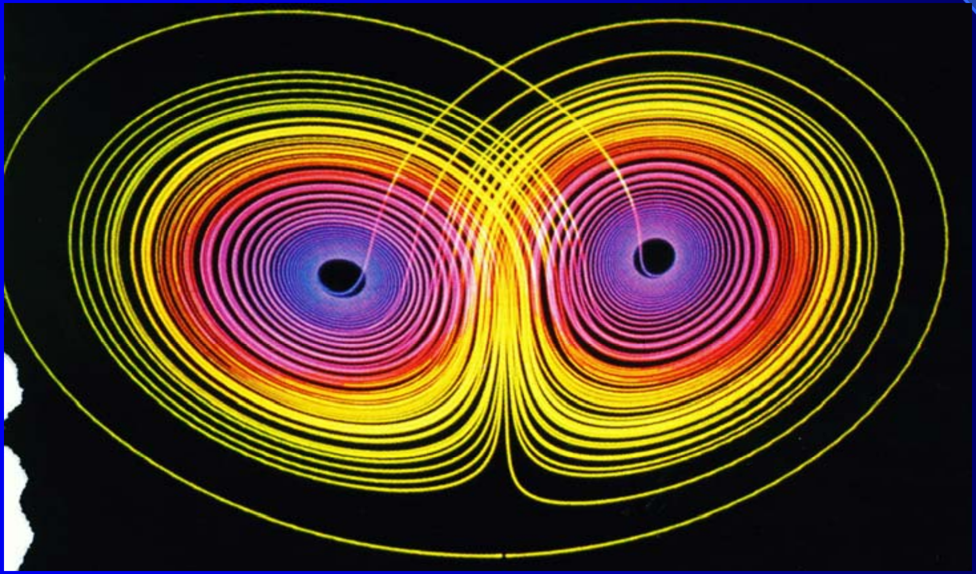
CHAOS



Lorenz Attractor [Gleick, 1987]

Outline

- Chaos Defined
- Characteristics of Chaos
- History of Chaos
- Edward Lorenz and the Butterfly Effect
- The Water Wheel
- The Lorenz Attractor
- Chaos in Action

Chaos Defined

- “Irregular motion of a dynamical system that is deterministic, sensitive to initial conditions, and impossible to predict in the long term with anything less than an infinite and perfect representation of analog values.” [Flake,]
- “Chaos is sustained and disorderly- looking long-term evolution that satisfies certain special mathematical criteria and that occurs in a deterministic non-linear system.” [Williams, 1997]
- “The property that characterizes a dynamical system in which most orbits exhibit sensitive dependence.” [Lorenz, 1993]

Characteristics of Chaos

- **Deterministic –** Chaotic systems are completely deterministic and not random.
- **Sensitive –** Chaotic systems are extremely sensitive to initial conditions, since any perturbation, no matter how minute, will forever alter the future of the chaotic system.
- **Ergodic –** Chaotic motion is ergodic, which means that the state space trajectory of a chaotic system will always return to the local region of a previous point in the trajectory.
- **Embedded –** Chaotic attractors are embedded with an infinite number of unstable periodic orbits.

History of Chaos

- The word chaos goes back to Greek mythology where it had two meanings:
  - The primeval emptiness of the universe before things came into being.
  - The abyss of the underworld.

History of Chaos (cont.)

- 1873 – British physicist James Clerk Maxwell said, “when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable…[and] renders impossible the prediction of future events…”

History of Chaos (cont.)

- 1898 – French mathematician Jacques Hadamard remarked that an error or discrepancy in initial conditions can render a system's long-term behavior unpredictable.
- 1908 – French mathematician Henri Poincare emphasized that slight differences in initial conditions eventually can lead to large differences, making prediction "impossible".

History of Chaos (cont.)

- It wasn’t until 1963 when meteorologist Edward Lorenz of MIT published his paper “Deterministic Nonperiodic Flow” that chaos really took off.
- As of 1991, the number of papers on chaos was doubling about every two years.
- Entire courses on chaos are taught at colleges and universities.
- Journals and magazines are devoted largely or entirely to chaos now.

Edward Lorenz and the Butterfly Effect

- Lorenz created a toy weather on his Royal McBee at MIT in 1960.
- Weather prediction was considered “less than science” at that time.
- Using a set of 12 equations his computer modeled the weather.
- In the winter of 1961, wanting to examine a sequence at greater length, Lorenz took a shortcut.

Edward Lorenz and the Butterfly Effect (cont.)

- Instead of starting the whole run over, he started midway through.
- To give the machine its initial conditions, he typed the numbers straight from the earlier printout.
- He went to get a cup of coffee.
- When he came back an hour later, the sequence had evolved differently.

Edward Lorenz and the Butterfly Effect (cont.)

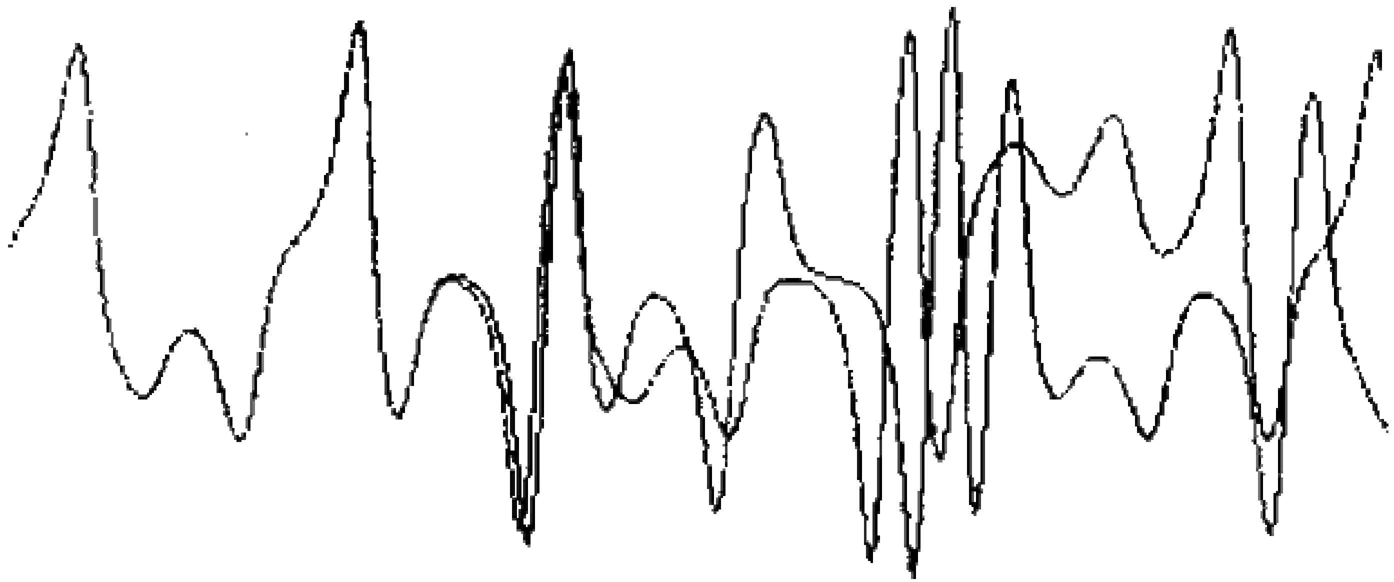


Figure 1: Lorenz's experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg 141)

Edward Lorenz and the Butterfly Effect (cont.)

- The new run should have exactly duplicated the original…why didn't it?
- In the computer's memory, six decimal places were stored: 0.506127.
- Lorenz entered 0.506 assuming that the difference – one part in a thousand – was inconsequential.

Edward Lorenz and the Butterfly Effect (cont.)

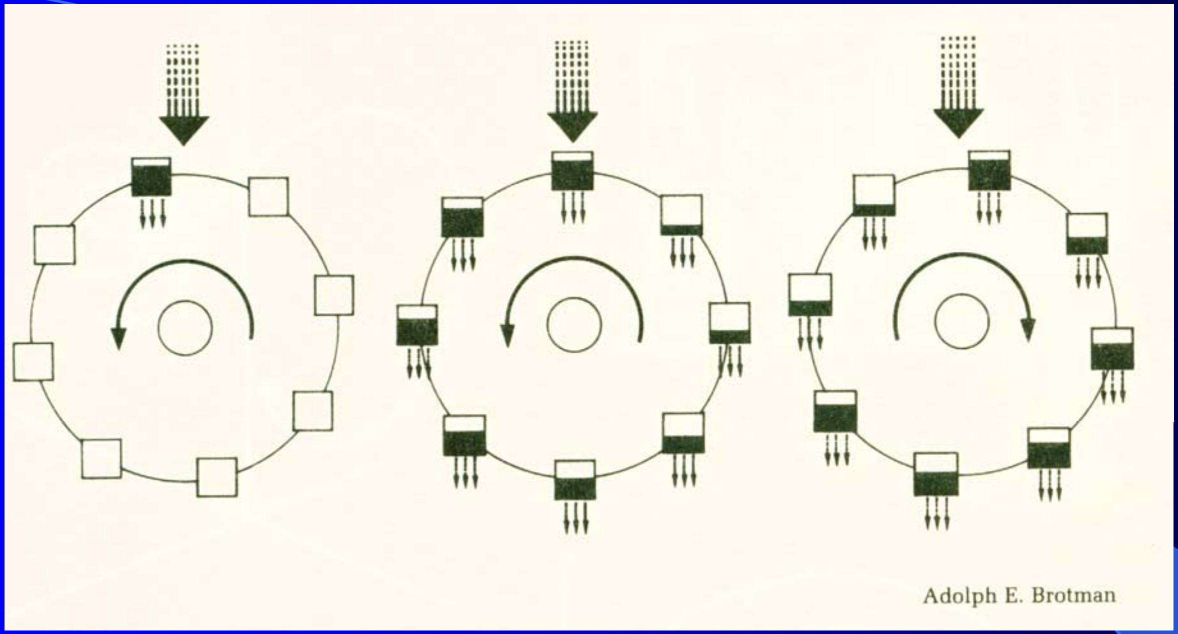
- This effect came to be known as the butterfly effect.

  “The flapping of a single butterfly’s wing today produces a tiny change in the state of the atmosphere. Over a period of time, what the atmosphere actually does diverges from what it would have done. So, in a month’s time, a tornado that would have devastated the Indonesian coast doesn’t happen. Or maybe one that wasn’t going to happen, does.” [Ian Stewart, *Does God Play Dice? The Mathematics of Chaos*]

- This phenomenon is also known as sensitive dependence on initial conditions.

The Water Wheel

- Lorenz put the weather aside and looked for even simpler ways to produce chaotic behavior.
- Using three equations for convection which he stripped down and made unrealistically simple, Lorenz was able to illustrate sensitive dependence on initial conditions.
- His equations correspond exactly to a mechanical device called the water wheel.

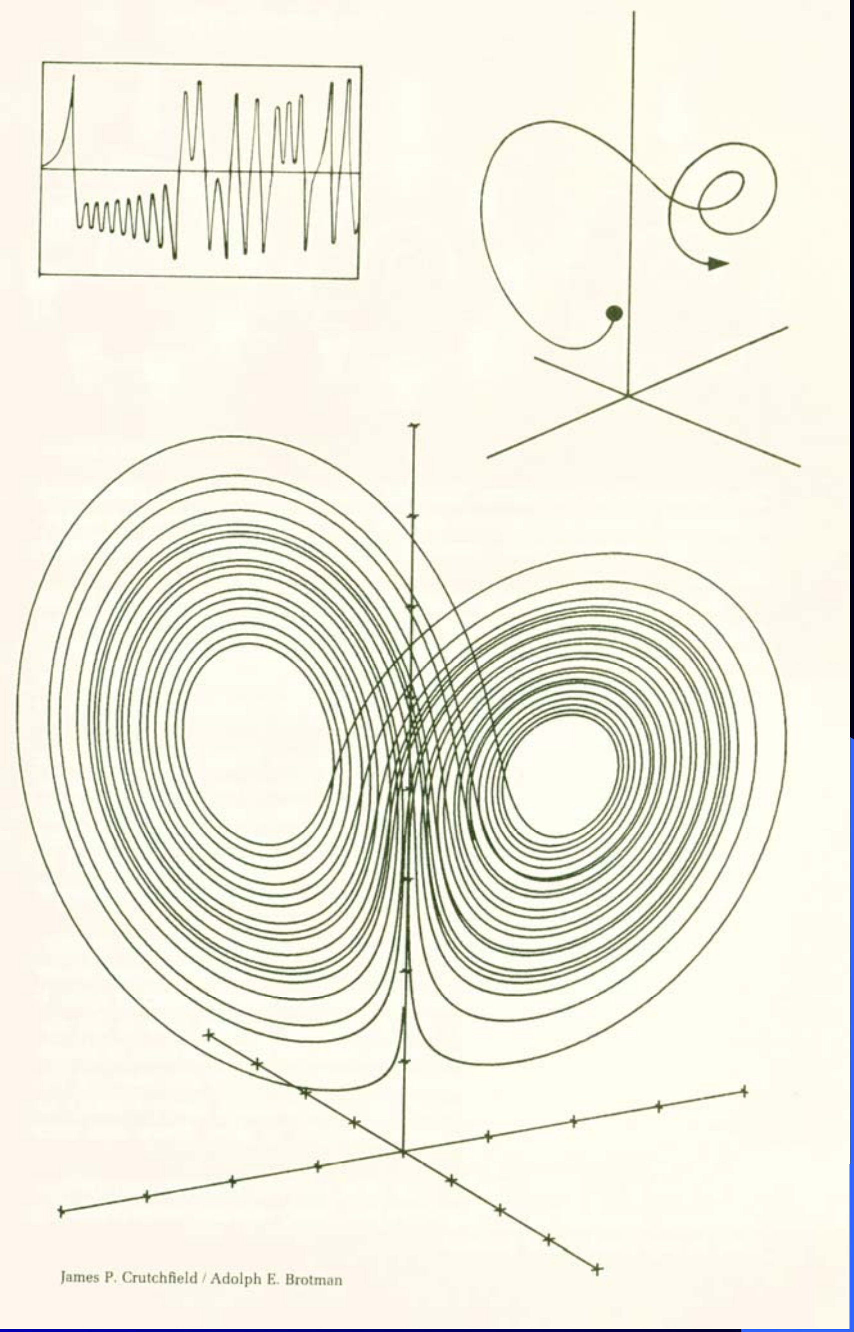


- Water drips steadily into containers hanging on the wheel’s rim.
- If the flow of the water is slow, the top bucket never fills up enough to overcome friction and the wheel never starts turning.
- If the flow is faster, the weight of the top bucket sets the wheel in motion (left).
- The waterwheel can set into a rotation that continues at a steady rate (center).
- If the flow is even faster, than the spin can become chaotic.

