Первое задание

Т-1

Дано: Сл. вел расп. равном. на $[0, \theta]$. По выборке объёма n найдены оценки параметра θ:

$$\tilde{\theta}_1 = 2\overline{X}$$

$$\tilde{\theta}_2 = X_{min}$$

$$\tilde{\theta}_3 = X_{max}$$

$$\tilde{\theta}_4 = \left(X + \frac{\sum_{k=2}^{n} X_k}{(n-1)} \right)$$

а) Проверить оценки на несмещенность и состоятельность.

Исправить эти оценки если необходимо

б) Какая из исправленных оценок более эффективна

Решение:

а) $\xi \sim R(0, \theta)$ θ - неизв. параметр

$\theta > 0$ - вероятностная модель

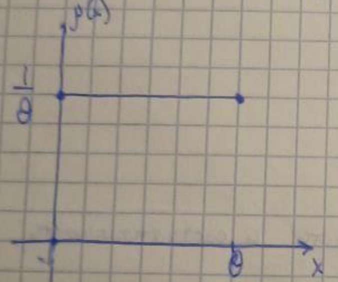
Пусть $\vec{X}_n$ - выборка объёма n

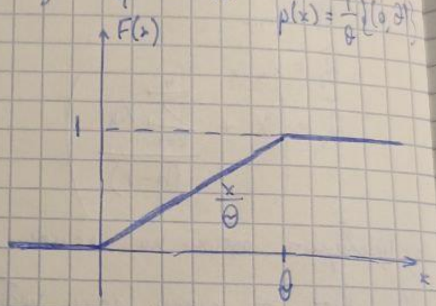
1) Найдём числовые характеристики сл. вел. ξ

$$M[\xi] = \int_{-\infty}^{+\infty} x\,p(x)\,dx = \int_0^{\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_0^{\theta} = \frac{\theta}{2}$$

$$M[\xi^2] = \int_0^{\theta} x^2 p(x)\,dx = \int_0^{\theta} \frac{x^2}{\theta}\,dx = \frac{x^3}{3\theta}\Big|_0^{\theta} = \frac{\theta^2}{3}$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$





2) Исследуем $\tilde{\theta}_1 = 2\bar{x} = 2\cdot\frac{1}{n}\sum_{i=1}^{n} x_i$

а) Несмещённость

Используем опр. несмещённости:

$\boxed{\forall\theta>0 \quad M[\tilde{\theta}_1] = \theta}$ $\quad M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \left\{x_i \text{ - незав.}\right\} =$

$= \frac{2}{n}\cdot n\,M[x_i] = 2M[x_i] = \left\{x_i \text{ - одинаково распред}\right\} =$

$= 2M[\xi] = 2\cdot\frac{\theta}{2} = \theta \Rightarrow \tilde{\theta}_1$ – несмещенная.

б) Состоятельность

Используем дост. усл. состоятельности

$$\left.\begin{array}{l}\tilde\theta \text{ - несмещённая} \\ D[\tilde\theta] \xrightarrow[n\to\infty]{} 0 \\ \forall \theta \in \Theta\end{array}\right\} \Rightarrow \tilde\theta \text{ - состоят.}$$

$$D[\tilde\theta_1] = D\left[\frac{2}{n}\sum x_i\right] = \frac{4}{n^2} D[x_i] = \left\{ x_i \text{ - независ. и один. распр.} \right\} =$$

$$= \frac{4}{n^2}\cdot n D[\xi_i] = \frac{4}{n} D[\xi_i] = \frac{4}{n}\cdot\frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$\Rightarrow$ $\tilde\theta_1$ - состоятельная

3) Исследуем $\theta_2 = \min(x_i)$

а) Несмещённость

$$M[\theta_2] = \int_{-\infty}^{+\infty} y\varphi(y)\,dy$$

$$\Phi(y) = 1 - (1 - F(y))^n$$

$$\Phi'(y) = \varphi(y) = n(1 - \underbrace{F(y)}_{= \frac{y}{\theta}})^{n-1} \underbrace{p(y)}_{= \frac{1}{\theta}\{(0,\theta)\}} = n\left(1 - \frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}$$

$$M[\theta_2] = \int_0^\theta n\left(1 - \frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}\,y\,dy = \left\{ t = 1 - \frac{y}{\theta} \right\} = \int_1^0 nt^{n-1}(1-t)\theta\,dt =$$

