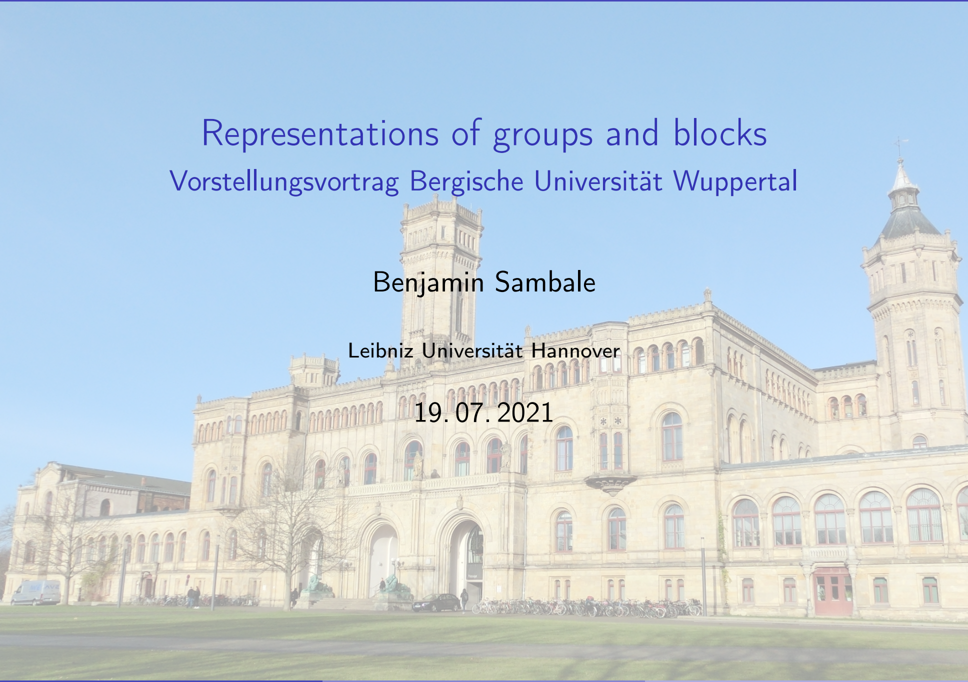

Representations of groups and blocks

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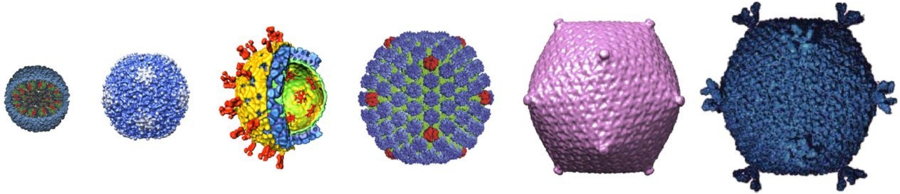
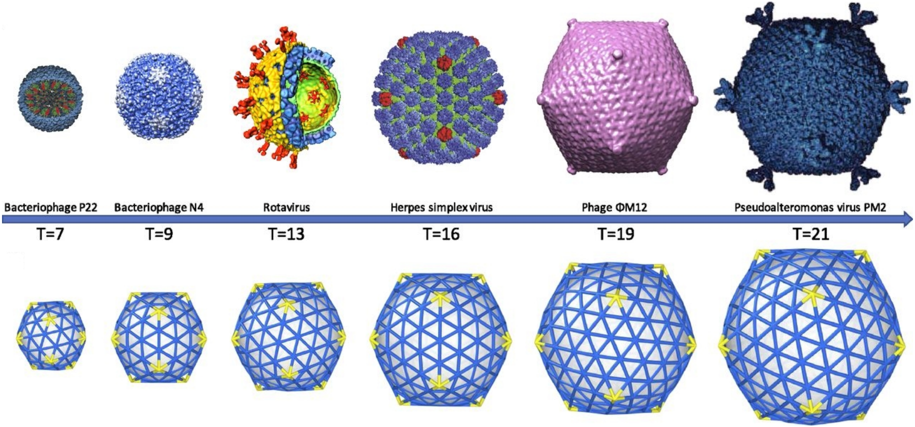
What is shown?



Viruses!



- Zandi et al., *Origin of icosahedral symmetry in viruses*, PNAS 101 (2004)
- Li et al., *Why large icosahedral viruses need scaffolding proteins*, PNAS 115 (2018)

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60
reflections	15
total	75?

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60
reflections	15
combinations	45
total	120!

Group theory simplifies counting!

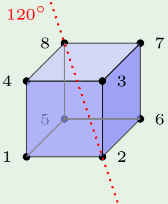
Introduction

Synopsis

In representation theory, mathematical objects are studied by their actions on sets, vector spaces, graphs, categories etc.

Example

- The symmetry group G of the cube permutes the 8 vertices.
- This gives rise to a group homomorphism $\varphi\colon G \to S_8$.

