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# Hydromagnetics of rotating fluids

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**Abstract**

This article reviews work on the dynamics of a rapidly rotating electrically conducting fluid in the presence of a corotating magnetic field. While the separate action of either rotation or a magnetic field produces strong dynamical constraints, their simultaneous action can result in comparatively weak net constraints and novel phenomena then arise. A systematic account of these phenomena is given and certain applications to natural systems (with emphasis on the dynamics of the Earth's liquid core) are outlined.

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## Contents

## 1. Introduction

### 1.1. *Scope*

The systematic study of the hydromagnetics (magnetohydrodynamics or MHD) of rapidly rotating fluids constitutes a comparatively recent development in theoretical fluid mechanics, with applications in astronomy and geophysics. A key parameter in the theory is the pure number

$$\mathscr{L} \equiv V^*/\Omega^* L^* \tag{1.1}$$

where $L^*$ is a typical length, $\Omega^*$ is the magnitude of the average angular velocity of rotation of the system relative to an inertial frame and

$$V^* \equiv B^*/(\mu\rho)^{1/2} \tag{1.2}$$

is the 'Alfvén speed' based on a typical magnetic field strength $B^*$, $\mu$ and $\rho$ being the magnetic permeability and density of the fluid respectively (all in rationalized MKS units). To $\mathscr{L}$ we must add the parameters

$$\mathscr{C} \equiv V^{*2}\mu\sigma/2\Omega^* \tag{1.3}$$

which is a dimensionless measure of the electrical conductivity $\sigma$, and

$$\mathscr{R} \equiv U^*/\Omega^* L^* \tag{1.4}$$

which is a dimensionless measure of the flow—typical speed $U^*$—relative to the rotating frame (see §§ 3.1 and 3.5 below).

**Table 1. Some rough typical values of the dimensionless parameter $V^*/L^*\Omega^*$ (see equation (1.1)) and the characteristic time scale $\Omega^*(L^*/V^*)^2$.**

| | Earth | Jupiter | Sun | Crab pulsar |
|---|---|---|---|---|
| $B^*$ (Wb m$^{-2}$) | $10^{-2}$ | $10^{-1}$ | $10^{-1}$ | $10^{9}$ |
| $\rho$ (kg m$^{-3}$) | $10^{4}$ | $10^{3}$ | $10^{3}$ | $10^{18}$ |
| $\mu$ (H m$^{-1}$) | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $\gtrsim 10^{-6}$ |
| $V^*$ (m s$^{-1}$) | $10^{-1}$ | $10^{0.5}$ | $10^{0.5}$ | $\lesssim 10^{3}$ |
| $L^*$ (m) | $10^{6}$ | $10^{7}$ | $10^{8}$ | $10^{4}$ |
| $\Omega^*$ (s$^{-1}$) | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | $10^{2}$ |
| $L^*\Omega^*$ (m s$^{-1}$) | $10^{2}$ | $10^{3}$ | $10^{2}$ | $10^{6}$ |
| $V^*/L^*\Omega^*$ | $10^{-3}$ | $10^{-2.5}$ | $10^{-1.5}$ | $\lesssim 10^{-3}$ |
| $\Omega^*(L^*/V^*)^2$ (s) | $10^{10}$ | $10^{9}$ | $10^{9}$ | $\gtrsim 10^{4}$ |

The most significant hydromagnetic phenomena occur in general under conditions that are barely achievable in laboratory systems, notably when the familiar 'magnetic Reynolds number' $\mathscr{S} \equiv U^* L^* \mu\sigma = \mathscr{C}\mathscr{R}/\mathscr{L}^2$ (see equation (2.24)) is so high that the lines of magnetic force move with the fluid (Alfvén's theorem, see § 3.3) and behave like elastic strings in their effects on the motion. These effects depend only weakly on $\Omega^*$ when the parameters $\mathscr{L}$ and $\mathscr{R}$ are very large, but they are strongly influenced by Coriolis forces when—as in the fluid interiors and atmospheres of certain planets and stars, including the Sun and pulsars (see table 1 and Gilman 1968, Hide 1966a,b, 1969a, 1971a, Lehnert 1954)—$\Omega^*$ is so large that $\mathscr{L} \ll 1$ and $\mathscr{R} \ll 1$, and novel phenomena then arise. It is with the basic processes that underlie these novel phenomena that the study of the hydromagnetics of rotating fluids is largely concerned. A characteristic timescale of these phenomena is $\Omega^*(L^*/V^*)^2$, which is directly proportional to $\Omega^*$ (see table 1) and, by virtue of the fact that $\mathscr{L} \ll 1$, is very much longer than either $L^*/V^*$ or $1/\Omega^*$ (but less than the free ohmic decay time $L^{*2}\mu\sigma$ if the Chandrasekhar number $\mathscr{C} \gtrsim 1$, see § 3.5).

The make-up of this article is evident from its table of contents. The selection of topics discussed after the presentation of the basic equations in §§ 2.1–2.6 is dictated largely by the interests of the authors. Particular attention is given to the important class of problems for which density variations are either negligible (see §§ 3.1–5.3) or so small (see §§ 6.1–6.5) that their sole effect is to produce buoyancy forces when a strong gravitational field is present ('Boussinesq approximation'), although references to work on effects associated with compressibility are included in the bibliography (see eg Carstoiu 1964, Roberts and Sozou 1971). We shall throughout the article mention various applications of the results derived and in most cases the application is to the liquid core of the Earth, where the geomagnetic field originates. A few comments on the hydromagnetics of the core are therefore in order before going on to the main part of the article.

### 1.2. *Comments on the hydromagnetics of the Earth's liquid core*

The dispersion relationship for hydromagnetic waves in a rapidly rotating system indicates that those planetary scale hydromagnetic oscillations of the Earth's liquid metallic core which are characterized by a 'magnetostrophic' balance between Coriolis torques and Lorentz torques (see §§ 3.5 and 4.1–4.5) could have periods comparable with the timescales found in the geomagnetic secular variation—namely decades to centuries. Moreover, these waves would propagate in a manner reminiscent of the westward drift of the geomagnetic field (see figure 1 and Acheson 1972a,b, Braginsky 1967a,b, 1970a,b, Hide 1966a, Hide and Stewartson 1972, Malkus 1967a, Roberts and Soward 1972, Stewartson 1967). These 'slow' oscillations might also play a key role in the production of the geomagnetic field by the 'hydromagnetic dynamo process' (Braginsky 1967a,b) but we note that, as with the other 'inverse problems' in geophysics (see Backus 1971), attempts to interpret specific features of the field in terms of this or any other hypothetical type of fluid motion in the core encounter fundamental mathematical difficulties associated with the lack of uniqueness of any particular solution.

Associated with these slow waves there exists, in principle at least, a general class of planetary scale hydromagnetic oscillations of the core (see §§ 4.1 and 4.5) with periods that are typically much shorter than the electromagnetic 'cut-off' period of a few years associated with the weak ($\sigma \lesssim 3 \times 10^2$ ohm$^{-1}$ m$^{-1}$) conductivity of the Earth's mantle. Although these fast waves would be undetectable in the magnetic record at the Earth's surface, concomitant eddy currents induced in the lower mantle might enhance the electromagnetic coupling between the core and mantle to the value required to provide the torques implied by the most pronounced irregular changes in the Earth's rotation rate (Hide 1966a). An alternative suggestion as to the nature of these torques is that they are largely due to hydromagnetic interactions between core motions and bumps on the core–mantle boundary with typical horizontal dimensions up to a few thousand kilometres and vertical dimensions of only a few kilometres (Hide 1969b).

Such bumps have not yet been resolved by seismological methods but possible evidence for their existence is provided by the recently discovered correlation between the Earth's gravitational and magnetic fields (Hide and Malin 1970, 1971a,b,c). It is conceivable that the pattern of temperature and height variations on the core–mantle boundary affects the frequency of reversals in sign of the geomagnetic dipole (as well as other properties of the field, such as the amplitude of the nondipole field, etc), variations in which are known from palaeomagnetic

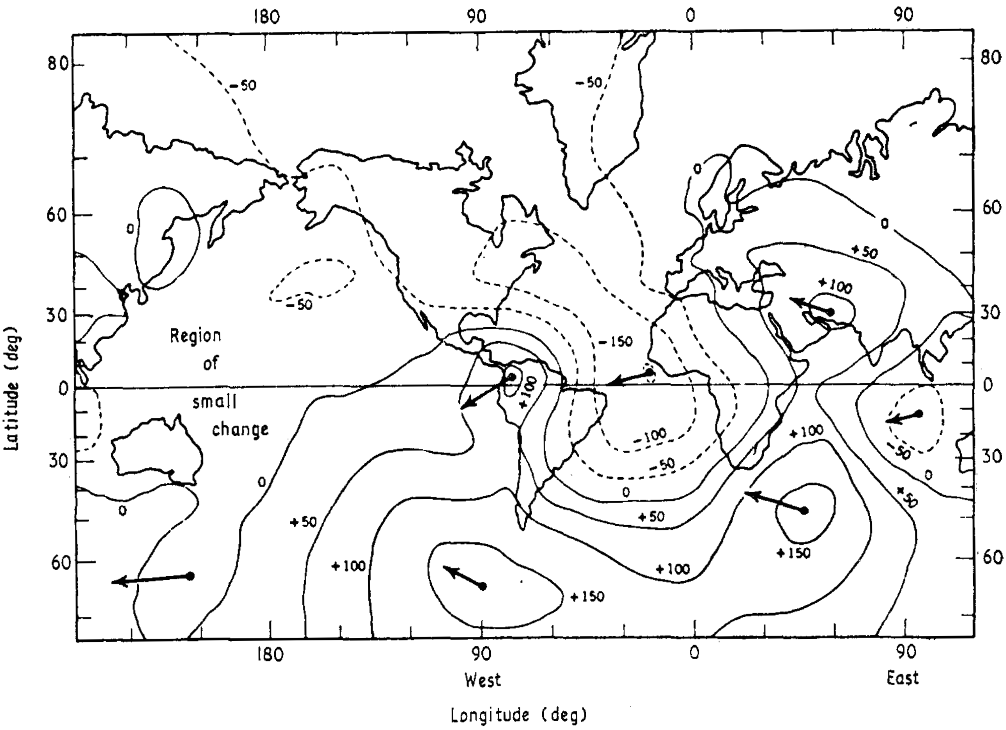


Figure 1. Schematic representation of the geomagnetic secular variation and westward drift. The contours are isopors (ie lines of equal rate of change) of the vertical component $B_z$ for 1922, the units being $3{\cdot}33 \times 10^{-6}$ gauss per year. Dotted isopors indicate regions where $\partial B_z/\partial t$ is negative. The arrow attached to each isoporic focus gives the magnitude and direction of its motion during the interval 1922 to 1945. Observe that the sense of the azimuthal component of motion is westward in all cases. For further diagrams see eg Akasofu and Chapman (1972). After Irving (1964).

data to be much greater than concomitant *amplitude* variations of the ancient geomagnetic field (see Cox 1968, Hide 1967, 1970). Studies of the hydromagnetics of an irregularly shaped core, taking into account the presence of the solid inner core and the possibility (Higgins and Kennedy 1971, cf Birch 1972, Busse 1972, Jacobs 1971, Malkus 1972 and see also Cox and Cain 1972) that the temperature gradient in the core might be subadiabatic, could be crucial steps in the interpretation of a variety of geomagnetic and other geophysical data. Indeed, it has become clear that further progress towards a satisfactory explanation of the Earth's magnetism will be inseparable from developments in the hydromagnetics of rotating fluids.

## 2. Basic equations

### 2.1. *Introduction*

The basic equations of hydromagnetics can be found in several textbooks and reviews (eg Alfvén and Fälthammar 1963, Cowling 1957, 1962, Roberts 1967, Shercliff 1965); here we follow the treatment of Hide and Roberts (1962). Non-relativistic equations suffice for most problems and as the relativistic equations are quite complicated their full presentation here would serve no useful purpose (but see §2.4 and Hide and Roberts 1962).

### 2.2. *Hydrodynamic and thermodynamic equations*

The continuity and momentum equations governing the flow of a Newtonian fluid of density $\rho$, coefficient of shear viscosity $\rho\nu$, and coefficient of bulk viscosity $\rho\nu'$ relate the values of pressure ($p$), fluid velocity ($\boldsymbol{u}$) and body force ($\boldsymbol{F}$) at a general point in space with position vector $\boldsymbol{r}$ at time $t$. They are, when $\boldsymbol{r}$ is measured relative to an inertial frame, the following:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} \equiv \frac{\partial \rho}{\partial t} + \boldsymbol{u}.\nabla\rho = -\rho\nabla.\boldsymbol{u} \tag{2.1}$$

and

$$\begin{aligned} \rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &\equiv \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\,\boldsymbol{u}\right) \\ &= -\nabla\{p + \rho(\nu' - \tfrac{2}{3}\nu)\operatorname{div}\boldsymbol{u}\} - \nabla\times(\rho\nu\nabla\times\boldsymbol{u}) + \boldsymbol{F}. \end{aligned} \tag{2.2}$$

We shall be largely concerned with conditions when the speed of fluid flow is much less than that of sound in the medium and when accelerations are slow compared with those associated with sound waves. The continuity equation (2.1) then reduces to that appropriate to the case of an 'incompressible' fluid, namely

$$\nabla.\boldsymbol{u} = 0 \tag{2.3}$$

and the first term on the right-hand side of equation (2.2) reduces to $-\nabla p$.

In the case of a conducting fluid carrying an electric current, density $\boldsymbol{j}$, in the presence of a magnetic field $\boldsymbol{B}$, to the usual body force we must add the Lorentz force $\boldsymbol{j}\times\boldsymbol{B}$. Thus

$$\boldsymbol{F} = -\rho\nabla\Phi + \boldsymbol{j}\times\boldsymbol{B} \tag{2.4}$$

where $\Phi$ is the gravitational potential (cf equation (2.19)).

The seven scalar equations to which (2.1), (2.2) and (2.4) are equivalent contain fourteen unknowns, and must therefore be supplemented with further mathematical relations. These relations stem from thermodynamic and electrodynamic considerations. The thermodynamic relations comprise an equation of state together with statements concerning irreversible transport processes, such as diffusion, thermal conduction and radiation. The electrodynamic relations are Maxwell's equations (or, more precisely, the 'pre-Maxwell' equations, see equations (2.10)–(2.13)) together with a statement concerning the dependence of the current on the electric field present (eg Ohm's law). We shall assume in this article that $\Phi$ is a known function of the space coordinates, thus excluding processes such as Jeans instability leading to gravitational condensation (but see Chandrasekhar 1954a, 1961 and Lynden-Bell 1966).

We can illustrate the thermodynamic relations required for a typical compressible fluid by considering a perfect gas for which

$$p/\rho T = \text{constant} \tag{2.5}$$

where $T$ denotes temperature. Two extreme cases will be cited to indicate the range of possibilities as regards entropy variations, without having to write down the field equations governing irreversible transport processes. The first is the isentropic case, in which changes of state are so rapid that transport processes can be ignored so that the entropy per unit mass

$$c_v \ln (p\rho^{-\gamma}) \tag{2.6}$$

(where $c_v$ is the specific heat at constant volume, $c_p$ is the specific heat at constant pressure, and $\gamma = c_p/c_v$) of a fluid element remains constant. At the other extreme, the isothermal case, transport processes are so efficient that the temperature of a fluid element remains constant.

It is convenient to consider two types of incompressible fluid, namely those which are 'barotropic' and those which are 'baroclinic'. Barotropic incompressible fluids have uniform density and in consequence the gravitational contribution to $\boldsymbol{F}$ cannot exert a torque. In the absence of hydromagnetic effects, hydrodynamic flow of a barotropic fluid has to be generated by applying forces at the bounding surfaces of the fluid. When $\rho$ is kept constant in equations (2.1) and (2.2), the thermodynamic relations are redundant.

Baroclinicity is associated with density variations, the action of gravity on which gives rise to buoyancy forces (see the first term on the right-hand side of equation (2.4)). Owing to the torque exerted by these forces, it is not generally possible to maintain hydrostatic equilibrium in their presence and hydrodynamic flow must ensue. Baroclinicity arises in a variety of ways; it may be due to variations in temperature, chemical composition, or both.

Differential heating produces temperature variations from place to place in the fluid and owing to thermal expansion they give rise to density variations (see § 6). If the heated incompressible fluid has a volume coefficient of thermal expansion $\vartheta$ the equation of state is

$$\rho = \rho_0(1-\vartheta T) \tag{2.7}$$

where $\rho_0$ is the density at $T = 0$. Entropy changes are taken into account by including the equation of heat transfer

$$\begin{aligned}\rho \frac{\mathrm{D}}{\mathrm{D}t}(c_v T) &\equiv \rho\left(\frac{\partial}{\partial t}(c_v T) + (\boldsymbol{u}.\nabla)(c_v T)\right) \\ &= \nabla.(K\rho c_v \nabla T) + Q \end{aligned} \tag{2.8}$$

where $K$ is the effective thermal diffusivity, equal to the coefficient of thermal conductivity divided by $\rho c_v$, and $Q$ is the rate of internal heating per unit volume, including radiative effects which could be negative (see Chandrasekhar 1961).

## 2.3. *Electrodynamic equations*

Now we must write down equations relating current density $\boldsymbol{j}$, magnetic field $\boldsymbol{B}$, electric field $\boldsymbol{E}$ and charge density $\varphi$. If

$$\boldsymbol{H} \equiv \boldsymbol{B}/\mu \tag{2.9a}$$

and

$$\boldsymbol{D} \equiv \epsilon \boldsymbol{E} \tag{2.9b}$$

where $\mu$ and $\epsilon$ denote, respectively, magnetic permeability and dielectric constant, then the equations are

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} \tag{2.10}$$

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \tag{2.11}$$

$$\nabla . \boldsymbol{B} = 0 \tag{2.12}$$

$$\nabla . \boldsymbol{D} = \varphi. \tag{2.13}$$

Equation (2.10) is Ampère's circuital law relating the magnetic field to its basic source, the electric current, Maxwell's displacement current being neglected (cf equation (2.17) below) and equation (2.12) expresses the fact that the magnetic field is solenoidal. Equation (2.11) is Faraday's law of induction which, in its differential form, contains many subtle difficulties of interpretation brought out clearly in relatively few standard texts (see Alfvén and Fälthammar 1963). Equations (2.13) and (2.9*b*) relate the electric field to the volume density of electric charge $\varphi$.

A unit electric charge moving at velocity $\boldsymbol{u}$ relative to a magnetic field $\boldsymbol{B}$ experiences a force $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$. Thus, if the conducting fluid satisfies Ohm's law with conductivity $\sigma$, then

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}). \tag{2.14}$$

This completes our set of equations governing seven unknown vector quantities, $\boldsymbol{u}$, $\boldsymbol{F}$, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and four unknown scalar quantities $p$, $\rho$, $T$ and $\varphi$, or twenty-five scalars in all.

## 2.4. *Range of validity*

Now we must consider the range of validity of our equations, and for this purpose we introduce a typical absolute flow speed $W^*$ (relative to an inertial frame, cf equation (1.4)), the ordinary sound speed $a_0$ and the speed of electromagnetic waves $c_0 = (\mu\epsilon)^{-1/2}$. The ratios $(W^*/c_0)^2$, $(a_0/c_0)^2$ and $(V^*/c_0)^2$ (see equation (1.2)) are respective measures of the ordered kinetic energy, thermal energy and magnetic energy in terms of the rest energy of the fluid.

When $W^*/c_0 \ll 1$ the $\mathrm{D}/\mathrm{D}t$ terms in equations (2.1) and (2.2) are nonrelativistic. To the same approximation, although the effective electric field $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$ depends on the local frame of reference in which it is measured, the magnetic field is frame independent. When $a_0/c_0 \ll 1$, that is to say when the root mean square speed of thermal motion is much less than the speed of light, the relativistic correction $p/c_0^2$ to the density $\rho$ is negligible.

If $\tau^*$ is a typical time scale associated with the hydromagnetic flow, $L^*$ being a typical length, then by equations (2.9), (2.10), (2.11) and (2.13)

$$\left.\begin{aligned} \left|\frac{\partial \boldsymbol{D}}{\partial t}\right| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}| \\ |\varphi \boldsymbol{E}| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}\times\boldsymbol{B}| \\ |\varphi \boldsymbol{u}| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}|. \end{aligned}\right\} \tag{2.15}$$

Thus, when $\tau^*$ is much greater than the time taken for electromagnetic waves to traverse a distance $L^*$ the neglect of the displacement current $\partial \boldsymbol{D}/\partial t$ in equation (2.10), of the electrostatic body force $\varphi \boldsymbol{E}$ in the expression for the body force (equation (2.4)), and of the advective contribution $\varphi \boldsymbol{u}$ to $\boldsymbol{j}$ in equation (2.14) is justified. However, when our equations are nonrelativistic (ie $W^*/c_0 \ll 1$ and $a_0/c_0 \ll 1$) but $V^*$ is not much less than $c_0$, then it may be shown that equations (2.4), (2.10) and (2.14) become

$$\boldsymbol{F} = -\rho \nabla \Phi + \boldsymbol{j} \times \boldsymbol{B} + \varphi \boldsymbol{E} \tag{2.16}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2.17}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) + \varphi \boldsymbol{u}. \tag{2.18}$$

We see that the neglect of terms of order $(V^*/c)^2$ in equations (2.4), (2.10) and (2.14) filters out electromagnetic waves. Plasma oscillations are automatically excluded by taking Ohm's law (equation (2.14), cf Spitzer 1956) as the relationship between the current and the effective electric field.

In addition to supposing that relativistic effects can be neglected we have assumed implicitly that the fluid can be regarded as a continuum with isotropic transport coefficients $\nu$, $K$ and $\sigma$. This procedure is valid for sufficiently dense media, but in the case of a tenuous fluid continuum theory breaks down (see Spitzer 1956). It would, however, take us too far away from our main topic even to summarize how criteria for the validity of continuum theory can be deduced from the equations governing the motions of individual ions, electrons and neutral particles.

## 2.5. *Equations referred to a rotating frame*

When referred to a frame that rotates with instantaneous angular velocity $\boldsymbol{\Omega}$ relative to an inertial frame, the equations of motion and continuity for an incompressible fluid of uniform coefficient of viscosity $\nu\rho$ become

$$\rho\{\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\,.\nabla)\,\boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u}\}$$
$$= -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \nu\rho \nabla^2 \boldsymbol{u} - \rho \nabla \Phi + \rho \boldsymbol{r} \times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t \tag{2.19}$$

and

$$\nabla . \boldsymbol{u} = 0. \tag{2.20}$$

Here $\Phi$ now includes the centrifugal term $-\frac{1}{2}|\boldsymbol{\Omega} \times \boldsymbol{r}|^2$ as well as gravitational effects, $\boldsymbol{r}$ being a vector from the origin to a general point P at which the eulerian fluid velocity vector, now measured relative to the rotating frame (see Kibble 1966), is $\boldsymbol{u}$, and the corresponding pressure, magnetic field and electric current density are $p$, $\boldsymbol{B}$ and $\boldsymbol{j}$ respectively. The last term in equation (2.19) vanishes when $\boldsymbol{\Omega}$ is steady.

If the density of the fluid depends on temperature only then equation (2.7) holds, and if variations in $K\rho c_v$ and $c_v$ are negligible, equation (2.8) reduces to

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}\,.\nabla)\,T = K \nabla^2 T + q \tag{2.21}$$

where $q = Q/\rho c_v$.

Equations (2.19) and (2.20) hold when $|\boldsymbol{\Omega}\times\boldsymbol{r}+\boldsymbol{u}|$, the speed of absolute motion, is everywhere much less than the speed of sound. Provided that $|\boldsymbol{\Omega}\times\boldsymbol{r}+\boldsymbol{u}|$ is much less than the speed of light (see § 2.4) the electrodynamic equations (2.10) to (2.14) are the same in the rotating frame as in the nonrotating frame (Trocheris 1949).