- Lorenz discovered that over long periods, the spin can reverse itself many times, never settling down to a steady rate and never repeating itself in any predictable pattern.
- Lorenz summarized his findings in his 1963 paper published in the *Journal of Atmospheric Sciences*.
- He ultimately states, “In view of the inevitable inaccuracy and incompleteness of weather observations, precise very long-range forecasting would seem to be non-existent.”
- In addition, the Lorenz Attractor is born, which became an emblem for early chaos theory.

Lorenz Attractor

- The changing values of one variable can be displayed in a time series (top).
- At any one instance in time the three variables fix the location of a point in 3-D space.
- Because the system never exactly repeats itself, the trajectory never intersects itself.



Chaos in Action

- Assume we have a ski slope with moguls.
- Imagine balls are allowed to roll down from the top of the slope. Will the balls follow the same path?
- Although the moguls on the right appear to be regularly spaced, they are not enough to fit a simple mathematical formula.

Figure 5. Moguls on a real-world ski slope.

- Assume a model ski slope with “squares” 2 meters wide and 5 meters long.
- Centers of the moguls are at the center of the dark squares.
- Let the slope face southward.
- The slope would have to be infinitely long to afford a perfect example of chaos.
- The moguls rise 1 meter above the pits to its east and west.

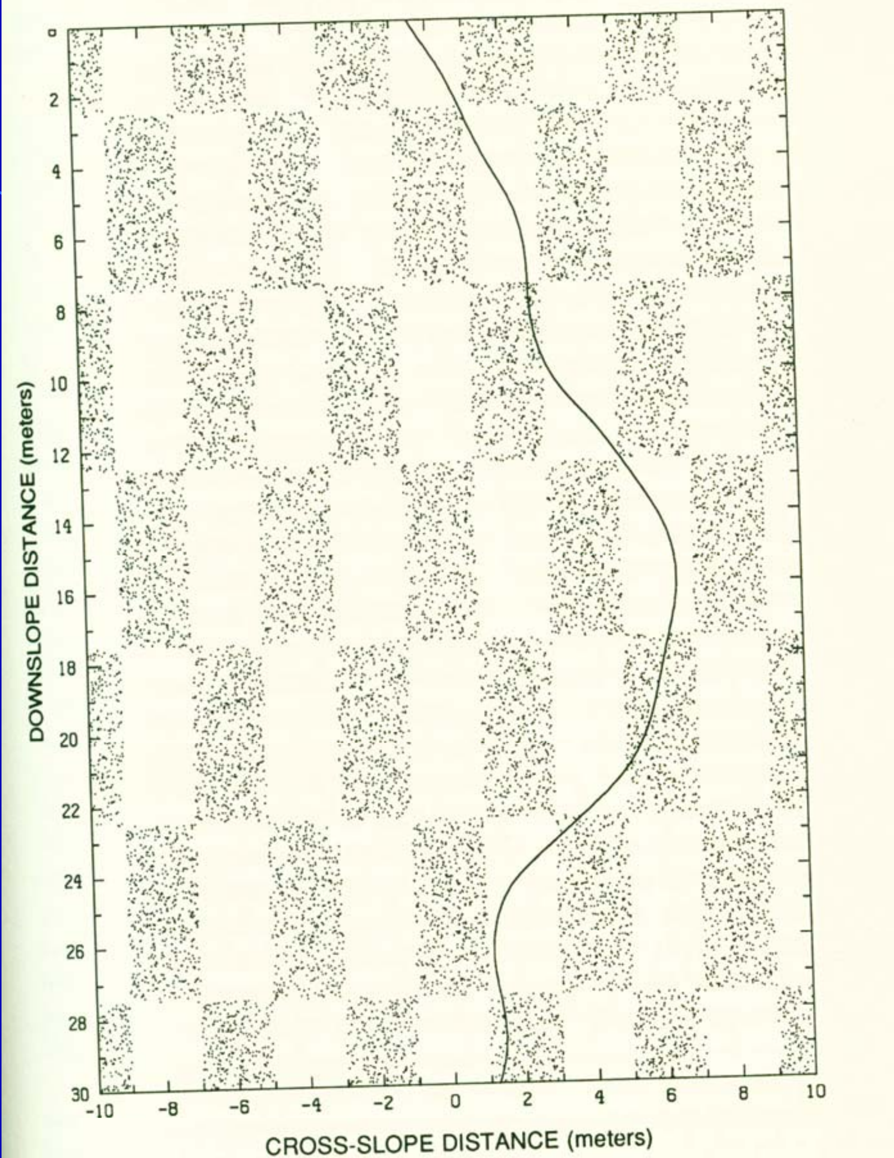


Figure 6. A top view of a section of the model ski slope, with the path of a single board sliding down it. The shaded rectangular areas of the slope project above a simple inclined plane, while the unshaded areas project below.

- Assume 7 boards all start from the same west-east starting line with the same speed and direction, but at points at 10-cm intervals.
- As seen in the figure (right) the paths are sensitively dependent on their starting points and the motion can be considered chaotic.

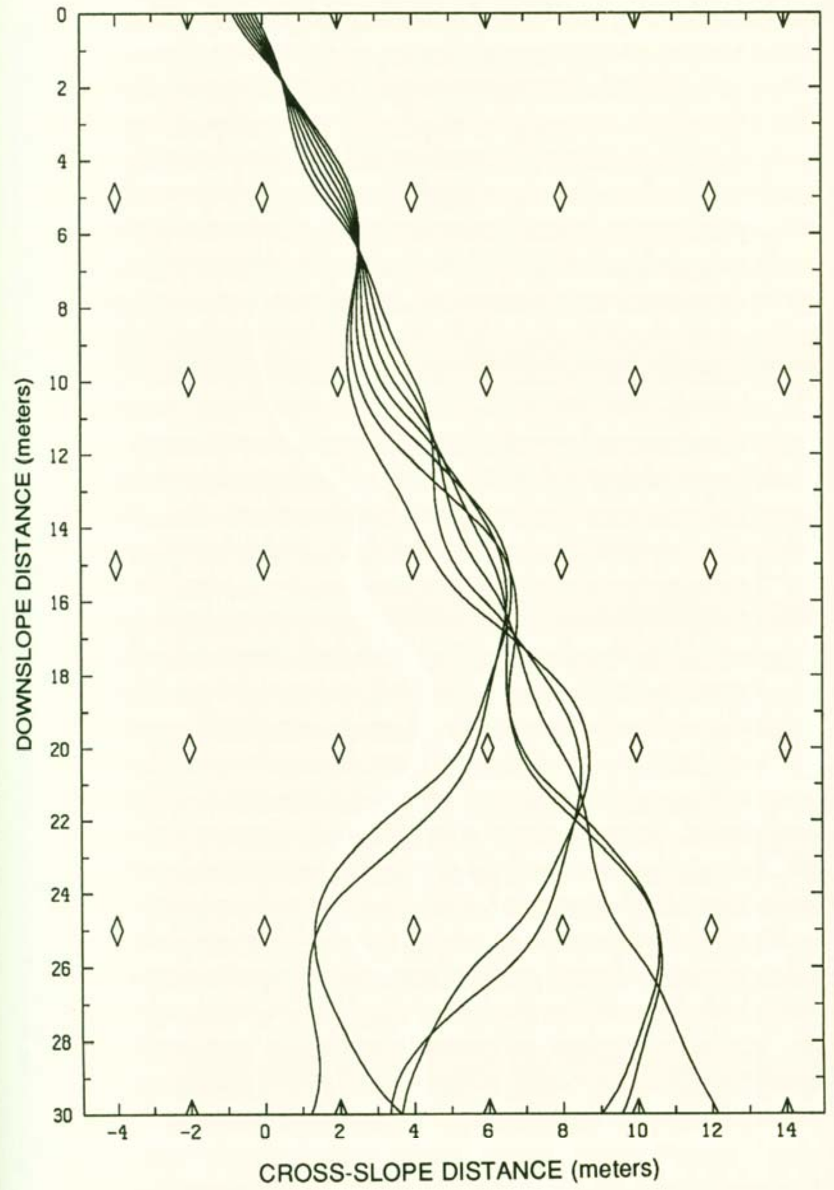


Figure 7. The paths of seven boards starting with identical velocities from points spaced at 10-centimeter intervals along a west-east line. The small diamonds indicate the locations of the centers of the moguls.

- An essential property of chaotic behavior is that nearby states will eventually diverge no matter how small the initial differences are.
- What if the boards are allowed to travel 60 meters down the slope, starting from points only 1 mm apart?
- At first the separation cannot be seen, but by 30 meters it is easily detectable.
- The initial separation can be as small as we want, provided the slope is long enough.

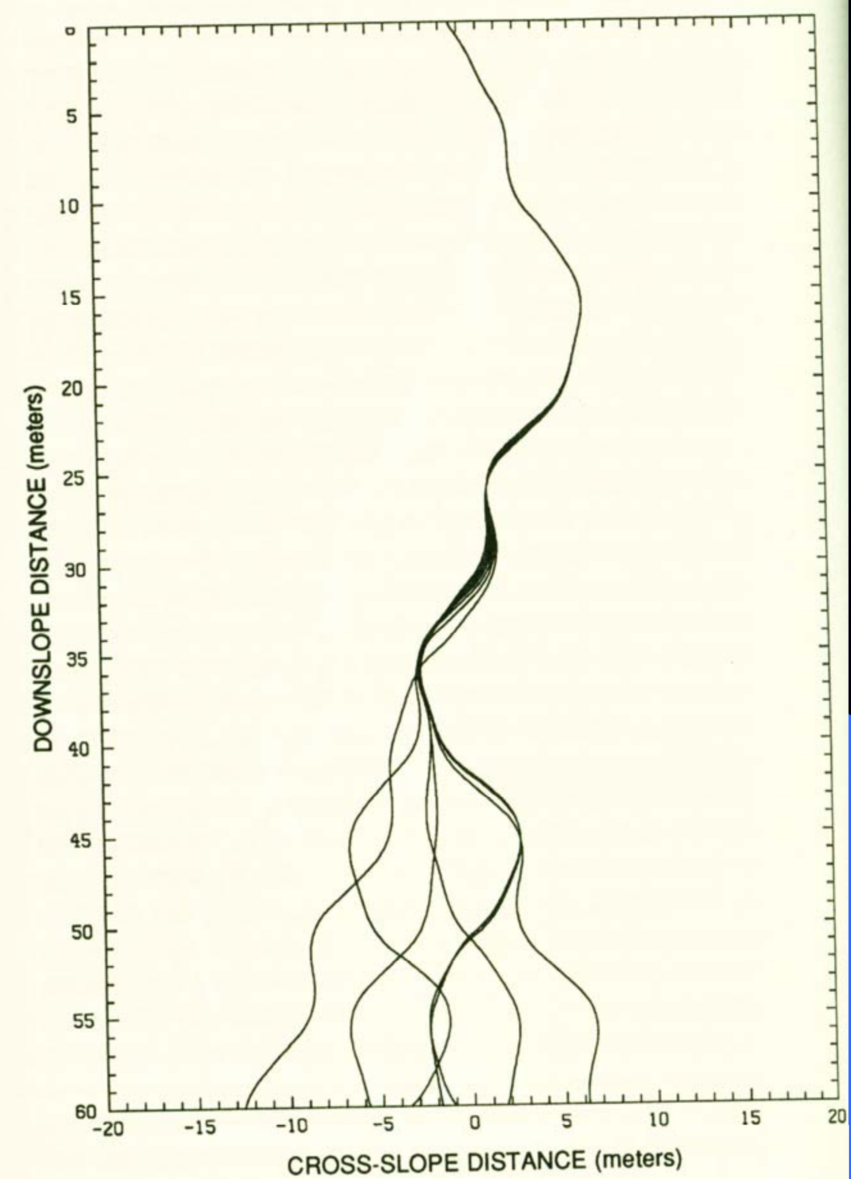


Figure 8. The paths of seven boards, starting with identical velocities from points spaced at 1-millimeter intervals along a west-east line.

"Now that science is looking, chaos seems to be everywhere. A rising column of cigarette smoke breaks into wild swirls. A flag snaps back and forth in the wind. A dripping faucet goes from a steady pattern to a random one. Chaos appears in the behavior of the weather, the behavior of an airplane in flight, the behavior of cars on an expressway, the behavior of oil flowing in underground pipes."

