm{C}_N(x)| < \frac{|N|}{q^{p-1}} \sum_{x\in R} |P : \mathrm{C}_P(x)| = \frac{1}{q^{p-1}}|G|.$$

(iii) Since P is non-cyclic, there exist maximal subgroups $P_1, \ldots, P_{p+1} \leq P$ such that $P = P_1 \cup \ldots \cup P_{p+1}$. Then $N_i := \mathrm{C}_N(P_i) \cong C_q^{p-1}$ for $i = 1, \ldots, p + 1$ by the argument of (ii). Since $P_j \trianglelefteq P$, each P_i acts on N_j. For $i \neq j$, we have $N_i \cap N_j = \mathrm{C}_N(\langle P_i, P_j\rangle) = \mathrm{C}_N(P) = 1$. By the Fitting decomposition (see [7, Theorem 4.34]), we obtain

$$N_j = \mathrm{C}_{N_j}(P_i) \times [P_i, N_j] = [P_i, N_j] \leq [P_i, N].$$

Since $N = N_i \times [P_i, N]$, it follows that

$$N_i \cap \prod_{j\neq i} N_j \leq N_i \cap [P_i, N] = 1.$$

Therefore, $N_1 \times \ldots \times N_{p+1} \le N$. We choose a basis $b_{i,1}, \ldots, b_{i,p-1}$ of N_i for every $i = 1, \ldots, p+1$. Then the elements $b_{i,j}$ are linearly independent and can be extended to a basis B of N. For $w \in N$ and $b \in B$ let w_b be the coefficient of w with respect to b. Define

$$T := \Bigl\{ w \in N : \forall j = 1, \ldots, p-1 : \sum_{i=1}^{p+1} w_{b_{i,j}} \equiv 0 \pmod q \Bigr\}.$$

Then $|T| = \frac{1}{q^{p-1}}|N|$. Let $n \in N$ and $x \in P$ be arbitrary. There exist i and $t \in T$ such that $x \in P_i$ and $t_b = n_b$ for all $b \in B \setminus \{b_{i,1}, \ldots, b_{i,p-1}\}$. It follows that $t^{-1}n \in N_i \le \mathrm{C}_N(x)$ and $nxn^{-1} = txt^{-1} \in tPt^{-1}$. Hence, G_p is covered by $\{tPt^{-1} : t \in T\}$. $\square$

If $q^{p-1} > |P|$ in the situation of Theorem 1, then $|G_p| < |N| = |G : \mathrm{N}_G(P)| = \nu_p(G)$ by (ii). If p or q goes to infinity, (iii) furnishes a counterexample to [11, Question 8.7]. At the same time, it provides some evidence for [11, Question 8.6]. If P contains a cyclic subgroup of index p, one can show that G_p cannot be covered by less than $\frac{1}{q^{p-1}}\nu_p(G)$ Sylow subgroups.

If P is the Klein four-group and $q = 3$, then the construction of the proof above yields the group $G \cong \texttt{SmallGroup}(108, 40)$ with $\nu_2(G) = 27$, which was mentioned in [11, Introduction]. Question 8.5 of [11] asks for the smallest possible value of $\nu_p(G)$ when G has a redundant Sylow p-subgroup. Our proof of Theorem 1 yields $\nu_p(G) = |N| = q^{|P|-1}$. We give a better bound, which only depends on p.

Theorem 2. *For every non-cyclic p-group P there exists a solvable group G such that P is redundant in G and $\nu_p(G) = q^{p+1}$, where $q > 1$ is the smallest prime power congruent to 1 modulo p.*

Proof. Since P is non-cyclic, there exist maximal subgroups $P_1, \ldots, P_{p+1} \le P$ such that $P = P_1 \cup \ldots \cup P_{p+1}$. Since $q \equiv 1 \pmod p$, the finite field $\mathbb{F}_q$ contains a primitive p-th root of unity. Hence, for $i = 1, \ldots, p+1$ there exists a 1-dimensional $\mathbb{F}_qP$-module N_i with kernel P_i. Define $N = N_1 \oplus \ldots \oplus N_{p+1}$. Since every $x \in P$ lies in some P_i, it follows that $\mathrm{C}_N(x) > 1 = \mathrm{C}_N(P)$. Now by the proof of Theorem 1(i) (or using [11, Corollary 3.2]), it follows that P is redundant in $G := N \rtimes P$ and $\nu_p(G) = |N| = q^{p+1}$ (we do not need that P acts faithfully on N). $\square$