When $\boldsymbol{j}$ and $\boldsymbol{E}$ are eliminated from equations (2.9*a*), (2.10), (2.11) and (2.14) we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\eta\nabla^2\boldsymbol{B}. \tag{2.22}$$

Here

$$\eta = (\mu\sigma)^{-1} \tag{2.23}$$

which has the same dimensions as $\nu$ and $K$, namely (length)$^2$(time)$^{-1}$. According to equation (2.22) the rate of change of $\boldsymbol{B}$ depends on two agencies, namely motional induction, represented by the term $\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$, and ohmic dissipation due to electrical resistance, represented by the term $\eta\nabla^2\boldsymbol{B}$. The magnetic Reynolds number

$$\mathscr{S}\equiv U^* L^* \mu\sigma = U^* L^*/\eta \tag{2.24}$$

is a measure of the relative importance of these two agencies. When $\mathscr{S}\ll 1$, equation (2.22) reduces to the diffusion equation, with solutions corresponding to magnetic fields decaying on the timescale $L^{*2}/\eta$, which is typically $\sim 3\times 10^{11}$ s ($\sim 10^4$ yr) for the liquid core of the Earth. When, on the other hand, $\mathscr{S}\gg 1$, diffusion effects are negligible and the magnetic lines of force move with the fluid (Alfvén's theorem, see § 3.3 below). The quantitative basis for the dynamo theory of the Earth's magnetism (Bullard 1949, Elsasser 1946a,b, 1947, Parker 1955), investigations of which have been largely though not entirely concerned with the so-called 'kinematic dynamo problem' in which $\boldsymbol{u}$ is specified *a priori* when solving equation (2.22) (for up to date references see Moffatt (1972, 1973) and reviews by Roberts (1971), Roberts and Stix (1971) and Weiss (1971)), is the recognition that motions as slow as $10^{-3}$ m s$^{-1}$ give $\mathscr{S}\sim 100$ in the core of the Earth. In these circumstances the magnetic energy increases at the expense of the kinetic energy of motion, the final balance being attained when the rate at which work is done on the system by the forces that drive the motion is offset by dissipation due to electrical resistivity and viscosity. For the Earth this dissipation rate is $\sim 3\times 10^{10}$ J s$^{-1}$, some two or three powers of ten less than the energy of thermal sources within the Earth (but evidently comparable, incidentally, with energy released by earthquakes and with the rate of working of the forces that move the continental plates). The source of energy for core motions has not yet been identified with certainty, but mechanical stirring associated with the Earth's precessional motion (cf the $\boldsymbol{r}\times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$ term in equation (2.19)) or thermal stirring due to radioactive heating or to the release of heat of crystallization might suffice (see Jacobs *et al* 1972, Malkus 1968, 1971 and Stacey 1969 for recent discussions of this question and references to earlier work).

## 2.6. *Energy and vorticity equations*

It is instructive to consider the energetics of hydromagnetic flows (see Chandrasekhar 1961, Hide 1956, Hide and Roberts 1962, Roberts 1967). Multiplying equation (2.22) scalarly by $\mu^{-1}\boldsymbol{B}$ we find, after some manipulation making use of equations (2.10) and (2.14), that

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}\right) = -\frac{j^2}{\sigma} - \frac{1}{\mu}\nabla.(E\times B) - (j\times B).u. \tag{2.25}$$

The left-hand side is the time rate of change of magnetic energy density, while on the right-hand side the first term represents decay of magnetic energy due to ohmic heating, the second term is the Poynting effect and represents the flux of electromagnetic energy across the boundaries of the (elementary) region considered, and the third term is a measure of effects due to motional induction. We observe that $\mathscr{S}$, the magnetic Reynolds number, is a measure of the third term divided by the first term.

We can deduce an expression for $(j\times B).u$ by multiplying equation (2.19) scalarly by $u$. On combining the resulting equation with equation (2.25) we find for the rate of change of total energy density in the system, magnetic plus kinetic, the following expression:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}+\frac{1}{2}\rho u^2\right) = &-\left\{\frac{j^2}{\sigma}+\frac{1}{\mu}\nabla.(E\times B)\right\}\\ &-\left\{\nu\rho\,|\nabla\times u|^2+\nu\rho\nabla.(\nabla\times u)\times u\right\}-\nabla.\left\{\left(p+\frac{1}{2}\rho u^2+\rho\Phi\right)u\right\}\\ &+\left\{\Phi u.\nabla\rho+\frac{1}{2}u^2\frac{\mathrm{D}\rho}{\mathrm{D}t}\right\}+\rho(u\times r).\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.\end{aligned} \tag{2.26}$$

The first group of terms on the right-hand side comprise the electromagnetic contributions already discussed. Combined in the second group are terms representing viscous dissipation and work done by tangential viscous forces. Terms representing the rate of working of normal pressure forces and the advection of kinetic and gravitational energy into the elementary fluid volume make up the third group. The first contribution to the fourth group of terms represents the internal release of gravitational potential energy, which vanishes when $\rho$ is uniform. This contribution is much greater than $\frac{1}{2}u^2\,\mathrm{D}\rho/\mathrm{D}t$ when the gravitational field is so large that density variations are negligible in all terms of the equation of motion (equation (2.19)) save the buoyancy term ('Boussinesq' approximation, see Spiegel and Veronis 1960). Finally, we have the only term directly involving $\boldsymbol{\Omega}$ (for $u.2\boldsymbol{\Omega}\times u = 0$), and this vanishes when $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = 0$. It represents the rate at which work is done on the system by the forces that produce time variations in $\boldsymbol{\Omega}$, and in the case of the Earth's core is a measure of the contribution to the hydromagnetic energy of the Earth made by the gravitational torques that cause the Earth's rotation axis to precess about an axis fixed in space.

The vorticity equation, obtained by taking the curl of equation (2.19) making use of the full continuity equation (2.1), may usefully be written for future reference in §§ 6.1–6.5 (where nonhomogeneous fluids are considered) as follows:

$$\begin{aligned}&(2\boldsymbol{\Omega}.\nabla)(\rho u)+\nabla\times(j\times B)+(\nabla\Phi)\times(\nabla\rho)\\ &\qquad = -2\boldsymbol{\Omega}\,\partial\rho/\partial t+\nabla\times\{\rho\,\partial u/\partial t+\rho(u.\nabla)\,u-\nu\rho\nabla^2 u-\rho r\times\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t\}.\end{aligned} \tag{2.27}$$

However, for the problems treated in §§ 3–5, *where attention is confined to fluids of constant density* (ie $\nabla\rho = \partial\rho/\partial t = 0$) *for which the angular velocity of basic rotation is* *steady* (ie $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = 0$), it follows from the last equation that the relative vorticity $\boldsymbol{\xi} = \nabla \times \boldsymbol{u}$ satisfies

$$\partial\boldsymbol{\xi}/\partial t + (\boldsymbol{u}.\nabla)\boldsymbol{\xi} - \{(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\boldsymbol{u}\} = \nabla \times \{(\boldsymbol{j} \times \boldsymbol{B})/\rho\} + \nu\nabla^2\boldsymbol{\xi}. \tag{2.28}$$

As we shall see in what follows next in §§ 3.1–3.5, equations (2.28) and (2.22) lead to several useful general theorems and concepts for rapidly rotating nonhydromagnetic systems (Proudman–Taylor theorem, Ertel's potential vorticity theorem, 'geostrophic flow') and for corresponding hydromagnetic systems (Alfven's theorem, 'aligned field' flows, a hydromagnetic potential vorticity theorem, 'magnetostrophic flow').

## 3. Some theorems

### 3.1. *Proudman–Taylor theorem and geostrophy*

If, in addition to making the foregoing simplifications, we can suppose that the motion (relative to the rotating frame) is so steady and slow and the hydromagnetic and viscous forces so tiny that all the following dimensionless parameters tend to zero

$$\mathscr{A} = \overline{\rho\,\partial\boldsymbol{u}/\partial t}/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \tag{3.1}$$

$$\mathscr{R} = \overline{\rho(\boldsymbol{u}.\nabla)\,\boldsymbol{u}}/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \tag{3.2}$$

(see equation (1.4))

$$\mathscr{M} = \overline{\boldsymbol{j}\times\boldsymbol{B}}/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \tag{3.3}$$

and

$$\mathscr{E} = \overline{\rho\nu\,\nabla^2\boldsymbol{u}}/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \tag{3.4}$$

(where the overbar denotes the root mean square value), then equation (2.19) takes the very simple form

$$2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u} = -\nabla p - \rho\nabla\Phi. \tag{3.5}$$

This expresses a 'geostrophic' balance between the Coriolis force (per unit volume) and the nonhydrostatic part of the pressure gradient. The corresponding vorticity equation (cf equation (6.8)) is, by equation (2.28),

$$2(\boldsymbol{\Omega}.\nabla)\,\boldsymbol{u} = 0. \tag{3.6}$$

This is the celebrated 'two-dimensional' theorem due originally to Proudman (1916) and Taylor (1921) (see Greenspan 1968), which shows that all components of $\boldsymbol{u}$ are independent of the coordinate parallel to $\boldsymbol{\Omega}$ (see § 6.2 for the extension of this result to the case $\nabla\rho \neq 0$). The Proudman–Taylor theorem together with certain compatibility conditions to be satisfied where an effectively inviscid 'interior' flow meets a viscous Ekman layer on a bounding surface (see equation (5.31)) are concepts of central importance which greatly simplify the theoretical analysis of complex rotating *non*hydromagnetic systems. (There is an analogy here with certain theorems satisfied by slow motions of a nonrotating fluid pervaded by a nearly uniform magnetic field (Hide 1956, Hide and Roberts 1962, Hunt and Ludford 1968, Kulikovsky 1968, Lundquist 1952; see also § 6.4) and compatibility conditions imposed by the presence of viscous Hartmann layers on bounding surfaces (see equation (5.25)).

The gyroscopic forces due to the rapid basic rotation thus have the effect of aligning elementary vortices so that their axes are parallel to the rotation axis. Displacement of these axes would produce restoring forces resulting in 'inertial' oscillations. In circumstances when these oscillations can be described in terms of a superposition of plane waves of small amplitude proportional to $\exp i(\boldsymbol{\kappa}.\boldsymbol{r}-\omega t)$, the corresponding dispersion relationship connecting the angular frequency $\omega$ to the wavenumber vector $\boldsymbol{\kappa}$ is

$$\omega^2 = (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2 \tag{3.7}$$

(see eg Batchelor 1967, Lighthill 1966, 1967 and §4.1). Particle orbits in these inertial oscillations are circular and lie in the plane of the wavefront (see figure 2).

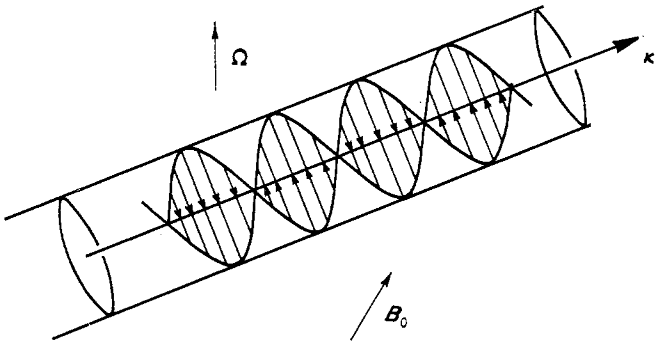


Figure 2. Illustrating the velocity field in a single plane wave of wavenumber $\boldsymbol{\kappa}$ in an unbounded rotating fluid with or without a corotating uniform magnetic field $\boldsymbol{B}_0$. Particle orbits are circular and lie in the plane of the wavefront. After Moffatt (1970).

Observe that when $\omega = 0$ (steady motion) the wavenumber vector $\boldsymbol{\kappa}$ is perpendicular to $\boldsymbol{\Omega}$, in keeping with the 'two-dimensional' theorem expressed by equation (3.6).

The foregoing results have proved useful in the discussion of effects of irregularly shaped boundaries on flows of rapidly rotating fluids. We conclude this section with a basic result concerning the effect of rotation on flows which, in virtue of the boundary conditions and method of generation, are strictly two-dimensional in planes perpendicular to the axis of rotation.

Consider the vorticity equation (2.28) when $(2\boldsymbol{\Omega}.\nabla)\boldsymbol{u}\equiv 0$. Thus we have (when $\boldsymbol{B} = 0$)

$$\partial\boldsymbol{\xi}/\partial t+(\boldsymbol{u}.\nabla)\boldsymbol{\xi}-(\boldsymbol{\xi}.\nabla)\boldsymbol{u}-\nu\nabla^2\boldsymbol{\xi} = 0 \tag{3.8}$$

which shows that *if the boundary conditions are independent of* $\boldsymbol{\Omega}$ then so is the two-dimensional field of relative motion. This result, which was first enunciated by Taylor (1917), and its extension to cases when it *cannot* be supposed that the boundary conditions are independent of $\boldsymbol{\Omega}$, have been the subject of several theoretical and experimental studies (Hide 1968, see also Greenspan 1968).

Since $\boldsymbol{\Omega}$ does not appear in the electromagnetic equations (see §2.4), Taylor's result carries over trivially to the hydromagnetic case provided, of course, that the electromagnetic as well as the mechanical boundary conditions are now independent of $\boldsymbol{\Omega}$. Thus, problems in which the Coriolis torque $(2\boldsymbol{\Omega}.\nabla)\boldsymbol{u}$ vanishes *by hypothesis* fall into a special category, for then there is no stretching or bending of fluid filaments that would couple the vorticity of the basic rotation to the vorticity of the relative flow and the novel phenomena with which this article is largely concerned do not then arise (see eg Causse and Poirier 1960, Poirier and Robert 1960, cf Loper and Benton 1970), although other interesting phenomena are by no means excluded (see eg Moffatt 1965).

3.2. *Ertel's potential vorticity theorem*

We define the 'potential vorticity' as

$$\mathscr{X} \equiv \rho^{-1}(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\Lambda \tag{3.9}$$

where $\Lambda$ is any scalar quantity that is conserved by individual fluid elements throughout their motion (ie $\mathrm{D}\Lambda/\mathrm{D}t \equiv \partial\Lambda/\partial t + \boldsymbol{u}.\nabla\Lambda = 0$). A theorem, due originally to Ertel (1942, see also Batchelor 1967, Ertel and Rossby 1949, Greenspan 1968, Lighthill 1966, Pedlosky 1971), can be obtained by multiplying the vorticity equation scalarly by $\nabla\Lambda$. When $\nabla\rho = 0$ and hydromagnetic effects vanish, this gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\{(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\Lambda\} = \nu\nabla\Lambda.\nabla^2\boldsymbol{\xi}. \tag{3.10}$$

When $\nu = 0$ the quantity $\mathscr{X}$ is conserved by individual fluid elements throughout their motion; an application of this result and of a hydromagnetic extension (see §3.4) is given in §4.5 below, where oscillations of spherically bounded rotating systems are considered.

3.3. *Alfvén's theorem*

We turn now to the hydromagnetic situation, but before examining the equation of motion it is instructive to consider the equation (2.22) for $\boldsymbol{B}$. In the limit when the magnetic Reynolds number $\mathscr{S}\to\infty$ (see equation (2.24)), equation (2.22) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t}+(\boldsymbol{u}.\nabla)\boldsymbol{B}-(\boldsymbol{B}.\nabla)\boldsymbol{u} = 0 \tag{3.11}$$

(since $\nabla.\boldsymbol{u} = \nabla.\boldsymbol{B} = 0$, see equations (2.20) and (2.12)). The line integral of equation (2.22) around a material circuit expresses the impossibility when $\sigma\to\infty$ of changing the flux linkage of such a circuit. The lines of force are then effectively 'frozen' into the fluid (Alfvén 1942a,b). So far as their mechanical effects are concerned (see below) the lines of force behave like elastic strings and permit wave motions which, in their simplest form (ie no Coriolis forces, etc), are plane 'Alfvén waves' satisfying the dispersion relationship

$$\omega^2 = (\boldsymbol{V}.\boldsymbol{\kappa})^2 \tag{3.12}$$

(cf equation (3.7) and §4.1). Here

$$\boldsymbol{V} = \frac{\boldsymbol{B}_0}{(\mu\rho)^{1/2}} \tag{3.13}$$

is the 'Alfvén velocity' based on the undisturbed magnetic field (cf equation (1.2)). Particle orbits in these Alfvén waves are linear and lie in planes parallel to the wavefronts. In contrast to inertial waves (see equation (3.7)), Alfvén waves are non-dispersive. Their group velocity $\partial\omega/\partial\boldsymbol{\kappa}$ is equal to $\pm\boldsymbol{V}$.

### 3.4. '*Aligned field*' *flows*

We now consider the *dynamical* aspects of hydromagnetic rotating fluid flows, and it is of some interest to examine first the restrictive but important class of motions for which the velocity and magnetic fields *as measured relative to the rotating frame* are everywhere parallel (Acheson 1971, cf Hasimoto 1959). Thus we write

$$\boldsymbol{B} = C\boldsymbol{u}(\mu\rho)^{1/2} \tag{3.14}$$

where $C$ is a dimensionless scalar. Such flows are necessarily steady when $C$ is uniform and the fluid is a perfect conductor of electricity, for then by equations (3.11) and (3.14) $\partial\boldsymbol{B}/\partial t = 0$. The equation of motion (2.19) then reduces to

$$(1-C^2)(\boldsymbol{u}.\nabla)\boldsymbol{u}+2\boldsymbol{\Omega}\times\boldsymbol{u} = -\rho^{-1}\nabla(p+\tfrac{1}{2}\mu^{-1}B^2)-\nabla\Phi+\nu\nabla^2\boldsymbol{u}. \tag{3.15}$$

Further, in view of equation (3.14), equations (2.3) and (2.12), $\nabla.\boldsymbol{u} = 0$ and $\nabla.\boldsymbol{B} = 0$ respectively, become identical. For such 'aligned field' flows the governing equations therefore reduce to a set *identical with the corresponding nonhydromagnetic equations* (consisting of equations (2.3) and (3.15) with $\boldsymbol{B} = C = 0$), provided only that we make the transformations

$$(\boldsymbol{\Omega}_*, p_*, \Phi_*, \nu_*) \equiv (\boldsymbol{\Omega}, p+\tfrac{1}{2}\mu^{-1}B^2, \Phi, \nu)/(1-C^2), \tag{3.16}$$

for equation (3.15) may then be written

$$(\boldsymbol{u}.\nabla)\boldsymbol{u}+2\boldsymbol{\Omega}_*\times\boldsymbol{u} = -\rho^{-1}\nabla p_*-\nabla\Phi_*+\nu_*\nabla^2\boldsymbol{u}. \tag{3.17}$$

For the Proudman–Taylor theorem to hold for such flows it is evidently necessary for $|\mathscr{R}(1-C^2)|$ and $\mathscr{E}$ to be much less than unity, a condition which may be much more restrictive than $\mathscr{R}\ll 1$ and $\mathscr{E}\ll 1$ (see equations (3.2) and (3.4)), especially when $C$, the ratio of the local Alfvén speed to the local fluid speed, is much greater than unity. The corresponding potential vorticity theorem takes the form

$$(\boldsymbol{u}.\nabla)\{(\boldsymbol{\xi}+2\boldsymbol{\Omega}_*).\nabla\Lambda\} = \nu_*\nabla\Lambda.\nabla^2\boldsymbol{\xi} \tag{3.18}$$

(cf equation (3.10)).

We have already noted that these 'aligned field' results do not in general apply to unsteady motions. Nevertheless, in cases when the unsteadiness can be removed by a simple transformation of coordinates (eg simple waves—see §§4.1 and 4.5) the above concepts can be both useful and illuminating.

### 3.5. '*Magnetostrophic*' *flows*

By analogy with geostrophic motion (see equation (3.5)) *magneto*strophic motion is defined as satisfying the equation

$$2\rho\boldsymbol{\Omega}\times\boldsymbol{u}-\boldsymbol{j}\times\boldsymbol{B} = -\nabla p-\rho\nabla\Phi. \tag{3.19}$$

In this case the dynamical pressure gradient (ie the actual pressure gradient plus $\rho\nabla\Phi$) is balanced by the difference between the Coriolis and Lorentz forces. The vorticity equation then becomes

$$2(\boldsymbol{\Omega}.\nabla)\boldsymbol{u} = -\frac{1}{\mu\rho}\nabla\times\{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\}, \tag{3.20}$$

expressing a balance between the gyroscopic and hydromagnetic torques acting on an individual fluid element.

In order to specify the conditions under which magnetostrophic flows can be expected to occur, we must consider the order of magnitude of the various terms in equation (2.22) for $\boldsymbol{B}$. Now

$$\left|\frac{\partial \boldsymbol{B}}{\partial t}\right| \sim \frac{B^*}{\tau^*},$$

$$|\nabla \times (\boldsymbol{u} \times \boldsymbol{B})| \sim \frac{U^* B^*}{L^*}$$

and

$$|\eta \nabla^2 \boldsymbol{B}| \sim \frac{\eta B^*}{L^{*2}}$$

if $B^*$ and $U^*$ are typical values of $|\boldsymbol{B}|$ and $|\boldsymbol{u}|$ respectively and $L^*$ and $\tau^*$ are corresponding length and time scales. Hence

$$U^* \sim \left(\frac{L^*}{\tau^*} + \frac{\eta}{L^*}\right)$$

so that the Coriolis acceleration is of order

$$2\Omega\left(\frac{L^*}{\tau^*} + \frac{\eta}{L^*}\right)$$

in magnitude. Thus when the flow is 'magnetostrophic' we infer from equation (3.20) that

$$2\Omega\left(\frac{1}{\tau^*} + \frac{\eta}{L^{*2}}\right) \sim \frac{V^{*2}}{L^{*2}} \tag{3.21}$$

where $V^*$ is given by equation (1.2). Hence

$$\frac{1}{\tau^*} \sim \frac{V^{*2}}{2\Omega L^{*2}} + \frac{\eta}{L^{*2}}$$

so that the neglect of $\partial \boldsymbol{u}/\partial t$, $(\boldsymbol{u}.\nabla)\boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$ in deriving equation (3.19) from (2.19) is valid provided that

$$\left(\frac{V^{*2}}{2\Omega^2 L^{*2}} + \frac{\eta}{\Omega L^{*2}}\right) \ll 1$$

$$\frac{U^*}{\Omega L^*} \ll 1$$

and

$$\nu \ll \Omega L^{*2}.$$

In terms of the dimensionless parameters already introduced, these criteria are satisfied when

$$\left.\begin{aligned} \mathscr{L}^2(1+\mathscr{C}^{-1}) &\ll 1 \\ \mathscr{R} &\ll 1 \\ \text{and} \qquad \mathscr{E} &\ll 1 \end{aligned}\right\} \tag{3.22}$$

(see equations (1.1)–(1.4) and equation (3.4)).

We see immediately from equation (3.22) that $\mathscr{C}^{-1}$ is the appropriate measure of dissipative effects due to ohmic heating. When $\mathscr{C} \gg 1$ (ie $\tau^* \ll L^{*2}/\eta$ as in the case of the geomagnetic secular variation; see § 1.2) the criteria for magnetostrophic flow become

$$\left.\begin{aligned} \mathscr{L}^2 &\ll 1 \\ \mathscr{R} &\ll 1 \\ \text{and} \qquad \mathscr{E} &\ll 1. \end{aligned}\right\} \tag{3.23}$$

When on the other hand $\mathscr{C} \not\gg 1$ (ie $\tau^* \not\ll L^{*2}/\eta$, as in the case of the geomagnetic dynamo problem, see eg Bullard and Gellman 1954, Elsasser 1956), the criteria reduce to

$$\left.\begin{aligned} \mathscr{L}^2 &\ll \mathscr{C} \\ \mathscr{R} &\ll 1 \\ \text{and} \qquad \mathscr{E} &\ll 1. \end{aligned}\right\} \tag{3.24}$$

We must note here that the satisfaction of these criteria is not sufficient to ensure that *all* modes of motion will be magnetostrophic, as evinced by the examples discussed below.

When dealing with wave motions we shall frequently suppose that the electrical resistivity of the fluid is so small that $\eta$ may be taken to be zero. In order to appreciate the interesting (and sometimes apparently paradoxical) results that emerge in this limit it is helpful to consider equation (3.21). If $\eta = 0$ it is clear that *no matter how small the magnetic field may be* magnetostrophic flow may always occur (inasmuch as viscous forces can be neglected; see § 5.2 especially equation (5.30) and cf § 4.3, especially equation (4.35)) *if the timescale of the fluid motions is sufficiently long.* More precisely, as $V^* \to 0$ hydromagnetic and Coriolis forces still remain equally important if $\tau^* \sim 2\Omega L^{*2}/V^{*2}$. This persistence of hydromagnetic effects in the dynamics is a direct consequence of the assumption of perfect conductivity, for Alfvén's theorem (see equation (3.11)) follows directly from that assumption and *the permanency of attachment of the lines of force to the fluid is then in no way dependent on the strength of the magnetic field.* As soon as the fluid is permitted a nonzero electrical resistivity, however small, it is clear from equation (3.21) that no matter how long the timescale of the fluctuations in the fluid motion they cannot be characterized by a magnetostrophic balance in the limit $V^* \to 0$.

Finally, we note (for reference when meeting phenomena described in §§ 5.1 and 6.4) that *steady* magnetostrophic motions are characterized by values of the parameter $\mathscr{C}$, the Chandrasekhar number, of order unity.