[Gleick, 1987]

Chaos' qualities

- Fractal functionality- self similarity
- Chaotic unpredictability

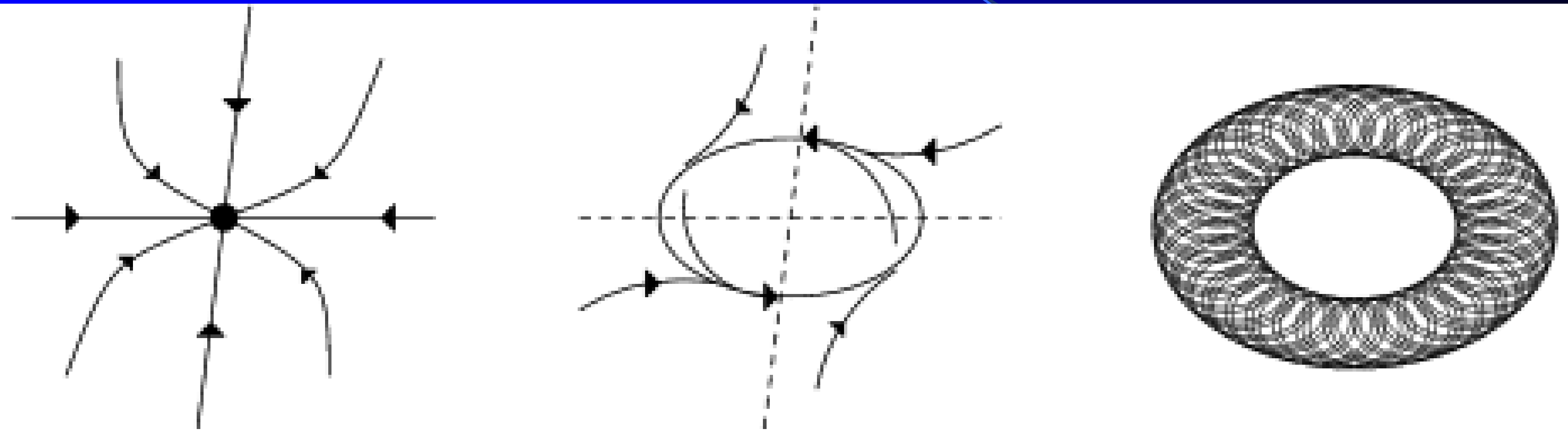
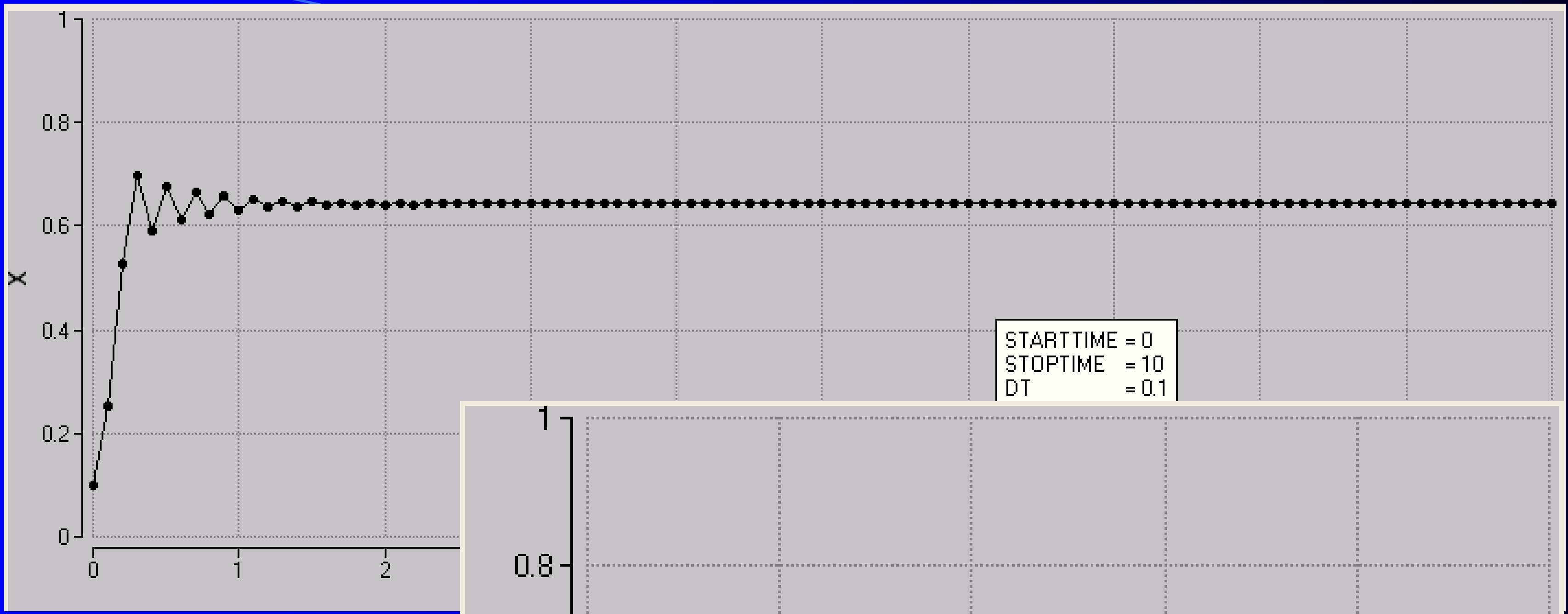
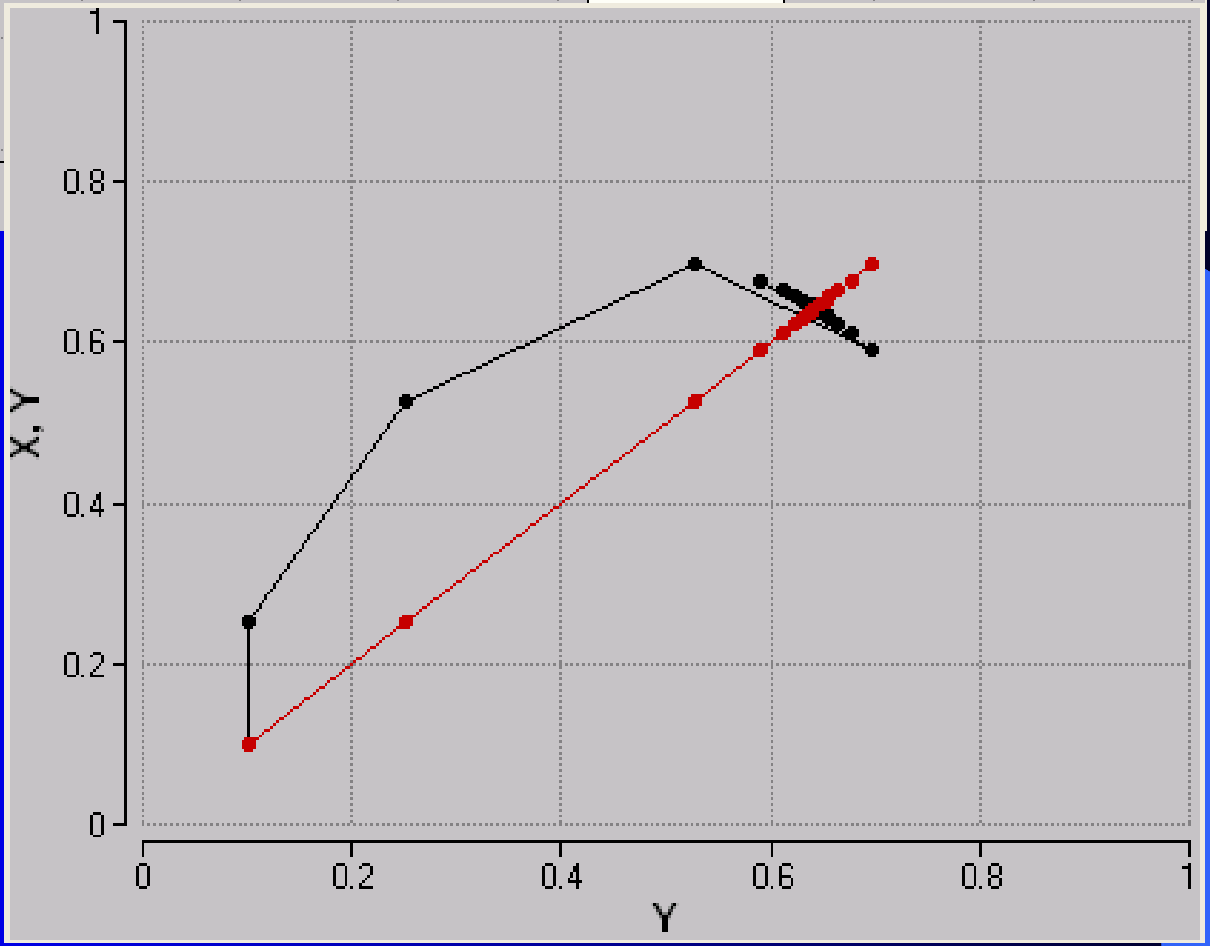

(a) (b) (c)

Figure 10.1 Different types of motion: (a) fixed point, (b) limit cycle (c) quasiperiodic





r = 0.7

Logistic map

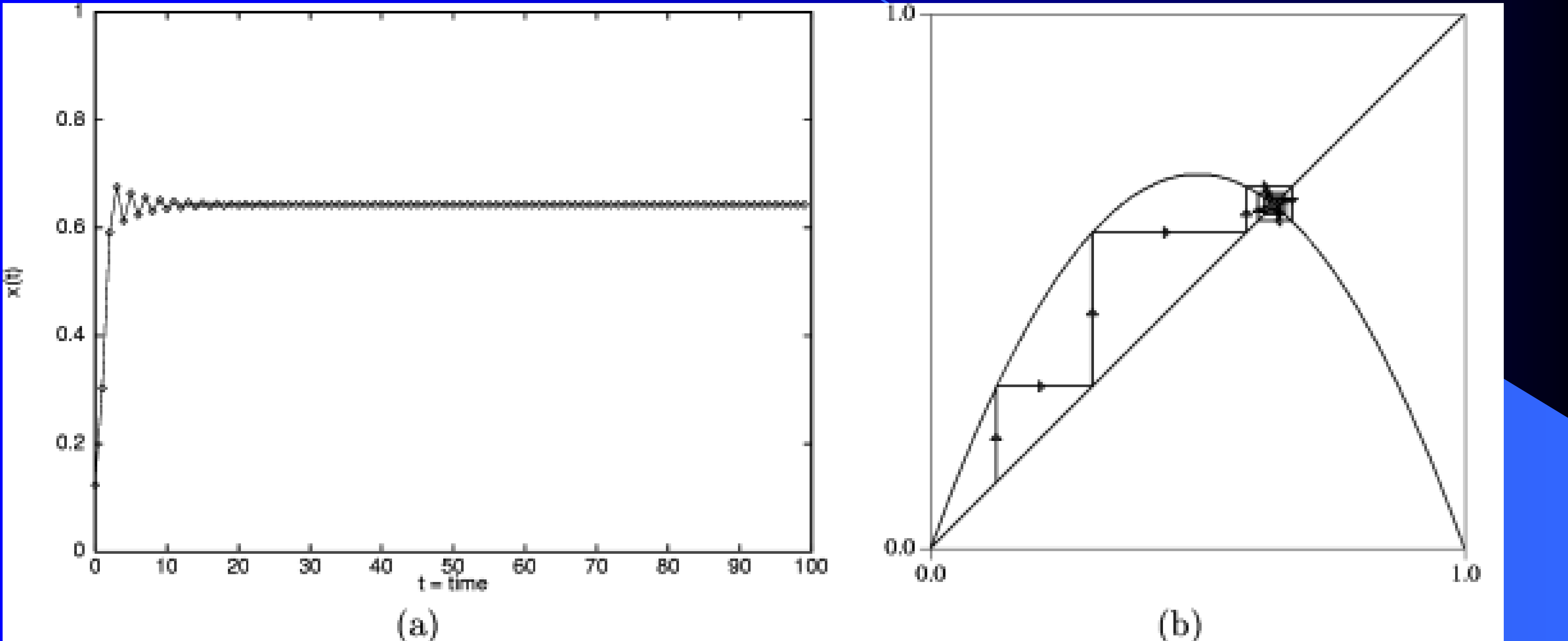


Figure 10.2 Logistic map with $r = \frac{7}{10}$: (a) The time series quickly stabilizes to a fixed point. (b) The state space of the same system shows how subsequent steps of the system get pulled into the fixed point.

