Т-6

Дано:

$$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases} \quad \theta > 1$$

Найти:

a) По выборке объёма n найти оценку параметра θ ММП

b) Построить доверительный интервал медианы

c) Построить асимптотический доверительный интервал для параметра θ

Решение:

a) $$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{(\theta-1)^n}{x_1^\theta \cdot x_2^\theta \cdot \ldots \cdot x_n^\theta}$$

$$\ln L = n\ln(\theta-1) - \theta\sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{n}{\theta-1} = \sum_{i=1}^{n} \ln x_i$$

$$\theta - 1 = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\frac{\partial^2 \ln(L(\theta))}{\partial \theta^2} = -\frac{n}{(\theta-1)^2} < 0$$

Выполнено дост. усл. max

Т.о. $$\boxed{\tilde{\theta} = 1 + \frac{n}{\sum\limits_{i=1}^{n} \ln x_i}}$$

b) 1) p – непр. диф-ма по θ

2) $$\frac{\partial}{\partial \theta} \int \frac{\theta-1}{x^{\theta}}\, dx = x^{1-\theta} \ln x$$

$$\int \frac{\partial}{\partial \theta} \left(\frac{\theta-1}{x^{\theta}} \right) dx = x^{1-\theta} \ln x$$

Модель сильно регулярная

$$\int_1^{\tilde{x}} \frac{\theta-1}{x^{\theta}}\, dx = -\frac{1}{\tilde{x}^{\theta-1}} + 1 = \frac{1}{2} \Rightarrow \tilde{x} = \sqrt[\theta-1]{2}$$

$$g(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$$

$$\sqrt{n} \cdot \frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma(\theta) \xrightarrow{p} \sigma(\tilde{\theta})$$

$$\sigma(\tilde{\theta}) = \sqrt{\nabla^T g(\tilde{\theta})\, I^{-1}(\theta)\, \nabla g(\tilde{\theta})}$$

$$I(\tilde\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = M\left[\left(\frac{\partial}{\partial\theta}\left(\ln(\theta-1) - \theta\ln x\right)\right)^2\right] =$$

$$= M\left[\left(\frac{1}{\theta-1} - \ln x\right)^2\right] = \int_1^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 p(x,\theta)\,dx =$$

$$= \int_1^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 \cdot \frac{\theta-1}{x^\theta}\,dx = \frac{1}{(\theta-1)^2}$$

$I(\tilde\theta)$ – непр при $\theta > 1$

$$\nabla g(\tilde\theta) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde\theta-1}}}{\tilde\theta - 1}$$

$$\sqrt{n}\cdot\frac{g(\tilde\theta) - g(\theta)}{\sigma(\tilde\theta)} \rightsquigarrow N(0,1)$$

$$\frac{1{,}96\cdot\sigma(\tilde\theta)}{\sqrt{n}} + g(\tilde\theta) < g(\theta) < -\frac{1{,}96\cdot\sigma(\tilde\theta)}{\sqrt{n}} + g(\tilde\theta)$$

c) $$\sqrt{n}\,\frac{\tilde\theta-\theta}{\sigma(\theta)} \rightsquigarrow N(0,1), \quad \sigma(\theta) \xrightarrow{p} \sigma(\tilde\theta)$$

$$\sigma(\tilde\theta) = \theta - 1 \Rightarrow \sqrt{n}\,\frac{\tilde\theta-\theta}{\theta-1} \rightsquigarrow N(0,1)$$

$$-\frac{1{,}96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_{i=1}^{n}\ln x_i} < \theta < \frac{1{,}96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_{i=1}^{n}\ln x_i}$$