Things left to prove

Characters and Blocks of Finite Groups Gabriel's conference 2025

Benjamin Sambale

31th January 2025

Gabriel likes (to propose) problems

- 1994: Some Open Problems on Coprime Action and Character Correspondences, Bull. London Math. Soc. 26, 513–522
- 2004: *Problems on characters and Sylow subgroups*, in: Finite groups, 275–281
- 2004: The McKay conjecture and Galois automorphisms, Ann. of Math. 160, 1129–1140 (71 citations)
- 2010: *Problems in character theory*, in: Character theory of finite groups, 97–125
- 2023: Problems on characters: solvable groups, Publ. Mat. 67, 173–198
- 2024: "All I would like to prove has been proved!", e-mail correspondence

Automorphisms

Let G be a finite group.

Definition (Automorphism tower)

Let
$$\operatorname{Aut}^1(G) := \operatorname{Aut}(G)$$
, $\operatorname{Aut}^2(G) := \operatorname{Aut}(\operatorname{Aut}(G))$, . . .

If
$$Z(G) = 1$$
, then $G \cong Inn(G) \le Aut^1(G) \le Aut^2(G) \le ...$

Theorem (WIELANDT)

If
$$Z(G) = 1$$
, then $\operatorname{Aut}^n(G) \cong \operatorname{Aut}^{n+1}(G)$ for some n .

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Automorphisms

Theorem (HAMKINS)

The transfinite automorphism tower of any group is bounded by some cardinal number.

Problem (Kourovka notebook 11.123)

- Is there a constant c such that $|\operatorname{Aut}^n(G)| < c$ for all n?
- Is $\operatorname{Aut}^n(G) \cong \operatorname{Aut}^{n+1}(G)$ for some n?
- Can $\operatorname{Aut}^n(G) \cong \operatorname{Aut}^m(G) \ncong \operatorname{Aut}^{n+1}(G)$ with n < m happen?

Automorphisms

Example

- If G is non-abelian simple, then $\operatorname{Aut}^2(G) \cong \operatorname{Aut}(G)$ (BURNSIDE).
- $\operatorname{Aut}(C_2^n) \cong \operatorname{GL}(n,2)$ is simple (for $n \geq 3$), so $\operatorname{Aut}^3(C_2^n) \cong \operatorname{Aut}^2(C_2^n)$.
- Center can stay non-trivial: $Aut(D_8) \cong D_8$.
- The sequence can decrease arbitrarily long: $\operatorname{Aut}(C_{2\cdot 3^n})\cong C_{2\cdot 3^{n-1}}$
- For G = SmallGroup(32, 13) we have

Homomorphisms

Theorem (FROBENIUS)

For every n, the number of $x \in G$ such that $x^n = 1$ is divisible by $\gcd(n, |G|)$.

Equivalently,

$$|\operatorname{Hom}(C_n, G)| \equiv 0 \pmod{\gcd(n, |G|)}.$$

Theorem (YOSHIDA)

For every finite abelian group A,

$$|\operatorname{Hom}(A,G)| \equiv 0 \pmod{\gcd(|A|,|G|)}.$$

Homomorphisms

Conjecture (ASAI-YOSHIDA)

For every finite group H,

$$|\operatorname{Hom}(H,G)| \equiv 0 \pmod{\gcd(|H/H'|,|G|)}.$$

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Conjugacy classes

- Let k(G) be the number of conjugacy classes of G.
- LANDAU: k(G) > f(|G|) for some increasing function f.

Theorem (BAUMEISTER-MARÓTI-TONG-VIET)

For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$k(G) > \frac{\delta \log |G|}{(\log \log |G|)^{3+\epsilon}}.$$

Things left to prove

Conjecture (Folklore?)

Is there a constant c such that $k(G) > c \log |G|$ for all G?

Characters

Constituents

Let Irr(G) be the set of irreducible complex characters of G.

Conjecture (HÉTHELYI–KÜLSHAMMER)

Let P be a p-group and $\chi \in Irr(P)$. Then the number of irreducible constituents of $\chi \overline{\chi}$ is 1 mod p-1.

Conjecture holds for $|P| \leq p^6$.

Problem (KNUTSON-MURRAY)

Let P be a p-group and $\chi \in Irr(P)$. Is there a generalized character ψ such that $\chi \psi$ is the regular character of P?

