Fusion invariant characters of p-groups

Benjamin Sambale*

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Abstract

We consider complex characters of a p-group P, which are invariant under a fusion system $\mathcal{F}$ on P. Extending a theorem of Bárcenas–Cantarero to non-saturated fusion systems, we show that the number of indecomposable $\mathcal{F}$-invariant characters of P is greater or equal than the number of $\mathcal{F}$-conjugacy classes of P. We further prove that these two quantities coincide whenever $\mathcal{F}$ is realized by a p-solvable group. On the other hand, we observe that this is false for constrained fusion systems in general. Finally, we construct a saturated fusion system with an indecomposable $\mathcal{F}$-invariant character, which is not a summand of the regular character of P. This disproves a recent conjecture of Cantarero–Combariza.

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1 Introduction

A fusion system $\mathcal{F}$ on a finite p-group P is a category, whose objects are the subgroups of P and whose morphisms are injective group homomorphisms satisfying certain technical conditions (we refer the reader to [1] for the details). For the moment, we do not require that $\mathcal{F}$ is saturated. Elements $x, y \in P$ are called *$\mathcal{F}$-conjugate* if there exists a morphism $f : \langle x \rangle \to P$ in $\mathcal{F}$ such that $f(x) = y$. We denote the number of $\mathcal{F}$-conjugacy classes of P by $k(\mathcal{F})$. A complex class function χ of P is called *$\mathcal{F}$-invariant* if χ is constant on the $\mathcal{F}$-conjugacy classes of P. These characters can often be used to construct new characters of finite groups via the Broué–Puig $*$-construction introduced in [3]. Further motivation and background can be found in the recent paper of Cantarero–Combariza [4].

We call an $\mathcal{F}$-invariant character of P *indecomposable* if it is not the sum of two non-zero $\mathcal{F}$-invariant characters (this is unrelated to the characters of indecomposable modules of the group algebra). Let $\mathrm{Ind}_{\mathcal{F}}(P)$ be the set of indecomposable $\mathcal{F}$-invariant characters of P. In the theory of lattices, $\mathrm{Ind}_{\mathcal{F}}(P)$ is sometimes called the *Hilbert basis* of the semigroup of $\mathcal{F}$-invariant characters. As a consequence, $\mathrm{Ind}_{\mathcal{F}}(P)$ is finite (see Lemma 3 below).

Our first theorem gives a lower bound on $|\mathrm{Ind}_{\mathcal{F}}(P)|$. This was previously proved by Bárcenas and Cantarero in [2, Lemma 2.1] for saturated fusion systems.

Theorem 1. *The space of $\mathcal{F}$-invariant class functions of P is spanned by* $\mathrm{Ind}_{\mathcal{F}}(P)$*. In particular,* $|\mathrm{Ind}_{\mathcal{F}}(P)| \geq k(\mathcal{F})$*.*

*Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, sambale@math.uni-hannover.de

Cantarero and Combariza have proven in [4, Lemma 2.17] that $|\mathrm{Ind}_{\mathcal{F}}(P)| = k(\mathcal{F})$ holds for controlled fusion systems (among other cases). A controlled fusion system is realized by a group of the form $P \rtimes H$ for some p'-group H. Our second theorem generalizes this result to the larger class of p-solvable groups.

Theorem 2. *Let $\mathcal{F}$ be the (saturated) fusion system on a Sylow p-subgroup P of a p-solvable group G. Then $|\mathrm{Ind}_{\mathcal{F}}(P)| = k(\mathcal{F})$.*

In the last section of this paper, we construct examples of saturated constrained fusion systems with $|\mathrm{Ind}_{\mathcal{F}}(P)| > |P|$ by making use of GAP [6]. Since there are only finitely many fusion systems on a given p-group P, it is clear that $|\mathrm{Ind}_{\mathcal{F}}(P)|$ can be bounded by a function in $|P|$. We do not know how to construct such a function explicitly. A related question for quasi-projective characters has been raised by Willems–Zalesski [14, Question 4.2].

In [4, Conjecture 2.19], Cantarero and Combariza have conjectured that for a saturated fusion system, every $\mathcal{F}$-invariant indecomosable character is a summand of the regular character of P. In the last section, we exhibit a counterexample to this conjecture.

2 The number of indecomposable $\mathcal{F}$-invariant characters

As in the introduction, $\mathcal{F}$ denotes a fusion system on a finite p-group P for the remainder of the paper. Our notation for characters follows Navarro's book [9]. In particular, if χ is a character of a group G and $P \le G$, then χ_P denotes the restriction of χ to P. Moreover, for characters χ, ψ of G, the usual scalar product is denoted by $[\chi, \psi]$.