r > 0.75

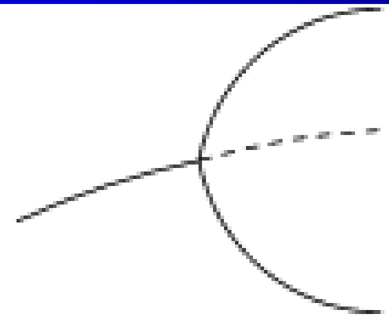
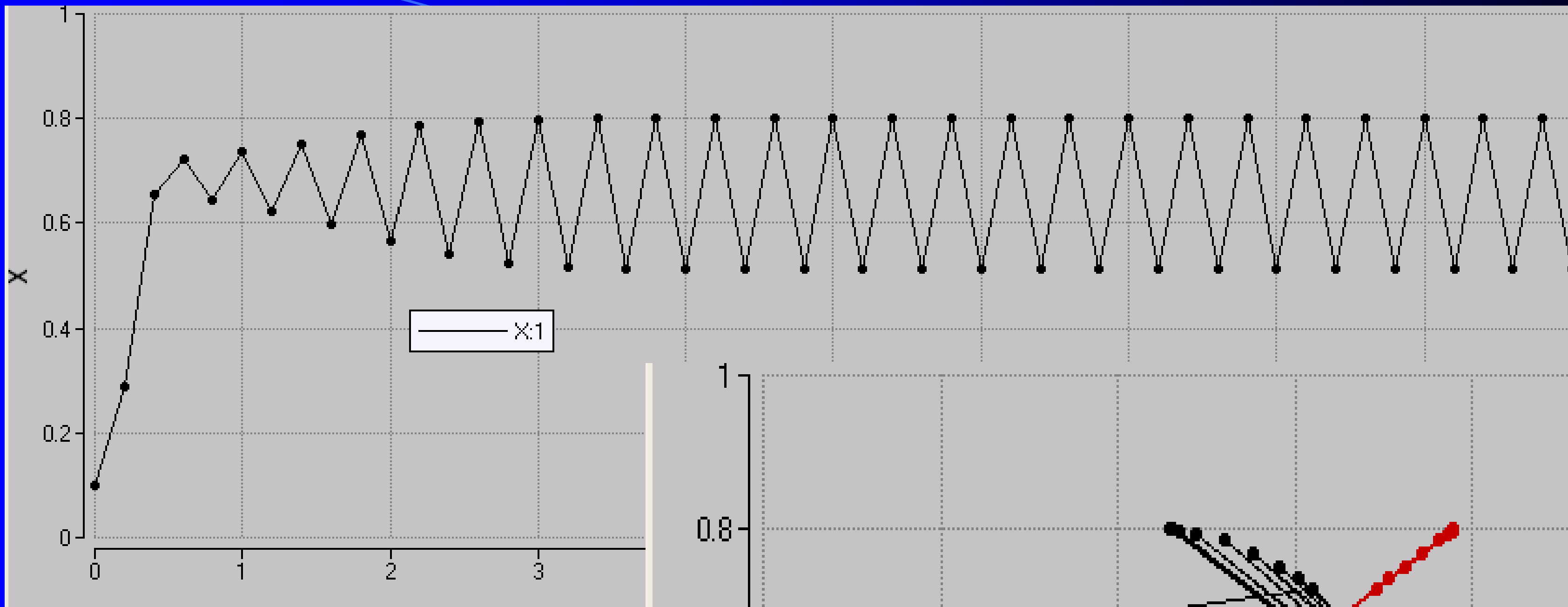
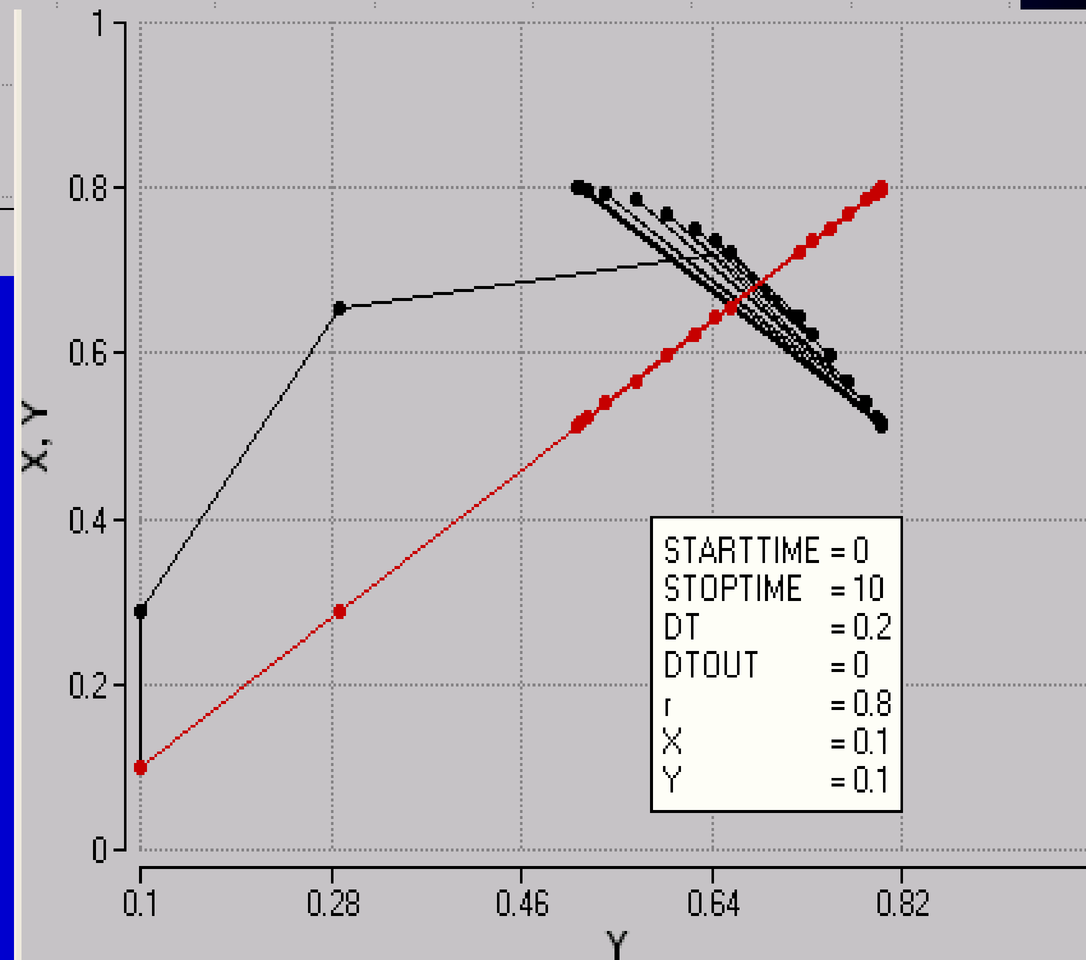


Figure 10.3 A single bifurcation

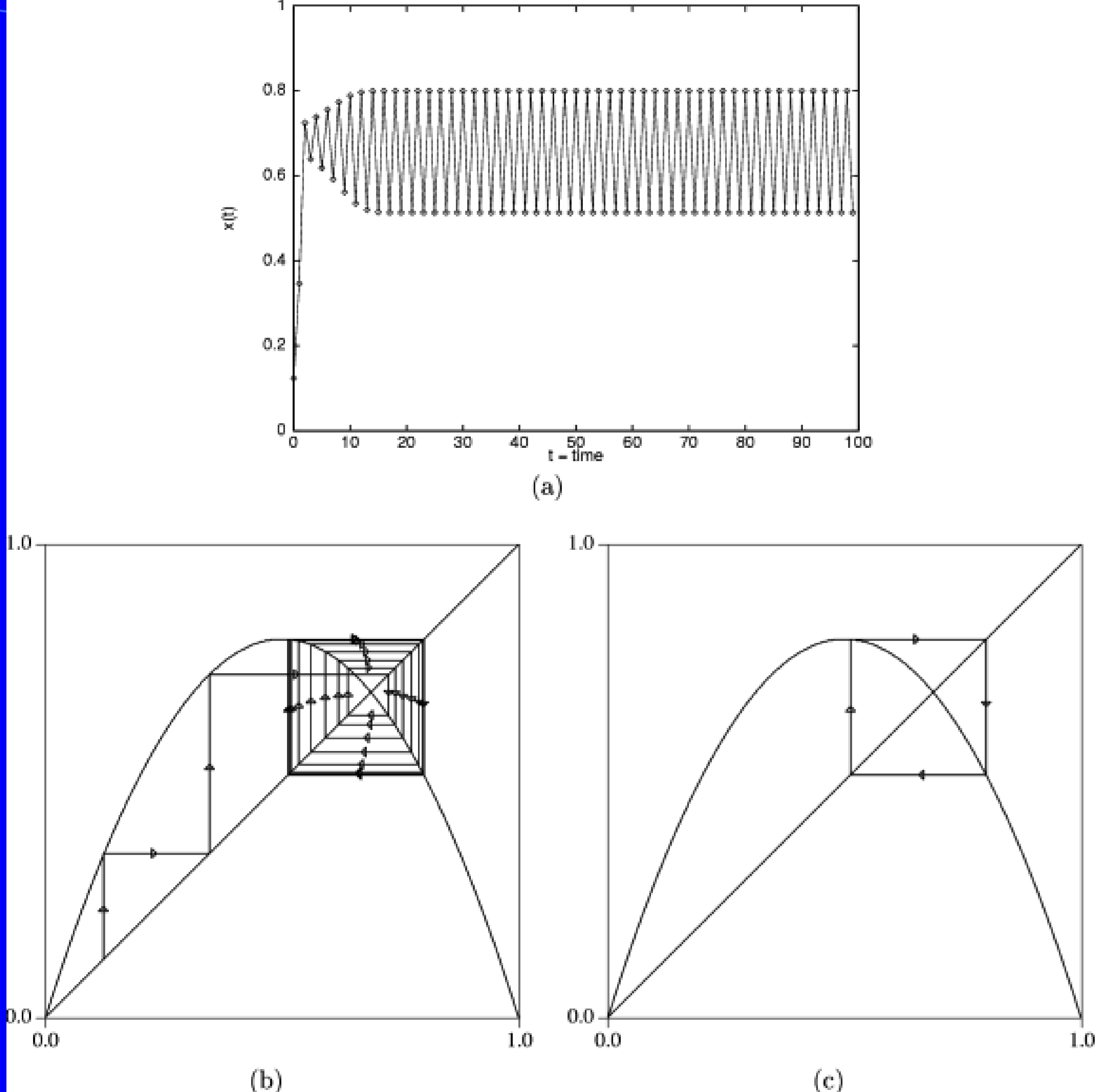


1
0.8
0.6
0.4
0.2
0
X
0
1
2
3
X:1

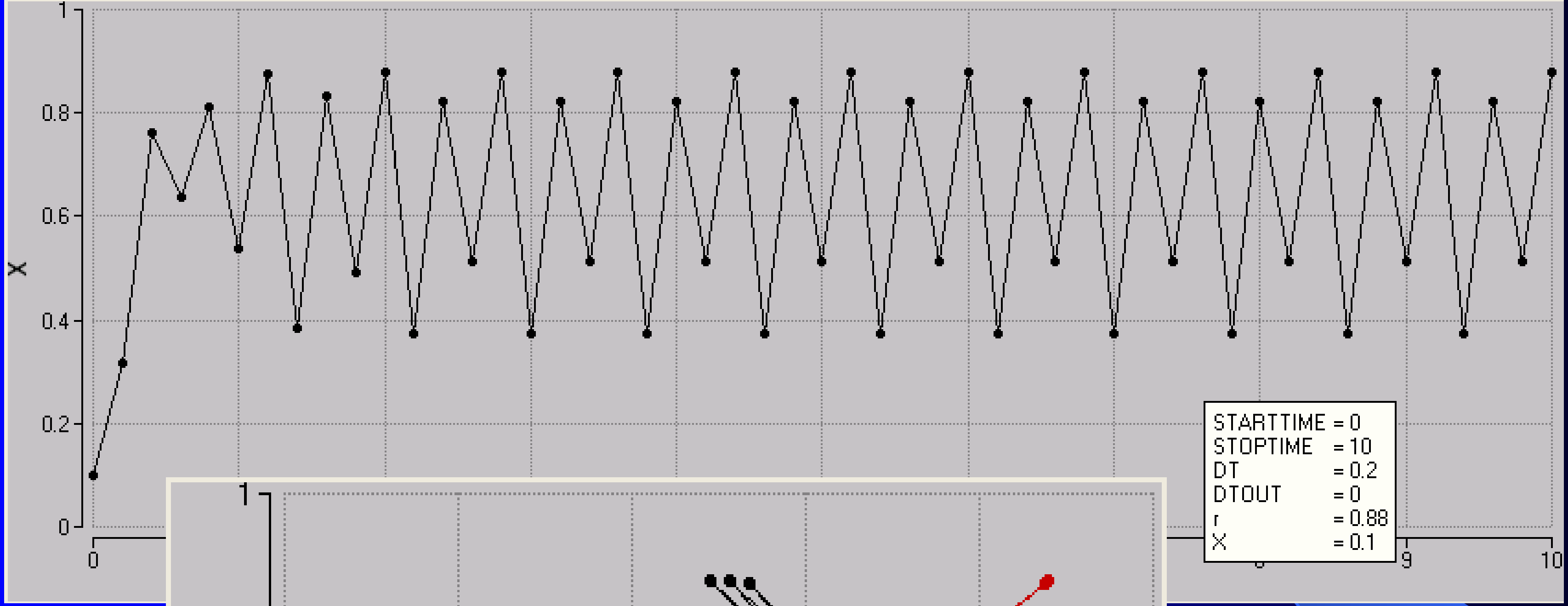


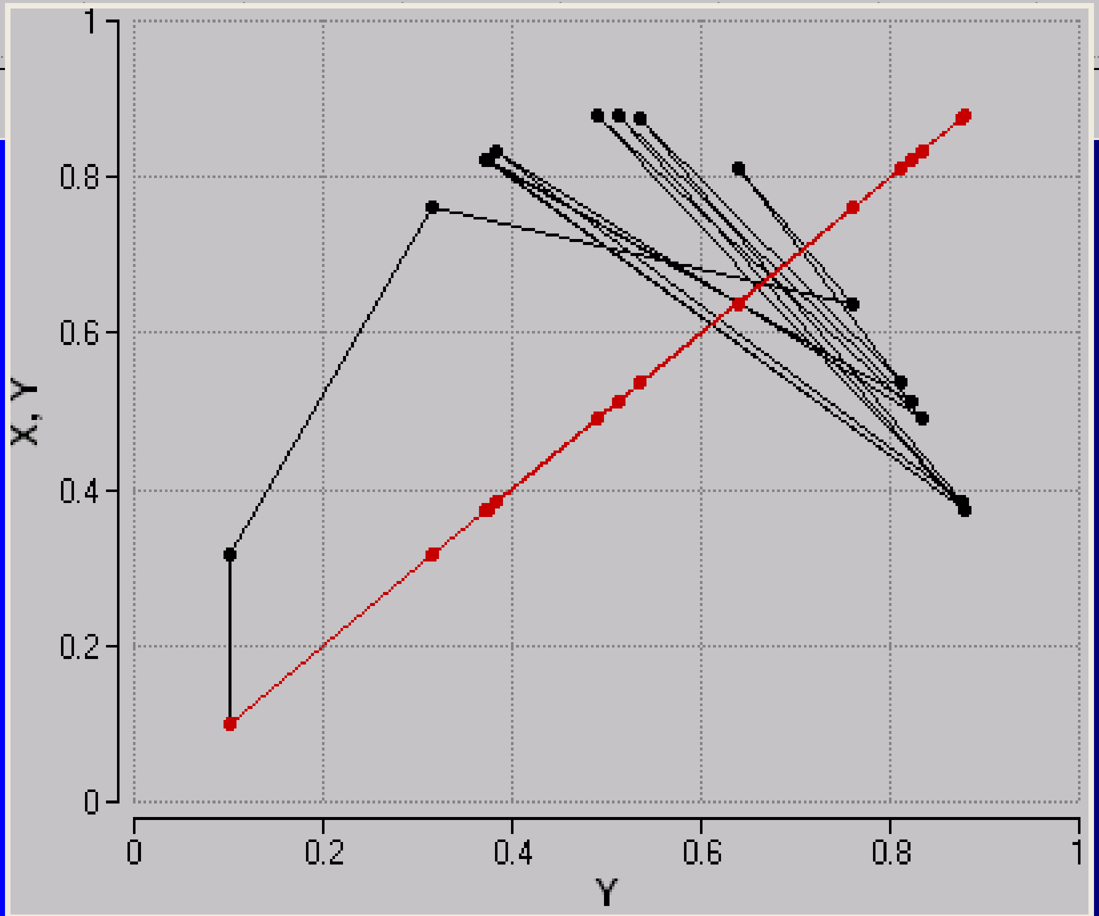
1
0.8
0.6
0.4
0.2
0
X, Y
0.1
0.28
0.46
0.64
0.82
1
Y
STARTTIME = 0
STOPTIME = 10
DT = 0.2
DTOUT = 0
r = 0.8
X = 0.1
Y = 0.1