The magnetic field in the liquid core of the Earth consists of two parts, a poloidal field $\boldsymbol{B}_{\text{pol}}$ with lines of force that penetrate the upper reaches of the solid Earth and pass through the Earth's surface into space, and a toroidal field $\boldsymbol{B}_{\text{tor}}$ which is largely confined to the core (as it would be completely if the surrounding mantle were a perfect insulator). Near the Earth's surface, where magnetic measurements are made, $B_{\text{pol}} \sim 5 \times 10^{-5}$ Wb m$^{-2}$ (0·5 gauss) in magnitude and largely dipolar in character, so that $B_{\text{pol}} \sim 5 \times 10^{-4}$ Wb m$^{-2}$ in the core. The toroidal field $\boldsymbol{B}_{\text{tor}}$ cannot be determined directly, but various lines of indirect evidence indicate that

$$B_{\text{tor}}/B_{\text{pol}} \sim 10^{1\cdot5\pm0\cdot5}.$$

Taking $\rho \sim 10^4$ kg m$^{-3}$ and $B \sim 10^{-2}$ Wb m$^{-2}$ we find that $V^* \sim 10^{-1}$ m s$^{-1}$ and

$$\mathscr{L} \sim 10^{-3} \tag{3.25}$$

for motions on the scale of the core (see table 1 and equation (1.1)) and $\mathscr{L} > 10^{-3}$ for any smaller scale motions that might be present. For comparison we note that for $U^* \sim 10^{-3}$ m s$^{-1}$, a crude estimate of the speed of core motions based on the displacement of the magnetic field pattern at the Earth's surface, we have

$$\mathscr{R} \gtrsim 10^{-5} \tag{3.26}$$

(see equation (1.4)). Finally we note that if the electrical conductivity of the core is $10^{5\cdot5\pm0\cdot5}$ ohm$^{-1}$ m$^{-1}$ (see Gardiner and Stacey 1971) then $\eta \sim 10^{0\cdot5\pm0\cdot5}$ m$^2$ s$^{-1}$ and

$$\mathscr{C} \sim 10^{1\cdot5\pm0\cdot5} \tag{3.27}$$

(see equation (1.3)).

## 4. Oscillations and instabilities

### 4.1. *Plane waves in a uniform magnetic field*

In the case when $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = 0$ and $\nabla\rho = \partial\rho/\partial t = 0$, the equation of motion (2.19) reduces to

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\,.\nabla)\,\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u}$$

$$= -\nabla\left(\frac{p}{\rho}+\Phi\right)+\frac{1}{\mu\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\nu\nabla^2\boldsymbol{u}, \tag{4.1}$$

the other equations being

$$\nabla\,.\,\boldsymbol{u} = 0 \tag{4.2}$$

$$\partial\boldsymbol{B}/\partial t = \nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\eta\nabla^2\boldsymbol{B} \tag{4.3}$$

and

$$\nabla\,.\,\boldsymbol{B} = 0 \tag{4.4}$$

(see equations (2.20), (2.22) and (2.12)).

Before discussing wave propagation and instabilities in nonuniform or bounded systems in which dissipative effects may be present (see §§ 4.2–4.5) it is instructive to consider small-amplitude plane wave solutions to equations (4.1)–(4.4) when the fluid extends indefinitely in all directions, the basic undisturbed magnetic field $\boldsymbol{B}_0$ is uniform, the basic motion is solid-body rotation ($\boldsymbol{u} = 0$) and dissipative effects vanish (ie $\nu = \eta = 0$, but see equation (4.16)). If $\boldsymbol{u}_1(\boldsymbol{r}, t)$, $p_1(\boldsymbol{r}, t)$ and $\boldsymbol{B}_1(\boldsymbol{r}, t)$ are the components of $\boldsymbol{u}$, $p$ and $\boldsymbol{B}$ that are associated with the wave motion then to first order of smallness in $\boldsymbol{u}_1$, $p_1$ and $\boldsymbol{B}_1$ equations (4.1)–(4.4) give

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + 2\boldsymbol{\Omega}\times\boldsymbol{u}_1 = -\nabla\left(\frac{p_1}{\rho}\right)+\frac{1}{\mu\rho}(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0 \tag{4.5}$$

$$\nabla\,.\,\boldsymbol{u}_1 = 0 \tag{4.6}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = (\boldsymbol{B}_0\,.\nabla)\,\boldsymbol{u}_1 \tag{4.7}$$

$$\nabla\,.\,\boldsymbol{B}_1 = 0. \tag{4.8}$$

We can eliminate $p_1$ and $\boldsymbol{B}_1$ between these equations and find that $\boldsymbol{u}_1$ satisfies

$$\left(\frac{\partial^2}{\partial t^2}-(\boldsymbol{V}.\nabla)^2\right)\nabla\times\boldsymbol{u}_1-(2\boldsymbol{\Omega}.\nabla)\frac{\partial\boldsymbol{u}_1}{\partial t}=0 \tag{4.9}$$

if $\boldsymbol{V}$ is now the Alfvén velocity based on $\boldsymbol{B}_0$ (see equation (3.13)). This equation admits plane wave solutions of the form

$$\boldsymbol{u}_1 \propto \exp \mathrm{i}(\boldsymbol{\kappa}.\boldsymbol{r}-\omega t) \tag{4.10}$$

and the angular frequency of the waves $\omega$ is related to the wavenumber vector $\boldsymbol{\kappa}=(k,l,m)$ by the dispersion relationship

$$\omega^2 \pm (2\boldsymbol{\Omega}.\boldsymbol{\kappa})\,\omega/\kappa-(\boldsymbol{V}.\boldsymbol{\kappa})^2=0 \tag{4.11a}$$

which has solutions

$$\omega^2=(\boldsymbol{V}.\boldsymbol{\kappa})^2+\frac{1}{2}\left\{\frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\pm\left(\frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^4}{\kappa^4}+\frac{4(\boldsymbol{V}.\boldsymbol{\kappa})^2(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\right)^{1/2}\right\} \tag{4.11b}$$

(see Chandrasekhar 1961, Lehnert 1954, 1955a, cf Carstoiu 1964, Hide 1969c, Kalra 1969, Kalra and Talwar 1967, Namikawa 1955, Nigam 1965, Nigam and Nigam 1963, Rao 1971, Sakurai 1967, also equation (6.14) below).

We shall denote by $\omega_+^2$ the solution of equation (4.11*b*) when the positive sign is taken and by $\omega_-^2$ the corresponding solution when the negative sign is taken. Clearly $\omega_-^2 \leqslant (\boldsymbol{V}.\boldsymbol{\kappa})^2 \leqslant \omega_+^2$. Observe that $\omega_+^2=\omega_-^2=(\boldsymbol{V}.\boldsymbol{\kappa})^2$ when $\Omega=0$ and that $\omega_+^2$ increases and $\omega_-^2$ decreases monotonically with increasing $\Omega$ in such a way that the product

$$\omega_+^2\,\omega_-^2=(\boldsymbol{V}.\boldsymbol{\kappa})^4 \tag{4.12}$$

remains independent of $\boldsymbol{\Omega}$. Thus, when $\kappa^2(\boldsymbol{V}.\boldsymbol{\kappa})^2/(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2 \sim \mathscr{L}^2$ (see equations (1.1) and (3.23)) is much greater than unity

$$\left.\begin{aligned}\omega_+^2 &\simeq (\boldsymbol{V}.\boldsymbol{\kappa})^2\left\{1+\left|\frac{2\boldsymbol{\Omega}.\boldsymbol{\kappa}}{\kappa\boldsymbol{V}.\boldsymbol{\kappa}}\right|\right\}\\ \text{and}\qquad & \\ \omega_-^2 &\simeq (\boldsymbol{V}.\boldsymbol{\kappa})^2\left\{1-\left|\frac{2\boldsymbol{\Omega}.\boldsymbol{\kappa}}{\kappa\boldsymbol{V}.\boldsymbol{\kappa}}\right|\right\}\end{aligned}\right\} \tag{4.13}$$

the 'frequency splitting' produced by the rotation being small, $\sim\mathscr{L}^{-1}$. At the other extreme, namely $\kappa^2(\boldsymbol{V}.\boldsymbol{\kappa})^2/(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2 \ll 1$, so that $\omega_+^2 \gg \omega_-^2$, we have

$$\omega_+^2 \doteqdot (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2 \tag{4.14a}$$

and

$$\omega_-^2 \doteqdot (\boldsymbol{V}.\boldsymbol{\kappa})^4\kappa^2/(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2. \tag{4.14b}$$

The former corresponds to an ordinary inertial wave (see equation (3.7)) and the latter to a 'hydromagnetic-inertial' wave (Lehnert 1954, see also Namikawa 1955), which is characterized by 'magnetostrophic' balance of forces (see § 3.5). Both these waves, unlike Alfvén waves, are highly dispersive, with group velocity $\partial\omega/\partial\boldsymbol{\kappa}$ not in general independent of $\boldsymbol{\kappa}$.

By virtue of equations (4.2) and (4.4), $\boldsymbol{u}_1.\boldsymbol{\kappa}=\boldsymbol{B}_1.\boldsymbol{\kappa}=0$, so that particle displacements and concomitant displacements of the magnetic field lines lie in planes parallel to the wavefronts. Particle orbits are in fact circular (as evinced by an examination of equation (4.9) bearing in mind (4.10); see also figure 2), and the wave is polarized in a right- or left-handed manner according as

$$(\mathbf{\Omega}\cdot\boldsymbol{\kappa})\,\omega\{(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2-\omega^2\}$$

is positive or negative.

A property of these waves of possible fundamental significance with regard to the geomagnetic dynamo problem (see eg Moffatt 1970) is that provided $\mathbf{\Omega}\cdot\boldsymbol{\kappa}\neq 0$ each individual mode possesses nonzero *helicity* $\overline{\boldsymbol{u}_1\cdot(\nabla\times\boldsymbol{u}_1)}$ (where the overbar denotes an average over a wavelength or period). The sign of this second-order quantity (magnitude $\kappa\overline{u_1^2}$) is, like the sense of polarization, positive or negative according to the sign of $(\mathbf{\Omega}\cdot\boldsymbol{\kappa})\,\omega\{(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2-\omega^2\}$.

From equation (4.7) we find that the ratio of magnetic to kinetic energy associated with the waves satisfies:

$$\tfrac{1}{2}\mu^{-1}\boldsymbol{B}_1^2/\tfrac{1}{2}\rho\boldsymbol{u}^2 = (\boldsymbol{V}\cdot\boldsymbol{\kappa})^2/\omega^2. \tag{4.15}$$

Thus while Alfvén waves, for which $\omega^2=(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2$, are characterized by equipartition between magnetic and kinetic energy, such equipartition does not obtain when the fluid is rotating. Indeed, when $\mathscr{L}\ll 1$ the ratio on the left-hand side of equation (4.15) $\sim\mathscr{L}^2$ for the inertial waves and $\sim\mathscr{L}^{-2}$ for the hydromagnetic-inertial waves (see equations (4.14*a*,*b*)).

When effects due to viscous and ohmic dissipation are taken into account, it is readily shown that waves are damped with exponential decay time $\tau$ given in the case of light damping (ie $\tau\gg 2\pi/\omega$) by the expression

$$\tau=\frac{\omega^2+(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2}{\{\nu\omega^2+\eta(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2\}\,\kappa^2}. \tag{4.16}$$

Thus, for inertial waves (ie $\boldsymbol{V}=0$, $\mathbf{\Omega}\neq 0$) we have $\tau=1/\nu\kappa^2$, while for Alfvén waves (ie $\mathbf{\Omega}=0$, $\boldsymbol{V}\neq 0$) we have $\tau=2/(\nu+\eta)\,\kappa^2$. In contrast, for the slow 'hydromagnetic-inertial' wave with frequency $\omega\simeq\pm(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2\kappa/(2\mathbf{\Omega}\cdot\boldsymbol{\kappa})$ (see equation (4.14*b*)) and magnetic energy equal to $(2\mathbf{\Omega}\cdot\boldsymbol{\kappa})^2/(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2\kappa^2$ times its kinetic energy, the relative importance of viscous to ohmic damping $\sim\mathscr{L}^2\nu/\eta$ and the decay time $\tau=\{\eta+\nu(\boldsymbol{V}\cdot\boldsymbol{\kappa})^2\kappa^2/(2\mathbf{\Omega}\cdot\boldsymbol{\kappa})^2\}^{-1}\kappa^{-2}\sim(\eta+\nu\mathscr{L}^2)^{-1}\kappa^{-2}$. Thus, when the 'magnetic Prandtl number' $\nu/\eta$ is much less than $\mathscr{L}^{-2}$, so that viscous dissipation can be neglected, the specific attenuation coefficient for the slow waves $\tau\omega/2\pi\sim\mathscr{C}/2\pi$ (see equations (1.3) and (3.22), cf equations (3.25)–(3.27)).

Several workers have noted that the timescale associated with the geomagnetic secular variation is comparable with the period of a typical hydromagnetic-inertial wave in the core, some $10^{10}$ s (300 yr), and that damping, which is mainly due to ohmic heating, is not of primary importance. Interest in the properties of such waves in more complicated systems than the one above, when the basic magnetic and density fields are not uniform (see §§ 4.2 and 6.3) and bounding surfaces are present (see §§ 4.3–4.5), and in generation mechanisms (see §§ 4.4 and 6.5) stems largely from work on this aspect of the Earth's magnetism.

### 4.2. *Plane waves in a nonuniform magnetic field*

Suppose now that the basic magnetic field is not uniform but varies in both magnitude and direction, taking the form $\boldsymbol{B}_0=\{B_x(z),B_y(z),0\}$ (see figure 3). Suppose further that at some locality hydromagnetic waves are generated in narrow frequency and wavenumber bands $(\omega, \omega+\Delta\omega)$ and $(\kappa, \kappa+\Delta\kappa)$. If the disturbances are sufficiently small, so that they satisfy linear equations of motion, the values of $\omega$, $k$ and $l$ (see equation (4.10)) will remain in their respective bands as the waves propagate but, owing to the dependence of $\boldsymbol{B}_0$ on $z$, their wavenumber component $m$ in the direction of the magnetic field gradient will change. We enquire into the consequences of such effects on the propagation of the waves.

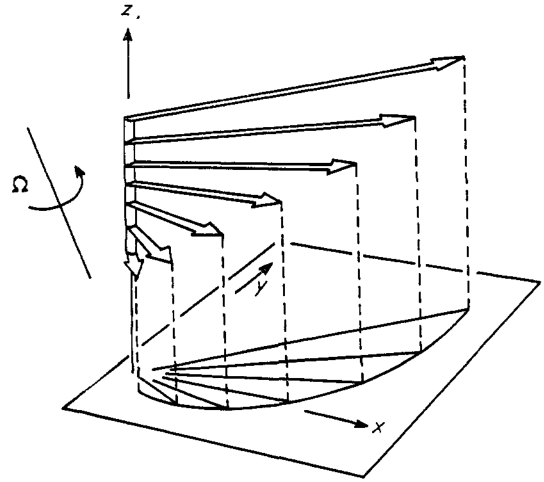


Figure 3. Schematic diagram in which the broad arrows illustrate the magnetic field configuration in the basic state. An observer rotating with angular velocity $\boldsymbol{\Omega}$ will see a steady basic magnetic field with no $z$ component but varying with $z$ in both magnitude and direction. From Acheson (1972c).

If the magnetic field, and hence the Alfvén velocity $V$, varies only slightly over distances of the order of a wavelength, then the uniform-field dispersion relation

$$\omega^2 \pm (2\boldsymbol{\Omega}\,.\,\boldsymbol{\kappa})\,\omega/\kappa - (\boldsymbol{V}\,.\,\boldsymbol{\kappa})^2 = 0 \tag{4.17}$$

may be used to compute the local vertical wavenumber $m$ at any station in terms of the local value of $V$; thus

$$[\{(\boldsymbol{V}\,.\,\boldsymbol{\kappa})^2 - \omega^2\}^2 - 4\omega^2\,\Omega_z^{\,2}]\,m^2 - 8\omega^2\,\Omega_z(\Omega_x k + \Omega_y l)\,m + (k^2+l^2)\{(\boldsymbol{V}\,.\,\boldsymbol{\kappa})^2 - \omega^2\}^2 - 4\omega^2(\Omega_x k + \Omega_y l)^2 = 0, \tag{4.18}$$

an equation quadratic in $m$. The group velocity $\boldsymbol{G}$ at every point can be computed directly from (4.17) and is given in terms of $m$ and $V$ by the usual expression

$$\boldsymbol{G} = (G_x, G_y, G_z) \equiv \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right). \tag{4.19}$$

By then combining (4.18) and (4.19) we find on selecting a particular wave group (characterized by particular values of $\omega$, $k$ and $l$) a relationship giving its propagation velocity at every point in terms of the local magnetic field.

If the magnetic field were uniform the vector $\boldsymbol{G}$ would, of course, be constant and wave groups would propagate in straight lines. Further, in the absence of rotation they would propagate along the lines of force, as evinced by setting $\boldsymbol{\Omega} = 0$ in (4.17) and computing $\boldsymbol{G}$ by means of (4.19). The presence of rotation, however, permits waves to propagate *across* the lines of force and if the magnetic field varies with $z$ then so will $G$ and the ray paths will be curved.

Note that as the group approaches a 'critical level' $z = z_c$ at which the Alfvén velocity takes a special value $V_c \equiv V(z_c)$ such that

$$\{(V_c . \kappa)^2 - \omega^2\}^2 = 4\omega^2 \Omega_z^2 \tag{4.20}$$

one root of (4.18) increases indefinitely and is given asymptotically by

$$[\{(V . \kappa)^2 - \omega^2\}^2 - 4\omega^2 \Omega_z^2]\, m \simeq 8\omega^2 \Omega_z(\Omega_x k + \Omega_y l). \tag{4.21}$$

The asymptotic behaviour of $G_z$ corresponding to this root is given by

$$m^2\{(V_c . \kappa)^2 + \Omega_z^2\}^{1/2} G_z = \omega(\Omega_x k + \Omega_y l) \operatorname{sgn} [\Omega_z\{(V_c . \kappa)^2 - \omega^2\}]. \tag{4.22}$$

On expanding $V(z)$ in a Taylor series about $z = z_c$ we find from (4.21) that $|m|$ increases indefinitely as $|z - z_c|^{-1}$. Thus $G_z$ decreases in the neighbourhood of the critical level as $|z - z_c|^2$. This group, which is (according to (4.22)) travelling in a direction such that

$$G_z \Omega_z(\Omega_x k + \Omega_y l)\, \omega\{(V_c . \kappa)^2 - \omega^2\} > 0,$$

consequently slows down in such a way that it cannot reach its critical level in a finite time and is in fact guided along the lines of force at the critical level. The other root of (4.18) tends to a finite value $m_c$ as the critical level is approached and the corresponding $G_z$ tends to a nonzero value such that

$$G_z \Omega_z(\Omega_x k + \Omega_y l)\, \omega\{(V_c . \kappa)^2 - \omega^2\} < 0.$$

As explained in the previous section the propagation of 'hydromagnetic-inertial' waves characterized by large values of $(V . \kappa)^2/\omega^2$ is of especial interest. According to the theory outlined above, as each wave propagates in a nonuniform magnetic field it can only penetrate its 'critical level' $z = z_c$, where the Alfvén velocity takes a value given by

$$(V_c . \kappa)^4 \simeq 4\omega^2 \Omega_z^2, \tag{4.23}$$

if it approaches $z = z_c$ from that side (ie $z > z_c$ or $z < z_c$) for which

$$G_z \Omega_z(\Omega_x k + \Omega_y l)\, \omega < 0 \tag{4.24}$$

(Acheson 1972c). If it approaches from the other side, rather than being reflected the wave is 'captured', its energy being guided along the lines of force at the critical level. Thus, so far as a given wave is concerned, the critical level acts like a valve, allowing energy to cross in one direction only (see figure 4).

The foregoing analysis provides no indication of how short the wavelengths $\lambda = 2\pi/\kappa$ involved should be (in comparison with the scale length $L$ characterizing magnetic field variations) for the approximate validity of this picture. Furthermore the method predicts that the energy density of a wave packet in the process of being 'captured' at a critical level increases without limit as $(z - z_c)^{-2}$, so that the assumption of small wave amplitude on which the whole approach is based ultimately breaks down. Nevertheless a more refined analysis (Acheson 1972c), using techniques developed by Booker and Bretherton (1967) confirms the broad aspects of the above picture and shows that the 'valve' effect persists even if $\epsilon \equiv \lambda/L$ is only marginally less than unity. Only a small fraction, notably

$$\exp\left(-2\pi\left|\frac{(\Omega_x k + \Omega_y l)}{\Omega_z} \frac{(V_x k + V_y l)^2}{\mathrm{d}(V_x k + V_y l)^2/\mathrm{d}z}\right|_{z=z_c}\right),$$

of the energy of a 'slow' wave approaching from the 'wrong' side such that $G_z \Omega_z(\Omega_x k + \Omega_y l)\omega > 0$ then penetrates the critical level.

It is important to note that in selecting a nonuniform magnetic field $\boldsymbol{B}_0 = \{B_x(z), B_y(z), 0\}$ we excluded a component $B_z$ in the direction of the field gradient. If we were to include such a component (4.18) would take the form of a quartic for $m$ and the roots would display no singular behaviour at any level. This does not, of course, automatically imply that the critical level properties described above are pathological, for (4.18) holds strictly only in the asymptotic limit $\epsilon \rightarrow 0$.

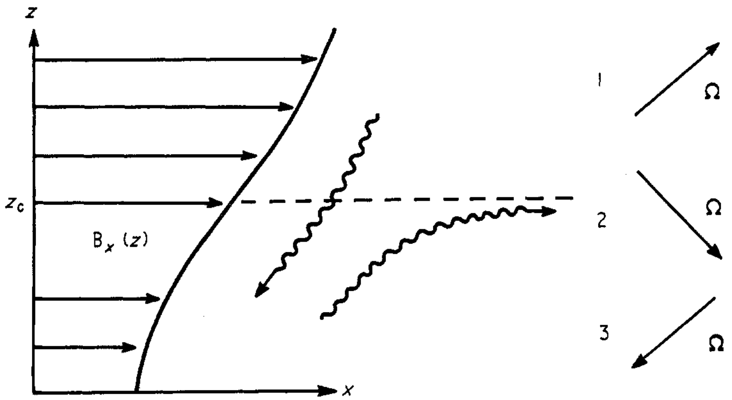


Figure 4. A simple example of a hydromagnetic-inertial wave group meeting a critical level. The system is supposed two-dimensional so that waves propagate in the $(x, z)$ plane only. A wave group with frequency $\omega$ and wavenumber $k$ in the $x$ direction will only penetrate its critical level if $G_z \Omega_z \Omega_x k\omega < 0$. Thus, when $\Omega_z \Omega_x > 0$ (as in systems 1 and 3) the wave can penetrate the critical level only from above if $\omega k > 0$ (the case illustrated) and only from below if $\omega k < 0$. These directions of penetration are reversed when $\Omega_z \Omega_x < 0$, as in system 2. From Acheson (1972c).

In practice one expects that if $B_z$ is sufficiently small compared with $B_x$ and $B_y$ it will enable *some* transmission in *both* directions (just as a field $\{B_x(z), B_y(z), 0\}$ does except in the short wavelength limit) but that the overall 'valve' effect will persist. An investigation of just *how* small $B_z$ must be would nevertheless appear to be one of the most important next steps in evaluating how significant these phenomena may be in the upper reaches of the Earth's core, say, where the strength of the toroidal field increases with depth from a value that is much less than the poloidal field of $\sim 5$ gauss at the core–mantle boundary to a value $\sim 100$ gauss deep within the core.

When considering the possibility of some such 'valve' effect in the core (see eg Moffatt 1972), or indeed in any other natural system, it is also important to recognize that both viscous and ohmic dissipation have been hitherto neglected, as in other recent investigations of hydromagnetic critical levels by McKenzie (1972 to be published) and Rudraiah and Venkatachalappa (1972). Elementary calculations by the authors based on the application of equation (4.16), for example, suggest that the ratio $|m|/(k^2 + l^2)^{1/2}$ associated with a 'slow' hydromagnetic wave in the process of being 'captured' at its critical level increases only to a value of order $\mathscr{C}^{1/2}$ before effects due to ohmic dissipation become important (cf equation (3.27)).

The sheared magnetic field in the above system, in which there is no shear in the basic flow, has field lines that are straight and it may be shown (Acheson 1972b) that this configuration is stable to small perturbations (inasmuch as 'resistive instabilities', in which critical layers are expected to play an altogether different role, can be neglected; see Baldwin and Roberts (1972)). In the following two sections we discuss some cases in which, owing in part to the curvature of the magnetic field lines, small perturbations can extract energy from the magnetic energy of the basic state. Attention is mainly confined to the case when the magnetic field is *azimuthal*, which is of especial geophysical interest (see § 3.5). At low values of the parameter $\mathscr{S}$ both the conditions under which such disturbances grow in amplitude and the character they subsequently assume (aperiodic or oscillatory) then depend crucially on whether or not they are symmetric about the rotation axis (although this will not necessarily be the case when the magnetic field configuration is more complicated, see Acheson (1972b)).