Theorem 2 provides the following upper bounds for the minimal values of $\nu_p(G)$:

p	2	3	5	7	11	13	17	19	23	29
$\min\limits_{q \equiv 1 \pmod p} q^{p+1}$	3^3	2^8	11^6	2^{24}	23^{12}	3^{42}	103^{18}	191^{20}	47^{24}	59^{30}

Now we work in the opposite direction by finding lower bounds on $\nu_p(G)$. We following result settles the case $p = 2$.

Theorem 3. *Let G be a finite group with a redundant Sylow 2-subgroup. Then $\nu_2(G) \ge 27$.*

Proof. Let N be the kernel of the conjugation action of G on $\mathrm{Syl}_2(G)$, i. e. N is the intersection of all Sylow normalizers. Let $P \in \mathrm{Syl}_2(G)$. Since P is the unique Sylow 2-subgroup of PN, the map $\mathrm{Syl}_2(G) \to \mathrm{Syl}_2(G/N)$, $P \mapsto PN/N$ is a bijection and P is redundant in $\mathrm{Syl}_2(G)$ if and only if PN/N is redundant in G/N. Hence, we may assume that $N = 1$. Then G is a transitive permutation group of degree $\nu_2(G)$. We run through the database of all transitive groups of odd degree up to 25 in GAP [3]. For each such group we can quickly check whether the stabilizer has a normal Sylow 2-subgroup. If this is the case, we check whether G has a redundant Sylow 2-subgroup using [11, Lemmas 2.1 and 2.6]. It turns out that there are no examples with $\nu_2(G) < 27$. $\square$

With the same method, we obtain $\nu_3(G) \geq 49$ and $\nu_5(G) \geq 51$ whenever G has a redundant Sylow p-subgroup for $p = 3$ or $p = 5$ respectively. The next lemma improves [11, Theorem 8.4] with an easier proof.

Lemma 4. *Let G be a finite group with a redundant Sylow p-subgroup. Then $\nu_p(G) \geq p^2 + p + 1$.*

Proof. Let $P \in \mathrm{Syl}_p(G)$ be covered by $P_1, \ldots, P_k \in \mathrm{Syl}_p(G) \setminus \{P\}$ such that k is as small as possible. Then $P \cap P_i \neq P \cap P_j$ for $i \neq j$. Since P is not the union of p proper subgroups, we must have $k \geq p+1$. Let $g \in \mathrm{N}_P(P \cap P_i) \setminus P_i$. Then $g \notin \mathrm{N}_G(P_i)$, since otherwise $P_i\langle g\rangle$ would be a p-subgroup larger than P_i. Hence, the Sylow subgroups $P_i^{g^j}$ for $j = 1, \ldots, p$ are pairwise distinct and

$$P \cap P_i^{g^j} = P^{g^j} \cap P_i^{g^j} = (P \cap P_i)^{g^j} = P \cap P_i.$$

In this way we obtain kp Sylow p-subgroups different from P. Hence, $\nu_p(G) \geq kp+1 \geq p^2+p+1$. $\square$

Lemma 5. *Let G be a finite group with a redundant Sylow p-subgroup. Then $\nu_p(G)$ is not a prime.*

Proof. Let G be a minimal counterexample with $P \in \mathrm{Syl}_p(G)$ redundant. As in the proof of Theorem 3, we may assume that G is a transitive permutation group of prime degree $q := |\,\mathrm{Syl}_p(G)|$. By a result of Burnside, G is a subgroup of the affine group $C_q \rtimes C_{q-1}$ or a 2-transitive almost simple group (see [2, Corollary 3.5B and Theorem 4.1B]). The first case is impossible, since P must be non-cyclic. The latter case can be investigated with the classification of the finite simple groups (see [2, p. 99] or [5]). More precisely, the socle N of G is one of the following simple groups:

(i) $N = A_q$. Since the stabilizer A_{q-1} must have a normal Sylow p-subgroup, it follows that $q = 5$ and $p = 2$. By Theorem 3, neither $G = A_5$ nor $G = S_5$ has a redundant Sylow 2-subgroup.