Things left to prove

Constituents

Call $\theta \in \operatorname{Irr}(\operatorname{Z}(G))$ fully ramified if θ^G is a multiple of an irreducible character.

Theorem (HOWLETT-ISAACS)

If $\theta \in Irr(Z(G))$ is fully ramified, then G is solvable.

Conjecture (HUMPHREYS, NAVARRO)

Let $\theta \in Irr(\mathbf{Z}(G))$ such that all constituents of θ^G have the same degree. Then G is solvable.

Conjecture holds whenever θ^G has at most two constituents (HIGGS).

Field of values

For $\chi \in Irr(G)$ define the abelian number fields

$$\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) : g \in G) \subseteq \mathbb{Q}_{|G|} \subseteq \mathbb{C},
\mathbb{Q}(G) := \mathbb{Q}(\chi(g) : \chi \in \operatorname{Irr}(G), g \in G) \subseteq \mathbb{Q}_{|G|}.$$

By the Kronecker-Weber theorem, every abelian number field lies in some \mathbb{Q}_n .

Theorem (FEIN-GORDON)

For every abelian number field F there exist a group G and $\chi \in Irr(G)$ such that $\mathbb{Q}(\chi) = F$.

Field of values

Conjecture

- Not every abelian number field has the form $\mathbb{Q}(G)$.
- \bullet For every d, there are only finitely many number fields $\mathbb{Q}(G)$ of degree d.

There are only finitely many fields $\mathbb{Q}(G)$ of degree d where G is solvable or simple (FARIAS E SOARES, FEIT-SEITZ).

Example (quadratic fields)

Let $d \mid 210$ or $d \in \{-231, -11, 13, 17\}$. Then there exists G with $\mathbb{Q}(G) \cong \mathbb{Q}(\sqrt{d})$.

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Characters

Field of values

Theorem (ROBINSON-THOMPSON)

For n > 24 we have

$$\mathbb{Q}(A_n) = \mathbb{Q}\Big(\sqrt{(-1)^{\frac{p-1}{2}}p}: p \text{ odd prime } \leq n, \ p \neq n-2\Big).$$

Things left to prove

In particular, $Gal(\mathbb{Q}(A_n)|\mathbb{Q})$ is an elementary abelian 2-group.

Problem

- Is every abelian group the Galois group of some $\mathbb{Q}(G)$?
- Is there a group G such that $|\mathbb{Q}(G):\mathbb{Q}|=7$?
- Is there a group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{11})$?
- Is there a solvable group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{-5})$?

Field of values

For $\chi \in Irr(G)$ let $f(\chi)$ be the smallest integer n such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$.

Conjecture (HUNG-TIEP)*

We have $|\mathbb{Q}_{f(\chi)}:\mathbb{Q}(\chi)| \leq \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

Conjecture holds whenever $\chi(1)$ is a prime (HUNG-TIEP-ZALESSKI).

Problem (HUNG-TIEP)*

Let $z=\zeta_1+\ldots+\zeta_n\in\mathbb{C}$ be a sum of roots of unity. Let c(z) be the smallest m such that $z\in\mathbb{Q}_m$. Is $|\mathbb{Q}_{c(z)}:\mathbb{Q}(z)|\leq n$?

*Added on 26.08.25: Both were disproved by C. Herbig: arXiv:2508.16732

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Algebraic integers

- Let \mathbb{Z}_G be the ring of integers of $\mathbb{Q}(G)$.
- Let $\mathbb{Z}[G] := \mathbb{Z}[\chi(g) : \chi \in \operatorname{Irr}(G), \ g \in G] \subseteq \mathbb{Z}_G$.
- Then $\mathbb{Z}_G/\mathbb{Z}[G]$ is a finite abelian group.

Conjecture (BÄCHLE-S.)

The exponent of $\mathbb{Z}_G/\mathbb{Z}[G]$ divides |G|.

Example

Conjecture holds for nilpotent groups. For $G = C_{15} \times D_{16}$,

$$\mathbb{Z}_G/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}.$$

Things left to prove

Weights

- Let π be a set of primes.
- Let l(G) be the number of conjugacy classes of π' -elements of G.
- Let $\chi \in Irr(G)$ of π -defect 0 if $|G|/\chi(1)$ is a π' -number.
- Let $k^0(G)$ be the number of π -defect 0 characters of G.