### 4.3. *Axisymmetric instabilities*

As a preliminary to the treatment of a certain class of hydromagnetic instabilities arising from the spatial variations of an azimuthal magnetic field we investigate first the related problem of the instability of a circular vortex ('Couette flow').

Consider the steady shear flow of an incompressible inviscid fluid between two rotating cylinders. Any azimuthal flow $(u_r, u_\theta, u_z) \equiv \{0, U_\theta(r), 0\}$ relative to a *stationary* set of cylindrical polar coordinates $(r, \theta, z)$ is then permissible, but in the absence of hydromagnetic effects the flow is stable if and only if

$$\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2 \geqslant 0 \tag{4.25}$$

everywhere, a result first obtained by Rayleigh (1920). A distribution of angular momentum increasing outwards is therefore stable and *vice versa.*

We now investigate the influence of hydromagnetic effects on this system, assuming that the fluid is a perfect conductor of electricity. In order not to obscure the comparison between the results of this section and those of the next (which provides several illustrations of the basic concepts introduced in § 3.5) we shall discuss only briefly effects due to viscous and ohmic dissipation. A full account of these effects would greatly complicate the discussion, even if attention were restricted to axisymmetric disturbances (see Chandrasekhar 1953, 1961, Edmonds 1958, Kurzweg 1963, Lai 1962, Pao 1966, Yih 1959), there being no comparable study of the nonaxisymmetric case (but see Chang and Sartory 1967 and Roberts 1964).

It is instructive to consider first the influence of an *axial* and *uniform* magnetic field $\boldsymbol{B}_0 = (0, 0, B_z)$, a problem treated by Chandrasekhar (1960) and Velikhov (1959). Both authors demonstrate that instability due to the nature of the shear flow, when it occurs, is aperiodic and that if

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta}{r}\right)^2 \geqslant 0 \tag{4.26}$$

everywhere, ie if the *angular velocity* increases outwards, then there are no amplifying modes (and this is, of course, also true in the absence of hydromagnetic effects, see equation (4.25)). Analysis of the case in which the gap width $d = r_2 - r_1$ between the cylinders is much less than their mean radius $R = \frac{1}{2}(r_1 + r_2)$ (Velikhov 1959, see also Acheson 1972b and § 4.4) shows that the (not necessarily real) frequency $\omega$ of a disturbance with radial and axial wavenumbers $l$ and $n$ respectively is given by

$$\left(1+\frac{l^2}{n^2}\right)\omega^4-\left[2V_z^2(l^2+n^2)+\left\{\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2\right\}_{r=R}\right]\omega^2$$

$$+n^2V_z^2\left[(l^2+n^2)V_z^2+\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta^2}{r^2}\right)\right\}_{r=R}\right]=0 \tag{4.27}$$

where $V_z$ is the Alfvén speed based on $B_z$ (see equation (1.2)) and (in view of the boundary conditions of zero normal velocity at the cylinder walls) $ld$ is an integral multiple of $\pi$. Such a disturbance will therefore either oscillate about the equilibrium position without amplifying or grow aperiodically according as

$$L\equiv(l^2+n^2)V_z^2+\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta^2}{r^2}\right)\right\}_{r=R} \tag{4.28}$$

is positive or negative. This illustrates the sufficient condition for stability (4.26).

In the main, therefore, the magnetic field plays a stabilizing role by imparting an 'elasticity' to the fluid through the hydromagnetic restoring force. Note, however, that an apparent paradox arises as the magnetic field tends to zero, for according to equation (4.28) the profile $U_\theta \propto r^{1/2}$, for example, will be unstable if $V_z$ is sufficiently small whereas according to Rayleigh's criterion (4.25) (which we recover, incidentally, on setting $V_z=0$ in equation (4.27)) such a flow is stable in the absence of a magnetic field! In view of the discussion presented in § 3.5 this apparent paradox is evidently a consequence of the possibility of 'magnetostrophic' balance, *however small* $V_z$, due to the perfect conductivity of the fluid. When $V_z$ is very small the roots of equation (4.27) become widely separated, taking the asymptotic form (provided neither $U_\theta/r$ nor $rU_\theta$ is constant)

$$\omega^2\simeq\left(\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2\right)_{r=R}\left(1+\frac{l^2}{n^2}\right)^{-1} \tag{4.29}$$

or

$$\omega^2\simeq n^2V_z^2\left\{r^4\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta^2}{r^2}\right)\bigg/\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2\right\}_{r=R} \tag{4.30}$$

(cf the 'inertial' and 'hydromagnetic-inertial' waves of § 4.1). Thus, while it is indeed possible for a disturbance to amplify provided only that the *angular velocity* decreases with radius, note that its *growth rate* decreases with $V_z$ so that in the limit $V_z\to 0$ any small but finite dissipation will suppress this particular mode of amplification. Such dissipative effects, if small, will however only slightly modify equation (4.29) and the stability or otherwise of the system will then be decided essentially by Rayleigh's criterion (4.25).

Chandrasekhar (1953) has examined the above problem taking viscous and ohmic effects into account. He finds that if $V_z$ is sufficiently large the radial gradient of angular momentum needed for instability is given asymptotically by

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2\right)_{r=R}=-D^2V_z^2\frac{\nu}{\eta} \tag{4.31}$$

where the constant of proportionality $D^2$ depends on the gap width $d$ and the mean radius $R$ (it being assumed here that $d\ll R$) as well as on the electromagnetic boundary conditions. Note that the angular momentum gradient needed for instability increases with both $V_z$ and the electrical conductivity $\sigma$. These results are consistent with the experiments of Donnelly and Ozima (1960). Equation (4.31) was derived by Chandrasekhar (1961) on the assumption that $\nu \ll \eta$. According to the work of Kurzweg (1963) the nondissipative results of Velikhov (1959) discussed above (see equation (4.28)) will approximately apply in the opposite extreme, namely $\nu \gg \eta$, provided that both $\nu$ and $\eta$ are sufficiently small.

As we might expect, the apparent paradox encountered above does not arise if the magnetic field $\boldsymbol{B}_0 = \{0, B_\theta(r), 0\}$ is purely *azimuthal*, for while the field lines can then be transported bodily with the fluid and stretched they cannot be *distorted* by axisymmetric disturbances. It is also natural that without the restoring force due to twisting of the 'equivalent elastic strings' the field should not necessarily play a general stabilizing role. Both these results emerge from the work of Michael (1954, cf Schubert 1968, Trehan and Nakagawa 1969), who showed that the flow is stable if and only if

$$\Psi \equiv \frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}(rU_\theta)^2 - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_\theta^2}{r^2}\right) \geqslant 0 \tag{4.32}$$

is satisfied everywhere in the fluid (cf equation (4.25)), the instability (when it occurs) being aperiodic. Here $V_\theta(r) \equiv B_\theta(r)/(\mu\rho)^{1/2}$ is the local Alfvén speed. (We note in passing an elegant extension of Michael's work by Howard and Gupta (1962) to include effects of a superimposed axial flow $U_z(r)$. A sufficient condition for stability is then that $r\Psi - \frac{1}{4}(\mathrm{d}U_z/\mathrm{d}r)^2 \geqslant 0$ everywhere, and there is also a simple upper bound on the growth rates $\omega_\mathrm{I}$, namely $\omega_\mathrm{I}^2 \leqslant \max\{\frac{1}{4}(\mathrm{d}U_z/\mathrm{d}r)^2 - r\Psi\}$. Instability may be oscillatory in nature when $U_z \neq 0$; the complex wave speed $\omega/n$ must then lie within the semicircle in the upper half-plane with the range of $U_z$ for diameter.)

A special case of particular interest is that of rigid-body rotation $U_\theta = \Omega r$, which is stable if and only if

$$\frac{4\Omega^2}{r} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_\theta^2}{r^2}\right) \geqslant 0 \tag{4.33}$$

everywhere. An increase of magnetic field with distance from the axis of rotation evidently has a destabilizing influence and *vice versa*, and it is natural to consider the physical interpretation of the second term in equation (4.32).

To isolate the instability mechanism due to spatial variations in the magnetic field let us first set $U_\theta = 0$. Consider the interchange of two thin rings of fluid whereby a ring at $r = r_a$ permeated by an azimuthal magnetic field $B_a$ is moved to $r = r_b$ and is replaced by a ring initially at $r = r_b$ permeated by a magnetic field $B_b$. If the cross-sectional area (in a meridional plane) of a ring be denoted by $\Delta$ the magnetic flux threading it is $B\Delta$, and this must remain constant by virtue of the perfect conductivity of the fluid. Since the fluid is incompressible the ring's volume $2\pi r\Delta$ must also remain constant as the ring moves, so that $B/r$ is a conserved quantity for each ring. The magnetic fields permeating the rings in their new positions are thus given by $B_a^\dagger = B_b r_a/r_b$ and $B_b^\dagger = B_a r_b/r_a$, and the concomitant increase in magnetic energy is proportional to $B_a^{\dagger 2} + B_b^{\dagger 2} - B_a^2 - B_b^2$, which may therefore be written $-(r_b^2 - r_a^2)(B_b^2/r_b^2 - B_a^2/r_a^2)$. This shows that unless $B_\theta^2/r^2$ is everywhere a decreasing function of $r$ it is possible to select rings for which the magnetic energy change is negative, corresponding to instability. This argument is, of course, entirely analogous to Rayleigh's (1920) well known physical explanation of the axisymmetric instability of a circular vortex $U_\theta(r)$ (see equation (4.25)), in which case it is angular momentum rather than magnetic flux that is the conserved quantity. Indeed, when the two arguments are combined—as they may be since axisymmetric disturbances in this case produce no azimuthal Lorentz force and hence (in the absence of diffusive effects, see below) no change in angular momentum of the material fluid ring—we can readily derive the criterion given by equation (4.32).

Rigid-body rotation evidently acts as a stabilizing influence to axisymmetric disturbances (as evinced by equation (4.33)) and we note in particular that a 'rapidly' rotating fluid ($\mathscr{L} \ll 1$) is not susceptible to instability resulting from spatial variations in an azimuthal magnetic field unless the field gradient is extremely large. The reason for this is that at low values of $\mathscr{L}$ 'magnetostrophic' balance cannot obtain (since the lines of force are not twisted and the hydromagnetic restoring torque not brought into play) and the stabilizing influence of the Coriolis force prevails (cf §6.4). This is again evinced by the result of a 'narrow-gap' analysis (Acheson 1972a,b, see also Schubert 1968)

$$\omega^2 \simeq \left[4\Omega^2 - \left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_\theta^2}{r^2}\right)\right\}_{r=R}\right](l^2+n^2)^{-1} \tag{4.34}$$

(cf equation (4.27)), for as $\Omega$ increases equation (4.34) steadily approaches the dispersion relationship for a pure 'inertial' wave for which only the Coriolis force is important (see equation (4.14*a*)). Equivalently, but in terms of energetics, the interchange of two material rings which conserve their respective angular momenta would require an input of kinetic energy greatly in excess (by a factor $\gtrsim \mathscr{L}^{-2}$) of the energy extractable from any unstable configuration of $B_\theta$ with moderate gradient (eg $B_\theta \propto r^2$).

Yih (1959), Lai (1962) and Pao (1966) have shown, however, that effects due to viscosity and electrical resistivity can be *destabilizing* (see also Schubert 1968) and that for arbitrary values of the magnetic Prandtl number $\nu/\eta$ stability to axisymmetric disturbances is only assured if $\mathrm{d}(V_\theta^2/r^2)/\mathrm{d}r \leqslant 0$ everywhere, a condition which places a much more severe constraint on the magnetic field profiles for which the system is stable than does the nondissipative constraint of Michael (1954, see equation (4.33)), especially when the fluid is rotating very rapidly (ie $\mathscr{L} \ll 1$). Pao (1966), for example, gives

$$4\Omega^2 - \frac{\nu}{\eta} r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_\theta^2}{r^2}\right) > 0 \tag{4.35}$$

as a sufficient condition for stability to axisymmetric disturbances (cf equation (4.31) and note, incidentally, how the combination $\mathscr{L}^2 \nu/\eta$ arises in both cases—cf §4.1). Thus the electrical resistance of the fluid opposes the destabilizing influence of a radial increase of the magnetic field, but viscous effects support it. Interpreting equation (4.35) in terms of the arguments of Lai (1962), the direct (damping) effect of viscosity on the disturbances would seem to be more than offset by its action as an agent for diffusion of momentum, which reduces the stabilizing effect of rigid-body rotation by enabling the circulation of a displaced ring of fluid to harmonize more readily with its surroundings.

In the next section we shall see how in the absence of such diffusive agencies *the azimuthal magnetic field itself* can significantly relax the stabilizing influence of a fluid's rapid uniform rotation *when the lines of force are twisted by the disturbances.*

*4.4. Nonaxisymmetric instabilities and wave generation*

We shall first focus attention on the case in which the undisturbed motion of the fluid is one of rigid-body rotation, so that the only source of energy for instability lies in the spatial distribution of the azimuthal magnetic field $\boldsymbol{B}_0 = \{0, B_\theta(r), 0\}$ (see figure 5). In contrast to the previous section, it will be convenient to view non-axisymmetric disturbances to this system from a *rotating* set of cylindrical polar coordinates $(r, \theta, z)$, relative to which the fluid appears stationary in its undisturbed state. We assume that the density of the fluid is uniform (but see § 6.5) and that both viscous and ohmic dissipation are negligible. Small perturbations to this

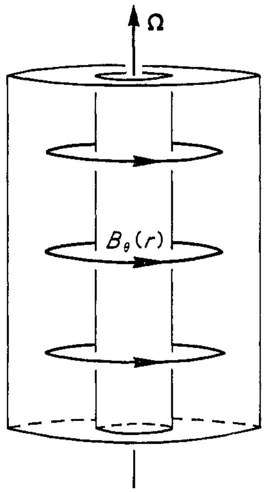


Figure 5. Illustrating the basic magnetic field configuration in a uniformly rotating fluid annulus when $B_r = B_z = 0$ and $B_\theta = B_\theta(r)$.

basic state are then governed by linear equations derived from the basic equations in § 2.5 and we can consider disturbances of the form (some function of $r$) $\times$ $\exp \mathrm{i}(m\theta + nz - \omega t)$ where, by hypothesis, $n$ and $m$ are real. Eliminating all perturbation variables except the radial velocity component $u$, we find (Acheson 1972a)

$$\left(\frac{r^3 P u'}{r^2 + m^2 n^{-2}}\right)' + \frac{r^3 Q u}{r^2 + m^2 n^{-2}} = 0 \tag{4.36}$$

where

$$P(r) \equiv \frac{m^2 V_\theta^2}{r^2 \omega^2} - 1 \tag{4.37}$$

$$V_\theta(r) \equiv B_\theta(r)/(\mu\rho)^{1/2} \tag{4.38}$$

$$\begin{aligned} Q(r) \equiv {} & \frac{n^2 r}{\omega^2}\left(\frac{V_\theta^2}{r^2}\right)' + 4\left(\frac{\Omega}{\omega} + \frac{m V_\theta^2}{r^2 \omega^2}\right)^2 \frac{n^2}{P} \\ & + 4\left(\frac{\Omega}{\omega} + \frac{m V_\theta^2}{r^2 \omega^2}\right) \frac{m}{r^2 + m^2 n^{-2}} \\ & - \frac{P\{r^2 n^2 + 1 + 2m^2 + m^2 n^{-2} r^{-2}(m^2 - 1)\}}{r^2 + m^2 n^{-2}} \end{aligned} \tag{4.39}$$

and primes denote differentiation with respect to $r$. Our aim is to examine the nature of the eigenvalues $\omega$ of (4.36) when coupled with the boundary conditions of zero normal velocity at the cylindrical boundaries, that is, $u(r_1) = u(r_2) = 0$. We wish, in particular, to elicit any properties of the eigenvalues which do not depend critically on the fine details of the magnetic field distribution.

Multiplying (4.36) by $u^*$ (the complex conjugate of $u$) and integrating between the boundaries $r = r_1$ and $r = r_2$ we thus find

$$\int_{r_1}^{r_2} \frac{r^3}{r^2+m^2 n^{-2}} (Q(r)|u|^2 - P(r)|u'|^2)\,\mathrm{d}r = 0. \tag{4.40}$$

Since the eigenvalues $\omega = \omega_{\mathrm{R}} + \mathrm{i}\omega_{\mathrm{I}}$ may be complex (where suffixes denote real and imaginary parts), so may $Q(r)$ and $P(r)$. Multiplying (4.40) by $\omega^2$ and equating the real and imaginary parts of the resulting left-hand side to zero we find (from the imaginary part)

$$\omega_{\mathrm{R}}\,\omega_{\mathrm{I}} \int_{r_1}^{r_2} \frac{r^3}{r^2+m^2 n^{-2}} (|u'|^2 + F(r)|u|^2)\,\mathrm{d}r = 0 \tag{4.41}$$

where

$$F(r) = \frac{r^2 n^2 + 1 + 2m^2 + m^2 n^{-2} r^{-2}(m^2-1) + 2\Omega m/\omega_{\mathrm{R}}}{r^2 + m^2 n^{-2}}$$
$$+ \frac{4V_\theta^2 m^2 n^2}{r^4 |\omega^2 P|^2} \left\{ V_\theta^2 + \Omega^2 r^2 + \frac{\Omega}{m\omega_{\mathrm{R}}} (m^2 V_\theta^2 + r^2 \omega_{\mathrm{R}}{}^2 + r^2 \omega_{\mathrm{I}}{}^2) \right\}. \tag{4.42}$$

Suppose now that a nonaxisymmetric disturbance is unstable (ie $\omega_{\mathrm{I}} > 0$) but that it does not propagate (ie $\omega_{\mathrm{R}} = 0$). Then (4.41) becomes

$$\Omega\omega_{\mathrm{I}}\, m \int_{r_1}^{r_2} \frac{r^3 |u|^2}{r^2+m^2 n^{-2}} \left( \frac{1}{r^2+m^2 n^{-2}} + \frac{2V_\theta^2 n^2}{V_\theta^2 m^2 + r^2 \omega_{\mathrm{I}}{}^2} \right) \mathrm{d}r = 0 \tag{4.43}$$

and we immediately have a contradiction, for the integrand is positive throughout the interval and the integral cannot therefore vanish. We thus find that (in contrast to the axisymmetric case) the nonaxisymmetric unstable modes must propagate. Further, from (4.41) it is evident that they must be characterized by negative values of $c_\theta\,\Omega$ (where $c_\theta = \omega_{\mathrm{R}}/m$ is the azimuthal angular phase velocity of the waves), for otherwise a similar contradiction arises from the fact that $F(r)$ would then be positive throughout the interval. Thus all unstable modes generated in a uniformly rotating fluid by nonuniformities of the basic azimuthal magnetic field must propagate against the basic rotation, that is, '*westward*'. This result is not only insensitive to the details of the magnetic field profile but also seems fairly insensitive to variations in *direction* as well as magnitude of the imposed field (Acheson 1972a). Thus if an axial magnetic component $B_z(r)$ is also present the above result still holds provided that $|mB_\theta/r| > |nB_z|$ everywhere. The existence of eastward propagating unstable modes in certain circumstances when this inequality is violated may, however, be readily demonstrated (Acheson 1972b).

We must now discuss the conditions required for the existence of such non-axisymmetric unstable modes (cf equation (4.33)). It is readily shown by the techniques used above, applied directly to (4.40) without first multiplying by $\omega^2$, that unstable nonaxisymmetric modes such that $|m| > 1$ and $V_\theta^2 m^2 > 3r^2 \omega_{\mathrm{R}}{}^2$ everywhere (which include, for example, the important 'hydromagnetic-inertial' waves) can only occur if $V_\theta^2/r^2$ *increases with radius somewhere in the interval* $r_1 \leqslant r \leqslant r_2$. A positive gradient of $V_\theta^2/r^2$ is, of course, also necessary for the existence of any amplifying *axisymmetric* modes (see equation (4.33)), but at low values of $\mathscr{L}$ the *magnitudes* of the gradient required in the two cases greatly differ (by a factor of order $\mathscr{L}^{-2}$), as demonstrated by the following specific example.

Suppose the radii of the cylinders are nearly equal so that the gap width $d = r_2 - r_1$ is very much less than the mean radius $R = \frac{1}{2}(r_1 + r_2)$ and suppose also that the magnetic field varies by a factor of order unity over a distance of order $R$ (so that it varies by only a small amount in the gap between the cylinders). It has been shown (Acheson 1972a,b) that in the special case when both radial *and axial* wavelengths are of order $d$ (the radial wavenumber $l$ satisfying $ld = N\pi$ where $N$ is an integer of order unity) the eigenvalues $\omega$ of the slow oscillations that take place when $\Omega^2 R^2 \gg V_\theta^2(R)$ are given by the expression

$$\frac{2\omega\Omega R^2}{mV_\theta^2(R)} \simeq -2 \pm \left\{m^2\left(1 + \frac{l^2}{n^2}\right) - \left(\frac{r(V_\theta^2/r^2)'}{(V_\theta^2/r^2)}\right)_{r=R}\right\}^{1/2}. \tag{4.44}$$

The azimuthal angular phase velocity is thus of order $V_\theta^2/\Omega R^2$ (characteristic of 'slow' hydromagnetic waves in a 'rapidly' rotating fluid—see § 4.1) and for unstable modes is 'westward' in accord with the general result outlined above. The destabilizing influence of a radial increase of $V_\theta^2/r^2$ is clearly illustrated by (4.44), but note that the modes with higher azimuthal wavenumbers $m$ are less likely to grow in amplitude, for the more the lines of force are twisted the greater the *restoring* force of the 'equivalent elastic strings'. It is also natural that disturbances with higher radial wavenumbers $l$ are less likely to be unstable, for in view of their smaller radial dimensions they will 'feel' the destabilizing effect of a radial magnetic field gradient correspondingly less.

We note especially that while (on a nondissipative theory) a rapidly rotating fluid annulus permeated by an azimuthal magnetic field is stable to axisymmetric disturbances unless

$$\frac{r(B_\theta^2/r^2)'}{B_\theta^2/r^2} > \frac{4\Omega^2 r^2}{V_\theta^2} \gg 1 \tag{4.45}$$

somewhere (cf equation (4.33)) it may evidently be unstable to *nonaxisymmetric* disturbances even if the field gradient is only strong enough to satisfy

$$\frac{r(B_\theta^2/r^2)'}{B_\theta^2/r^2} > \mathrm{O}(1) \tag{4.46}$$

(cf (4.44)). Thus there is no analogue of Squire's (1933) theorem for the stability of plane parallel shear flows operating here—the system is much more unstable to three-dimensional disturbances than it is to two-dimensional disturbances (cf Ludwieg 1962, Pedley 1968). Equation (4.44) shows that whether or not the 'slow' modes increase in amplitude with time is independent of $\Omega$ (except inasmuch as $\mathscr{L}$ must be small for the oscillations to have such a long period). Increasing the rotation rate indefinitely will produce a continuing decrease in the phase velocity of the waves but will not necessarily guarantee stability! Once again the hydromagnetic effects are not necessarily masked, no matter how rapid the rotation, for they can always adjust themselves (through the mechanism of slow wave propagation) so that the Lorentz and Coriolis forces remain comparable. In practice it is likely that any small but finite electrical resistance of the fluid will (if the rotation rate is large enough) damp the disturbances so heavily that the system will be stabilized, for the parameter $\mathscr{C} \equiv V^2/2\Omega\eta$ must eventually become small compared with unity (see § 3.5). The role played by transport processes in this problem of stability with respect to *nonaxisymmetric* disturbances has, however, not yet been fully studied (but see Chang and Sartory 1967 and Roberts 1964).

In the next section we shall discuss various properties of hydromagnetic waves in spherically, rather than cylindrically, bounded rotating systems and it is therefore of interest to consider the extent to which the results derived above carry over to such systems. Malkus (1967a), in his investigation of the hydromagnetic oscillations of a rotating fluid sphere, took as the basic state $(B_\theta^2/r^2)' = 0$ corresponding to a uniform electric current along the rotation axis. If the criterion stated above that unstable nonaxisymmetric '*slow*' modes *with* $|m| > 1$ can only occur if $(B_\theta^2/r^2)'$ is somewhere positive were to hold for a spherically bounded system we would anticipate that any unstable modes found by Malkus would have to break one of these two restrictions. In fact they break both—Malkus showed that only the $|m| = 1$ modes could be unstable and that they could not take the form of 'slow' oscillations for they could only occur if $\Omega^2 r^2/V_\theta^2 \leqslant 1$. Inspection of the equation for the eigenvalues derived by Malkus (see (4.67)) shows that such modes propagate '*westward*', providing further indication that the effects of cylindrical and spherical boundaries on these unstable waves may not be too different (but cf § 4.5).