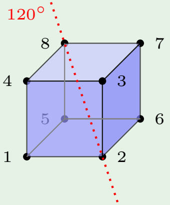


$\xrightarrow{\varphi} (1,3,6)(4,7,5)$

Introduction

Example

- There is also a linear action $\psi\colon G \to \mathrm{GL}(3,\mathbb{R})$.



$$\xrightarrow{\psi} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

Advantage: Computations are easier inside S_8 or $\mathrm{GL}(3,\mathbb{R})$ than in G.

Applications

Representation theory has numerous applications

- within mathematics:
  - group theory (Frobenius kernels, Odd order theorem)
  - combinatorics (Young diagrams, graph automorphisms)
  - number theory (Langlands program, Artin L-series)
  - geometry (Coxeter groups, Lie groups)
  - topology (fundamental groups, classifying spaces)
- outside mathematics:
  - chemistry (crystallography, spectroscopy)
  - physics (particle physics, quantum mechanics)
  - computer science (cryptography, coding theory)

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Goal

Find a representation $\Delta\colon G \to \mathrm{GL}(d, F)$ such that

- degree d is small (efficient computation).
- kernel $\mathrm{Ker}(\Delta)$ is small (preserving information).

Extreme examples

- The trivial representation $\Delta_{\mathrm{tr}}\colon G \to \mathrm{GL}(1, F)$, $g \mapsto 1$ contains no information on G.
- The regular representation $\Delta_{\mathrm{reg}}\colon G \to \mathrm{GL}(|G|, F)$, $g \mapsto (\delta_{x,gy})_{x,y\in G}$ is injective, but $d = |G|$ is large.

Irreducible representations

The regular representation decomposes with respect to a suitable basis:

$$G \to \mathrm{GL}(d_1, F) \times \ldots \times \mathrm{GL}(d_k, F),$$
$$g \mapsto \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

Study the irreducible representations $\Delta_i\colon G \to \mathrm{GL}(d_i, F)$, $g \mapsto A_i$.
Extend linearly to a representation of algebras:

$$\widehat{\Delta}_i\colon FG \to F^{d_i \times d_i}$$

where $FG = \sum_{g \in G} Fg$ is the group algebra of G.

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).
- Then FG is semisimple by Maschke's Theorem, i. e.

$$\mathrm{Ker}(\widehat{\Delta}_1) \cap \ldots \cap \mathrm{Ker}(\widehat{\Delta}_k) = 0.$$

- If additionally F is algebraically closed (e. g. $F = \mathbb{C}$), then $\widehat{\Delta}_i$ is surjective and we obtain the Artin–Wedderburn isomorphism

$$\boxed{FG \cong F^{d_1 \times d_1} \times \ldots \times F^{d_l \times d_l}}$$

(not all $\widehat{\Delta}_i$ are needed).

- This situation is well-understood.

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.
- Decompose FG into indecomposable algebras

$$\boxed{FG = B_1 \times \ldots \times B_n.}$$

- Call $B_1, \ldots, B_n$ the (p-)blocks of FG.
- Each irreducible representation belongs to exactly one block.
- The block containing Δ_{tr} is called the principal block.

A comparison

Example

- For the symmetry group of the cube $G \cong S_4 \times C_2$ we have

$$\mathbb{C}G \cong \mathbb{C}^4 \times \left(\mathbb{C}^{2\times 2}\right)^2 \times \left(\mathbb{C}^{3\times 3}\right)^4.$$

- On the other hand, $\overline{\mathbb{F}_2}G$ is just the principal block.
- For $G = S_{20}$ and $F = \overline{\mathbb{F}_2}$ not even the degrees $d_1, \ldots, d_k$ are known!

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Theorem (Brauer)

B is a simple algebra iff $D = 1$. In this case, $B \cong F^{d\times d}$ for some $d \geq 1$.

- The defect group of the principal block is a Sylow p-subgroup of G. In particular, not all blocks are simple.
- In general the isomorphism type of B (even its dimension) cannot be described by D alone.
- Instead, classify blocks up to Morita equivalence, i. e. determine the module category B-mod.

Finiteness conjectures

Motivation:

Conjecture (Donovan)

For every p-group D there exist only finitely many Morita equivalence classes of blocks with defect group D.

Conversely, many features of D can be read off from B-mod. However:

Theorem (García–Margolis–Del Río, 2021)

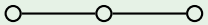
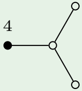
There exist p-groups $P \ncong Q$ such that $FP \cong FQ$.

Representation type

Theorem (Hamernik, Dade, Janusz, Kupisch)

B has finite representation type iff D is cyclic. In this case, B-mod is determined by the Brauer tree of B.

Example

- The principal 3-block of $G = S_4$ has Brauer tree 
- No block with Brauer tree 

is known!

Tame blocks

Theorem (Bondarenko–Drozd)

B has tame representation type iff $p = 2$ and D is a dihedral, semidihedral or quaternion group.

Erdmann described tame blocks as path algebras. For dihedral D, mod-B was determined by Macgregor (2021).