r = 0.8



x(t)
t = time
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100
(a)
1.0
0.0
0.0
1.0
(b)
1.0
0.0
0.0
1.0
(c)





r = 0.88

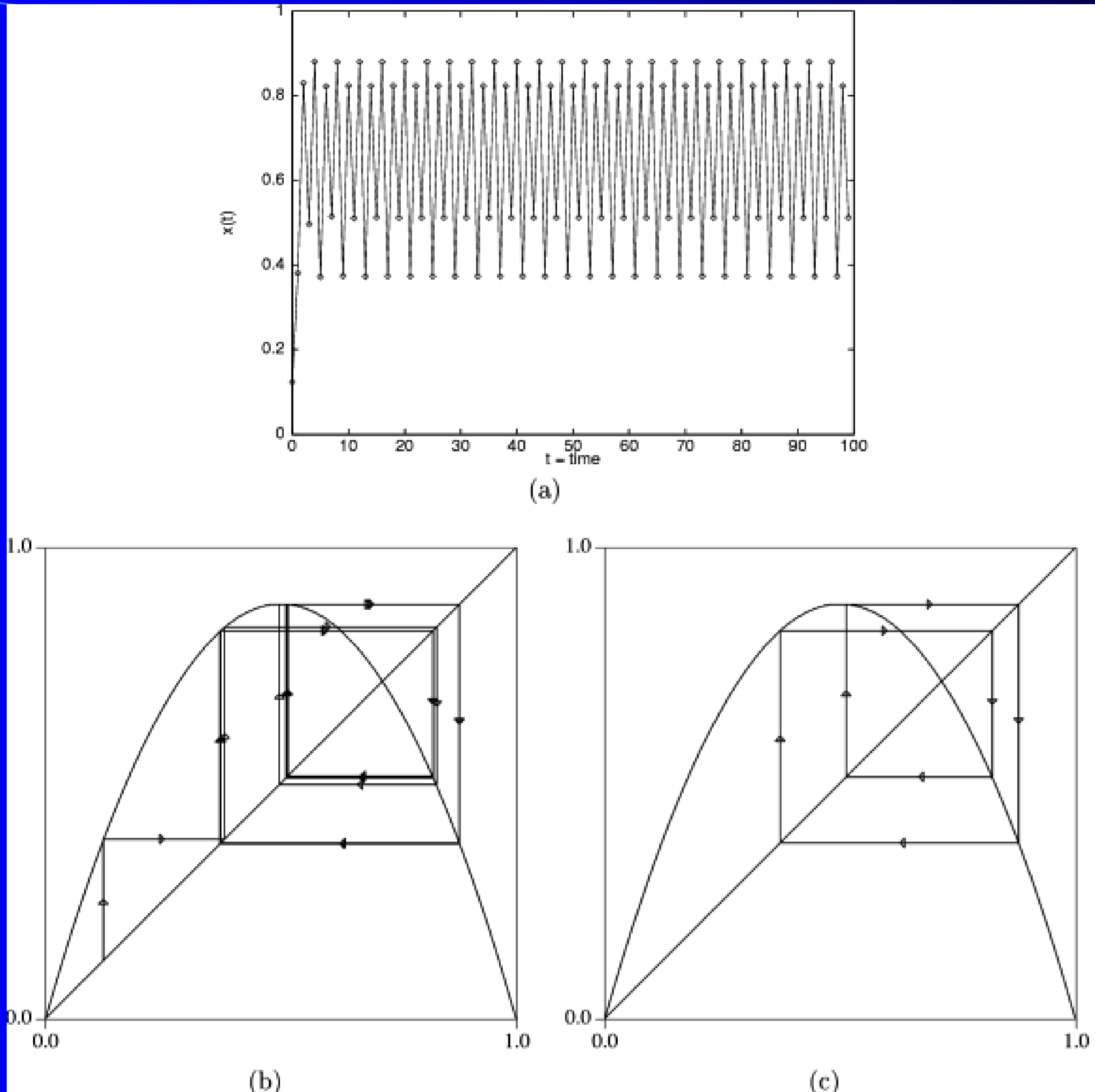
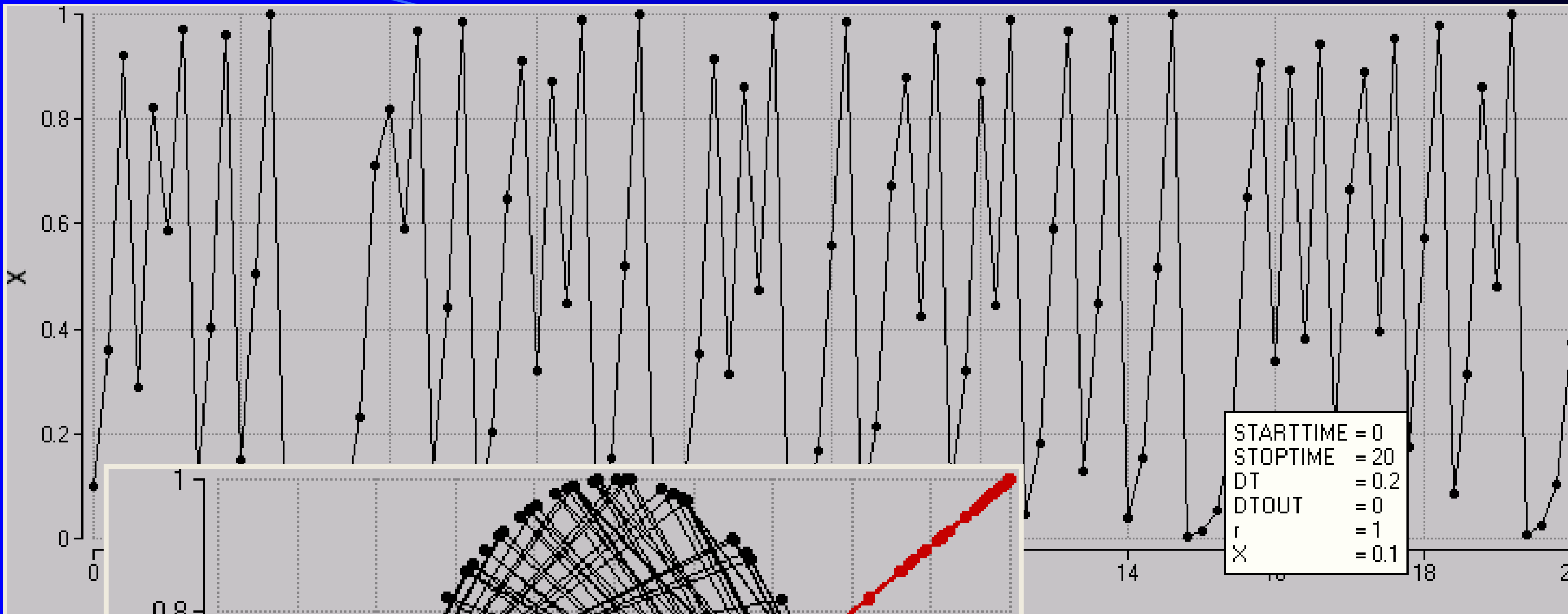
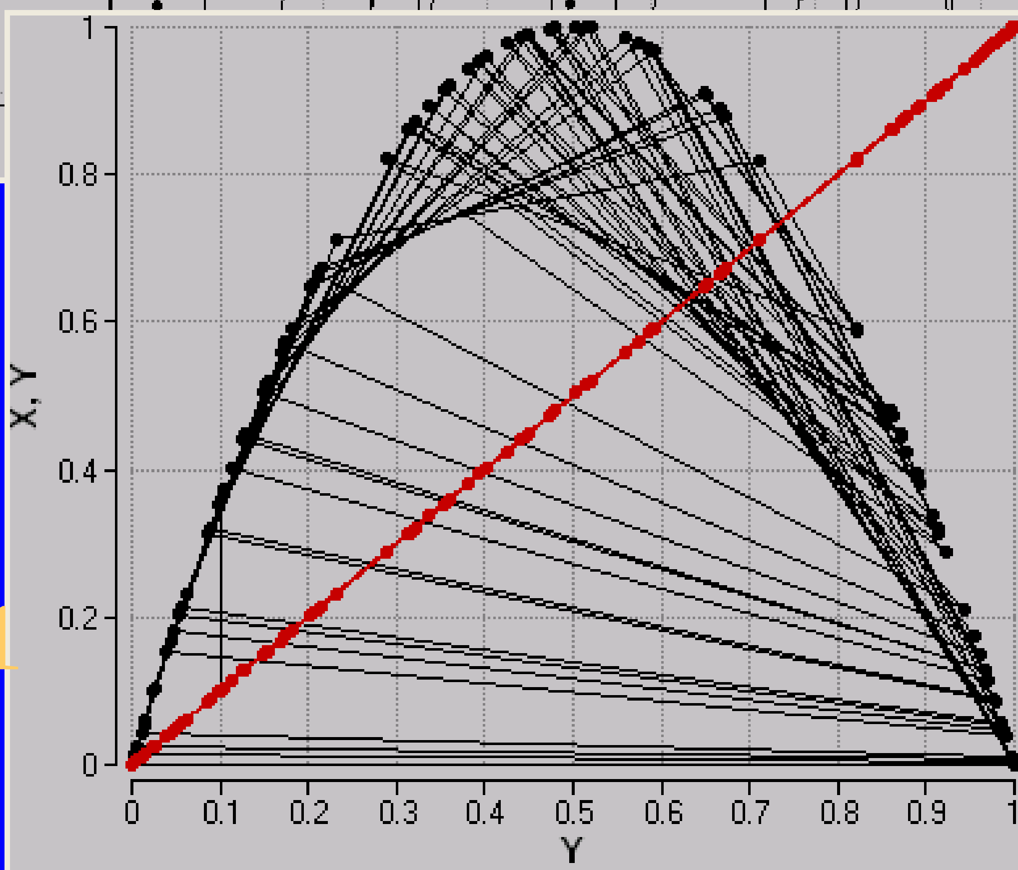


Figure 10.5 Logistic map with $r = \frac{88}{100}$: (a) The time series quickly stabilizes to a