Instabilities in a rotating sphere permeated by an azimuthal magnetic field, including certain features of the finite-amplitude development of the waves, have been considered further by Roberts and Soward (1972), following an approach due originally to Braginsky (1967b). They take into account the additional effects of nonuniform rotation and variations with both $r$ *and* $z$ in the equilibrium state. Thus relative to the rotating frame there is both an azimuthal magnetic field $B_\theta(r, z)$ and an azimuthal flow $U_\theta(r, z)$. To attempt a correlation between their results and those described above we first set $U_\theta = 0$ in which case it is found that

$$\frac{\partial}{\partial r}(rB_\theta^2) < 0 \tag{4.47}$$

everywhere is a sufficient condition for stability.

When the basic flow departs from uniform rotation by an azimuthal velocity profile $U_\theta(r, z)$ it is found that the stability of the flow to nonaxisymmetric disturbances is largely dependent on the radial gradient of

$$\frac{V_\theta^2}{r^2} - \frac{2\Omega U_\theta}{r} - \frac{U_\theta^2}{r^2} \tag{4.48}$$

(Roberts and Soward 1972). If the basic state departs only slightly from rigid body rotation (so that $|U_\theta| \ll |2\Omega r|$) then it is the radial gradient of $V_\theta^2/r^2 - 2\Omega U_\theta/r$ which is important. The radial gradient of (4.48) is also crucial to the character of the equilibrium of a *cylindrically* bounded system (Acheson 1972b, cf Pathak and Pathak 1971). An outward increase of the angular velocity evidently has a stabilizing influence and *vice versa*. It has been shown that none of the unstable modes in the cylindrical system (whether generated by an unstable magnetic field configuration, velocity configuration or both—see also § 6.5) can propagate azimuthally faster than the maximum fluid angular velocity (Acheson 1972b), and this also appears to be true for certain spherically bounded systems (Booker 1972 private communication).

When the angular velocity is uniform this result reduces to the 'westward' propagation demonstrated above. Clearly one may only continue to speak of 'eastward' and 'westward' propagation *relative to the fluid as a whole* if the angular velocity deviations $U_\theta/r$ from rigid body rotation are somewhat smaller than the azimuthal phase speeds of the slow waves. In that case (ie $|U_\theta/r|$ rather less than $|V_\theta^2/2\Omega r^2|$) we see from (4.48) that the sign of the radial gradient of $V_\theta^2/r^2$ will continue to be the crucial factor upon which the stability of the system depends.

Strong internal *shear layers* with large local gradients of $U_\theta/r$ are characteristic general features of flows in rapidly rotating fluids, and these layers can in some circumstances develop quasi-two-dimensional wave-like instabilities (Busse 1968a,b, Hide and Titman 1967, Greenspan 1968). Malkus (1967b) has shown that in the hydromagnetic case the waves that arise in this way would at low values of $\mathscr{L}$ be of the hydromagnetic-inertial type and propagate slowly in a westward direction relative to the local flow.

The general westward drift of the pattern of the geomagnetic field at a fraction of a degree of longitude per year makes a significant contribution to the geomagnetic secular variation (see eg Yukutake 1970 and also figure 1). The proposal that this drift is a manifestation of slow waves in the core gains support from the main result of this section (see also §4.1), namely that one plausible generation mechanism would selectively produce westward propagating waves. The same result applies when instability is due to buoyancy forces (see §6.5), but effects due to nearly spherical boundaries must be fully considered before the theory can be applied with any confidence to the core. Such boundaries can introduce selection mechanisms of a different kind, as the study of free hydromagnetic oscillations of a rotating spherical shell of inner radius $a$ and outer radius $b$ has shown (see §4.5 below), but in the case of a thick shell, for which $b-a \not\ll b$, it is possible that the main effect of the boundaries would be to give a quasi-cylindrical structure (see figure 8($b$)) to the basic velocity and magnetic fields (Hide 1966a).

Little is known about oscillations of spherical fluid shells of finite thickness but the above mentioned work and studies by Stewartson (1967, see also Hide and Stewartson 1972) indicate that they are likely to be very complicated. As Stewartson points out, further progress will require basic mathematical studies of the governing differential equation (4.64), which is hyperbolic in type, under boundary conditions of the Dirichlet–Neumann type (see equation (4.66)). These mathematical difficulties have led Stewartson to question the validity of concepts such as group velocity when dealing with oscillations governed by such equations and boundary conditions. References to such concepts in the following section are therefore to be treated with some caution.

### 4.5. *Oscillations of spherical systems*

It will be convenient to consider first the problem of a thin shell ($b-a \ll b$), for which Hide (1966a) has proposed a local dispersion relationship on the basis of a model equivalent (see below) to the Rossby–Haurwitz 'beta plane' (see figure 6) used in dynamical meteorology and oceanography. Suppose that the 'beta plane' is tangential to the sphere at latitude $\psi = \bar{\psi}$ and denote by $(x, y)$ the local eastward and northward cartesian coordinates. When the basic magnetic field $\boldsymbol{B}$ is independent of $x$ and $y$ and any shear of the basic fluid flow relative to the rotating frame is negligible, then the dispersion relationship for two-dimensional waves propagating in the $xy$ plane relative to that basic flow is

$$\omega^2 + \beta k\omega/\kappa^2 - (V.\boldsymbol{\kappa})^2 = 0 \tag{4.49}$$

as illustrated in figure 7. Here $\omega$ is the angular frequency of the wave, $\boldsymbol{\kappa} = (k, l)$ is the wavenumber vector (so that $\kappa = (k^2 + l^2)^{1/2}$ is the total wavenumber), and

$$\beta \equiv (2\Omega/b)\cos\bar{\psi} \tag{4.50}$$

is twice the rate of change with respect to latitude of the radial component of $\boldsymbol{\Omega}$ at $\psi = \bar{\psi}$, $\boldsymbol{\Omega}$ being the basic angular velocity of rotation of the system, and

$$V = (B_x, B_y)/(\mu\rho)^{1/2} \tag{4.51}$$

(cf equations (1.2) and (3.13)).

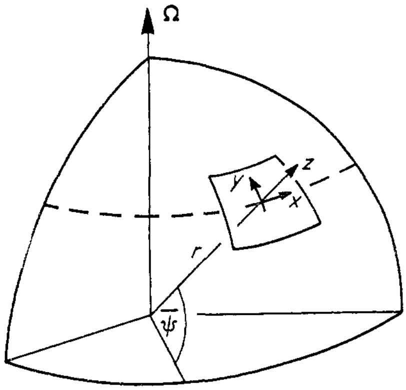


Figure 6. Illustrating the Rossby–Haurwitz 'beta-plane' (see text).

Equation (4.49) should be valid when $2\pi/\kappa$ is much less than the length of a great circle $2\pi b$, and evidence that this is so in the nonhydromagnetic case $V = 0$ can be found in accurate studies of the eigenmodes of a thin spherical shell (eg Longuet–Higgins 1968), when equation (4.49) reduces to the well known Rossby–Haurwitz formula for ordinary 'planetary waves'

$$\omega = -\beta k/\kappa^2 \tag{4.52}$$

(see Hide 1966a, Hide and Jones 1972). Comparable studies of the general hydromagnetic case are not available, but the expression

$$\omega = V^2 k\kappa^2/\beta \tag{4.53}$$

to which equation (4.49) reduces when $\chi = 0$ and $\omega^2 \ll (V.\boldsymbol{\kappa})^2$, where

$$\chi = \tan^{-1}(B_y/B_x) \tag{4.54}$$

has been shown (see Hide and Jones 1972) to be compatible with accurate analyses by Stewartson (1967) and Rickard (1970).

We denote the two roots of equation (4.49) by $\omega_+$ and $\omega_-$, taking, as in the case of plane waves in an unbounded fluid discussed in §4.1, $\omega_+$ to correspond to the wave that is speeded up by rotation and $\omega_-$ to the wave that is slowed down (ie $\omega_+| \geqslant |\omega_-|$), and note that

$$\omega_+\omega_- = -(V.\boldsymbol{\kappa})^2 \tag{4.55}$$

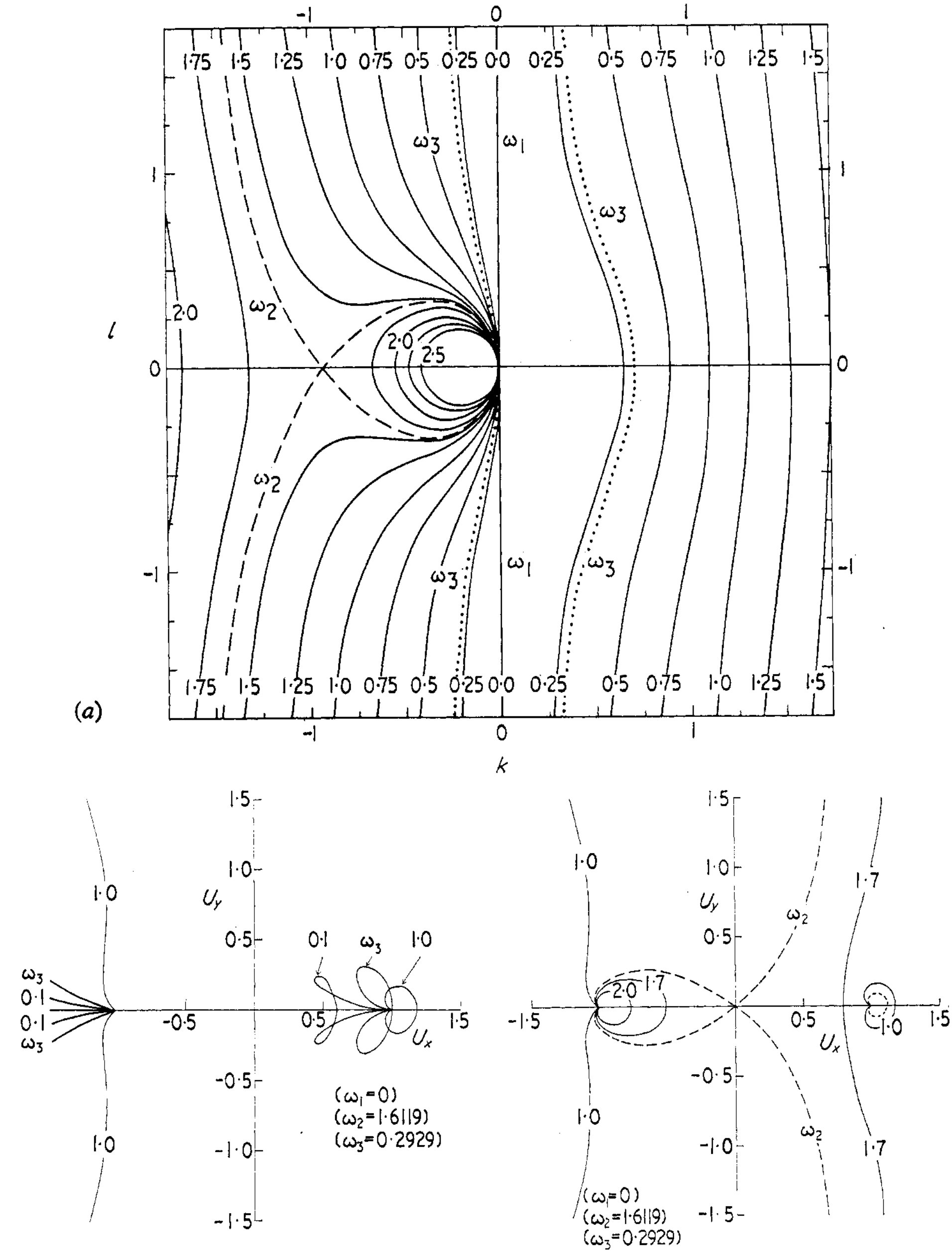


Figure 7. (*a*) 'Normal' curves of constant $\omega$ in the $kl$ plane, illustrating the dispersion relationship given by equation (4.49) when the basic magnetic field has no $y$ component (ie $\chi = 0$, see equation (4.54)). These curves form a unique set when $(V/\beta)^{1/2}$ is taken as the unit of length and $(V\beta)^{-1/2}$ as the unit of time. $\omega_1$, $\omega_2$ and $\omega_3$ are critical frequencies (nondimensionalized in this way) equal in general to $(27/4)^{1/4}\,|\sin\frac{1}{2}\chi\,|$, $(27/4)^{1/4}\,|\cos\frac{1}{2}\chi\,|$ and $(1-2^{-1/2})\,|\cos\frac{1}{2}\chi\,|$. (*b*) and (*c*) Group velocity curves based on the dispersion relationship illustrated by (*a*). Each curve is the locus traced by the tip of the group velocity vector $(U_x, U_y) \equiv (\partial\omega/\partial k, \partial\omega/\partial l)$ as one proceeds along the corresponding curve, characterized by a given value of $\omega$, in (*a*). The two points $(\pm 1, 0)$ represent ordinary Alfvén waves, which arise when $\kappa \gg (V/2\beta)^{1/2}$ (in ordinary units) or $\kappa \gg 2^{-1/2}$ in the units used in the diagram. From Hide and Jones (1972).

(cf equation (4.12)). Equation (4.49) leads to quite simple expressions for $\omega$ when a 'hydromagnetic Rossby number'

$$\gamma \equiv |2\kappa^2(\boldsymbol{V}.\boldsymbol{\kappa})/\beta k| \tag{4.56}$$

(see equation (1.1)) is either very much greater than unity or very much less than unity. In the former case, $\gamma \gg 1$, we have

$$\omega_+ \simeq -(\boldsymbol{V}.\boldsymbol{\kappa})(1+\gamma^{-1}) \tag{4.57$a$}$$

and

$$\omega_- \simeq (\boldsymbol{V}.\boldsymbol{\kappa})(1-\gamma^{-1}) \tag{4.57$b$}$$

(cf equation (4.13)), while in the latter case, $\gamma \ll 1$,

$$\omega_+ \simeq -\beta k/\kappa^2 \tag{4.58$a$}$$

and

$$\omega_- \simeq (\boldsymbol{V}.\boldsymbol{\kappa})^2\,\kappa^2/\beta k \tag{4.58$b$}$$

(cf equations (4.14), (4.52) and (4.53)). We note in passing that $V$ and $\beta$ can be eliminated from the foregoing equations in this section by taking $(V/\beta)^{1/2}$ as the unit of length and $(V\beta)^{-1/2}$ as the unit of time (see Hide and Jones 1972). Equations (4.57a,b), which apply when Coriolis forces are very small, are (to a first approximation) the dispersion relationships for ordinary Alfvén waves propagating in opposite directions. Equations (4.58*a*,*b*), which apply when Coriolis forces are very important, are the respective dispersion relationships for ordinary planetary waves and 'hydromagnetic-planetary' waves; these have their plane counterparts in the inertial waves and hydromagnetic-inertial waves discussed in § 4.1. The dispersion relationship (4.58*b*) for hydromagnetic-planetary waves in a thin shell reduces to equation (4.53) when the lines of force of the basic magnetic field are parallel to latitude circles (ie $\chi = 0$, see equation (4.54)). Both $\partial\omega/\partial k$ and $\omega/k$ are positive, so that the $x$ component of the group velocity is in the same sense as that of phase propagation parallel to the $x$ axis, namely eastward. It may be shown (Acheson 1971, 1972*b*), that this eastward propagation is insensitive to latitudinal variations of the magnetic field and that, in contrast to the previous section, such variations give rise to no unstable modes if the shell is thin. The reasons for this are not yet well understood (but see § 6.5).

To a first approximation the Earth's liquid core is a *thick* spherical shell of inner radius $a = 1250$ km and outer radius $b = 3475$ km. Thin shell modes of oscillation could occur in a thick shell and under certain conditions, notably if the density distribution (which is unknown for the core, but see Busse 1972, Higgins and Kennedy 1971, Jacobs 1971) were so strongly 'bottom heavy' that radial motions were negligibly small (see § 6.3), these 'toroidal' modes would dominate the spectrum. Otherwise toroidal modes would be expected to be much less important than modes with significant radial motions—'nontoroidal' modes. The satisfactory elucidation of the properties of these 'nontoroidal' modes will require further theoretical work, but a little progress has been made in recent years (see Malkus 1967a, Stewartson 1967, 1971, also Hide and Stewartson 1972). Some useful results can be obtained by means of the following simple theoretical model (Hide 1966a) in which filaments of fluid aligned with the rotation axis move more or less as coherent units (see figure 8(*b*)), so that the coupling of the relative vorticity field to the vorticity of basic rotation arises solely from axial changes in filament length brought about by motion away from or towards the rotation axis.

Denoting by $z$ the space coordinate of a general point measured parallel to $\mathbf{\Omega}$ and by $r$ its distance from the rotation axis we let $H = H(r)$ be the length of the fluid filament. We consider first the nonhydromagnetic case, making use of the potential vorticity theorem given by equation (3.10), which reduces to

$$\frac{\mathrm{D}}{\mathrm{D}t}\{(\boldsymbol{\xi}+2\mathbf{\Omega}).\nabla\Lambda\} = 0 \tag{4.59}$$

when $\nu = 0$. A property of a material particle that remains constant throughout such a motion is its axial distance from a bounding sphere divided by $H$, so we

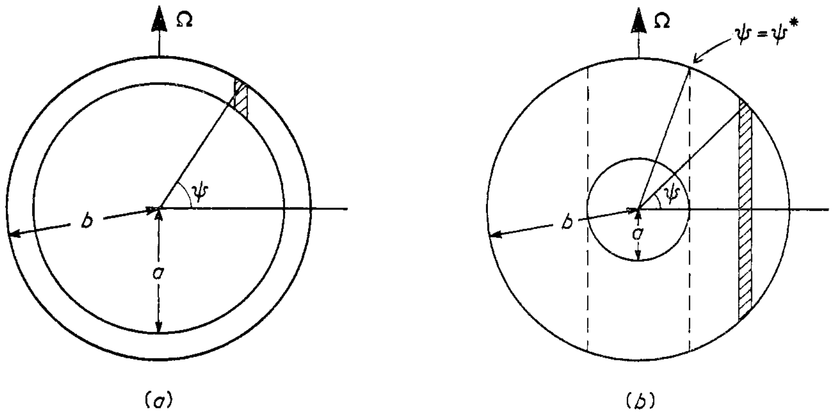


Figure 8. Illustrating axial filaments of fluid (hatched) ($a$) outside the equatorial region of a thin spherical shell, and ($b$) outside the polar regions of a thick shell. From Hide (1966a).

take $\Lambda$ equal to this ratio and find that

$$(\boldsymbol{\xi}+2\mathbf{\Omega}).\nabla\Lambda = (\zeta+2\Omega)/H$$

is conserved throughout the motion of the filament, which has relative vorticity $\boldsymbol{\xi} = (0, 0, \zeta)$. For small amplitude motions about a basic state of no motion relative to the rotating frame equation (4.59) reduces to

$$\frac{\partial\zeta}{\partial t} = \frac{2\Omega}{H}\frac{\mathrm{d}H}{\mathrm{d}r}u_r \tag{4.60}$$

where $u_r$ is the $r$ component of $\boldsymbol{u}$.

While equation (4.60) is valid whether the shell is thin or thick in the former case (see figure 8($a$)) it may be locally simply transformed into its 'beta plane' form (see figure 6) by noting that

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} = \zeta\sin\bar{\psi}$$

and

$$u_r = -v\sin\bar{\psi}$$

where $x, y$ denote the local eastward and northward 'beta plane' coordinates and $u, v$ the associated velocity components, whence

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = -\beta^* v \tag{4.61}$$

where

$$\beta^* \equiv \sin^2\bar{\psi}\left(\frac{2\Omega}{H}\frac{\mathrm{d}H}{\mathrm{d}r}\right)_{r=b\cos\bar{\psi}}. \tag{4.62}$$

For a thin shell the quantity $\beta^*$ is the same as $\beta$ given by equation (4.50), and the simultaneous solution of equation (4.61) and the continuity equation

$$\nabla . \boldsymbol{u} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

leads directly to the dispersion relationship for planetary waves given by equation (4.52) (see Hide 1966a).

The westward propagation of these planetary waves in a thin shell may be interpreted in terms of the vorticity changes encountered by a filament of fluid (see figure 8(*a*)) as it changes latitude. The instantaneous velocity profile of the motion at a time $t_1$ is shown by the upper curve of figure 9, it being supposed for

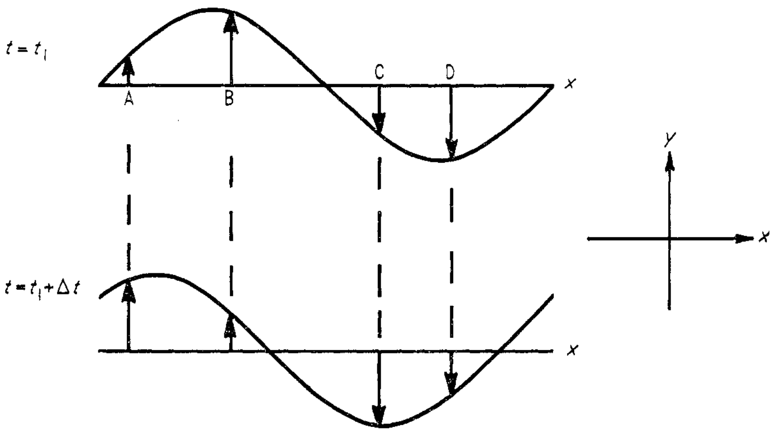


Figure 9. Illustrating the westward phase propagation of ordinary planetary waves in a thin spherical shell (see text and figure 7(*a*) ).

simplicity of illustration that the wave is propagating *purely* in the azimuthal direction, thus being characterized by purely north–south fluid motions. As we have seen above $(\zeta + 2\Omega)/H$ is conserved by an individual fluid filament throughout its motion. The filament at A has positive vorticity relative to the rotating frame (if, as is supposed in the diagram, the filament is in the northern hemisphere) and is moving northwards. At a time $\Delta t$ later its length will therefore have decreased slightly (see figure 8(*a*)), and so will its relative vorticity. This results (for this particular filament) in a slight flattening of the velocity profile at A, and the new profile may evidently be identified with that at time $t_1$ slightly to the *east* of A. When similar arguments have been applied at B and (with only slight modification) to the southward moving filaments at C and D it is clear that the whole wave pattern shifts *westward* with time.

Let us now turn to the hydromagnetic problem, in particular to the propagation of the slow hydromagnetic-planetary waves, which we know (see equation (4.58*b*)) in a thin shell to be *eastward.* Hide (1966a) also considered the motion of axial filaments in a thick shell (see figure 8(*b*)). Noting that (except within polar caps

$\psi > \psi_*$ that vanish when $a = 0$, the full sphere case) $H^{-1}\,\mathrm{d}H/\mathrm{d}r$ has exactly the same magnitude as in the thin shell case, namely $r/(b^2 - r^2)$, but is *opposite in sign*, he argued that rough values of typical eigenfrequencies of these nontoroidal modes of oscillation of a thick shell or a full sphere should be obtainable from equation (4.49) by changing the sign of the second term. The corresponding nontoroidal hydromagnetic-planetary waves would then propagate *westward*, unlike their toroidal counterparts which, as we have seen, propagate eastward.

To impart further precision to these arguments it is desirable to have an extension to the hydromagnetic case of the result concerning the conservation of $(\zeta + 2\Omega)/H$ for an individual fluid filament. This has proved possible in one particularly simple (but not necessarily unrepresentative) situation. Suppose that the fluid is in *steady* motion (within a spherical shell of arbitrary thickness) and that the fluid velocity $\boldsymbol{u}$ is everywhere aligned with, and proportional to, the local magnetic field $\boldsymbol{B}$. The 'aligned fields' results of § 3.4 then imply that conservation of $(\zeta + 2\Omega)/H$ for an individual fluid filament is replaced in this hydromagnetic case by conservation of

$$\frac{1}{H}\left(\zeta + \frac{2\Omega}{1 - V^2/U^2}\right) \tag{4.63}$$

where $V/U$ is the constant of proportionality between the Alfvén speed and fluid speed at any point in the fluid (see equations (3.14), (3.18)). It is immediately evident that as a fluid filament changes its latitude (and hence its length $H$) it acquires cyclonic or anticyclonic vorticity according as $(1 - V^2/U^2)H^{-1}\,\mathrm{d}H/\mathrm{d}r$ is positive or negative. Thus, in a given change of latitude, a fluid filament in a thin shell $(H^{-1}\,\mathrm{d}H/\mathrm{d}r > 0)$ moving faster than the Alfvén speed $(U^2 > V^2)$ acquires vorticity of the same sense as that acquired in the same change in distance from the rotation axis by a filament in a *thick* shell $(H^{-1}\,\mathrm{d}H/\mathrm{d}r < 0)$ moving *slower* than the Alfvén speed $(U^2 < V^2)$. We note the parallel between the properties of these steady motions and those of the wave motions discussed above, for Hide (1966a) indeed argues that a slow 'hydromagnetic-planetary' wave in a thick shell propagates in the same azimuthal direction as an ordinary planetary wave in a thin shell, namely *westward*. The parallel is incomplete only inasmuch as the *fluid* velocity $U$ in these aligned field motions takes the place of the *phase speed* of the wave motions. Although the two approaches can be more precisely related by changing to a moving set of coordinates and Doppler shifting the wave velocity to zero (Acheson 1971) we shall not labour the point here, but will rather turn attention to an exact analysis of a related problem by Malkus (1967a) which, without contradicting the above arguments, sheds further light upon the propagation of hydromagnetic waves in bounded rotating systems.