Example

The principal 2-block of $G = S_4$ has defect group $D \cong D_8$ and quiver/relations

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & \circ \arrow[l, bend left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd} \qquad \begin{gathered} \beta\eta = \eta\gamma = \gamma\beta = \alpha^2 = 0, \\ \alpha\beta\gamma = \beta\gamma\alpha, \qquad \eta^2 = \gamma\alpha\beta. \end{gathered}$$

Some wild blocks

Very little is known for blocks of wild representation type.
A cyclic extension of a cyclic group is called metacyclic.

Theorem (Eaton–Kessar–Külshammer–S.)

If D is a metacyclic 2-group, then one of the following holds:

1. *B has tame representation type.*
2. *B is nilpotent. Then $B \cong (FD)^{d\times d}$ for some $d \geq 1$.*
3. *$D \cong C_{2^d} \times C_{2^d}$ with $d \geq 2$ and B is Morita equivalent to $F[D \rtimes C_3]$.*

Numerical invariants

Since Morita equivalent algebras have isomorphic centers, we investigate

$$k(B) := \dim_F \mathrm{Z}(B).$$

Brauer's $k(B)$-Conjecture (1946)

For every block B with defect group D we have $k(B) \le |D|$.

Theorem (Brauer–Feit)

In general, $k(B) \le |D|^2$.

Theorem (S.)

- *If $|D| \le p^3$, then $k(B) \le |D|$ and D is determined by B-mod.*
- *If D is abelian, then $k(B) \le |D|^{3/2}$.*

Abelian defect groups

The Brauer correspondence is a bijection:

$$\text{blocks } B \text{ of } G \longleftrightarrow \text{blocks } b_D \text{ of } \mathrm{N}_G(D).$$

Unfortunately, B and b_D are not Morita equivalent in general.

Conjecture (Broué)

If D is abelian, then B and b_D are derived equivalent.

Theorem (Eaton–Livesey)

Donovan's Conjecture holds for all abelian 2-groups.

Theorem (Eaton, Livesey, Ardito–S.)

Broué's Conjecture holds if $p = 2$ and $|D| \leq 32$.

Characters

- The "shadow" of a complex representation $\Delta\colon G \to \mathrm{GL}(d,\mathbb{C})$ is its character $\chi\colon G \to \mathbb{C}$, $g \mapsto \mathrm{tr}(\Delta(g))$.
- Although we lose information, Δ is determined by χ up to basis choice.
- Characters are more convenient than representations since they are class functions providing inner products, orthogonality relations, Frobenius reciprocity, Mackey decomposition, perfect isometries, ...

Theorem (Brauer's induction theorem)

Every character is an integer linear combination of linear characters induced from elementary subgroups.

Fusion systems

For every subgroup $S \leq D$ there is a (non-unique) Brauer correspondent b_S of B in $\mathrm{N}_G(S)$.

Definition

The fusion system $\mathcal{F}$ of B is a category with

- $\mathrm{Ob}(\mathcal{F}) = \{S : S \leq D\}$,
- $\mathrm{Hom}_{\mathcal{F}}(S, T) = \{$conjugation maps $S \to T$ sending b_S to $b_T\}$.

Theorem (Alperin)

$\mathcal{F}$ is determined by (very few) essential subgroups $S \leq D$.

Nilpotent blocks

We call B nilpotent if all morphisms come from $\mathrm{Inn}(D)$ (no essential subgroups).

Theorem (Frobenius)

The principal block is nilpotent iff G is p-nilpotent, i. e. G has a normal p-complement.

Theorem (Puig)

Every nilpotent block with defect group D is isomorphic to $(FD)^{d\times d}$ for some $d \geq 1$.

Puig's Theorem generalizes Brauer's Theorem for $D = 1$.

Cartan matrices

- The regular representation can also be decomposed into indecomposable summands:

$$\Delta_{\text{reg}}\colon G \to \mathrm{GL}(e_1, F) \times \ldots \times \mathrm{GL}(e_l, F), \quad g \mapsto \begin{pmatrix} A_1' & & 0 \\ & \ddots & \\ 0 & & A_l' \end{pmatrix}$$

- The multiplicities of the Δ_i as constituents of the indecomposable representations are encoded in the Cartan matrix $C \in \mathbb{Z}^{l\times l}$ of B.
- It gives rise to a positive definite quadratic form $q(x) = xCx^{\mathrm{t}}$.
- By Minkowski reduction or the LLL algorithm there exists $S \in \mathrm{GL}(l, \mathbb{Z})$ such that SCS^{t} has "small entries".
- Apply $k(B) \leq \mathrm{tr}(SCS^{\mathrm{t}})$ and refinements thereof.

Simple groups

Some problems reduce to (quasi)simple groups by Clifford theory.
They can be checked via the classification of finite simple groups:

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,*
- *alternating groups of degree* ≥ 5*,*
- *matrix groups of Lie type,*
- *26 sporadic groups.*