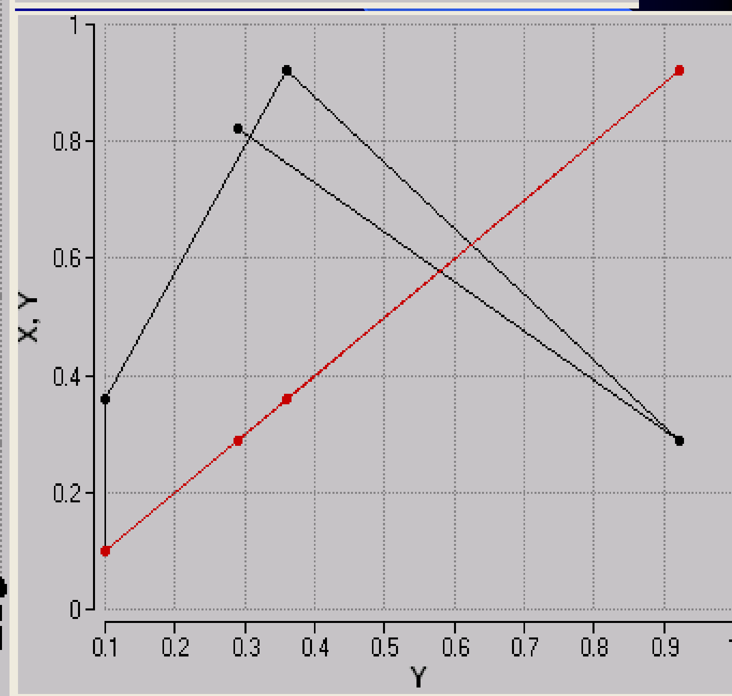


X
0
0.2
0.4
0.6
0.8
1
14
18
20
STARTTIME = 0
STOPTIME = 20
DT = 0.2
DTOUT = 0
r = 1
X = 0.1

r = 1



X, Y
Y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



X, Y
Y
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

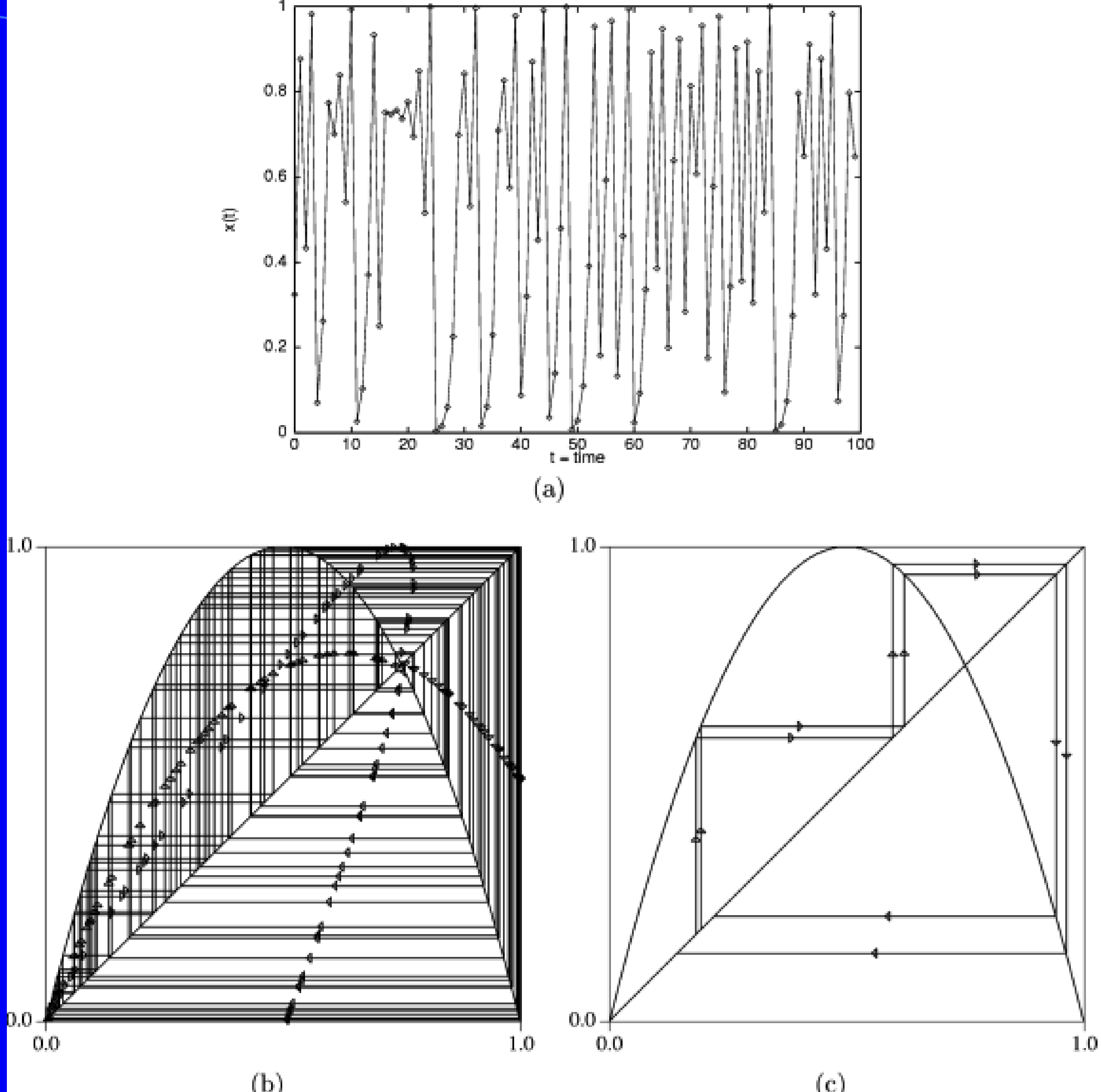
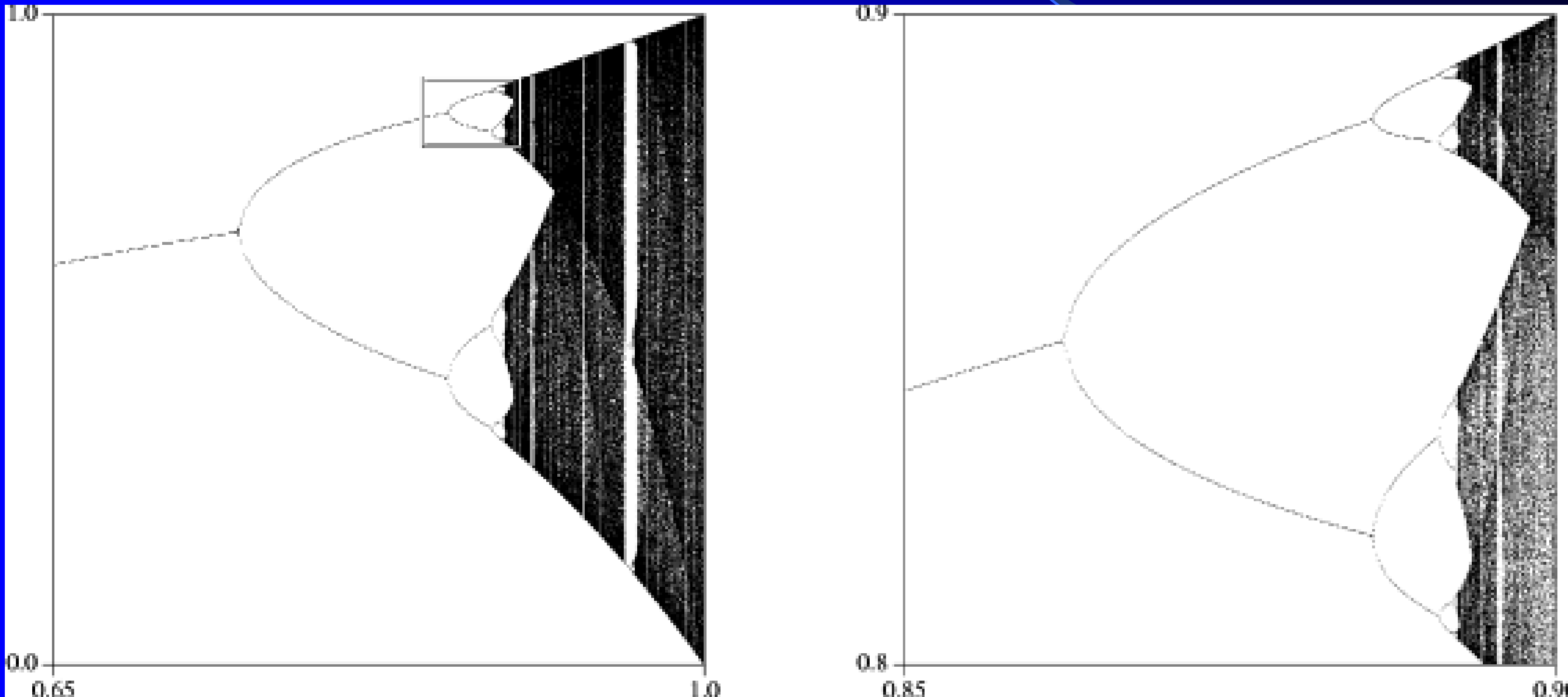


Figure 10.6 Logistic map with $r = 1$: (a) The time series is chaotic and has the appearance of noise. (b) The state space of the same system, which illustrates how the system's trajectory visits every local region. (c) The state space of the same system with only four

Bifurcation

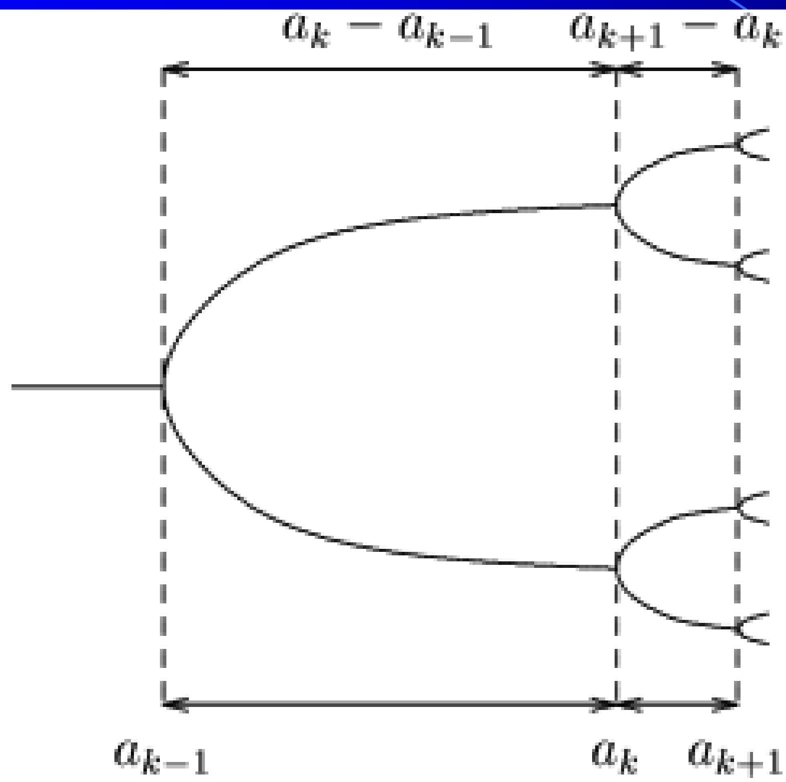


(a)

(b)

Figure 10.7 Bifurcation diagrams for the logistic map: (a) This image has values of r such that fixed points, limit cycles, and chaos are all visible. (b) This image shows the detail of the boxed section of (a).

Feigenbaum constant



$$d_k = \frac{a_k - a_{k-1}}{a_{k+1} - a_k}$$

Figure 10.8 Detail of a bifurcation diagram to show the source of the Feigenbaum constant

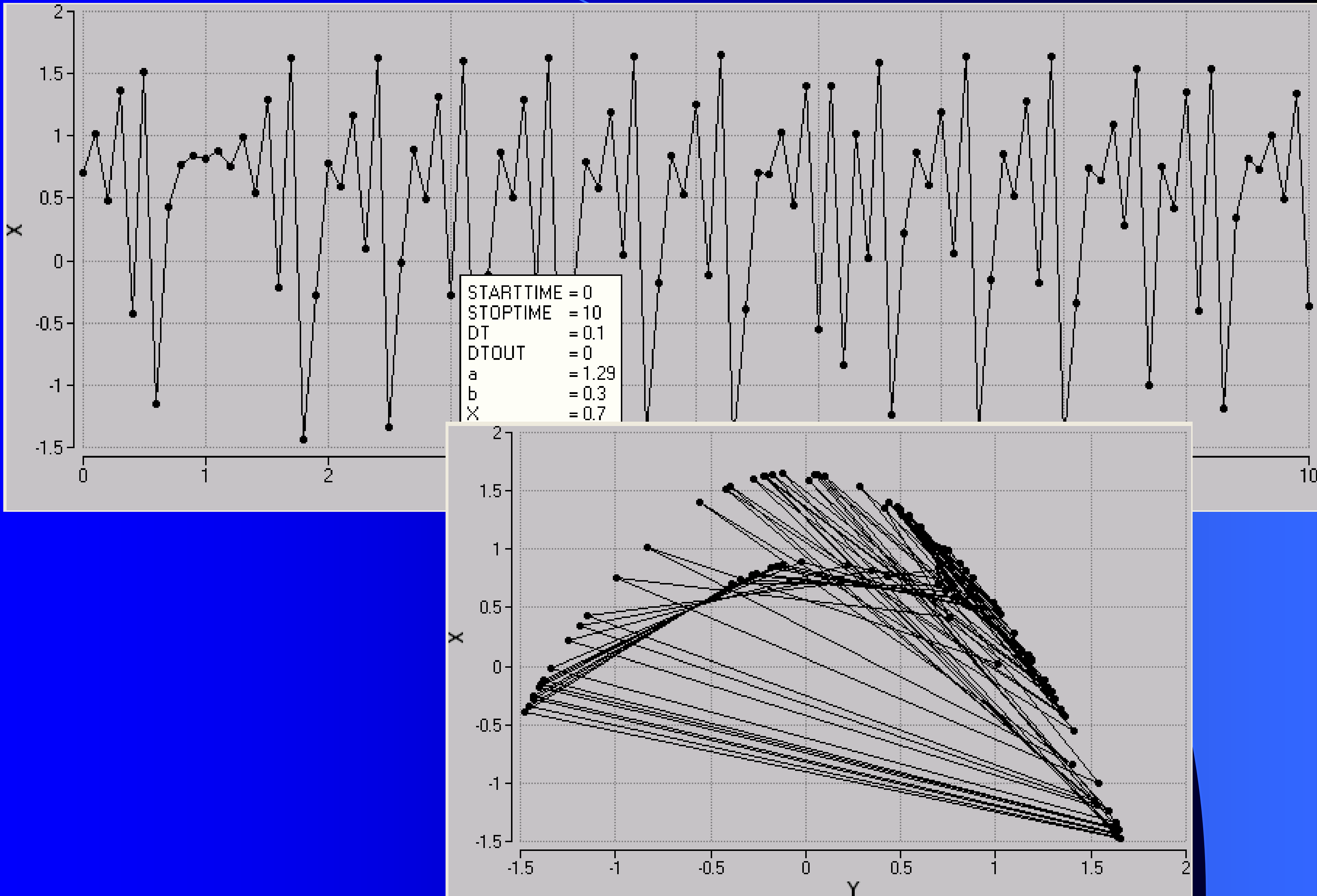
Chaotic vs stochastic

- One of the most important differences between chaotic processes and truly stochastic processes is that the future behavior of a chaotic system can be predicted in the short term, while stochastic processes can be characterized only statistically

Strange attractors

- The Henon map – discrete
- The Lorenz attractor – continuous
- Mackey-Glass system – discrete and continuous