The following lemma is well-known among experts in lattice theory (it follows from *Gordan's lemma*, see [13, Theorem 16.4]), but perhaps less known among representation theorists.

Lemma 3. *There are only finitely many indecomposable $\mathcal{F}$-invariant characters of P.*

Proof. Let $\mathrm{Irr}(P) = \{\chi_1, \ldots, \chi_k\}$. For $\psi \in \mathrm{Ind}_{\mathcal{F}}(P)$ let $c(\psi) = ([\psi, \chi_i] : i = 1, \ldots, k) \in \mathbb{N}_0^k$. We define a partial order on $\mathbb{N}_0^k$ by $a \le b :\Longleftrightarrow b - a \in \mathbb{N}_0^k$. For distinct characters $\psi, \psi' \in \mathrm{Ind}_{\mathcal{F}}(P)$, we have $c(\psi) \not\le c(\psi')$, since otherwise $\psi' = (\psi' - \psi) + \psi$ would be a non-trivial decomposition of $\mathcal{F}$-invariant characters. Therefore, $\{c(\psi) : \psi \in \mathrm{Ind}_{\mathcal{F}}(P)\}$ is an antichain in $\mathbb{N}_0^k$ with respect to $\le$, i. e. no two distinct elements are comparable. Therefore, it is enough to show that every antichain in $\mathbb{N}_0^k$ is finite.

By way of contradiction, suppose that $c^{(1)}, c^{(2)}, \ldots$ is an infinite antichain in $\mathbb{N}_0^k$. We may replace this sequence by an infinite subsequence such that $c_1^{(1)} \le c_1^{(2)} \le \ldots$. This sequence can in turn be replaced by a subsequence such that $c_2^{(1)} \le c_2^{(2)} \le \ldots$. Repeating this process k times yields an infinite sequence $c^{(1)} \le c^{(2)} \le \ldots$. But this is impossible since the original sequence was an antichain. □

Since for every $k \ge 2$, the poset $\mathbb{N}_0^k$ contains antichains of arbitrary finite lengths (e. g. $(n, 1, *, \ldots, *)$, $(n-1, 2, *, \ldots, *), \ldots$ for any $n \in \mathbb{N}$), it is not easy to give an upper bound on $|\mathrm{Ind}_{\mathcal{F}}(P)|$.

We now prove the first theorem stated in the introduction.

Proof of Theorem 1. By a theorem of Park [10], there exists a finite group G such that $P \leq G$ and the morphisms of $\mathcal{F}$ are induced by conjugation in G. In particular, $k(\mathcal{F})$ is the number of G-conjugacy classes which intersect P. Let T be the part of the character table of G, whose columns correspond to elements in P. Since the character table is invertible, T has full rank. Hence, the (G-invariant) restrictions χ_P for $\chi \in \mathrm{Irr}(G)$ span the space of G-invariant class functions on P. Since each χ_P can be decomposed into G-invariant indecomposable characters, the claim follows. □

Next we restrict ourselves to saturated fusion systems arsing from a finite group with Sylow p-subgroup P (those fusion systems are sometimes called *non-exotic*). Here we can prove a stronger theorem, which resembles the fact that Brauer characters are restrictions of generalized characters (see [8, Corollary 2.16]).

Theorem 4. *Let G be a finite group with Sylow p-subgroup P. Then every G-invariant character ζ of P is the restriction of a generalized character of G.*

Proof. We extend ζ to a class function $\hat{\zeta}$ of G in the following way: Every $g \in G$ is conjugate to an element of the form $xy = yx$ where $x \in P$ and y is a p'-element. We define $\hat{\zeta}(g) := \zeta(x)$ (this is well-defined since ζ is G-invariant). Now we use Brauer's characterization of characters to show that $\hat{\zeta}$ is a generalized character of G (see [9, Corollary 7.12]). To this end, let $N \leq G$ be a nilpotent subgroup with Sylow p-subgroup $Q \trianglelefteq N$. After conjugation, we may assume that $Q \leq P$. Then $\hat{\zeta}_Q = \zeta_Q$ is a character of $Q \cong N/\mathrm{O}_{p'}(N)$ and $\hat{\zeta}_N$ is the inflation of ζ_Q to N. In particular, $\hat{\zeta}_N$ is a (generalized) character of N. Hence, $\hat{\zeta}$ is a generalized character of G, which restricts to ζ. □

Obviously, every G-invariant character of P is a summand of a restriction of a character of G. However, an indecomposable character is not necessarily a summand of a restriction of an irreducible character of G. A counterexample will be given in the last section of the paper.