$$= \int_0^1 n\theta t^{n}\,dt - \int_0^1 n\theta t^{n-1}\,dt = n\theta\left(\frac{t^{n+1}}{n+1}\Big|_0^1 - \frac{t^{n}}{n-1}\Big|_0^1\right) =$$

$$= n\theta\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = n\theta\left(\frac{n-1-n-1}{n^2-1}\right) = n\theta\cdot\frac{2}{1-n^2} \Rightarrow$$

$$\| \quad \theta\cdot\frac{2n}{1-n^2}$$

$\Rightarrow$ $\tilde{\theta}_2$ – смещённая

$\tilde{\theta}_2' = \frac{1-n^2}{2n}\tilde{\theta}_2$ – несмещённая оценка $M[\tilde{\theta}_2'] = \theta$

$$= \int_0^1 n\theta t^{n-1}\,dt - \int_0^1 n\theta t^{n}\,dt = \theta\left[1 - \frac{n}{n+1}\right] = \frac{\theta}{n+1} \Rightarrow$$

$\Rightarrow$ $\tilde{\theta}_2$ – смещённая

$\tilde{\theta}_2' = (n+1)x_{\min:n} = (n+1)\tilde{\theta}_2$ – несмещ. $M[\tilde{\theta}_2'] = \theta$

б) Состоятельность

$$M[\tilde{\theta}_2^2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}y^2\,dy = \left\{t = 1-\frac{y}{\theta}\right\} =$$

$$= \int_1^0 n t^{n-1}\theta^2(1-t)^2\,dt = n\theta^2\left[\int_0^1 \left(t^{n-1} - 2t^{n} + t^{n+1}\right)dt\right] =$$

$$= n\theta^2\left[\frac{1}{n} - 2\frac{1}{n+1} + \frac{1}{n+2}\right] = n\theta^2\left[\frac{(n+1)(n+2) - 2n(n+2) + n(n+1)}{(n+1)(n+2)}\right] =$$

$$= \theta^2\frac{n^2+3n+2-2n^2-4n+n^2+n}{(n+1)(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \xrightarrow[n\to\infty]{} 0$$

$D[\tilde{\theta}_2] \xrightarrow[n\to\infty]{} 0$

$\tilde{\theta}_2$ – смещённая

/ $\Rightarrow$ про $\tilde{\theta}_2$ ничего не известно

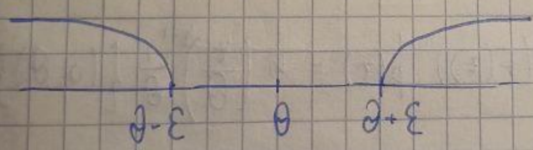
• Посчитаем дисперсию $D[\tilde{\theta}_2']$

$$D[\tilde{\theta}_2'] = (n+1)^2 D[\tilde{\theta}_2] = \frac{\theta^2 n}{n+2} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$\Rightarrow$ Достаточное усл. не работает

• Исследуем $\tilde{\theta}_2'$ и $\tilde{\theta}_2$ на состоятельность по определению

$$\forall\theta>0 \quad \forall\varepsilon>0 \quad P(|\tilde{\theta}_2' - \theta| > \varepsilon) \xrightarrow[n\to\infty]{} 0$$



$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta+\varepsilon) = P((n+1)x_{min} \geq \theta+\varepsilon) =$$

$$= P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - \underbrace{P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right)}_{\varphi\left(\frac{\theta+\varepsilon}{n+1}\right)} =$$

$$= 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) =$$

$$= \left(1 - \left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0 \Rightarrow \tilde{\theta}_2' \text{ не состоятельная}$$

$\tilde{\theta}_2: P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$\forall \varepsilon > 0$
$\forall \theta > 0$ $: P(\tilde{\theta}_2 < \theta - \varepsilon) + \underbrace{P(\tilde{\theta}_2 > \theta + \varepsilon)}_{\overset{\shortparallel}{\text{нет}}}$

$\overset{\shortparallel}{0}$

$P(x_{min} < \theta - \varepsilon) = \Phi(\theta - \varepsilon) = 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n = 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1 \neq$