The Henon map



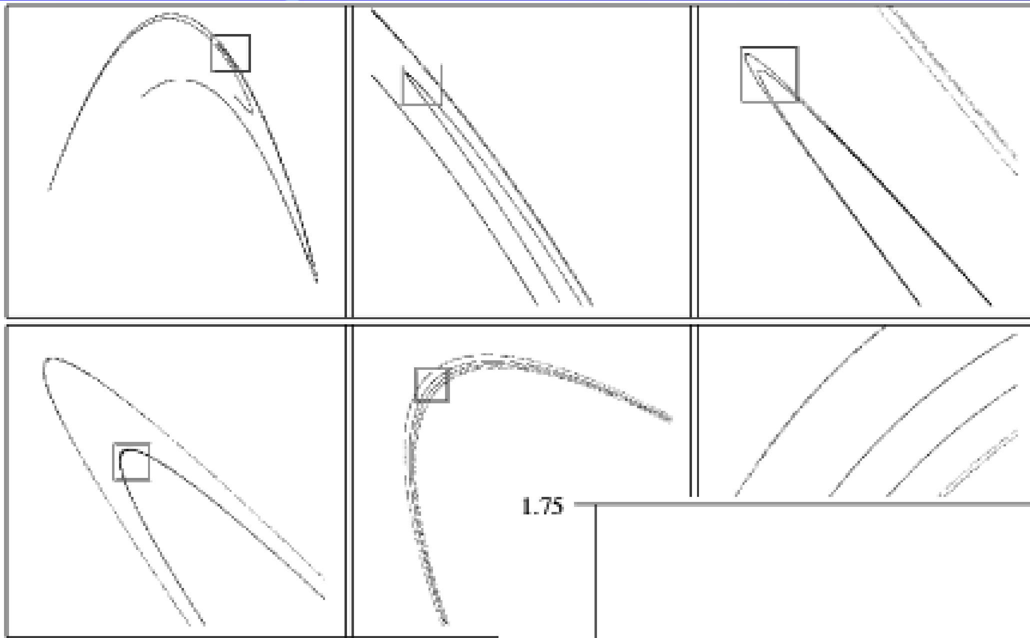


Figure 11.2 The Hénon attractor viewec

$a = 0.5 \sim 1.4$

$b = 0.3$

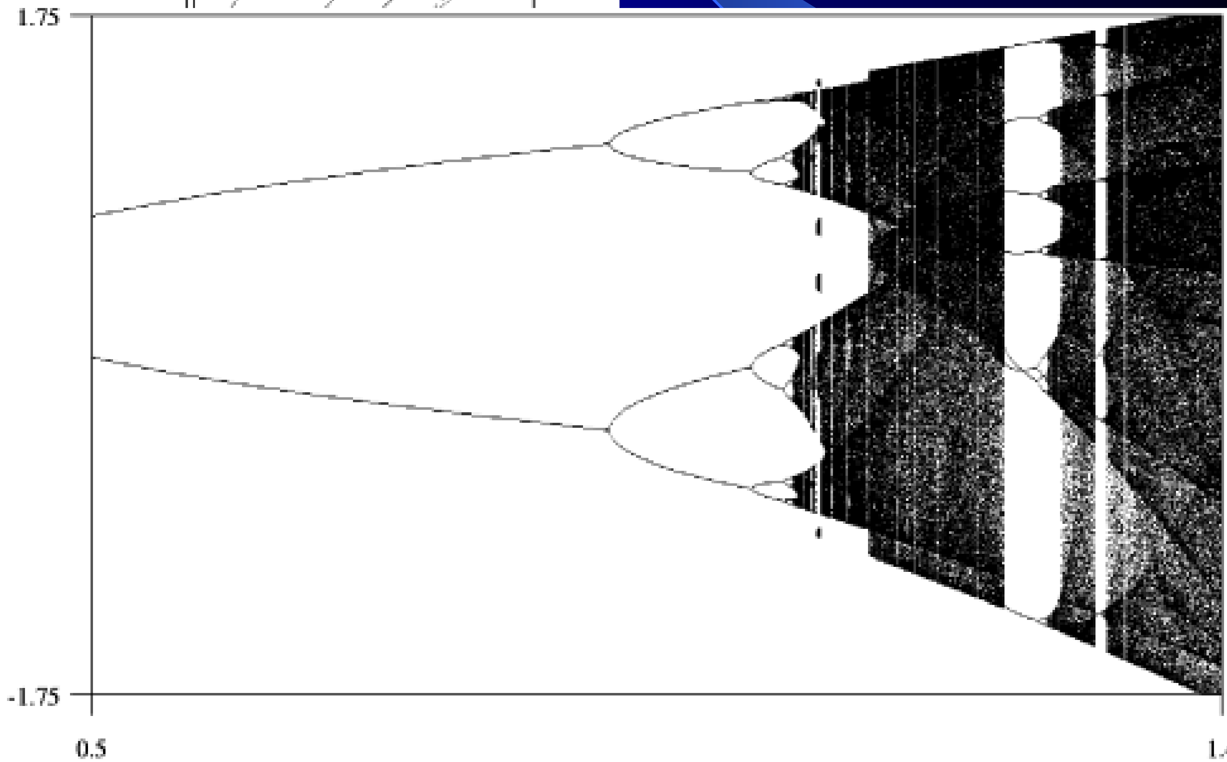


Figure 11.3 A bifurcation diagram of the Hénon map for a varying from 0.5 to 1.4, and b set to 0.3

dt change – time for prediction

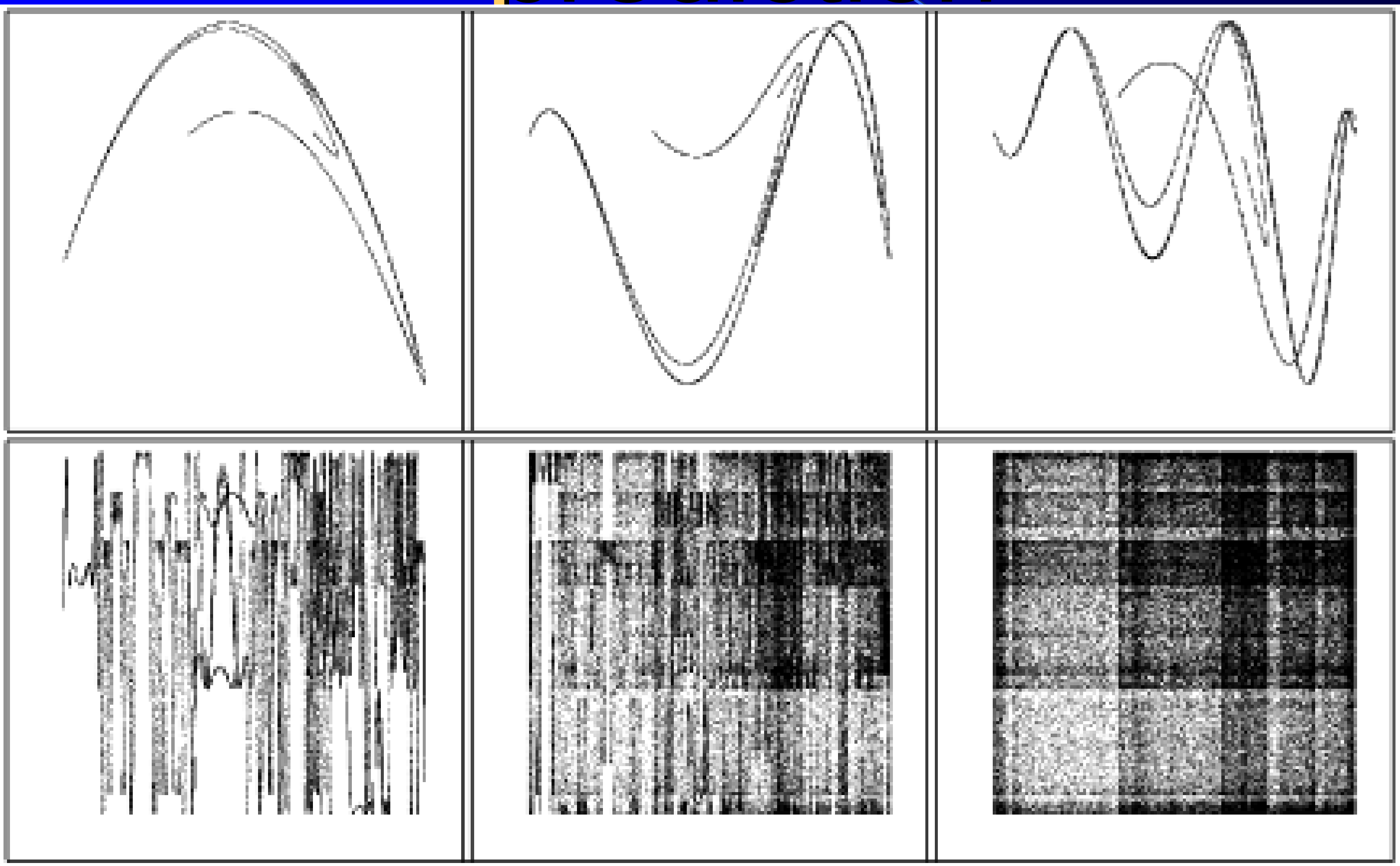
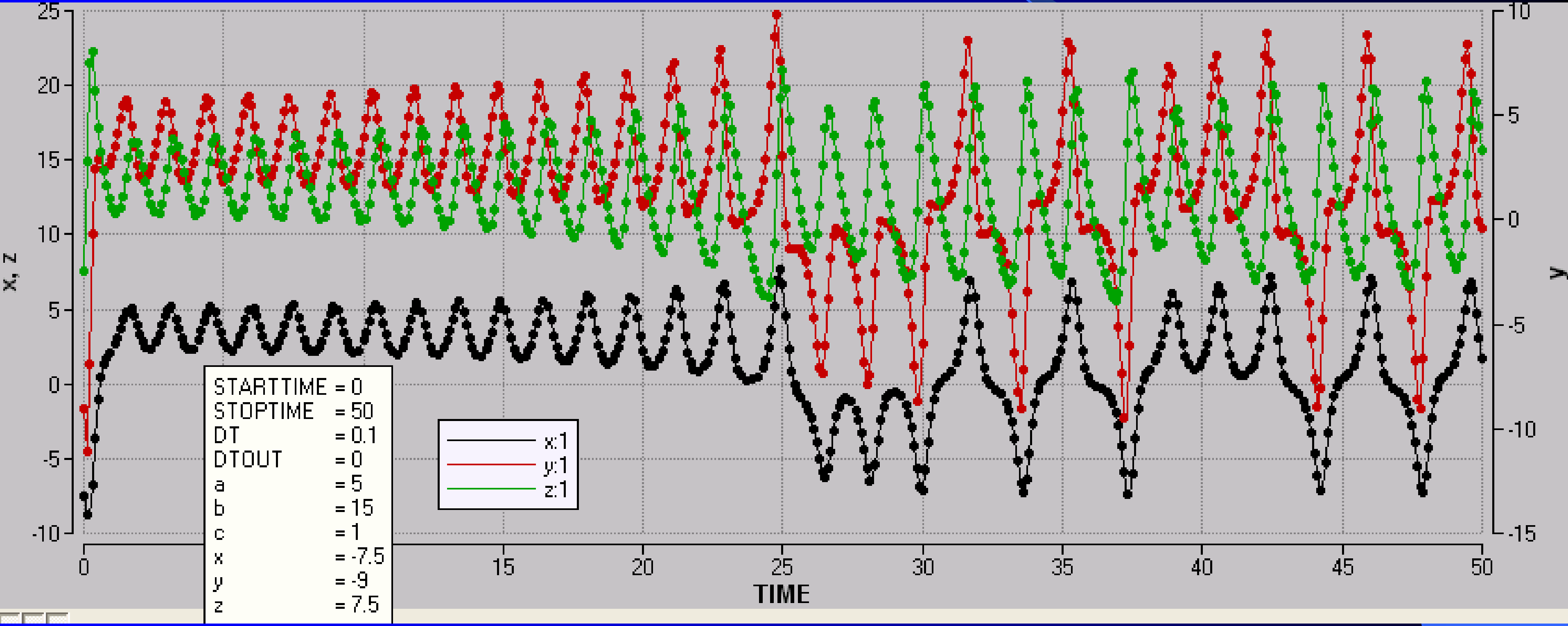
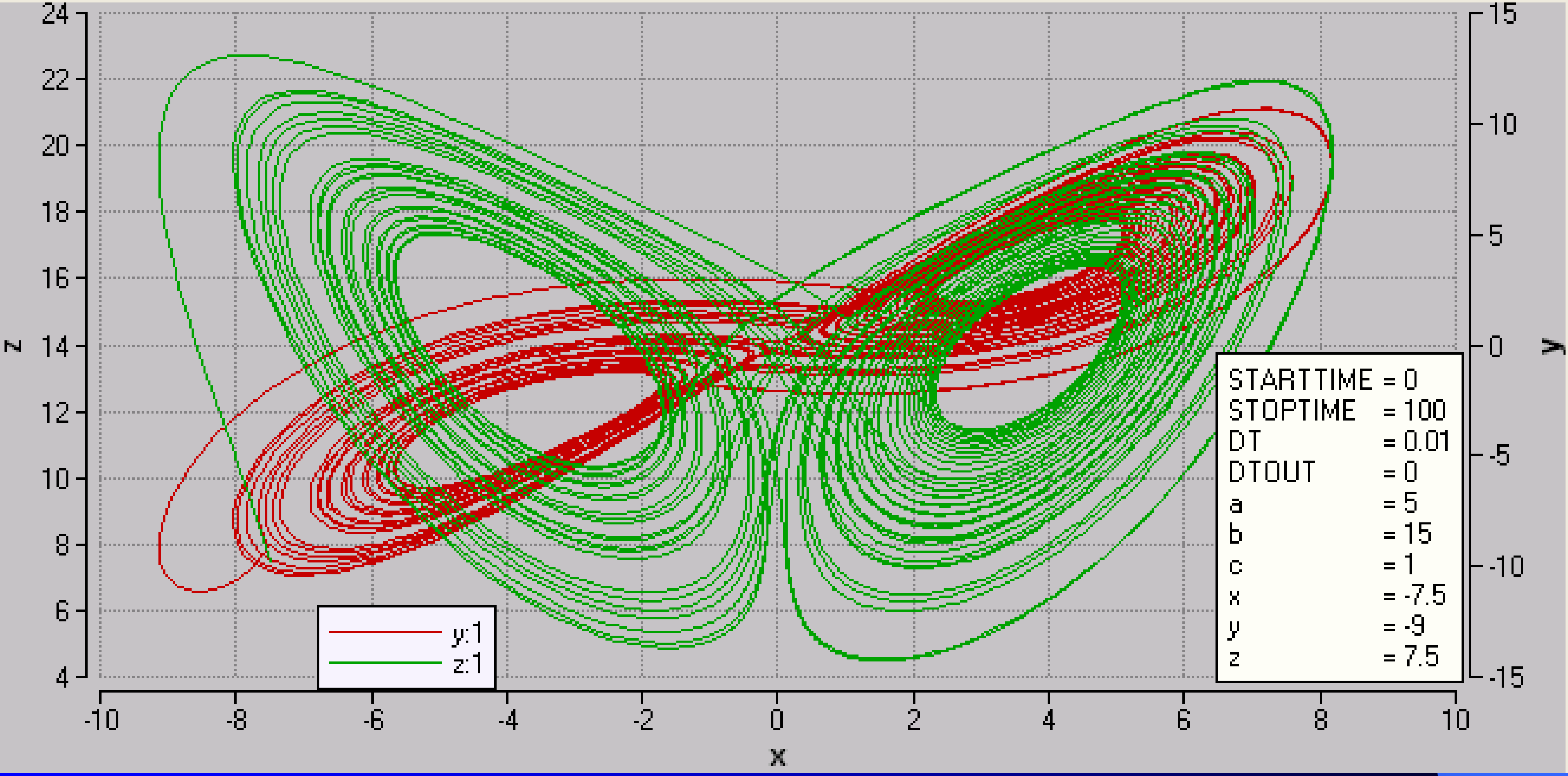


Figure 11.5 Plotting x_t versus x_{t+m} with m equal to 1, 2, 3, 10, 20, and 100

Lorenz attractor





z
y
x
y:1
z:1
STARTTIME = 0
STOPTIME = 100
DT = 0.01
DTOUT = 0
a = 5
b = 15
c = 1
x = -7.5
y = -9
z = 7.5

