T-5

Дано: $\xi \sim R[\theta, 2\theta]$ $\quad p(x,\theta) = \frac{1}{\theta}\{\theta, 2\theta\}$

Найти:

а) По выборке объёмом n найти оценки параметра $\theta$ ММ и ММП

б) Проверить эти оценки на несмещ. и сост.

в) Сравнить эффект. испр. оценок

г) Построить точный доверит. интервал для $\theta$

д) Построить асимпт. довер. интервал для параметра $\theta$.

Решение:

а) Метод моментов

$$\alpha_1 = M[\xi] = \int_{-\infty}^{+\infty} x p(x)\,dx = \int_{\theta}^{2\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} =$$

$$= 2\theta - \frac{\theta}{2} = \frac{3}{2}\theta$$

$$\alpha_2 = M[\xi^2] = \int_{-\infty}^{+\infty} x^2 p(x)\,dx = \int_{\theta}^{2\theta} \frac{x^2}{\theta}\,dx = \frac{x^3}{3\theta}\Big|_{\theta}^{2\theta} =$$

$$= \frac{8}{3}\theta^2 - \frac{1}{3}\theta^2 = \frac{7}{3}\theta^2$$

$$D[\xi] = \alpha_2 - \alpha_1^2 = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{28-27}{12}\theta^2 = \frac{1}{12}\theta^2$$

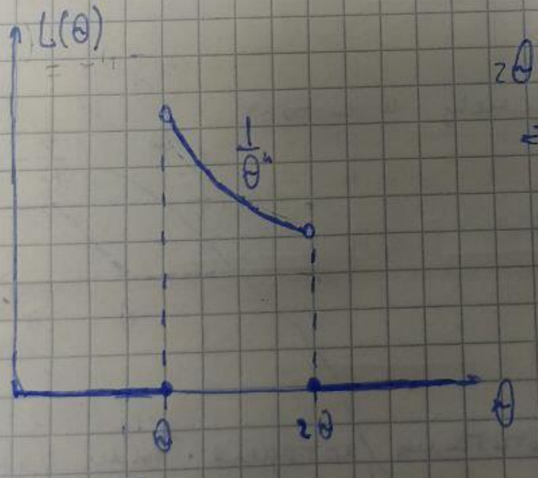
$$\alpha_1 = \tilde{\alpha}_1 = \bar{x}$$

$$\frac{3}{2}\theta = \bar{x} \Rightarrow \boxed{\tilde{\theta}_1 = \frac{2}{3}\bar{x}}$$

$$\alpha_2 = \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

Метод максимального правдоподобия

$$L(\theta) = \prod_{i=1}^{n} p(x, \theta) = \frac{1}{\theta^n}$$




$$2\theta = x_{max} \Rightarrow$$

$$\Rightarrow \tilde{\theta}_2 = \frac{x_{max}}{2}$$

$$\boxed{\tilde{\theta}_2 = \frac{x_{max}}{2}}$$

5) Проверим $\tilde{\theta}_1$ на несмещ. и состоят.

Несмещ.:

$$M[\tilde{\theta}_1] = \frac{2}{3}M[\bar{x}] = \frac{2}{3}M[y] = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta \Rightarrow$$

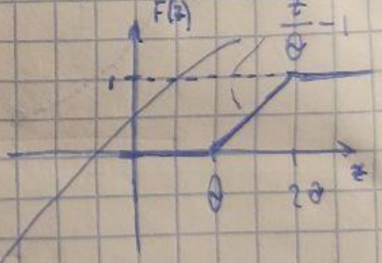
$\Rightarrow$ $\tilde{\theta}_1$ – несмещённая.

Состоят.:

$$D[\tilde{\theta}_1] = \frac{4}{9}D[\bar{x}] = \frac{4}{9n}D[y] = \frac{\cancel{4}}{9n}\cdot\frac{1}{\cancel{4}\cdot 3}\theta^2 = \frac{\theta^2}{27}\cdot\frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ $\tilde{\theta}_1$ – состоятельная.

Проверим $\tilde{\theta}_2$ на несмещ. и состоят.




