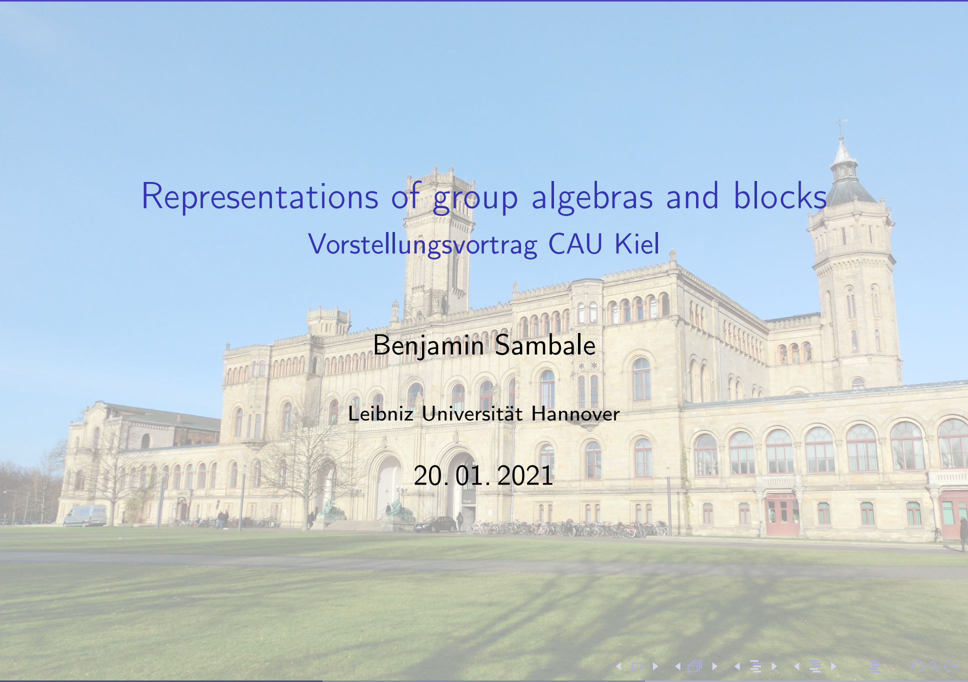

Representations of group algebras and blocks

Vorstellungsvortrag CAU Kiel

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Introduction

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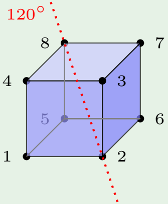
- This gives rise to a group homomorphism $\varphi : G \to S_8$.



$$\xrightarrow{\varphi} (1,3,6)(4,7,5)$$

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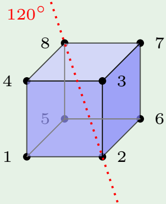
- There is also a linear action $\psi : G \to \mathrm{GL}(3, \mathbb{R})$.



$$\xrightarrow{\psi} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

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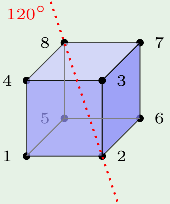

$$\xrightarrow{\psi} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

Advantage: It is easier to compute inside S_8 or $\mathrm{GL}(3, \mathbb{R})$ than in the abstract group G.

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From now on let G be an abstract finite group.
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- The regular representation $\Delta_{\mathrm{reg}} : G \to \mathrm{GL}(|G|, F)$, $g \mapsto (\delta_{x,gy})_{x,y \in G}$ is injective, but $d = |G|$ is large.

Irreducible representations

The regular representation decomposes with respect to a suitable basis:

$$
\begin{aligned}
G &\to \mathrm{GL}(d_1, F) \times \ldots \times \mathrm{GL}(d_k, F), \\
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Study the irreducible representations $\Delta_i : G \to \mathrm{GL}(d_i, F)$, $g \mapsto A_i$.
Extend linearly to a representation of algebras:

$$\hat{\Delta}_i : FG \to F^{d_i \times d_i}$$

where $FG = \sum_{g\in G} Fg$ is the group algebra of G.