The following lemma of Cantarero–Combariza [4, Corollary 2.9] characterizes equality in Theorem 1. We include the short proof for the convenience of the reader.

Lemma 5. *For every fusion system $\mathcal{F}$ on P we have $|\mathrm{Ind}_{\mathcal{F}}(P)| = k(\mathcal{F})$ if and only if every $\mathcal{F}$-invariant character of P can be decomposed* uniquely *into indecomposable characters.*

Proof. If $|\mathrm{Ind}_{\mathcal{F}}(P)| = k(\mathcal{F})$, then $\mathrm{Ind}_{\mathcal{F}}(P)$ is a basis of the space of $\mathcal{F}$-invariant class functions and the result follows. Now assume that $|\mathrm{Ind}_{\mathcal{F}}(P)| > k(\mathcal{F})$. Since the dimension of the $\mathbb{Q}$-vectorspace spanned by $\mathrm{Ind}_{\mathcal{F}}(P)$ is bounded by $k(\mathcal{F})$, the set $\mathrm{Ind}_{\mathcal{F}}(P)$ is linearly dependent over $\mathbb{Q}$. Hence, there exist integers $c_\psi \in \mathbb{Z}$ (not all zero) such that

$$\sum_{\psi \in \mathrm{Ind}_{\mathcal{F}}(P)} c_\psi \psi = 0.$$

Since the degree of each character is positive, not all c_ψ can have the same sign. If we bring the negative coefficients to the right hand side, we end up with two distinct decompositions of an $\mathcal{F}$-invariant character. □

We turn to the proof of our second main theorem.

Proof of Theorem 2. We apply Isaacs' theory of π-partial characters, where $\pi = \{p\}$ (see [7, p. 71]). Every indecomposable $\mathcal{F}$-invariant character χ of P extends uniquely to a class function $\hat{\chi}$ on the set of p-elements of G. By [7, Corollary 3.5], $\hat{\chi}$ is an irreducible p-partial character of G. The number of those characters is exactly $k(\mathcal{F})$ by [7, Theorem 3.3]. $\square$

We remark that every fusion system of a p-solvable group is constrained. Conversely, by the model theorem [1, Theorem I.4.9], every constrained fusion system is realized by a p-constrained group. However, Theorem 2 does not hold for constrained fusion systems in general as we will see in the next section.

As a consequence of Theorem 2, we obtain the following extension of some results in [4].

Theorem 6. *Let $\mathcal{F}$ be the (saturated) fusion system on a Sylow p-subgroup P of a p-solvable group. Then every indecomposable $\mathcal{F}$-invariant character of P is a summand of the regular character of P.*

Proof. This follows from Theorem 2 and [4, Remark 2.18]. For the convenience of the reader we repeat the short proof of the latter result: Let ψ be an indecomposable $\mathcal{F}$-invariant character of P. Let

$$m := \max\bigl\{[\psi, \chi] : \chi \in \mathrm{Irr}(P)\bigr\}.$$

Then ψ is a summand of $m\rho$, where ρ is the regular character of P. By the hypothesis and Lemma 5, $m\rho$ has a unique decomposition into indecomposable $\mathcal{F}$-invariant characters. Since ρ itself is $\mathcal{F}$-invariant (remember that $\rho(x) = 0$ for all $x \in P \setminus \{1\}$), ψ must appear as a summand of ρ. $\square$

3 Counterexamples

In [4, table on p. 5206] and [5], the authors list some fusion systems $\mathcal{F}$ where $|\mathrm{Ind}_{\mathcal{F}}(P)| > k(\mathcal{F})$, including the system on $P \cong D_{16}$ of the group $\mathrm{PSL}(2, 17)$. This fusion system has two conjugacy classes of essential subgroups. The authors seem to have overlooked the "smaller" fusion system of $\mathrm{PGL}(2, 7)$ with only one class of essential subgroups (still on D_{16}). With the notation

$$P = \langle x, y \mid x^8 = y^2 = 1,\ x^y = x^{-1}\rangle,$$

the character table of P is:

	1	x	x^3	x^2	x^4	y	xy
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1	-1
χ_3	1	-1	-1	1	1	-1	1
χ_4	1	1	1	1	1	-1	-1
χ_5	2	0	0	-2	2	0	0
χ_6	2	$\sqrt{2}$	$-\sqrt{2}$	0	-2	0	0
χ_7	2	$-\sqrt{2}$	$\sqrt{2}$	0	-2	0	0

We may assume that x^4 and y are $\mathcal{F}$-conjugate, but the other classes of P are not fused. The $\mathcal{F}$-invariant characters of P must agree on the fifth and sixth column of the character table. Hence, we are looking for non-negative integral vectors orthogonal to $(0, 0, 1, 1, 1, -1, -1)$. Now it is easy to see that

$$\mathrm{Ind}_{\mathcal{F}}(P) = \{\chi_1,\ \chi_2,\ \chi_3 + \chi_6,\ \chi_3 + \chi_7,\ \chi_4 + \chi_6,\ \chi_4 + \chi_7,\ \chi_5 + \chi_6,\ \chi_5 + \chi_7\}.$$

In particular, $|\mathrm{Ind}_{\mathcal{F}}(P)| = 8 > 6 = k(\mathcal{F})$.