Несмещ.

$$M[\tilde{\theta}_2] = \frac{1}{2}M[x_{max}]$$

Воспользуемся результатами, которые были получены в Т-1 для $M[x_{max}]$

$$M[\tilde{\theta}_2] = \frac{1}{2}\int_{\theta}^{2\theta}\frac{nz}{\theta}\left(\frac{z}{\theta}-1\right)^{n-1}dz = \left\{\begin{matrix} \frac{z}{\theta}=t & \text{в.г.} = 2 \\ dz = \theta dt & \text{н.г.} = 1 \end{matrix}\right\} =$$

$$= \frac{\theta n}{2} \int_1^2 t (t-1)^{n-1} dt = \left| \begin{array}{l} u = t \\ du = dt \\ dv = (t-1)^{n-1} dt \\ v = \frac{(t-1)^n}{n} \end{array} \right| =$$

$$= \frac{\theta n}{2} \left[ \frac{t(t-1)^n}{n} \Big|_1^2 - \int_1^2 \frac{(t-1)^n}{n} dt \right] =$$

$$= \frac{\theta}{2} \left[ 2 - \frac{(t-1)^{n+1}}{n+1} \Big|_1^2 \right] = \frac{\theta}{2} \left[ 2 - \frac{1}{n+1} \right] =$$

$$= \frac{\theta}{2} \left[ \frac{2(n+1) - 1}{n+1} \right] = \frac{\theta}{2} \left[ \frac{2n+1}{n+1} \right]$$

Т.о. $\tilde{\theta}_2$ – смещённая

$$\tilde{\theta}_2' = \frac{2n+2}{2n+1} \cdot \frac{X_{max}}{2} = \frac{n+1}{2n+1} X_{max} \text{ – несмещ.}$$

Состоят.

$D[\tilde{\theta}_2] = \alpha_2 - \alpha_1^2$

$$\alpha_2 = M[\tilde{\theta}_2^2] = \frac{1}{4} \int_\theta^{2\theta} \frac{n z^2}{\theta} \left( \frac{z}{\theta} - 1 \right)^{n-1} dz = \left| \begin{array}{l} \frac{z}{\theta} = t \\ dz = \theta dt \\ В.г. = 2 \\ Н.г. = 1 \end{array} \right| =$$

$$= \frac{\theta^2}{4} \int_1^2 n t^2 (t-1)^{n-1} dt = \left| \begin{array}{l} u = t^2 \\ du = 2t\,dt \\ dv = (t-1)^{n-1} dt \\ v = \frac{(t-1)^n}{n} \end{array} \right| =$$

$$= \frac{\cancel{n}\theta^2}{4}\left[\frac{t^2(t-1)^n}{\cancel{n}}\Big|_1^2 - \int_1^2 2t\cdot\frac{(t-1)^n}{\cancel{n}}dt\right] =$$

$$= \frac{\theta^2}{4}\left[4 - 2\int_1^2 t(t-1)^n dt\right] = \left\{\begin{array}{l} u = t \\ du = dt \\ dv = (t-1)^n dt \\ v = \frac{(t-1)^{n+1}}{n+1} \end{array}\right\} =$$

$$= \frac{\theta^2}{4}\left[4 - 2\left(t\frac{(t-1)^{n+1}}{n+1}\Big|_1^2 - \int_1^2 \frac{(t-1)^{n+1}}{n+1}dt\right)\right] =$$

$$= \frac{\theta^2}{4}\left[4 - 2\cdot\left(\frac{2}{n+1} - \frac{(t-1)^{n+2}}{(n+1)(n+2)}\Big|_1^2\right)\right] =$$

$$= \frac{\theta^2}{4}\left[4 - \frac{4}{n+1} - \frac{2}{(n+1)(n+2)}\right] =$$

$$= \frac{\theta^2}{4}\left[\frac{4n^2+12n+8-4n-8+2}{(n+1)(n+2)}\right] = \frac{\theta^2(4n^2+8n+2)}{4(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{\theta^2(4n^2+8n+2)}{4(n+1)(n+2)} - \frac{\theta^2(4n^2+4n+1)}{4(n+1)^2} =$$

$$= \frac{\theta^2}{4}\left[\frac{\cancel{4n^3}+\cancel{8n^2}+2n+\cancel{4n^2}+\cancel{8n}+\cancel{2}-\cancel{4n^3}-\cancel{4n^2}-n-\cancel{8n^2}-\cancel{8n}-\cancel{2}}{(n+1)^2(n+2)}\right] =$$

$$= \frac{n\theta^2}{4(n+1)^2(n+2)}$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 \cdot \frac{n\theta^2}{\cancel{4(n+1)^2}(n+2)} = \frac{n\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$