$\Rightarrow$ $\tilde{\theta}_2$ не состоятельная

4) Исследуем $\tilde{\theta}_3 = x_{max}$

$$M[\tilde{\theta}_3] = \int_{-\infty}^{+\infty} z\psi(z)\,dz = \int_0^\theta n \frac{z^n}{\theta^n}\,dz = \frac{n}{n+1}\theta$$

$$\Psi = (F(z))^n$$

$$\psi(z) = \Psi'(z) = n(F(z))^{n-1} p(z) = n\left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta} \{(0,\theta)\}$$

$\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$ — несмещённая оценка

$$M[\tilde{\theta}_3^2] = \int_0^\theta n \frac{z^{n+1}}{\theta^n}\,dz = \frac{n}{n+2}\theta^2$$

$$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2\left[\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right] =$$

$$= \theta^2\left[\frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2}\right] = \frac{n\theta^2}{(n+2)(n+1)^2} \xrightarrow[n\to\infty]{} 0$$

$D[\tilde{\theta}_3] \xrightarrow[n\to\infty]{} 0$

$\tilde{\theta}_3$ – смещенная $\Big/ \Rightarrow$ про $\tilde{\theta}_3$ ничего сказать нельзя

· Исследуем $\tilde{\theta}_3'$ на состоятельность

$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} \cdot D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$

$\Rightarrow \underbrace{\tilde{\theta}_3' \text{ – состоятельная}}$

· Иссл. $\tilde{\theta}_3$ на состоятельность по опред.

$\forall \varepsilon > 0$
$\forall \theta > 0$: $P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(x_{max} < \theta - \varepsilon) + \overset{0}{\overbrace{+ P(x_{max} > \theta + \varepsilon)}} = (F(\theta - \varepsilon))^n$

$\varepsilon < \theta$: $\left(\frac{\theta - \varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$

$\varepsilon > \theta$ $\quad (0)^n \xrightarrow[n\to\infty]{} 0$

5) Исследуем $\tilde{\theta}_4 = x_1 + \frac{\sum\limits_{k=2}^{n} x_k}{(n-1)}$

a) Несмещённость

$$M[\tilde{\theta}_1] = M[x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i] = M[x_1] + \frac{1}{n-1}\sum_{i=2}^{n} M[x_i] =$$

$$= \frac{\theta}{2} + \frac{\theta}{2} = \theta \Rightarrow \underbrace{\tilde{\theta}_1 \text{ - несмещенная}}$$

б) Состоятельность

$$D[\tilde{\theta}_3] = D[x_1 + \frac{1}{n-1}\sum x_i] = D[\xi] + \left(\frac{1}{n-1}\right)^2 (n-1) D[\xi] =$$

$$= \frac{\theta^2}{12} + \frac{\theta^2}{12(n-1)} = \frac{\theta^2}{12} \cdot \frac{n}{n-1} \xrightarrow[n \to \infty]{} 0$$

Достаточное усл. не работает

$$\tilde{\theta}_n \xrightarrow{p} \theta$$

Св-во: Если $\xi_n \xrightarrow{p} \xi$, $\eta_n \xrightarrow{p} \eta$

$$\xi_n + \eta_n \longrightarrow \xi + \eta$$

Разобьём $\tilde{\theta}_n$ на две части

$$x_1 \xrightarrow{p} x_1$$

$$\frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{p} \left\{ \begin{array}{l} \text{ЗБЧ Хинчина} \\ \xi_1 \dots \xi_n \text{ незав, случ велич.} \\ \frac{1}{n}\sum_{i=1}^{n} \xi_i \xrightarrow{p} M[\xi] \end{array} \right\} \longrightarrow M[\xi] =$$

$$= \xi_1, \dots, \xi_n$$

$$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i \longrightarrow x_1 + \frac{\theta}{2}$$

$\left.\begin{matrix}\tilde{\theta}_1 \\ \tilde{\theta}_3'\end{matrix}\right]$ несмещ и сост

$$\tilde{\theta}_1 = 2\bar{x}$$

$$\tilde{\theta}_3 = \frac{n+1}{n} x_{max}$$

$$\left.\begin{aligned} D[\tilde{\theta}_1] &= \frac{\theta^2}{3n} \\ D[\tilde{\theta}_3'] &= \frac{\theta^2}{n(n+2)} \end{aligned}\right] \forall \theta > 0: \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n} \Rightarrow$$

$\Rightarrow$ $\tilde{\theta}_3'$ – эффективнее $\tilde{\theta}_1$