Malkus (1967a) studies the oscillations of a complete sphere $(a = 0)$ when the basic magnetic field is everywhere parallel to latitude circles and given by $\boldsymbol{B}_0 = \mu \boldsymbol{j}_0 \times \boldsymbol{r}$, where the axial electric current $\boldsymbol{j}_0$ is a constant. (Related analyses of *axisymmetric* oscillations when the basic magnetic field is uniform and parallel to the rotation axis will not be considered here, but see Cowling 1955, Kendall 1960, Plumpton and Ferraro 1955, Rikitake 1956, 1966, Stewartson 1957, also Hide and Stewartson 1972). On linearizing the equations, assuming that all independent variables are of the form

$$(\text{function of } r \text{ and } z) \times \exp \mathrm{i}(m\theta - \omega t)$$

where $(r, \theta, z)$ are cylindrical polar coordinates, $m$ is proportional to the azimuthal wavenumber and $\omega$ is the eigenfrequency, the equation for the effective pressure $p$ reduces to

$$\left(\nabla^2 - \frac{4}{\lambda^2}\frac{\partial^2}{\partial z^2}\right) p = 0 \tag{4.64}$$

where

$$\lambda \equiv (V^2 m^2 - \omega^2 b^2)/(V^2 m + \Omega\omega b^2) \tag{4.65}$$

if $V = \mu^{1/2} j_0 b/\rho^{1/2}$ (see equation (1.2)), a representative Alfvén speed. The boundary conditions, including the requirement that the radial component of fluid motion vanishes on the bounding sphere, reduce in this case to

$$r\frac{\partial p}{\partial r} + \left(1 - \frac{4}{\lambda^2}\right) z \frac{\partial p}{\partial z} + \frac{2m}{\lambda} p = 0 \tag{4.66}$$

on $r^2 + z^2 = b^2$.

Equation (4.64), coupled with the boundary condition (4.66), forms an eigenvalue problem for $\lambda$ which is mathematically equivalent to that encountered in studies of the *non*hydromagnetic oscillations of a rotating fluid sphere (see Greenspan 1968), and it has been shown that the eigenvalues $\lambda$ have the property $|\lambda| \leqslant 2$. The eigenvalues of the corresponding hydromagnetic problem may be expressed in terms of $\lambda$ by rearranging equation (4.65), whence

$$\omega^2 + \Omega\lambda\omega - V^2 b^{-2} m(m - \lambda) = 0 \tag{4.67}$$

(cf equation (4.49)). Thus, when $\mathscr{L} \ll 1$, to each (essentially) purely inertial oscillation

$$\omega \simeq -\Omega\lambda \tag{4.68}$$

there corresponds a slow hydromagnetic-planetary oscillation with frequency

$$\omega \simeq V^2 m(m - \lambda)/\lambda\Omega b^2 \tag{4.69}$$

(cf equations (4.58*a*,*b*)). The slow waves have eastward phase velocity when $0 < \lambda < m$ and westward phase velocity when $\lambda < 0$ or $\lambda > m$. Malkus was able to evaluate $\lambda$ in a few special cases, when the corresponding eigenfunction $p$ is particularly simple in form, and these cases revealed no net preference for either eastward or westward propagation, although a comprehensive discussion of equation (4.64) will require further work. We note in passing that there is also one special case, namely $B_z =$ constant and $B_\theta \propto r$ for which the problem of the hydromagnetic oscillations of a rotating *cylinder* may be reduced to finding the eigenvalues of the corresponding nonhydromagnetic problem (Acheson 1971, 1972b, Chandrasekhar 1961, Gans 1970). No obvious westward or eastward selection mechanism for the slow hydromagnetic-planetary waves appears in this case either, although in the absence of the generation mechanisms of § 4.4 or variations of container height (see above) this is perhaps not surprising.

When the basic magnetic field is entirely azimuthal, so that modes with motions that are both axisymmetric (ie $m = 0$) and independent of the axial coordinate (see § 3.1, especially equation (3.8)) experience neither Lorentz nor Coriolis torques (since no change in filament length is then possible), the frequency $\omega$ vanishes. Several workers (eg Gans 1971, Roberts 1971, Roberts and Soward 1972) have considered these 'pseudogeostrophic' modes in more general magnetic fields and they find, as one might expect in view of the discussion given in § 3.1, that $\omega$ is determined by the nonazimuthal component of $\boldsymbol{B}_0$ and is independent of $\Omega$. This behaviour is exemplified by the case $k = 0$ in equation (4.49).

The theory of the attenuation of oscillations of bounded systems lies beyond the scope of the present article (but see Gans 1971 and §4.1). Suffice it to remark here that an unpublished study by Hide indicates that hydromagnetic-planetary waves (see equation (4.58*b*)) in the core would be attenuated largely by ohmic effects in the core and ordinary planetary waves (see equation (4.58*a*)) by eddy currents induced in the weakly conducting lower reaches of the overlying solid mantle, viscous effects being comparatively weak in both cases. The radial component of the magnetic field at the core–mantle interface plays an essential role in the production of eddy currents in the lower mantle, but we anticipate here (see §5.1) that its effect on the structure of the viscous boundary layer is only important near the equator, as we shall see in the next section where the theory of such boundary layers is considered.

## 5. Boundary layers

### 5.1. *Ekman–Hartmann layers*

The boundary layer that arises near the rigid bounding surface of a rapidly rotating incompressible fluid of low but nonzero viscosity in which slow steady relative motions are occurring is of the Ekman type (Ekman 1905; see also Prandtl 1952, Greenspan 1968), with thickness $(\nu/|\boldsymbol{\Omega}.\boldsymbol{n}|)^{1/2}$ where $\nu$ is the coefficient of kinematic viscosity, $\boldsymbol{\Omega}$ the angular velocity of basic (uniform) rotation, and $\boldsymbol{n}$ a unit vector normal to the surface and directed toward the fluid. The corresponding boundary layer at a rigid bounding surface of an electrically conducting, nonrotating, incompressible fluid pervaded by a magnetic field is of the Hartmann type (Hartmann 1937; see also Alfvén and Fälthammar 1963, Cowling 1957, Ferraro and Plumpton 1966, Hide and Roberts 1962, Roberts 1967, Shercliff 1965), with thickness $(\eta\nu\rho\mu/B_0^2)^{1/2}$, where $\rho$ is the density of the fluid, $\mu\eta$ its electrical resistivity and $B_0$ the strength of the normal component of the magnetic field at the bounding surface. Complications arise when the flow is not steady and combined rotational and hydromagnetic effects occur (see Backus 1968, Gilman and Benton 1968, Hide 1969a, Hide and Roberts 1960, Vidyanidhi 1969, Vidyanidhi and Rao 1968, Nanda and Mohanty 1971; see also Chandrasekhara and Rudraiah 1971, Datta 1964, Datta and Biswas 1971, Govindarajulu 1971, Gupta 1968, Ingham 1969, Kakutani 1962, Khan and Srivastava 1970, Pop 1969, Soundalgekar and Pop 1970 for treatment of the special case when $\mathscr{S} \ll 1$).

Assuming that the conducting fluid is in contact over the whole or part of its boundaries with rigid impermeable, solid surfaces that are fixed in the rotating frame, we consider an area $\Sigma$ of the interface between the solid and the fluid; at that interface the boundary conditions

$$\boldsymbol{u}.\boldsymbol{n} = 0 \tag{5.1}$$

and

$$\nu\boldsymbol{u}\times\boldsymbol{n} = 0 \tag{5.2}$$

must be satisfied.

When $\nu$, though nonzero, is not too large, viscous forces will be weak except in a thin boundary layer of thickness $\Delta$, where by hypothesis

$$\Delta \ll L^* \tag{5.3}$$

separating the interface $\Sigma$ from the effectively inviscid 'interior' region of the fluid, $L^*$ being a length characteristic of the scale of the 'interior' motions.

We denote by $\phi$ a dummy variable representing $p$ or any component of $\boldsymbol{u}$ or $\boldsymbol{B}$ and separate $\phi$ into its 'interior' part and its 'boundary layer' part by writing

$$\phi = \bar{\phi} + \hat{\phi} \tag{5.4}$$

where $\hat{\phi}$ is that part of $\phi$ which undergoes significant variations in the boundary layer, but only in directions parallel to $\boldsymbol{n}$, and vanishes in the 'interior' where $\phi = \bar{\phi}$. The gradients of $p$ and $\boldsymbol{B}.\boldsymbol{n}$ will be small within the boundary layer, so that we can set (see Hide 1969a)

$$\hat{\boldsymbol{B}}.\boldsymbol{n} = \hat{p} = 0. \tag{5.5}$$

On making use of the usual boundary layer approximations, in which gradients in directions parallel to $\Sigma$ are treated as negligible in comparison with gradients normal to $\Sigma$, and introducing the further assumption that the curvature of $\Sigma$ is so small (ie $\ll 1/L^*$) that the term $(\boldsymbol{u}.\nabla)\,\boldsymbol{u}$ is everywhere negligible within the boundary layer, we find that equations (4.1)–(4.4) give

$$\frac{\partial(\hat{u}, \hat{v})}{\partial t} + f(-\hat{v}, \hat{u}) = \nu \mathrm{D}^2(\hat{u}, \hat{v}) + V\mathrm{D}(\hat{B}_x, \hat{B}_y) \tag{5.6}$$

$$\frac{\partial(\hat{B}_x, \hat{B}_y)}{\partial t} = V(\mu\rho)^{1/2}\,\mathrm{D}(\hat{u}, \hat{v}) + \eta \mathrm{D}^2(\hat{B}_x, \hat{B}_y). \tag{5.7}$$

Here

$$\left.\begin{aligned} \hat{\boldsymbol{u}} &= (\hat{u}, \hat{v}, \hat{w}) \\ \hat{\boldsymbol{B}} &= (\hat{B}_x, \hat{B}_y, \hat{B}_z) \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} f &\equiv |2\boldsymbol{\Omega}.\boldsymbol{n}| \\ \mathrm{D} &\equiv \frac{\partial}{\partial z} = \boldsymbol{n}.\nabla \end{aligned}\right\} \tag{5.9}$$

and

$$V \equiv (\bar{\boldsymbol{B}}(z=0).\boldsymbol{n})/(\mu\rho)^{1/2} \tag{5.10}$$

where the modulus of $V$ is the Alfvén speed based on $B_0$, $(x, y, z)$ being the coordinates of a point fixed in the rotating frame, referred to a coordinate system whose $x$ and $y$ axes lie within the surface $\Sigma$, whose $z$ axis is normal to $\Sigma$ and directed into the fluid and which is conveniently taken as right-handed or left-handed according as $\boldsymbol{\Omega}.\boldsymbol{n}$ is greater or less than zero. $\boldsymbol{n}$ is a unit vector along the positive $z$ axis.

We can eliminate any three of the four dependent variables from equations (5.6) and (5.7) and show that each of them satisfies

$$\left\{\left(\frac{\partial}{\partial t} - \eta \mathrm{D}^2\right)\left(\frac{\partial}{\partial t} - \nu \mathrm{D}^2\right) - V^2\mathrm{D}^2\right\}^2 \hat{\phi} + f^2\left(\frac{\partial}{\partial t} - \eta \mathrm{D}^2\right)^2 \hat{\phi} = 0. \tag{5.11}$$

Neither the velocity nor the magnetic field can undergo discontinuous changes, so that

$$\left.\begin{aligned} &(a)\ \ \hat{\boldsymbol{u}} = -\bar{\boldsymbol{u}} && \text{at } z = 0 \\ &(b)\ \ \boldsymbol{B} \text{ continuous} && \text{at } z = 0 \end{aligned}\right\} \tag{5.12}$$

are boundary conditions under which equation (5.11) must be solved. According to their definitions $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{B}}$ vanish in the interior region, that is to say

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{B}} = 0 \qquad \text{when } z \gg \Delta. \tag{5.13}$$

If time variations of the 'interior' flow are harmonic, with positive angular frequency $\omega$, and the $z$ variation of $\hat{\phi}$ has the form $\exp(\gamma z)$, then

$$\hat{\phi} \propto \exp(\mathrm{i}\omega t + \gamma z) \tag{5.14}$$

and (5.11) and (5.13) are satisfied if

$$\{(\mathrm{i}\omega - \eta\gamma^2)(\mathrm{i}\omega - \nu\gamma^2) - V^2\gamma^2\}^2 + f^2(\mathrm{i}\omega - \eta\gamma^2)^2 = 0 \tag{5.15}$$

provided that we choose those roots for which

$$\mathrm{Re}(\gamma) < 0. \tag{5.16}$$

If we interpret the boundary layer thickness $\Delta$ as the distance from $\Sigma$ in which $|\hat{\boldsymbol{u}}|$ decreases to $(1 - \mathrm{e}^{-1})$ of its value at $z = 0$ (see (5.18)) then

$$\Delta^{-1} = -\mathrm{Re}(\gamma). \tag{5.17}$$

Though in general quite complicated, in the absence of hydromagnetic effects (ie when $V/\eta = 0$) acceptable solutions of (5.15) are

$$\gamma = \left(\frac{\omega \pm f}{2\nu}\right)^{1/2}(-1-\mathrm{i}) \qquad \text{when } \omega > f$$

$$\gamma = \left(\frac{f \pm \omega}{2\nu}\right)^{1/2}(-1 \mp \mathrm{i}) \qquad \text{when } \omega < f$$

which when $\omega = 0$ reduce to the case first discussed by Ekman in connection with ocean currents and when $f = 0$ to the case first discussed by Stokes in an investigation of the attenuation of sound waves. On the other hand, when $f = 0$ but $V/\eta \neq 0$, we have a variety of hydromagnetic cases discussed in detail by a number of authors (see eg Hide and Roberts 1960, 1962; Roberts 1967). When $\omega = 0$ and $f = 0$ we have the Hartmann case for which

$$\gamma = -V/(\eta\nu)^{1/2}$$

(see Stewartson 1960).

In what follows we shall discuss in detail the case of *steady* flows (ie $\omega = 0$) subject to the combined influence of rotation and a magnetic field (ie $f \neq 0$ and $V/\eta \neq 0$). Equation (5.15) then has acceptable solutions

$$\gamma = -(f/\nu)^{1/2}(1 + \alpha^2)^{1/4}(\cos\vartheta \mp \mathrm{i}\sin\vartheta) \tag{5.18}$$

where

$$\vartheta \equiv \tfrac{1}{2}\cot^{-1}\alpha \tag{5.19}$$

and

$$\alpha \equiv V^2/f\eta = B_0^2/\eta\mu\rho|2\boldsymbol{\Omega}.\boldsymbol{n}|. \tag{5.20}$$

The parameter $\alpha$ is the same as the dimensionless ratio $\mathscr{C}$ (see equation (1.3) and §3.5)). It is the speed with which electromagnetic disturbances are transmitted by hydromagnetic-inertial waves (see §4.1), proportional to $V^2/f$, divided by the speed at which they are transmitted by diffusive processes, proportional to $\eta$.

We denote by $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{B}}$ the interior values of $\boldsymbol{u}$ and $\boldsymbol{B}$ evaluated at $z = 0$ (see (5.4)) ie $\tilde{\boldsymbol{u}} \equiv \bar{\boldsymbol{u}}$ $(z = 0)$ etc and choose the orientation of the axes of coordinates so that $\tilde{v}$, the $y$ component of $\tilde{\boldsymbol{u}}$, is zero. By (5.6), (5.7), (5.12), (5.13) and (5.18)

$$\hat{u}+\mathrm{i}\hat{v} = -\tilde{u}\exp-\{(1+\alpha^2)^{1/4}(\cos\vartheta+\mathrm{i}\sin\vartheta)\,\zeta\} \tag{5.21}$$

$$\hat{B}_x+\mathrm{i}\hat{B}_y = -\frac{\tilde{u}}{\eta}\left(\frac{\nu}{f}\right)^{1/2}\frac{\tilde{\boldsymbol{B}}.\boldsymbol{n}}{(1+\alpha^2)^{1/4}}\exp-\{(1+\alpha^2)^{1/4}(\cos\vartheta+\mathrm{i}\sin\vartheta)\,\zeta+\mathrm{i}\vartheta\} \tag{5.22}$$

where we have introduced the stretched coordinate

$$\zeta\equiv z/(\nu/f)^{1/2}. \tag{5.23}$$

The dependence on $\alpha$ of the factor multiplying $\zeta$ in the exponential function is illustrated by figure 10.

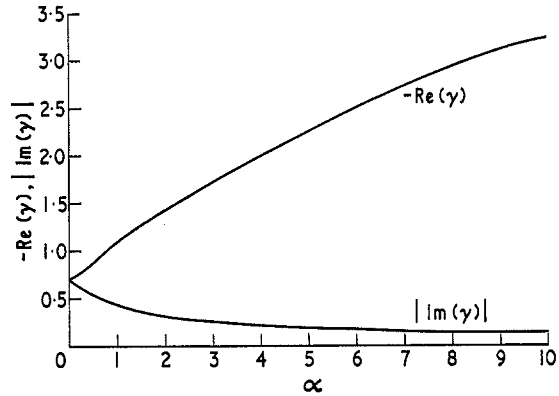


Figure 10. Illustrating the $\alpha$ dependence of the reciprocal of the boundary layer thickness, $-\mathrm{Re}\,(\gamma)$, and of $\mathrm{Im}\,(\gamma)$ (in units of $(f/\nu)^{1/2}$, see equations (5.17) and (5.18)). After Hide (1969a).

Equations (5.21) and (5.22) reduce to well known cases when $\alpha\ll 1$ and when $\alpha\gg 1$. In the former case, $\alpha\ll 1$, hydromagnetic effects are negligible and we have the Ekman boundary layer, for which

$$\left.\begin{aligned}\hat{u}+\mathrm{i}\hat{v} &= \tilde{u}\exp(-\zeta 2^{-1/2})\{-\cos(\zeta 2^{-1/2})+\mathrm{i}\sin(\zeta 2^{-1/2})\}\\ \hat{B}_x+\mathrm{i}\hat{B}_y &= 0.\end{aligned}\right\} \tag{5.24}$$

In the latter case, $\alpha\gg 1$, rotational effects are negligible and we have the Hartmann boundary layer, for which

$$\left.\begin{aligned}\hat{u} &= -\tilde{u}\exp(-\alpha^{1/2}\,\zeta)\\ \hat{v} &= 0\\ \hat{B}_x &= -(\nu\mu\rho/\eta)^{1/2}\,\tilde{u}\exp(-\alpha^{1/2}\,\zeta)\\ \hat{B}_y &= 0.\end{aligned}\right\} \tag{5.25}$$

Observe that the argument of the exponential function, $\zeta\alpha^{1/2}$, is equal to $zV/(\nu\eta)^{1/2}$.

For the boundary layer at the core–mantle interface in mid-latitudes we take $B_0\sim B_{\mathrm{pol}}\sim 5$ gauss $(=5\times 10^{-4}\,\mathrm{Wb\,m^{-2}})$ and $f\sim 10^{-4}\,\mathrm{rad\,s^{-1}}$, so that if $\eta = 10^{0\cdot 5\pm 0\cdot 5}\,\mathrm{m^2\,s^{-1}}$ then $\alpha\sim 10^{0\cdot 2\pm 0\cdot 5}$ (cf equation (3.27)). The function

$$\{\sec(\tfrac{1}{2}\cot^{-1}\alpha)\}/(1+\alpha^2)^{1/4}$$

is then close to unity and varies only slowly with $\alpha$, so that $\Delta\sim(2\nu/f)^{1/2}$, the Ekman thickness. The coefficient of kinematical viscosity $\nu$ is not accurately known for the core, estimates varying over a wide range (Rochester 1970, Gans 1972). If the recently discovered correlation between global features of the Earth's gravitational and magnetic fields (Hide and Malin 1970, 1971c) is a manifestation of interactions between core motions and bumps on the core–mantle interface then $\Delta$ is unlikely to exceed $10^3$ m, so that $\nu \lesssim 10^6 f \sim 10^2\ \mathrm{m^2\,s^{-1}}$ (Hide 1971b). According to Bullard (1949) $\nu$ might be as low as $10^{-7}\ \mathrm{m^2\,s^{-1}}$ in which case $\Delta$ is typically no more than 0·03 m!

The extent to which the theory of the steady Ekman–Hartmann layer can be applied depends *inter alia* on whether or not the flow is stable to small perturbations, a problem considered by Gilman (1971) who found the critical Reynolds number for instability to be a rapidly increasing function of $\mathscr{C}$ when $\mathscr{C} \sim 1$. Gilman's work would indicate that the boundary layer at the core–mantle interface is stable except possibly near the equator.

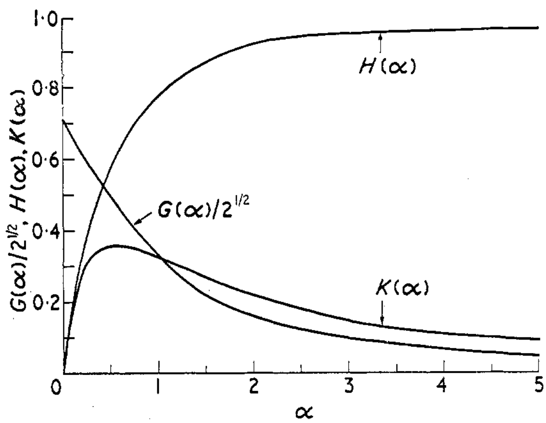


Figure 11. Illustrating the functions $H(\alpha)$, $K(\alpha)$ and $G(\alpha)$ (see equations (5.28) and (5.32)). $G(\alpha)$ is a measure of the influence of hydromagnetic effects on boundary layer suction and $H(\alpha)$ is a measure of the normal component of the current density at the edge of the boundary layer. From Hide (1969a).

### 5.2. *Boundary layer suction and 'jump' conditions*

The theory of the Ekman–Hartmann layer can be applied to the formulation of appropriate boundary conditions on $\boldsymbol{u}$ and $\boldsymbol{B}$ for problems in which $\nu$ and $\eta$ are taken as zero in the main body of the fluid. We denote the change in $\hat{\phi}$ across the boundary layer by

$$[\hat{\phi}] \equiv \hat{\phi}(z \gg \Delta) - \hat{\phi}(z = 0). \tag{5.26}$$

By (5.21) and (5.22) the jump in $\hat{\boldsymbol{B}}$ is related to the jump in $\hat{u}$ as follows:

$$[\hat{B}_x + \mathrm{i}\hat{B}_y] = [\hat{u}]\,(\mu\rho)^{1/2}\,\mathrm{sgn}\,(\tilde{\boldsymbol{B}}\,.\,\boldsymbol{n})\,(\nu/\eta)^{1/2}(H(\alpha) - \mathrm{i}K(\alpha)) \tag{5.27}$$

where

$$H(\alpha) - \mathrm{i}K(\alpha) = \frac{\alpha^{1/2}}{(1+\alpha^2)^{1/4}} \exp\left(\frac{-\mathrm{i}}{2}\cot^{-1}\alpha\right). \tag{5.28}$$

As shown in figure 11 $H(\alpha) = 1$ and $K(\alpha) = 0$ when $\alpha^{-1} = 0$. Rotational effects are then absent and the boundary layer is of the Hartmann type, with the velocity and magnetic field in the boundary layer parallel to the local interior flow. The expression for $[\hat{B}_x]$ then reduces to one found by Stewartson (1960) in his discussion of the boundary conditions to be used at a rigid surface when dealing with problems in which the interior fluid is regarded as inviscid (ie $\nu = 0$) and perfectly conducting (ie $\eta = 0$), namely

$$[\hat{B}_x] = [\hat{u}]\,\mathrm{sgn}\,(\tilde{\boldsymbol{B}}\,.\,\boldsymbol{n})\,(\mu\rho)^{1/2}(\nu/\eta)^{1/2}. \tag{5.29}$$

In the limit when $\eta/\nu \to 0$ a current sheet can occur, across which $[\hat{B}_x] \neq 0$, but the 'no slip' condition on the velocity must then be satisfied. At the other extreme limit, $\nu/\eta \to 0$, a vortex sheet can occur, across which $[\hat{u}] \neq 0$, but there can then be no jump in $\hat{B}_x$. A current sheet and a vortex sheet cannot occur simultaneously; only one or the other may occur.