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).

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- This situation is well-understood.

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- Each irreducible representation belongs to exactly one block.
- The block containing Δ_{tr} is called the principal block.

A comparison

Example

- For the symmetry group of the cube $G \cong S_4 \times C_2$ we have

$$\mathbb{C}G \cong \mathbb{C}^4 \times \left(\mathbb{C}^{2\times 2}\right)^2 \times \left(\mathbb{C}^{3\times 3}\right)^4.$$

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- On the other hand, $\mathbb{F}_2 G$ is just the principal block.
- For $G = S_{20}$ and $F = \mathbb{F}_2$ not even the degrees $d_1, \ldots, d_k$ are known!

Defect groups

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- In general the isomorphism type of B (even its dimension) cannot be described by D alone.
- Instead, classify blocks up to Morita equivalence, i. e. determine the module category B-mod.

Finiteness conjectures

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For every p-group D there exist only finitely many Morita equivalence classes of blocks with defect group D.

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Conjecture (Blockwise modular isomorphism problem)

B-mod determines the isomorphism type of D.

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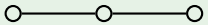
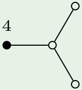
- The principal 3-block of $G = S_4$ has Brauer tree 
- No block with Brauer tree 

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Example

The principal 2-block of $G = S_4$ has defect group $D \cong D_8$ and quiver/relations

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & \circ \arrow[l, bend left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd} \qquad \begin{array}{c} \beta\eta = \eta\gamma = \gamma\beta = \alpha^2 = 0, \\ \alpha\beta\gamma = \beta\gamma\alpha, \qquad \eta^2 = \gamma\alpha\beta. \end{array}$$

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Theorem (Eaton–Kessar–Külshammer–S.)

Let B be a 2-block with metacyclic defect group D. Then one of the following holds:

1. *B has tame representation type.*
2. *B is nilpotent. Then $B \cong (FD)^{d\times d}$ for some $d \geq 1$.*
3. *$D \cong C_{2^d} \times C_{2^d}$ for some $d \geq 2$ and B is Morita equivalent to $F[D \rtimes C_3]$.*

Numerical invariants

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- *If $|D| \le p^3$, then $k(B) \le |D|$ and D is determined by B-mod.*
- *If D is abelian, then $k(B) \le |D|^{3/2}$.*

Abelian defect groups

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Theorem (Eaton, Livesey, Ardito–S.)

Broué's Conjecture holds if $p = 2$ and $|D| \leq 32$.

Characters

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Theorem (Brauer's induction theorem)

Every character is an integer linear combination of linear characters induced from elementary subgroups.

Fusion systems

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The fusion system $\mathcal{F}$ of B is a category with

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Theorem (Alperin)

$\mathcal{F}$ is determined by (very few) essential subgroups $S \leq D$.

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Puig's Theorem generalizes Brauer's Theorem for $D = 1$.

Cartan matrices

- The regular representation can also be decomposed into indecomposable summands:

$$\Delta_{\mathsf{reg}} : G \to \mathrm{GL}(e_1, F) \times \ldots \times \mathrm{GL}(e_l, F), \quad g \mapsto \begin{pmatrix} A_1' & & 0 \\ & \ddots & \\ 0 & & A_l' \end{pmatrix}$$

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- It gives rise to a positive definite quadratic form $q(x) = xCx^{\mathrm{t}}$.

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- It gives rise to a positive definite quadratic form $q(x) = xCx^{\mathrm{t}}$.
- By Minkowski reduction or the LLL algorithm there exists $S \in \mathrm{GL}(l, \mathbb{Z})$ such that SCS^{t} has "small entries".

Cartan matrices

- The regular representation can also be decomposed into indecomposable summands:

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- Apply $k(B) \leq \mathrm{tr}(SCS^{\mathrm{t}})$ and refinements thereof.

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Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,*
- *alternating groups of degree* ≥ 5*,*
- *matrix groups of Lie type,*
- *26 sporadic groups.*