(ii) $N = \mathrm{PSL}(2, 11)$ with $q = 11$. Here $|G : N| \leq 2$ and the stabilizer is isomorphic to A_5, so it cannot have a normal Sylow p-subgroup.

(iii) $N = M_{11} = G$ with $q = 11$. Again the stabilizer M_{10} has no normal Sylow p-subgroup.

(iv) $N = M_{23} = G$ with $q = 23$. Here the stabilizer M_{22} is simple.

(v) $N = \mathrm{PSL}(n, r)$ with $q = \frac{r^n-1}{r-1}$ where n is a prime. Suppose that $n \mid r-1$. Then $q = 1 + r + \ldots + r^{n-1} \equiv n \equiv 0 \pmod{n}$ and $q = n$. But this contradicts $q > r-1$. Hence, $\gcd(n, r-1) = 1$ and $N = \mathrm{SL}(n, r)$. Here G acts on the set of lines or hyperplanes of $\mathbb{F}_r^n$. In both cases the stabilizer, say N_v contains a copy of $\mathrm{GL}(n-1, r)$. If $n > 2$, then $|\,\mathrm{GL}(n-1, r)|$ is divisible by $r\frac{r^{n-1}-1}{r-1} = q-1$. In particular, N_v has a non-trivial Sylow p-subgroup, which cannot be normal since $\mathrm{GL}(n-1, r)$ is involved in N_v. Consequently, $n = 2$ and $q = r+1$ is a Fermat prime. Now G/N is a cyclic 2-group. For $p > 2$ it is well-known that the Sylow p-subgroup of N and G are cyclic (see [10, 8.6.9]). Hence, $p = 2$ and $G = PN$. The upper unitriangular matrices constitute a Sylow 2-subgroup $Q \leq P$ of N. We consider $x := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in Q$. It is easy to see that $\mathrm{C}_N(x) = Q$. In particular, Q is the only Sylow 2-subgroup of N containing x. Since $\mathrm{N}_G(P) \leq \mathrm{N}_G(P \cap N) = \mathrm{N}_G(Q)$ and $\nu_p(G) = q$ is a prime, we have

$$\nu_p(N) = |N : \mathrm{N}_N(Q)| = |N : N \cap \mathrm{N}_G(Q)| = |N\mathrm{N}_G(Q) : \mathrm{N}_G(Q)| \;\big|\; |G : \mathrm{N}_G(P)| = \nu_p(G)$$

and $\nu_p(N) = \nu_p(G)$. Therefore, P is the only Sylow 2-subgroup of G containing Q and x. Thus, P is not redundant and we derived a contradiction. $\square$

Now we consider p-solvable groups. For $H \leq P \in \mathrm{Syl}_p(G)$ let $\lambda_G(H)$ be the number of Sylow p-subgroups of G containing H. The following result was proved using Wielandt's subnormalizers.

Lemma 6 (Casolo)**.** *Let G be a p-solvable group and $H \leq P \in \mathrm{Syl}_p(G)$. Let $\mathcal{M}$ be the set of p'-quotients in a normal series of G whose quotients are p-groups or p'-groups. Then*

$$\lambda_G(H)|\mathrm{N}_G(P) : P| = \prod_{Q \in \mathcal{M}} |\mathrm{C}_Q(H)|.$$

Proof. See Theorems 2.6 and 2.8 in [1]. □

Theorem 7. *Let G be a p-solvable group with a redundant Sylow p-subgroup. Then $\nu_p(G) \geq q^{p+1}$, where $q > 1$ is the smallest prime power congruent to 1 modulo p.*