Turning now to the case when rotational effects are so important that $\alpha \ll 1$, we find that $H(\alpha) = K(\alpha) = (\alpha/2)^{1/2}$, so that

$$\mathrm{sgn}\,(\tilde{\boldsymbol{B}}\,.\,\boldsymbol{n})\,[\hat{B}_x + \mathrm{i}\hat{B}_y] = [\hat{u}]\,(\mu\rho)^{1/2}\left(\frac{\nu}{\eta}\right)^{1/2}\left(\frac{V^2}{2f\eta}\right)^{1/2}(1-\mathrm{i}). \tag{5.30}$$

This shows that $[\hat{B}_x]$ is very much less, by a factor $(\alpha/2)^{1/2}$, than in the Hartmann case $\alpha^{-1} = 0$, and that when $f \neq 0$ there can be a jump in the *direction* as well as in the magnitude of the tangential component of the magnetic field.

The normal component of $\boldsymbol{u}$ may, as a result of 'boundary layer suction', undergo a significant variation across the boundary layer. Because $w = 0$ when $z = 0$, the speed $\tilde{w}(\equiv \bar{w}\ (z = 0))$ with which fluid passes out of the boundary layer into the interior region is equal to $[\hat{w}]$ which, by (4.2), (5.4), (5.12), (5.21) and (5.26), is given by

$$[\hat{w}] = -\frac{\partial}{\partial y}\int_0^\infty \hat{v}\,\mathrm{d}z = \left(\frac{\nu}{2f}\right)^{1/2} G(\alpha)\,(\nabla \times \tilde{\boldsymbol{u}})_z \tag{5.31}$$

where

$$G(\alpha) \equiv \{2^{1/2}\sin\left(\tfrac{1}{2}\cot^{-1}\alpha\right)\}/(1+\alpha^2)^{1/4}. \tag{5.32}$$

When $\alpha = 0$ the function $G(\alpha) = 1$ (see figure 11) and equation (5.31) then reduces to the well known Ekman boundary layer suction formula relating the normal component of the flow velocity at the edge of the interior region to $\boldsymbol{n}\,.\,(\nabla \times \bar{\boldsymbol{u}})_{z=0}$, the component of relative vorticity of the interior flow normal to the bounding surface (Prandtl 1952, Greenspan 1968). The magnetic field reduces this suction; thus $G(\alpha)$ falls steadily to zero with increasing $\alpha$, slowly at first, then more rapidly, and eventually, when $\alpha \to \infty$, as $(2\alpha^3)^{-1/2}$.

The effect of the boundary layer on the magnetic field in the interior region and in the region outside the fluid depends on the constraints imposed by side walls at large $x$ and $y$ on the electric current induced by the flow; this aspect of the problem will not concern us here (see Hide and Roberts 1962). It is of interest, however, to evaluate the jump in the normal component of the electric current density $[\hat{j}_z]$ across the boundary layer. Since

$$\begin{aligned}\mu[\hat{j}_z] &= -\mu\int_0^\infty \left(\frac{\partial \hat{j}_x}{\partial x} + \frac{\partial \hat{j}_y}{\partial y}\right)\mathrm{d}z \\ &= \frac{\partial}{\partial x}[\hat{B}_y] - \frac{\partial}{\partial y}[\hat{B}_x]\end{aligned} \tag{5.33}$$

by (5.30) we have

$$[\hat{j}_z] = \left(\frac{\rho\nu}{\mu\eta}\right)^{1/2} H(\alpha)\,(\nabla\times\tilde{\boldsymbol{u}})_z\,\mathrm{sgn}\,(\tilde{\boldsymbol{B}}.\boldsymbol{n}). \tag{5.34}$$

This shows that $|[\hat{j}_z]|\to(\rho\nu/\mu\eta)^{1/2}|(\nabla\times\boldsymbol{u})_z|$ when $\alpha\to\infty$, the Hartmann limit, and vanishes when $\alpha=0$, the Ekman limit. Note that both $[\hat{w}]$ and $[\hat{j}_z]$ are proportional to $(\nabla\times\tilde{\boldsymbol{u}})_z$; thus

$$[\hat{j}_z] = (\rho V^2/\mu\eta)^{1/2}\,\mathrm{sgn}\,(\tilde{\boldsymbol{B}}.\boldsymbol{n})\,\{(1+\alpha^2)^{1/2}+\alpha\}\,[\hat{w}].$$

### 5.3. '*Spin-up*'

The prototype problem illustrating the role of 'Ekman suction' (see equation (5.31)) in transient motions of a rapidly rotating fluid is that of 'spin-up'. Consider a homogeneous viscous fluid confined between two infinite parallel plates separated by a distance $L$, the whole system rotating about an axis normal to the plates with uniform angular velocity $\Omega$. At some time $t=0$ the rotation speeds of both plates are simultaneously increased by the same small amount $\epsilon\Omega$, and the aim is to find the process by which the fluid adjusts to its new equilibrium state of rigid-body rotation with angular velocity $\Omega(1+\epsilon)$. We present here only a brief summary of the process, referring the reader to Greenspan and Howard (1963) (see also Greenspan 1968) for further details.

Within a few revolutions quasi-steady Ekman layers of thickness $\sim(\nu/\Omega)^{1/2}$ (see equation (5.36)) develop on both plates. Since the increase in rotation rate is $\epsilon\Omega$ then $\epsilon\Omega r$ is a characteristic relative flow speed. Since the boundaries rotate faster than the main body of the fluid equation (5.31) shows that fluid is sucked from the effectively inviscid interior into the boundary layers with typical velocity of order $\epsilon(\Omega\nu)^{1/2}$. This suction into both boundary layers must (by conservation of mass) induce radially inward flow $\sim\epsilon(\Omega\nu)^{1/2}r/L$ in the interior region. The interior flow is practically inviscid, so that a ring of fluid moving inwards conserves its angular momentum and thus acquires an increase in angular velocity. The radial displacement required to change the angular velocity in this way from $\Omega$ to $\Omega(1+\epsilon)$ is $\sim\frac{1}{2}\epsilon r$ and this is achieved in the 'spin-up' time

$$\tfrac{1}{2}\epsilon L/\epsilon(\nu\Omega)^{1/2} = (2\Omega E^{1/2})^{-1} \tag{5.35}$$

where

$$E\equiv\nu/\Omega L^2\ll 1. \tag{5.36}$$

A precise analysis (Greenspan and Howard 1963) shows that the azimuthal velocity $u_\theta$ in the main body of the fluid is given by

$$u_\theta = \Omega r[1+\epsilon\{1-\exp(-2\Omega E^{1/2}t)\}] \tag{5.37}$$

and is therefore effectively two-dimensional throughout the spin-up process, in keeping with the Proudman–Taylor theorem for slow quasi-steady motions of a rapidly rotating inviscid fluid. Note how radically the whole process, with weak three-dimensional secondary motions playing a fundamental role (see figure 12), differs from straightforward diffusion of vorticity, and how the spin-up time is much shorter (by a factor of order $E^{1/2}$) than the timescale of viscous diffusion $L^2/\nu$.

Suppose now that the fluid is an electrical conductor permeated by a magnetic field $\boldsymbol{B}$ which in the absence of relative flow would be axial and uniform. If the only effect of the Lorentz forces associated with the magnetic field were to reduce the suction into the viscous boundary layer (see equation (5.31)), which is now of the Ekman–Hartmann type, then the spin-up time would be lengthened. However, the Lorentz forces contribute directly to the torque acting on the fluid and this results, even when the conductivity $\sigma_b$ of the boundary surfaces is zero, in a net *reduction* of the spin-up time.

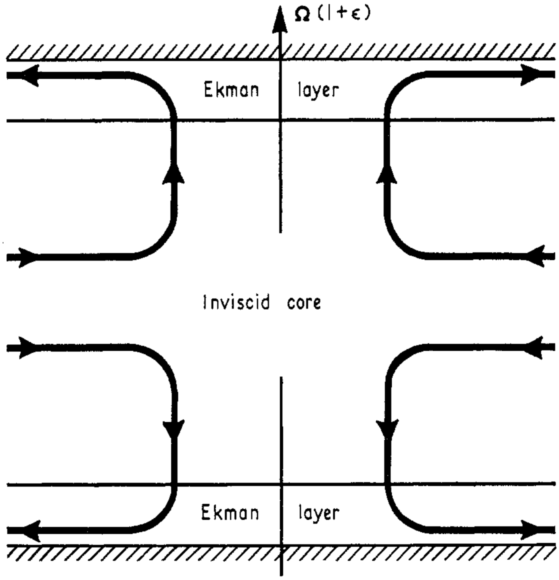


Figure 12. Schematic diagram illustrating the secondary circulation that 'spins-up' the interior flow when hydromagnetic effects are absent (see text).

Thus, for the case $\sigma_b = 0$ Loper and Benton (1970) found that outside the Ekman–Hartmann boundary layers the azimuthal velocity adjusts to the new rotation speed of the plates according to the formula

$$u_\theta = \Omega r[1 + \epsilon\{1 - \exp(-2s\Omega E^{1/2} t)\}] \tag{5.38}$$

where

$$s = \{\alpha + (1 + \alpha^2)^{1/2}\}^{1/2} \tag{5.39}$$

(see equation (5.20), cf equation (5.37)). Equation (5.38) is valid for small Ekman number $E$, small magnetic Prandtl number $\nu/\eta$ and for values of $\alpha$ of order unity or less.

A lucid account of this problem has been given by Benton and Loper (1969), Gilman and Benton (1968) and Loper and Benton (1970). Outside each Ekman–Hartmann layer of thickness $\sim s^{-1}(\nu/\Omega)^{1/2}$, attained rapidly before $t \sim \Omega^{-1}$, is an intermediate magnetic diffusion layer whose thickness grows comparatively slowly, as $2(t\eta)^{1/2}$, eventually attaining the value $\sim LE^{1/4}(s\nu/\eta)^{-1/2}$ or $L$, whichever is the smaller.

The former case, when $E^{1/4}(s\nu/\eta)^{-1/2} < 1$, is illustrated in figure 13. As found in § 5.2 the suction *at the edge of the Ekman–Hartmann layers* (EHL) is reduced by the action of the magnetic field. The imposed axial magnetic field is distorted by the velocity shear in the boundary layer so that the field develops an azimuthal component, associated with which is an axial 'Hartmann current' which passes between the Ekman–Hartmann layer and the magnetic diffusion region (MDR) (cf equation (5.34)). This current turns into the radial direction within the magnetic diffusion region and interacts with the imposed axial magnetic field to produce an axial Lorentz torque within the diffusion layer. Fluid is there propelled outwards by centrifugal force, and by continuity radial motion is induced in the opposite sense in the current-free part of the inviscid core. The suction velocity at the outer edge of the magnetic diffusion region must evidently therefore be greater than the corresponding suction into the Ekman–Hartmann layer; detailed analysis shows that the former is greater by a factor $s(\alpha)$ and the latter smaller by a factor $G(\alpha)$ (see equation (5.32)) than the suction into an ordinary Ekman layer (see equation (5.31)).

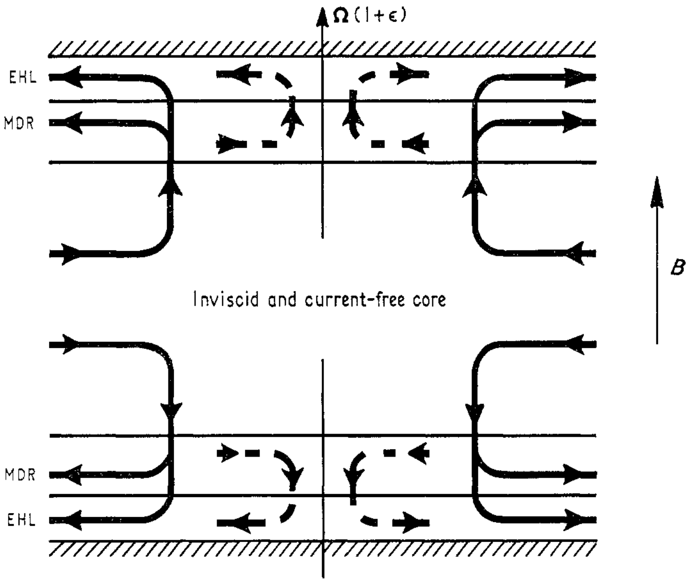


Figure 13. Schematic diagram illustrating the secondary circulation (full lines) and induced electric current (broken lines) that 'spin up' the interior flow when hydromagnetic effects are present. After Loper and Benton (1970).

Since the main part of the inviscid core is current-free it spins up by the process outlined at the beginning of this section, but faster than in the nonhydromagnetic case by a factor $s$ in view of the net *enhancement* of the suction into the double-decked boundary layers. Loper and Benton (1970) have in fact shown that outside the Ekman–Hartmann layers the fluid spins up in a time equal to $(2s\Omega E^{1/2})^{-1}$ (see equation (5.38)), irrespective of whether the magnetic diffusion regions penetrate the whole or only a limited part of the inviscid interior during the spin-up process.

So far nothing has been said about effects on 'spin-up' due to the presence of side walls parallel to $\boldsymbol{\Omega}$ (and $\boldsymbol{B}$). These effects are not fully understood but they are expected to be small, this being the justification for their neglect in the foregoing discussion. Their treatment involves the analysis of the very complex structure of the side wall boundary layers, whose overall maximum thickness will in general be greater than the thickness of the Ekman–Hartmann layer. The structure of such layers in related steady problems has been discussed by Ingham (1969) for the low magnetic Reynolds number case $\mathscr{S} \ll 1$ (see equation (2.24)) when nonlinear effects are negligible, and by several authors for various nonhydromagnetic cases. When in the latter case nonlinear inertial effects are negligible (ie when $\epsilon \ll E^{1/4}$) the side wall boundary layers have the Stewartson thickness $\sim LE^{1/4}$ (see Greenspan 1968); otherwise the thickness depends in a complicated way on both $\epsilon$ and $E$ (see Barcilon 1970, Hide 1968).

Loper (1970, 1971) has extended the work described above to include the effects of conducting boundaries (ie $\sigma_b \neq 0$) and has shown that when $\sigma_b$ is an arbitrary even function of $z$ (the distance along the axis of rotation, measured from midway between the plates), the spin-up mechanism depends crucially on the ratio

$$\phi \equiv \frac{2}{\sigma L} \int_{L/2}^{\infty} \sigma_b(z)\, dz \tag{5.40}$$

(see Rochester 1970 for references to related work on electromagnetic coupling of the Earth's liquid core to the overlying solid mantle). Provided that $\phi$ is much less than unity the spin-up process has the same aperiodic character as found in the cases discussed above (ie $\phi = 0$), the spin-up time being $(2s_* \Omega E^{1/2})^{-1}$, where

$$s_* = \{\alpha + (1+\alpha^2)^{1/2}\}^{1/2} + 2\alpha\left(\frac{\Omega}{\nu}\right)^{1/2} \int_{L/2}^{\infty} \frac{\sigma_b(z)\, dz}{\sigma} \tag{5.41}$$

which reduces to equation (5.39) when $\sigma_b = 0$. If, on the other hand, $\phi \not\ll 1$ there are circumstances in which the spin-up process takes on an essentially oscillatory character (Loper 1971, cf Chawla 1972). When $\sigma_b$ is so large that $\phi \gg E^{1/2} \alpha^{-1}$ (but $\phi$ is much less than unity), the second term of equation (5.41) greatly exceeds the first and the spin-up is largely brought about not by viscous stresses, as in the Greenspan–Howard mechanism, but by 'electromagnetic coupling' due to electric currents leaking out of the fluid into the conducting walls of the container. The spin-up time $T_s$ is then given by

$$T_s \simeq \left(2V^2 \mu \int_{L/2}^{\infty} \frac{\sigma_b\, dz}{L}\right)^{-1} \tag{5.42}$$

and is independent of $\Omega$, $\nu$ and $\eta$.

For the Earth we have $\phi \lesssim 10^{-4}$ and $E \lesssim 10^{-6}$, so that electromagnetic coupling is probably more important than viscous. Unfortunately neither mechanism seems adequate to account for the observed variations in the length of the day unless high frequency magnetic variations occur in the core (Hide 1966a, see also Rochester 1970). It is partly for this reason that effects due to topographic features of the core–mantle interface have been invoked (Hide 1969b, 1970).

## 6. Some effects due to density inhomogeneities

### 6.1. *Preamble*

In this final section we shall consider how some of the phenomena discussed in §§ 3 and 4 are affected by buoyancy forces produced by the action of gravity on density inhomogeneities. The appropriate momentum equation for an incompressible fluid referred to a frame rotating with uniform angular velocity $\mathbf{\Omega}$ is then

$$\rho(\partial \boldsymbol{u}/\partial t + \boldsymbol{u} . \nabla \boldsymbol{u} + 2\mathbf{\Omega} \times \boldsymbol{u}) = -\nabla p + \mu^{-1}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu\rho\nabla^2 \boldsymbol{u} + \rho \boldsymbol{g} \tag{6.1}$$

where $\boldsymbol{g} = -\nabla\Phi$ represents the gravitational and centrifugal body force per unit mass (cf equations (2.19) and (4.1)). As before, the continuity equation is

$$\nabla . \boldsymbol{u} = 0 \tag{6.2}$$

and the electromagnetic equations are

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \tag{6.3}$$

and

$$\nabla . \boldsymbol{B} = 0 \tag{6.4}$$

(see equations (2.20), (2.22), (2.12)). These must now be supplemented by an equation of state, which for a liquid whose density depends on temperature alone takes the form

$$\rho = \rho_0(1 - \vartheta T) \tag{6.5}$$

and a heat transfer equation such as

$$\frac{\mathrm{D}T}{\mathrm{D}t} = K\nabla^2 T + q \tag{6.6}$$

(see equations (2.7) and (2.21)). When $q = 0$, so that the change in density of a moving fluid element is due solely to thermal conduction, we have by equations (6.5) and (6.6)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = K\nabla^2 \rho. \tag{6.7}$$

## 6.2. *Some theorems*

When $\Omega$ is so large that the dimensionless parameters $\mathscr{A}$, $\mathscr{R}$, $\mathscr{M}$, $\mathscr{E}$ (see equations (3.1)–(3.4)) all tend to zero, then by taking the curl of equation (6.1) and making use of equation (2.1) (the full continuity equation, cf equation (6.2), the validity of the main result of this section not being restricted to incompressible fluids) we obtain

$$2(\boldsymbol{\Omega} . \nabla)(\rho \boldsymbol{u}) = \boldsymbol{g} \times (\nabla \rho) - 2\boldsymbol{\Omega} \frac{\partial \rho}{\partial t}. \tag{6.8}$$

When the motion is steady (so that $\partial \rho / \partial t = 0$) this equation expresses an exact balance between the gyroscopic and gravitational torques acting on unit volume of an individual fluid element. In contrast to the case when $\nabla \rho = 0$, the motion of a nonhomogeneous fluid is evidently not constrained by rotation to be the same in all planes normal to the rotation axis (cf equation (3.6)). Nevertheless a careful study of equation (6.8) shows that steady flows can exhibit certain important two-dimensional features even when $\nabla \rho \neq 0$ (Hide 1971c), for equation (6.8) may then be written

$$2\Omega \frac{\partial}{\partial s}(\rho u, \rho v, \rho w) = \left(-g\frac{\partial \rho}{\partial y}, g\frac{\partial \rho}{\partial x}, 0\right) \tag{6.9}$$

where $\boldsymbol{g} = (0, 0, g)$ and $\mathrm{d}s$ denotes an element of length parallel to $\boldsymbol{\Omega}$. Thus

$$2\Omega \frac{\partial}{\partial s}\left(\frac{v}{u}\right) = \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right)\frac{g}{\rho u^2}. \tag{6.10}$$

In the isentropic limit ($q \to 0$, $K \to 0$, see Hide 1971c) individual fluid elements conserve their density as they move (see equation (6.7)), so that since the motion is steady

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \tag{6.11}$$

and equation (6.10) may be written in the form

$$\frac{\partial}{\partial s}\left(\frac{v}{u}\right) = -\frac{g}{\rho}\frac{\partial \rho}{\partial z}\frac{w}{2\Omega u^2}. \tag{6.12}$$

Thus, in the case of wide practical interest when

$$\left|\frac{g}{\rho}\frac{\partial \rho}{\partial z}\frac{wL}{2\Omega uv}\right| \ll 1$$

where $L$ is a typical linear dimension of the fluid container in the direction of $\boldsymbol{\Omega}$, the horizontal flow *pattern* changes very slowly with $s$, notwithstanding the possibility of large gradients of flow *speed* in the $s$ direction (Hide 1971c).

We were prompted by equation (6.12) to investigate the possibility that analogous theorems exist for *magneto*strophic flows (ie those satisfying equation (2.27) with the right-hand side set equal to zero). So far nothing particularly straightforward has been found, although it is possible that success might ultimately be achieved in this direction by carrying out formal mathematical transformations of the governing equations (see eg Calkin 1963, Elsasser 1956). We should, however, mention a result given by J B Taylor (1963) and subsequently generalized by Soward (1970 private communication), namely that an inviscid incompressible rotating fluid enclosed within a rigid boundary which is a surface of revolution can execute magnetostrophic motions only if the Lorentz couple in the direction of $\boldsymbol{\Omega}$ exerted over any imaginary cylindrical surface in the fluid coaxial with $\boldsymbol{\Omega}$ vanishes. This important result has proved useful in various 'dynamo' studies (see §§ 1.2 and 2.5).

### 6.3. *Plane waves*

Theoretical treatments of fluid motion taking density stratification into account are greatly simplified when variations of density are so weak that the 'Boussinesq approximation' can be used. Thus, when $\boldsymbol{g}$ greatly exceeds all the acceleration terms ($\partial \boldsymbol{u}/\partial t$ etc) on the left-hand side of equation (6.1), variations in $\rho$ can be neglected in all but the buoyancy term $\rho \boldsymbol{g}$.

Let the undisturbed steady state be one of rigid-body rotation ($\boldsymbol{u} = 0$) in the presence of a uniform magnetic field $\boldsymbol{B}_0$, with a density distribution with no horizontal component (ie $\rho_0 = \rho_0(z)$ if $\boldsymbol{g} = (0, 0, g)$). In the first instance we shall neglect dissipative effects, so that $\nu = \eta = K = 0$ (but see equation (6.22)). Small perturbations $\boldsymbol{u}_1(\boldsymbol{r}, t), p_1(\boldsymbol{r}, t), \boldsymbol{B}_1(\boldsymbol{r}, t), \rho_1(\boldsymbol{r}, t)$ are then related by linear equations (obtained from equations (6.1)–(6.7)). Furthermore, the coefficients of these equations are constant if the 'Brunt–Väisälä frequency'

$$N = \left(\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\right)^{1/2} \tag{6.13}$$

(which is real when, as we shall assume throughout this section, the density distribution is 'bottom heavy') is uniform. They accordingly admit constant-amplitude plane wave solutions for which all perturbation quantities are proportional to $\exp \mathrm{i}(\boldsymbol{\kappa} . \boldsymbol{r} - \omega t)$, the angular frequency $\omega$ and wavenumber vector $\boldsymbol{\kappa} = (k, l, m)$ being related by

$$\omega^4 - \left(2(\boldsymbol{V}.\boldsymbol{\kappa})^2 + \frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{\kappa^2} + \frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\right)\omega^2 + (\boldsymbol{V}.\boldsymbol{\kappa})^2\left(\frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{\kappa^2} + (\boldsymbol{V}.\boldsymbol{\kappa})^2\right) = 0 \tag{6.14}$$

where $\boldsymbol{N} = (0, 0, N)$ (Hide 1969c, Kalra 1969, cf Ariel 1970, Talwar 1960, 1962).

For a homogeneous fluid $N = 0$ and equation (6.14) reduces to (4.11), namely $\omega^2 \pm (2\boldsymbol{\Omega}.\boldsymbol{\kappa})\,\omega/\kappa - (\boldsymbol{V}.\boldsymbol{\kappa})^2 = 0$. In the absence of both rotation and hydromagnetic effects, on the other hand, the solutions are

$$\omega^2 = \frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{\kappa^2} \tag{6.15}$$

representing 'internal' waves with buoyancy effects alone providing the restoring forces (see eg Veronis 1970, Yih 1965).