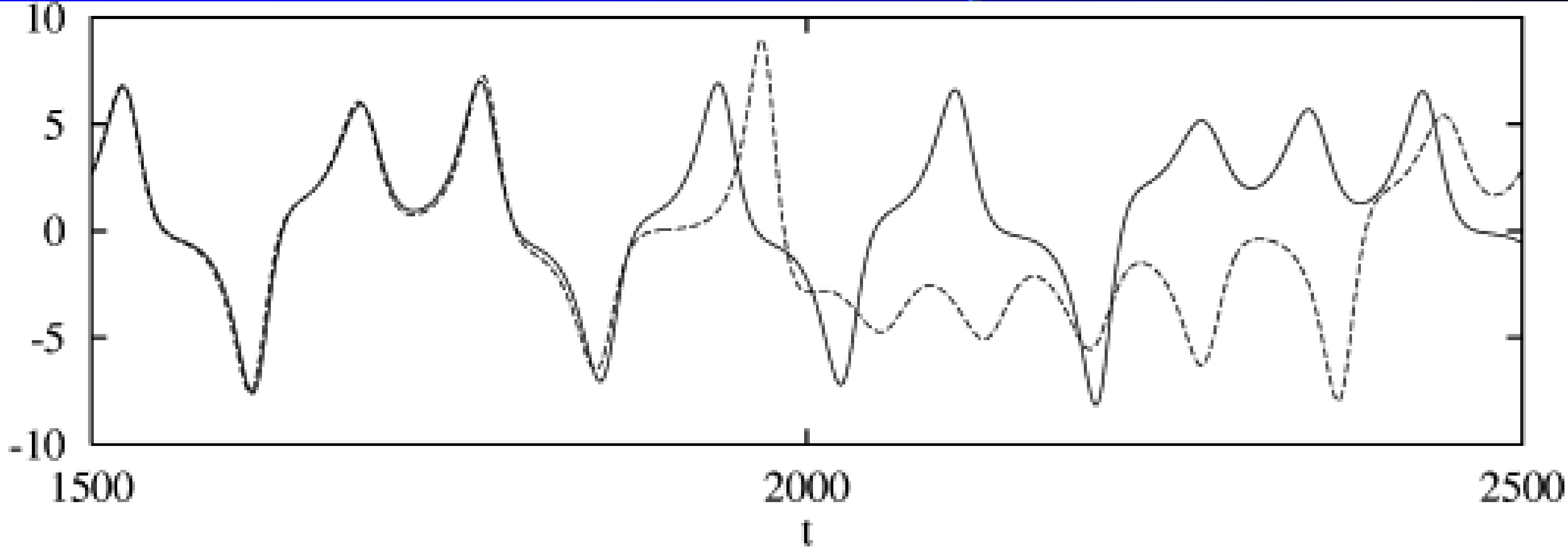


Figure 11.7 Two time evolutions of x with an infinitesimal initial difference

Mackey-Glass system

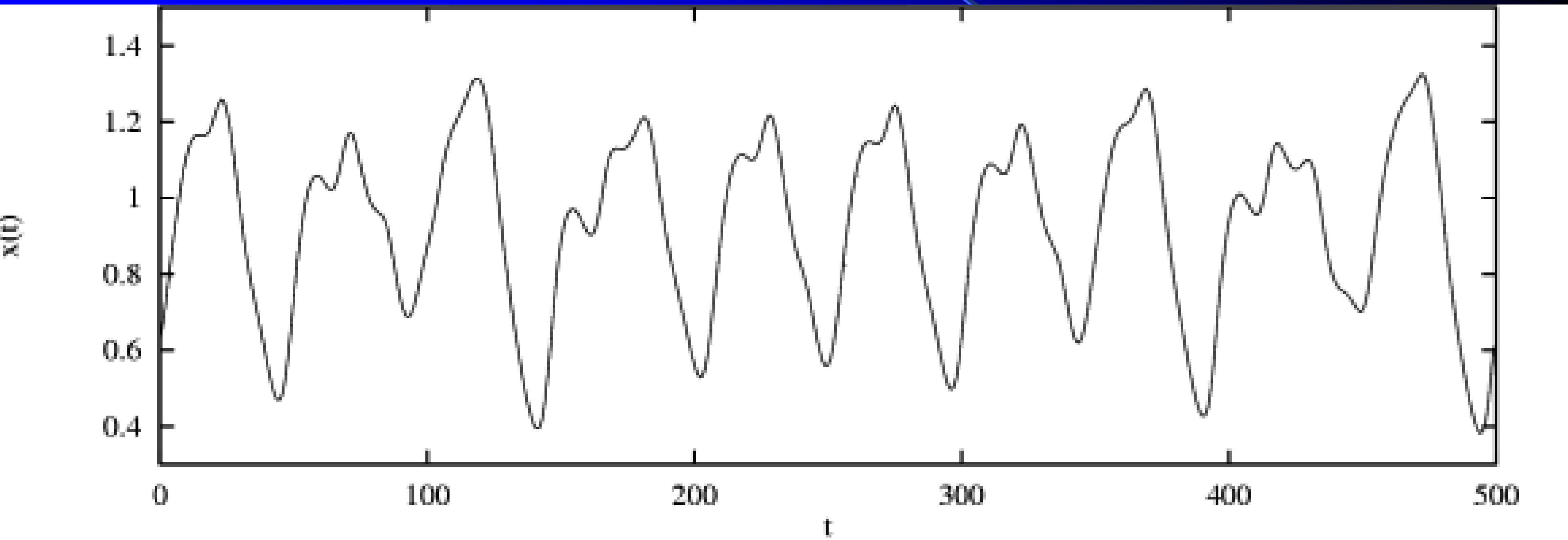
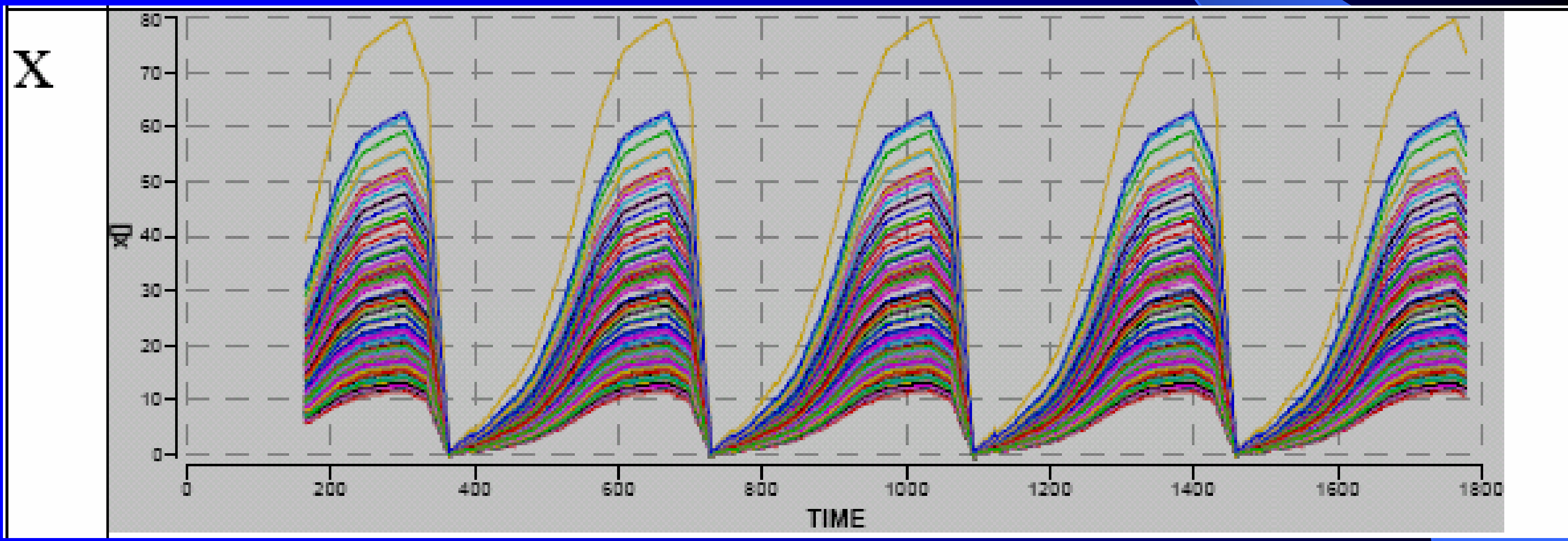


Figure 11.11 The time evolution of x from the Mackey-Glass system

$$\frac{dx}{dt} = e^{-\mu_x \cdot \tau_x} \cdot B_i\left(t-\tau_x\right) \cdot x_i\left(t-\tau_x\right) - \mu_x \cdot x_i$$

$$B_i\left(t-\tau_x\right) = B_m \cdot e^{\left(-c \cdot \left(T_2\left(t-\tau_x\right) - T_{2m}\right)^2\right)}$$



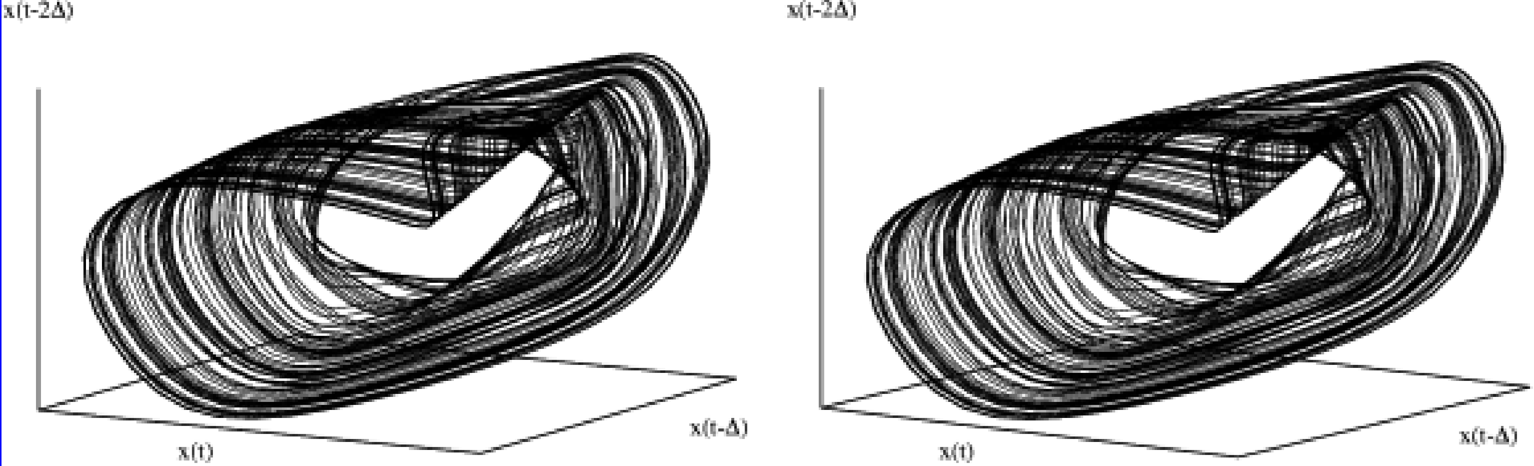


Figure 11.12 A dual-image stereogram of the a three-dimensional embedding of the Mackey-Glass system. Here Δ is set to 6. To view, stare at the center of the two images and cross your eyes until the two images merge. Allow your eyes to relax so that they can refocus.

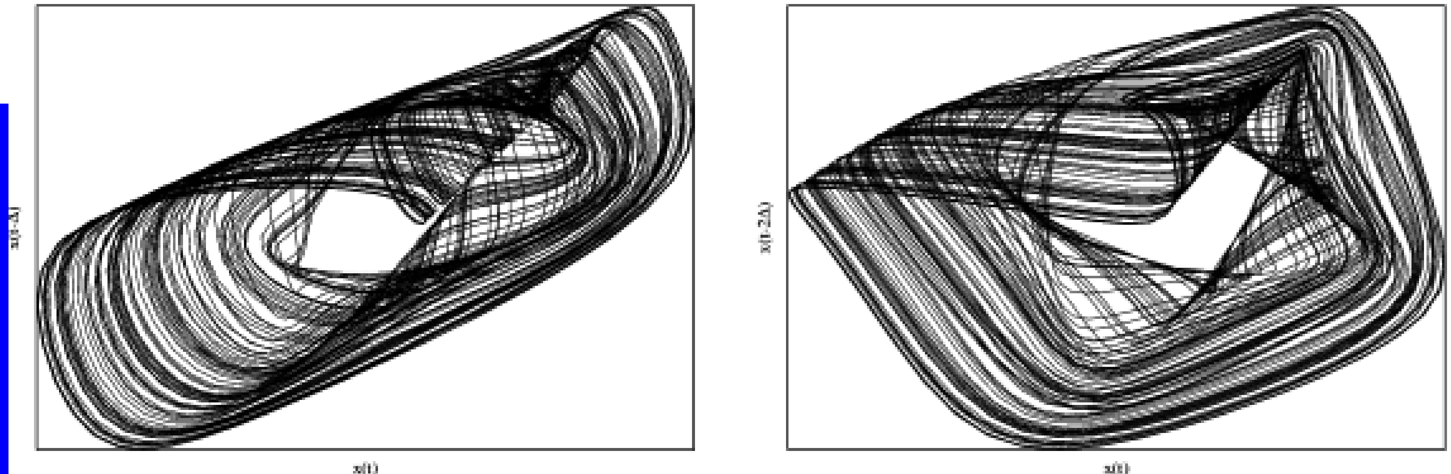


Figure 11.13 Two dimensional embeddings of the Mackey-Glass system, with Δ set to 6

The day after tomorrow

Climatologist Jack Hall (Dennis Quaid) discovers erratic changes on the weather patterns on the climate.It is from here that the earth,finally rebelling against years of abuses from the greenhouse effect and global warming,unleashes a cataclysmic onslaught of hurricanes,floods and other natural disasters before everything freezes over.Hall must try and save his son Sam (Jake Gyllenhaal) who is trapped over in New York.