# MERU UNIVERSITY OF SCIENCE AND TECHNOLOGY

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**University Examinations 2021/2022**

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

**SMA 3413: FIELD THEORY**

**DATE: OCTOBER 2022** **TIME: 2 HOURS**

**INSTRUCTIONS:** *Answer question **one** and any other **two** questions*

## QUESTION ONE (30 MARKS)

a) Define what is meant by a field E is a field extension of a field F. (1 mark)

b) For each of the given numbers $\alpha \in \mathbb{C}$ , find $\deg(\alpha, Q)$

(i) $\sqrt{2}+i$ (3 marks)

(ii) $\sqrt{i+(2)^{\frac{1}{3}}}$ (3 marks)

c) Give a basis for each of the following vector spaces over the indicated fields

(i) $Q(\sqrt[i]{2})$ over $Q$ (2 marks)

(ii) $\mathbb{R}(\sqrt{5})$ over $\mathbb{R}$ (2 marks)

d) Find the degree of the extension $Q(\sqrt{8}, \sqrt{24})$ over $Q$ (4 marks)

e) Find all conjugates of each of the given number over the given fields.

(i) $\sqrt{3}+i$ over $Q$ (2 marks)

(ii) $\sqrt{1+\sqrt{3}}$ over $Q(\sqrt{3})$ (4 marks)

f) Show that $\sqrt{2}+\sqrt{5}$ is algebraic over $Q$ (5 marks)

g) Given $f(x)=x^4-1$, find the degree over $Q$ of the splitting field over $Q$ of $f(x)$ (4 marks)

**QUESTION TWO (20 MARKS)**

a) (i) Define an integral domain (2 marks)

(ii)Show that every field is an integral domain (4 marks)

b) Let $Q(\sqrt{2})=\{a+b\sqrt{2}|a,b\in Q\}$. Show that $Q(\sqrt{2})$is a subfield of $\mathbb{R}$ (7 marks)

c) Using a suitable example, show that a polynomial may be irreducible over a field A but reducible over a field B where $A\subseteq B$ (3 marks)

d) Show that $i\in\not\subset$ is algebraic over $\mathbb{R}$ (4 marks)

**QUESTION THREE (20 MARKS)**

a) For each of the given numbers $\alpha\in\not\subset$, find $irr(\alpha,Q)$ and $\deg(\alpha,Q)$

(i) $\sqrt{2}+\sqrt{6}$ (4 marks)

(ii) $\sqrt{2+\sqrt{3}}$ (3 marks)

b) Find the dimension and a basis of the vector space $Q(\sqrt[2]{2})$ over $Q$ (3 marks)

c) Show that $\not\subset$ is algebraic over $\mathbb{R}$ (5 marks)

d) Prove that a finite extension E of a field F is an algebraic extension of F (5 marks)

**QUESTION FOUR (20 MARKS)**

a) Find the degree of the following field extensions:

(i) $Q\left(\sqrt{2+\sqrt{3}}\right)$ over $Q(\sqrt{3})$ (5 marks)

(ii) $Q(\sqrt{2},\sqrt[3]{5})$ over $Q$ (5 marks)

b) If E is an extension field of F and is algebraic over F with $\beta\in F(\alpha)$, then $\deg(\beta,F)$ divides $\deg(\alpha,F)$. Prove (4 marks)

c) Find the splitting field of the polynomial $x^4-5x^2+6$ over $Q$ (6 marks)

**QUESTION FIVE (20 MARKS)**

a) Let E be a finite extension of degree $n$ over a finite field F. If F has $q$ elements then E has $q^n$ elements. Prove (4 marks)

b) (i) Construct the finite field $GF(8)$ (9 marks)

(ii)Construct the multiplication table of the non-zero elements in $GF(8)$ (7 marks)