Theorem (NAVARRO-S.)

If G is π -separable with solvable Hall π -subgroups, then

$$l(G) = \sum_{P} k^{0}(N_{G}(P)/P)$$

where P runs through the nilpotent π -subgroups of G up to conjugation.

For $\pi = \{p\}$, this is Alperin's weight conjecture for p-solvable groups.

Characters

Weights

Definition

Define the weight of a group P by

$$\mu(P) = 1 - \sum_{\substack{Q < P \\ \mathcal{N}_P(Q) = Q}} \mu(Q) \qquad (Q \text{ up to conjugation}).$$

Conjecture (NAVARRO-S.)

For every π -separable group G,

$$l(G) = \sum_{P} \mu(P)k^{0}(N_{G}(P)/P)$$

Things left to prove

where P runs through the π -subgroups of G up to conjugation.

Weights

Example

- If P is nilpotent, then $\mu(P) = 1$.
- If P is solvable, but not nilpotent, then $\mu(P) = 0$ (Carter subgroups).
- On the other hand,

Characterization of blocks

Let G^0 be the set of p-regular elements of G.

Theorem (OSIMA)

A subset $J \subseteq Irr(G)$ is a union of p-blocks if and only if

$$\sum_{\chi \in I} \chi(g)\chi(h) = 0 \qquad (\forall g \in G^0, \ h \in G \setminus G^0).$$

Conjecture (HARADA)

It is enough to fix g=1 in Osima's theorem.

Basic sets

- Let B be a p-block of G.
- For $\chi \in Irr(G)$ let χ^0 be the restriction to G^0 .
- The Brauer characters in p-solvable groups have the form χ^0 (FONG-SWAN).

Conjecture (ordinary basic set, GECK)

There exist $\chi_1, \ldots, \chi_l \in \operatorname{Irr}(B)$ such that $\chi_1^0, \ldots, \chi_l^0$ is a \mathbb{Z} -basis for the ring of generalized Brauer characters of B.

This holds in a strong sense for "most" groups of Lie type (Brunat-Dudas-Taylor).

Number of characters

• For $\chi, \psi \in Irr(G)$ let

$$[\chi, \psi]^0 = \frac{1}{|G|} \sum_{g \in G^0} \chi(g) \overline{\psi(g)}.$$

• Let d be the defect of B. Then $p^d[\chi,\chi]^0 \in \mathbb{N}$ for all $\chi \in \operatorname{Irr}(B)$.

Conjecture (NAVARRO-S.)

We have $p^d[\chi,\chi]^0 \ge l(B)$ for all $\chi \in Irr(B)$.

- This implies Brauer's conjecture $k(B) \leq p^d$ with equality only if B has abelian defect groups.
- The case $\chi = 1_G$ was conjectured by MURAI and implies FROBENIUS conjecture: G is p-nilpotent iff $|G^0| = |G|_{p'}$ (known via CFSG).

Blocks

Fusion numbers

Let

$$X_B := (\chi(g_i) : \chi \in \operatorname{Irr}(B), \ i = 1, \dots, k(G))$$

be a submatrix of the character table.

• The non-zero elementary divisors of $X_B^{\rm t} \overline{X_B}$ over a complete discrete valuation ring are p-powers e_1, \ldots, e_t .

Conjecture (S.)

We have $\gamma(B) := \frac{1}{e_1} + \ldots + \frac{1}{e_k} \ge 1$ with equality if and only if B is nilpotent.

Example

Conjecture holds for symmetric groups and "ATLAS groups". For the principal 2-block B of the Monster, $\gamma(B) \approx 39.5$.

Things left to prove

Some memories

• I read your question just before going to bed and you have ruined my night. Good question. I dreamt that I had an example.

- I am two days, 18 hours each, trying to clean up the inductive McKay, no waving hands. [finishing his second book]
- If these two guys are not Morita equivalent then I won't say the word Morita again! [they weren't]
- Silly? I am in the middle of a restaurant.
- Seriously, I have thought on this for 3 minutes.
- Zero. [answer to: How many examples did you check?]
- I am also amazed by my intuition, sorry to say!
- 25th anniversary of my wedding. I have promised my wife not to think on mathematics for a couple of days. Now you are making my promise easy to break...