Proof. Let $P \in \mathrm{Syl}_p(G)$ and $\mathcal{M}$ as in Lemma 6. Choosing $H = P$ in Lemma 6 yields

$$|\mathrm{N}_G(P) : P| = \prod_{Q \in \mathcal{M}} |\mathrm{C}_Q(P)|.$$

Let $N := \bigtimes_{Q \in \mathcal{M}} Q$ and $\tilde{G} := N \rtimes P$. Then $\nu_p(G) = |G : \mathrm{N}_G(P)| = |\tilde{G} : \mathrm{N}_{\tilde{G}}(P)| = \nu_p(\tilde{G})$. Now let $H \leq P$ be a cyclic subgroup. Since P is redundant in G, we have $\lambda_G(H) > 1$. In this situation Lemma 6 shows that $\lambda_{\tilde{G}}(H) > 1$. Hence, P is redundant in $\tilde{G}$ and we may assume that $G = \tilde{G}$ is p-nilpotent and $N = \mathrm{O}_{p'}(G)$. Then $\mathrm{C}_N(x) > \mathrm{C}_N(P)$ for all $x \in P$. We consider N as a P-set via the conjugation action. By a theorem of Hartley–Turull [6] (see also [7, Theorem 3.31]), there exists an abelian group A and an isomorphism of P-sets $\varphi \colon N \to A$, i. e. $\varphi(n^x) = \varphi(n)^x$ for all $x \in P$ and $n \in N$. In particular, $\mathrm{C}_A(x) = \varphi(\mathrm{C}_N(x)) > \varphi(\mathrm{C}_N(P)) = \mathrm{C}_A(P)$. Hence, P is redundant in $A \rtimes P$ and

$$\nu_p(A \rtimes P) = |A : \mathrm{C}_A(P)| = |N : \mathrm{C}_N(P)| = \nu_p(G).$$

Thus, we may assume that $N = A$ is abelian. Then $\mathrm{C}_N(P) \trianglelefteq G$. Going over to $G/\mathrm{C}_N(P)$, we may assume that $\mathrm{C}_N(P) = 1$. Let $P_1, \ldots, P_{p+1} \leq P$ be maximal subgroups of P such that $P = P_1 \cup \ldots \cup P_{p+1}$. If $\mathrm{C}_N(P_i) = 1$ for some i, then P_i is redundant in P_iN and $\nu_p(P_iN) = |N| = \nu_p(G)$. Arguing by induction on $|G|$, we can assume that $N_i := \mathrm{C}_N(P_i) > 1$ for $i = 1, \ldots, p+1$. Using the Fitting decomposition as in the proof of Theorem 1(iii), we obtain $N_1 \times \ldots \times N_{p+1} \leq N$. Since P acts non-trivially on each N_i, it is clear that $|N_i| \geq q$. In total, $|N| \geq q^{p+1}$. □

We remark that $G := \mathrm{PSL}(2, 11)$ has a redundant Sylow 2-subgroup by [11, Theorem D] and $\nu_2(G) = 55$ is a product of only two primes. This indicates that Theorem 7 may not hold for arbitrary groups.

For $x \in P \in \mathrm{Syl}_p(G)$ let $\lambda_G(x) = \lambda_G(\langle x \rangle)$. Gheri [4] has introduced the following condition:

$$\nu_p(G)^{|P|/p} \geq \prod_{x \in P} \lambda_G(x). \tag{2.1}$$

He has shown in [4, Theorem B] that (2.1) holds for all finite groups if and only it holds for all almost simple groups. No counterexamples are known to exist. This yields a conjectural bound for $\nu_p(G)$.