To turn this into a constrained fusion system, we set $G := \mathrm{PGL}(2,7)$ and choose an irreducible faithful $\mathbb{F}_2G$-module V of dimension 6. Then

$$\hat{G} := V \rtimes G = \mathrm{PrimitiveGroup}(64, 64) = \mathrm{TransitiveGroup}(16, 1802)$$

(notation from GAP [6]) is a 2-constrained group with Sylow 2-subgroup $\hat{P} := V \rtimes P$. Let $\hat{\mathcal{F}}$ be the corresponding constrained fusion system. The inflations of the eight G-invariant indecomposable characters of P are $\hat{\mathcal{F}}$-indecomposable. By the proof of Theorem 1, we may construct further $\hat{\mathcal{F}}$-indecomposable characters by restricting characters $\chi \in \mathrm{Irr}(G)$ with $V \nsubseteq \mathrm{Ker}(\chi)$ to $\hat{P}$. The space spanned by those restrictions has dimension at least $k(\hat{\mathcal{F}}) - k(\mathcal{F}) = k(\hat{\mathcal{F}}) - 6$. In particular, $|\mathrm{Ind}_{\mathcal{F}}(\hat{P})| > k(\hat{\mathcal{F}})$.

Finally, we provide a counterexample to [4, Conjecture 2.19] as claimed in the introduction. Let $\mathcal{F}$ be the fusion system on a Sylow 2-subgroup P of the automorphism group of the Mathieu group $G = \mathrm{Aut}(M_{22}) \cong M_{22} \rtimes C_2$. Then $|P| = 2^8$. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots \chi_{21}\}$ and $\mathrm{Irr}(P) = \{\lambda_1, \ldots, \lambda_{34}\}$. It can be checked with GAP that $k(\mathcal{F}) = 10$. Let

$$A := \big([(\chi_j)_P, \lambda_i]\big)_{i,j} \in \mathbb{Z}^{34 \times 21}.$$

By Theorem 4, every $\zeta \in \mathrm{Ind}_{\mathcal{F}}(P)$ is the restriction of some generalized character ψ of G. Setting $x := ([\psi, \chi_i])_i \in \mathbb{Z}^{21}$, we obtain $Ax = ([\zeta, \tau_i])_i \geq 0$. Hence, x belongs to the semigroup

$$S := \{x \in \mathbb{Z}^{21} : Ax \geq 0\}.$$

Moreover, since ζ is indecomposable, x is a member of a Hilbert basis H of S. We remark that H is not unique, because there exist vectors y with $Ay = 0$. However, if $y \in H$ satisfies $Ax = Ay$, then $x = y$, since otherwise $x = (x - y) + y$ would be a non-trivial decomposition of x in S. In this way, H corresponds to $\mathrm{Ind}_{\mathcal{F}}(P)$. Using the `nconvex`-package [11] in GAP, we compute H and obtain $|H| = |\mathrm{Ind}_{\mathcal{F}}(P)| = 25$. The source code is available at [12]. Moreover, 14 indecomposable $\mathcal{F}$-invariant characters are not summands of the regular character of P and six are not summands of restrictions of irreducible characters of G. It would take too much space to print these characters here, but we exhibit at least one indecomposable character for illustration:

$$\zeta := \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + 2\lambda_8 + \lambda_9 + 2\lambda_{10} + \lambda_{11} + 2\lambda_{12}.$$

The labeling is chosen in such a way that $\lambda_1, \ldots, \lambda_4$ have degree 1, $\lambda_5, \ldots, \lambda_8$ have degree 2, λ_9, λ_{10} have degree 4, and $\lambda_{11}, \lambda_{12}$ have degree 8. Since λ_4 occurs with multiplicity 2, ζ is not a summand of the regular character of P.

The symmetric group $G = S_{12}$ is a counterexample for $p = 2, 3$. As promised in the introduction, $G = S_{10}$ for $p = 2$ provides an example where $|\mathrm{Ind}_{\mathcal{F}}(P)| = 266 > 256 = |P|$.

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