$\tilde{\theta}_2^1$ – состоят

$D[\tilde{\theta}_1] = \frac{\theta^2}{27n}$

$D[\tilde{\theta}_2^1] = \frac{n\theta^2}{(n+2)(2n+1)^2}$

$n=1: \quad \frac{\theta^2}{27} = \frac{\theta^2}{27}$

$n=2: \quad \frac{\theta^2}{54} < \frac{2\theta^2}{4\cdot 25} = \frac{\theta^2}{50}$

$n=3: \quad \frac{\theta^2}{81} > \frac{3\theta^2}{5\cdot 49} = \frac{3\theta^2}{245}$

$n=4: \quad \frac{\theta^2}{108} > \frac{\overset{2}{4}\theta^2}{3\cancel{6}\cdot 81} = \frac{2\theta^2}{243}$

$\vdots$

Таким образом при $n \geq 3$ $D[\tilde{\theta}_1] > D[\tilde{\theta}_2^1]$

Значит, $\tilde{\theta}_2^1$ – эффективнее.

г) Точный доверит. интервал

$\xi \sim R[\theta, 2\theta]$

Пусть $\vec{X}_n$ – выборка

$$f(\vec{x}_n, \theta) = \frac{x_{max}}{\theta} - 1$$

$$P(f < t) = P(x_{max} < \theta t + \theta) = \left(F(\theta t + \theta)\right)^n$$

$$F(x) = \begin{cases} 0, & x < \theta \\ \frac{x}{\theta} - 1, & x \in [\theta, 2\theta] \\ 1, & x > 2\theta \end{cases} \Rightarrow \begin{cases} 0, & t \leq 0 \\ t^n, & 0 < t \leq 1 \\ 1, & t > 1 \end{cases}$$

$$t_1 = q_{\frac{\alpha}{2}} = \sqrt[n]{\frac{\alpha}{2}} = \sqrt[n]{\frac{1-\beta}{2}} \qquad \alpha + \beta = 1$$

$$t_2 = q_{1-\frac{\alpha}{2}} = \sqrt[n]{1 - \frac{\alpha}{2}} = \sqrt[n]{\frac{1+\beta}{2}}$$

$$P\left(t_1 < \frac{x_{max}}{\theta} - 1 < t_2\right) = \beta$$

$$t_1 + 1 < \frac{x_{max}}{\theta} < t_2 + 1$$

$$P\left(\frac{x_{max}}{1 + \sqrt[n]{\frac{1+\beta}{2}}} < \theta < \frac{x_{max}}{1 + \sqrt[n]{\frac{1-\beta}{2}}}\right) = \beta$$

д) Асимптотический доверительный интервал

Оценка метода моментов $\quad (g(\tilde{\alpha}) - g(\alpha))\sqrt{n} \rightsquigarrow$

$$\tilde{\theta}_1 = \frac{2}{3}\overline{x} = \frac{2}{3}\tilde{\alpha}_1 = g(\tilde{\alpha}_1) \qquad \rightsquigarrow N(\vec{0}, \vec{\nabla}g(\alpha) K \vec{\nabla} g(\alpha))$$

$$g(\alpha_1) = \frac{2}{3}\alpha_1 = \theta \qquad \nabla g = g'_\theta = \frac{2}{3}$$

$$\frac{\sqrt{n}\left(g(\tilde{\alpha}) - g(\alpha)\right)}{\nabla^T g(\tilde{\alpha})\, K(\tilde{\alpha})\, \vec{\nabla} g(\tilde{\alpha})} \sim N(0,1)$$

$$K_{11} = \alpha_2 - \alpha_1^2$$

$$\tilde{\mu}_2 = \tilde{\alpha}_2 - \tilde{\alpha}_1^2 = \frac{S^2(n-1)}{n}$$

$$\frac{\sqrt{n}\,(\tilde{\theta}_1 - \theta)}{\frac{2S}{3}\sqrt{\frac{n-1}{n}}} = \frac{3n\,(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} \rightsquigarrow N(0,1)$$

$$t_1 = -1{,}96$$

$$t_2 = 1{,}96$$

$$P\left(t_1 < \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} < t_2\right) = \beta$$

$$\frac{2St_1\sqrt{n-1}}{3n} < \tilde{\theta}_1 - \theta < \frac{2St_2\sqrt{n-1}}{3n}$$

$$P\left(\tilde{\theta}_1 - \frac{2St_2\sqrt{n-1}}{3n} < \theta < \tilde{\theta}_1 - \frac{2St_1\sqrt{n-1}}{3n}\right) = \beta$$