Theorem 8. *Suppose that G has a redundant Sylow p-subgroup of order p^n. If G satisfies (2.1), then*

$$\nu_p(G) \geq (p+1)^{\frac{p^n - 1}{p^{n-1} - 1}} > (p+1)^p.$$

Proof. Let $x \in P \in \mathrm{Syl}_p(G)$. Since P is redundant, there exists a Sylow p-subgroup $Q \neq P$ such that $x \in P \cap Q$. As in the proof of Lemma 4, we may choose $g \in \mathrm{N}_P(P \cap Q) \setminus Q$ such that $Q^g, Q^{g^2}, \ldots, Q^{g^p}$ are distinct Sylow p-subgroups containing x. Hence, $\lambda_G(x) \geq p + 1$. Moreover, $\lambda_G(1) = \nu_p(G)$. Now (2.1) implies

$$\nu_p(G)^{p^{n-1}} \geq \lambda_G(1) \prod_{x \in P \setminus \{1\}} \lambda_G(x) \geq \nu_p(G)(p+1)^{p^n-1}.$$

Since P is non-cyclic, $n \geq 2$ and $\frac{p^n-1}{p^{n-1}-1} > p$. $\square$

If $n = 2$ in Theorem 8, then $\nu_p(G) \geq (p+1)^{p+1}$. This coincides with Theorem 7, whenever, p is a Mersenne prime or $p = 2$. The proof of [4, Theorem B] reduces (2.1) to an almost simple group S such that $\nu_p(S) \leq \nu_p(G)$. Then S is a primitive permutation group of degree $\leq \nu_p(S)$. If $\nu_p(G)$ is small, say $\nu_p(G) < 2^{12}$, we can check (2.1) by running through the library of primitive groups in GAP [3]. We did not find examples among non-solvable groups improving the values in Theorem 2.

Next we answer [11, Question 8.8].

Theorem 9. *For every $n \in \mathbb{N}$ there exists a constant $\delta_n < 1$ with the following property: For every set of Sylow p-subgroups $P_1, \ldots, P_n$ of a finite group G we have $G_p = P_1 \cup \ldots \cup P_n$ or*

$$|P_1 \cup \ldots \cup P_n| < \delta_n |G_p|.$$

Proof. We assume that $G_p \neq P_1 \cup \ldots \cup P_n$ and argue by induction on n. Let $P \in \mathrm{Syl}_p(G) \setminus \{P_1, \ldots, P_n\}$. A well-known theorem of Frobenius asserts that $|G_p| = a|P|$ for some integer $a \geq 2$ (see e. g. [8]). If $n = 1$, then the claim holds with $\delta_1 = \frac{1}{2}$. Now let $n \geq 2$ and assume that δ_{n-1} is already given. Let ρ_n be the smallest positive integer such that $\delta_{n-1} + \frac{1}{\rho_n} < 1$. If $a \geq \rho_n$, then induction yields

$$|P_1 \cup \ldots \cup P_n| \leq |P_1 \cup \ldots \cup P_{n-1}| + |P| \leq \delta_{n-1}|G_p| + \frac{1}{a}|G_p| \leq \Big(\delta_{n-1} + \frac{1}{\rho_n}\Big)|G_p|.$$

Now suppose that $a \leq \rho_n$. We may assume that $P \nsubseteq P_1 \cup \ldots \cup P_n$. Hence, by [12, Theorem 1], there exists a constant $\gamma_n < 1$ such that

$$\begin{aligned} |P_1 \cup \ldots \cup P_n| &\leq |G_p \setminus P| + |(P \cap P_1) \cup \ldots \cup (P \cap P_n)| \leq \frac{a-1}{a}|G_p| + \gamma_n |P| \\ &= \Big(1 - \frac{1-\gamma_n}{a}\Big)|G_p| \leq \Big(1 - \frac{1-\gamma_n}{\rho_n}\Big)|G_p|. \end{aligned}$$

Finally, the claim holds with

$$\delta_n := \max\Big\{\delta_{n-1} + \frac{1}{\rho_n},\ 1 - \frac{1-\gamma_n}{\rho_n}\Big\}.$$

$\square$

We finally remark that the prime p in Theorem 1 can be replaced by a set of primes. In fact the proof easily generalizes to the following theorem:

Theorem 10. *For every finite group H there exists a finite group G such that H is a Hall π-subgroup of G (where π is a set of primes) and every π-element of G lies in at least two Hall π-subgroups of G.*

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