Of especial interest here is the effect of buoyancy forces on the magnetostrophic hydromagnetic-inertial waves. When $\Omega$ is so large that

$$\frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2} \gg (\boldsymbol{V}.\boldsymbol{\kappa})^2 + \frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{\kappa^2}$$

the solutions of equation (6.14) are

$$\omega_+{}^2 \simeq \frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2} \tag{6.16a}$$

and

$$\omega_-{}^2 \simeq \frac{(\boldsymbol{V}.\boldsymbol{\kappa})^2\{(\boldsymbol{V}.\boldsymbol{\kappa})^2\kappa^2 + (\boldsymbol{N}\times\boldsymbol{\kappa})^2\}}{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2} \tag{6.16b}$$

(cf equations (4.14*a,b*)). As in the homogeneous case, rapid rotation produces strong 'frequency splitting'. The roots $\omega_+{}^2$ represent inertial waves propagating very rapidly compared with the Alfvén speed $V$ and the roots $\omega_-{}^2$ represent waves propagating very slowly compared with $V$. Equation (6.16*b*) illustrates the significance of the parameter

$$\mathscr{B} \equiv N^* L^*/V^* \tag{6.17}$$

as a measure of the importance of density stratification in magnetostrophic motion. When $\mathscr{B} \ll 1$ the slow hydromagnetic waves are almost unaffected by the action of the buoyancy force and equations (6.16*b*) and (4.14*b*) become identical. When, on the other hand, $\mathscr{B} \gg 1$ the 'slow' wave dispersion relationship takes the hybrid form

$$\omega_-{}^2 \simeq \frac{(\boldsymbol{V}.\boldsymbol{\kappa})^2(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}. \tag{6.18}$$

The action of the buoyancy force evidently has the effect of speeding up the slow hybrid waves, and that described by equation (6.18) propagates considerably faster than a 'hydromagnetic-inertial' wave in the homogeneous case by a factor of order $\mathscr{B}$ (cf equation (4.14*b*)). A stable density gradient also has the effect of opposing vertical motion, that associated with a wave described by equation (6.18) being smaller than the horizontal motion by a factor of order $\mathscr{B}$ (Acheson 1972b, cf Cowling 1957). As in the homogeneous case the ratio of magnetic to kinetic energy associated with any of the above waves is given by

$$\boldsymbol{B}_1{}^2/\mu\rho\boldsymbol{u}_1{}^2 = (\boldsymbol{V}.\boldsymbol{\kappa})^2/\omega^2. \tag{6.19}$$

Thus the kinetic energy of the slow waves (see equation (6.16*b*)) is negligible in comparison with their magnetic energy. While the gravitational potential energy is also negligible by comparison when $\mathscr{B} \ll 1$, it becomes more important as $\mathscr{B}$ increases, and when $\mathscr{B} \gg 1$ the waves that result (see equation (6.18)) are characterized by *equipartition* between magnetic and gravitational potential energy (Acheson 1972b).

If we set $V = 0$ in equation (6.14) we recover the dispersion relationship for waves in a rotating stratified fluid, namely

$$\omega^2 = \frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2+(2\boldsymbol{\Omega}\,.\,\boldsymbol{\kappa})^2}{\kappa^2}. \tag{6.20}$$

Observe that equation (6.20) goes formally to equation (6.14) if we replace $\boldsymbol{\Omega}$ by $\boldsymbol{\Omega}/\{1-(\boldsymbol{V}.\boldsymbol{\kappa})^2/\omega^2\}$ and $N^2$ by $N^2/\{1-(\boldsymbol{V}.\boldsymbol{\kappa})^2/\omega^2\}$. We also note in passing that the common denominator $1-(\boldsymbol{V}.\boldsymbol{\kappa})^2/\omega^2$ is the same as the denominator on the right-hand side of equation (3.16) for 'aligned field' flows if the velocity $\boldsymbol{U}$ with which it is necessary to move in order to render the flow pattern of an individual wave steady (ie Doppler shifted frequency $\omega-\boldsymbol{U}.\boldsymbol{\kappa} = 0$) turns out to be parallel to $\boldsymbol{V}$. We note also that the analogy between rotating fluids and stratified fluids that several authors have discussed and exploited in various ways (eg Veronis 1970, Yih 1958) breaks down when hydromagnetic effects are present. This breakdown is illustrated by equation (6.20), which has no nonhydromagnetic counterpart to the 'slow' waves whose dispersion relations are given by equation (6.16*b*) (Hide 1969c, see also Chandrasekhar 1953).

When effects due to viscous and ohmic dissipation and thermal conduction are taken into account the dispersion relationship takes the form

$$\left(\omega+\mathrm{i}\nu\kappa^2-\frac{(\boldsymbol{V}.\boldsymbol{\kappa})^2}{\omega+\mathrm{i}\eta\kappa^2}\right)^2-\frac{(\boldsymbol{N}\times\boldsymbol{\kappa})^2}{\kappa^2(\omega+\mathrm{i}K\kappa^2)}\left(\omega+\mathrm{i}\nu\kappa^2-\frac{(\boldsymbol{V}.\boldsymbol{\kappa})^2}{\omega+\mathrm{i}\eta\kappa^2}\right)-\frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}=0. \tag{6.21}$$

When the exponential decay time $\tau$ is much longer than the wave period $2\pi/\omega$ it is given by

$$\tau \simeq \frac{\omega_V{}^2(\omega_V{}^2+\omega_N{}^2)-\omega^4}{\{(\omega_V{}^2-\frac{1}{2}\omega_N{}^2-\omega^2)(\omega^2\nu+\omega_V{}^2\eta)+\frac{1}{2}\omega_N{}^2(\omega_V{}^2-\omega^2)K\}\kappa^2} \tag{6.22}$$

where $\omega_V{}^2 \equiv (\boldsymbol{V}.\boldsymbol{\kappa})^2$ and $\omega_N{}^2 \equiv (\boldsymbol{N}\times\boldsymbol{\kappa})^2/\kappa^2$ (cf equation (4.16)). We limit our discussion to the case of a 'rapidly' rotating system such that $\omega_\Omega{}^2 \gg \omega_V{}^2+\omega_N{}^2$ (where $\omega_\Omega{}^2 \equiv (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2$), for which there are two possible classes of wave motion (see equations (6.16*a*,*b*)), namely $\omega^2 \simeq \omega_\Omega{}^2$ and $\omega^2 \simeq \omega_V{}^2(\omega_V{}^2+\omega_N{}^2)/\omega_\Omega{}^2$. In the former case, that of the inertial wave,

$$\tau \simeq \left(\nu+\frac{\omega_V{}^2}{\omega_\Omega{}^2}\eta+\frac{1}{2}\frac{\omega_N{}^2}{\omega_\Omega{}^2}K\right)^{-1}\kappa^{-2} \tag{6.23}$$

the small factors multiplying $\eta$ and $K$ in this equation being equal respectively to the magnetic energy and the gravitational potential energy in the wave measured in terms of the kinetic energy. For the slow wave, on the other hand,

$$\tau \simeq \frac{\omega_V{}^2+\omega_N{}^2}{\omega_V{}^2+\frac{1}{2}\omega_N{}^2}\left\{\eta+\left(\frac{\omega_V{}^2+\omega_N{}^2}{\omega_\Omega{}^2}\right)\nu+\left(\frac{\frac{1}{2}\omega_N{}^2}{\omega_V{}^2+\omega_N{}^2}\right)K\right\}^{-1}\kappa^{-2}. \tag{6.24}$$

### 6.4. *Thermal instability*

Consider a horizontal layer of fluid of depth $d$ under the influence of gravity $\boldsymbol{g} = (0, 0, g)$. When heated from below it has a natural tendency to overturn, for on account of thermal expansion the fluid at the bottom will be lighter than the fluid on the top, the corresponding Brunt–Väisälä frequency $N$ then being imaginary (see equation (6.13)). The direct effect of viscosity and thermal conduction is to inhibit this tendency to overturn; thus thermal instability cannot arise unless

$$R \equiv |N^2| d^4/\nu K \tag{6.25}$$

(the Rayleigh number) exceeds a certain critical value $R_c$. The purpose of this section is to outline some effects on this system of rotation about the vertical axis with angular velocity $\mathbf{\Omega}$ and/or a vertical magnetic field $\boldsymbol{B}$.

Instability may set in aperiodically (convection) or as growing oscillations (overstability). Which mode of instability has the lower critical value of $R$ and therefore prevails will in general depend in a complicated way on the parameters of the problem. This has been discussed in detail by Chandrasekhar (1961) and more recently by Eltayeb (1972). We shall confine our attention here to aperiodic instabilities (but see § 6.5), which are expected to be more important than overstable modes when $K/\eta$ and $K/\nu$ are not too large (Chandrasekhar 1961, Eltayeb 1972, Namikawa 1961).

It is readily shown (Chandrasekhar 1961) that when both boundaries are free the dispersion relation (6.21) (which, though equally valid for $N$ imaginary, was derived above in a quite different context) is applicable, and on setting $\omega = 0$ in order first to obtain an expression for the critical Rayleigh number at the onset of aperiodic instability when $V = 0$ we find

$$R = \left((k^2+l^2+m^2)^3 + \frac{4\Omega^2 m^2}{\nu^2}\right)\frac{d^4}{(k^2+l^2)}. \tag{6.26}$$

The boundary conditions imply that $m = n\pi/d$ where $n$ is an integer, and our task is to find the minimum value of $R$ when considered a function of $k$, $l$ and $n$. Clearly this will arise when $n = 1$ and we find by differentiating equation (6.26) with respect to the square of the total horizontal wavenumber $k_*{}^2 = k^2 + l^2$ that $R$ achieves its minimum value when

$$(k_*{}^2 + \pi^2/d^2)^2(2k_*{}^2 - \pi^2/d^2) = 4\Omega^2\pi^2/\nu^2 d^2. \tag{6.27}$$

Thus when $\Omega = 0$, $2k_*{}^2 = \pi^2/d^2$ and the critical Rayleigh number $R_c$ is $27\pi^4/4 \simeq 628$. Inspection of equations (6.26) and (6.27) reveals that as $\Omega$ increases from zero both $R_c$ and the critical value of $k_*$ steadily increase, so that rotation exerts a stabilizing influence on the system. Indeed, defining the 'Taylor number'

$$\mathscr{T} \equiv 4\Omega^2 d^4/\nu^2 \tag{6.28}$$

(cf equation (5.36)), we find that as $\mathscr{T} \to \infty$, $k_*$ and $R_c$ take the asymptotic form

$$\left.\begin{aligned} k_* &\sim (\tfrac{1}{2}\mathscr{T}\pi^2)^{1/6} d^{-1} \\ R_c &\sim 3(\tfrac{1}{2}\mathscr{T}\pi^2)^{2/3}. \end{aligned}\right\} \tag{6.29}$$

The stabilizing influence of rotation may evidently be attributed to its tendency to impose a 'two-dimensionality' on the system, thus making it more difficult for the fluid to overturn. We note in passing that in this asymptotic limit viscosity acts as a *destabilizing* influence, precisely because its opposition to the gyroscopic constraints more than offsets its direct effect as a dissipative agency (cf equation (4.35)).

In view of the tendency for a strong uniform impressed magnetic field to render (nonrotating) hydromagnetic flows two-dimensional when the 'Hartmann number'

$$M \equiv Vd/(\nu\eta)^{1/2} \tag{6.30}$$

is much greater than unity, as exemplified by channel flows (see Shercliff 1965), it is natural to expect that in the absence of rotation a vertical magnetic field will oppose convection. This expectation is confirmed by an analysis similar to that above. Setting $\Omega = 0$ and $V \neq 0$ in equation (6.21) and letting $M \to \infty$, the total horizontal wavenumber at marginal instability $k_*$ and the corresponding critical Rayleigh number $R_c$ satisfy

$$\left.\begin{aligned} k_* &\sim (\tfrac{1}{2}M^2\pi^4)^{1/6}\,d^{-1} \\ R_c &\sim \pi^2 M^2. \end{aligned}\right\} \tag{6.31}$$

Chandrasekhar (1954b, 1956, 1961) was the first to consider the combined effects of a magnetic field and rotation. His calculation of the critical Rayleigh number $R_c$ as a function of $M^2$ for various values of the parameter $\mathscr{T}$ is illustrated in figure 14.

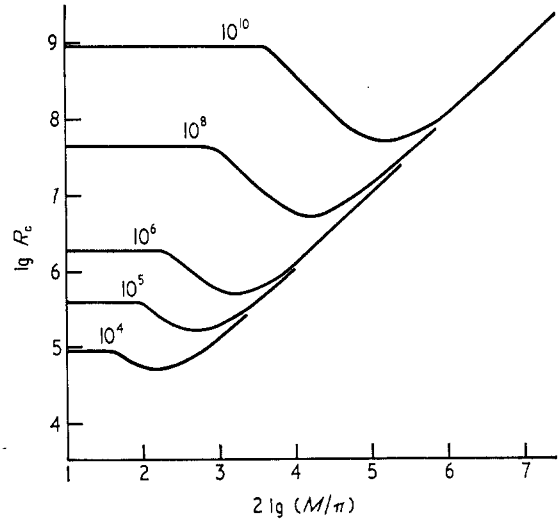


Figure 14. Illustrating the dependence on Hartmann number $M$ of the critical Rayleigh number $R_c$ for aperiodic instabilities at various values of $\mathscr{T}/\pi^4$, where $\mathscr{T}$ is the Taylor number (see equations (6.30), (6.25) and (6.28)). After Chandrasekhar (1961).

The most striking feature revealed by the calculation is the occurrence when $\mathscr{T} \neq 0$ of a range of values of $M$ within which $R_c$ *decreases* with increasing $M$, a result that seems insensitive to the orientation of $\boldsymbol{B}_0$ and $\boldsymbol{\Omega}$ as well as to the mechanical and electromagnetic boundary conditions (Eltayeb 1972, see also Eltayeb and Roberts 1970, Lehnert 1955b, Malkus 1959, Murphy and van de Borght 1972, Nakagawa 1957, 1959, 1962; see Lynden-Bell 1966, Tassoul 1967 for similar phenomena in *self-gravitational* (Jeans) instability, cf Chandrasekhar 1954a, Sengar and Khare 1972, Stephenson 1961).

Thus, the minima in the critical Rayleigh number curves correspond to a delicate balance between Coriolis and Lorentz forces, while elsewhere a strong 'two-dimensionality' prevails, being provided at lower values of $M$ by Coriolis forces and at higher values of $M$ by Lorentz forces. By the arguments given in § 3.5 we expect that at the minima the Chandrasekhar number

$$\mathscr{C} \equiv \frac{V^2}{2\Omega\eta} = \frac{M^2}{\mathscr{T}^{1/2}} \tag{6.32}$$

will be of order unity and that $k_*$ will be of the order of $d^{-1}$ (cf (6.29) and (6.31)).

Detailed calculations show that for large values of $M$ and $\mathscr{T}$ the minima occur when $\mathscr{C} = \sqrt{3}$ and $k_* d = \pi\sqrt{2}$ (Acheson 1972b). The latter result further reflects the breakdown of 'two-dimensionality' due to the magnetostrophic nature of the motion, for unlike the case of high rotation alone ($\mathscr{T} \gg 1$, $M = 0$) and high field alone ($M \gg 1$, $\mathscr{T} = 0$) the horizontal and vertical scales of motion are comparable at the onset of instability.

When the possibility of overstable modes is taken into account the dip in the curves of marginal stability in the $R_c$ against $M$ diagram is much less pronounced than in figure 14 but has nevertheless been observed experimentally by Nakagawa (1957, 1959). These experiments also confirm another prediction of the theory concerning the behaviour near the minima, where $\mathscr{C} \sim 1$, namely a sudden increase (followed by a more gradual decrease) of the cell width with increasing magnetic field.

### 6.5. *Wave generation*

In §4.4 we found that the instability of a nonuniform and curved magnetic field could generate slow hydromagnetic waves in a uniformly rotating fluid. The generation of such waves by an unstable *density* distribution has been demonstrated by Braginsky (1967b), and we now discuss some of the properties of the waves when both effects are present within the framework of the theory set out in §4.4 (see Acheson 1972a,b).

Consider the cylindrical system discussed in §4.4 but suppose now that the fluid density $\rho_0(r)$ varies with distance from the rotation axis, and let there be a net radial body force per unit mass $\boldsymbol{g} = \{g(r), 0, 0\}$ acting on the fluid. If the density gradient is very weak compared with the magnetic field gradient and variations of density over the range $r_1 \leqslant r \leqslant r_2$ are very small then the Boussinesq approximation is appropriate (see §6.3) and the basic equations of the cylindrical stability problem (4.36)–(4.39) then undergo simple modification. The first term in equation (4.39)

$$\frac{n^2}{\omega^2}\frac{r}{\mu\rho_0}\left(\frac{B_\theta{}^2}{r^2}\right)' \tag{6.33}$$

is simply replaced by

$$\frac{n^2}{\omega^2}\left\{\frac{r}{\mu\rho_0}\left(\frac{B_\theta{}^2}{r^2}\right)' - \frac{\rho_0' g}{\rho_0}\left(1+\frac{m^2}{r^2 n^2}\right)\right\}. \tag{6.34}$$

It is important to note that the (complex) eigenvalue $\omega$ comes out as a common factor. Thus the analysis of the stability of the system with respect to axisymmetric disturbances is identical in form to that of Michael (1954) for the case of constant density and it is easily shown that the system is stable to axisymmetric disturbances if *and only if*

$$4\Omega^2 - \frac{r}{\mu\rho_0}\left(\frac{B_\theta{}^2}{r^2}\right)' + \frac{\rho_0' g}{\rho_0} > 0 \tag{6.35}$$

everywhere in the fluid (Acheson 1972b, cf equation (4.33), see also Rudraiah and Murthy 1971, Murthy *et al* 1972, Rudraiah 1964). (We note that when magnetic effects are negligible and $\rho_0' g < 0$ equation (6.35) gives the critical value of $\Omega$ required to prevent Rayleigh–Taylor instability in the case when $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{g}$; see Chandrasekhar 1961). As in the homogeneous case such axisymmetric instabilities do not take the form of growing waves (for the eigenvalues $\omega$ are either real or purely imaginary) and furthermore the instabilities are unlikely to occur for 'rapid' rotation rates (ie when both parameters $\mathscr{L}$ and $|N^2|/\Omega^2$ are very much less than unity). This is once again a feature only of axisymmetric disturbances, which do not twist the lines of force of the azimuthal magnetic field.

The analysis of nonaxisymmetric disturbances follows that indicated by equations (4.40)–(4.43) and it is immediately evident by inspection that (i) all unstable modes propagate westward, regardless of the details of *both* magnetic field *and* density profiles, and (ii) such unstable modes having $|m|>1$ and $V_\theta^2 m^2 > 3r^2\omega_R^2$ everywhere can only occur if

$$\frac{r}{\mu\rho_0}\left(\frac{B_\theta^2}{r^2}\right)' - \frac{\rho_0' g}{\rho_0}\left(1+\frac{m^2}{r^2 n^2}\right) > 0 \tag{6.36}$$

is satisfied somewhere in the fluid. Once again the term $4\Omega^2$, which so far as axisymmetric modes are concerned represents a potent stabilizing influence at 'rapid' rotation rates, is not present in the inequality (6.36) for the nonaxisymmetric case (cf equation (6.35)). In §4.4 an explicit expression for the eigenvalues $\omega$ was given for modes of very short radial and axial wavelengths in a homogeneous fluid; the corresponding result for a nonhomogeneous fluid is

$$\frac{2\omega\Omega R^2}{mV_\theta^2(R)} \simeq -2 \pm \left\{m^2\left(1+\frac{l^2}{n^2}\right) - \left(\frac{r(B_\theta^2/r^2)'}{B_\theta^2/r^2} - \frac{\rho_0' g r^2}{\rho_0 V_\theta^2}\right)_{r=R}\right\}^{1/2} \tag{6.37}$$

(cf equation (4.44), see also Braginsky 1967b).

We therefore now have two mechanisms for generating slow hydromagnetic waves in a uniformly rotating fluid, namely (i) spatial variations in the magnetic field such that $(B_\theta^2/r^2)' > 0$ and (ii) a 'top heavy' density gradient such that $\rho_0' g/\rho_0 = N^2 < 0$. It is clearly seen that the effects of a destabilizing magnetic field configuration can be overcome by a sufficiently strong 'bottom heavy' density gradient, which, if very large, considerably shortens the periods of the slow oscillations (see equation (6.18)).

We recall here (as noted in §4.5) that analyses of hydromagnetic wave motion in a thin spherical shell bounded by latitude circles, either by use of the '$\beta$ plane' or by taking the spherical geometry more fully into account (Acheson 1971, 1972b), have shown that such a system is stable for any azimuthal magnetic field varying with latitude (cf Suffolk and Allan 1969). The reason for this is not entirely clear, for magnetic field variations in a 'fat' body are evidently capable of giving rise to instabilities (see §4.4). We note here, however, one possible cause, for both (*a*) shrinking a thick spherical shell until it becomes thin and (*b*) imposing a spherically symmetric 'bottom heavy' density distribution have a stabilizing influence and they have one further element in common—a tendency to suppress vertical motion. The absence of this instability mechanism in a thin spherical shell has also been noted by Gilman (1967a) in the course of an extensive study of hydromagnetic effects on *baroclinic* instability associated with horizontal density gradients in a rapidly rotating fluid, and it is illuminating to consider finally the major conclusion of that work pertinent to the generation of 'slow' hydromagnetic waves.

Consider a fluid contained in a cylindrical annulus of rectangular cross section which rotates about the axis of symmetry, the $z$ axis, when the acceleration of gravity $\boldsymbol{g}$ has no radial component (ie $\boldsymbol{g} = (0, 0, g)$) and the axially symmetric impressed density field $\rho_0(r, z)$ is 'bottom heavy', that is to say the Brunt–Väisälä frequency

$N = (g\rho_0^{-1}\partial\rho_0/\partial z)^{1/2}$ is real. Associated with the radial horizontal density gradient $\partial\rho_0/\partial r$ is an azimuthal 'thermal wind' with velocity $U_\theta$ given by the equation

$$2\Omega\frac{\partial U_\theta}{\partial z} = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial r} \tag{6.38}$$

when $\Omega$ is so large that geostrophic balance obtains. The problem of the stability of such flows to nonaxisymmetric disturbances ('baroclinic waves'), of central importance in dynamical meteorology, has been the subject of numerous theoretical, experimental and numerical studies (see Lorenz 1967, Hide 1969d).

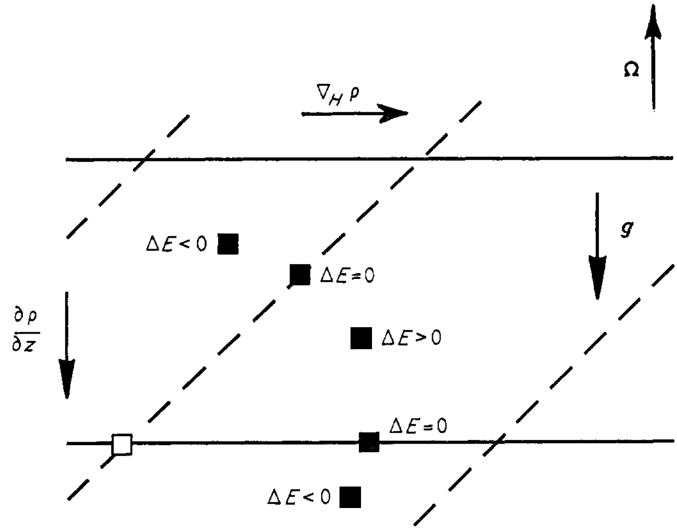


Figure 15. Illustrating the energetics of baroclinic instability. The full lines are geopotentials and the broken lines are surfaces of equal density. By hypothesis the system is vertically stable, that is to say $\partial\rho/\partial z < 0$. Thus, the energy $\Delta E$ released when the fluid element indicated by a full black square is interchanged with the element indicated by an open square is negative or zero except when the surface containing the two elements slopes at an angle less than the slope $\tan^{-1}\{(\partial\rho/\partial z)/|\nabla_H \rho|)\}$ of the surfaces of equal density. Baroclinic instability arises when $\Omega$ is so large that typical trajectories of individual fluid elements are constrained by Coriolis forces to slope at such small angles to the horizontal, an effect which is diminished when Lorentz forces act in opposition to Coriolis forces.

Theory shows and experiments confirm that when the end walls are horizontal and dissipative effects are negligible it is the value of the parameter

$$\mathscr{F} \equiv \frac{N^2}{4\Omega^2}\frac{d^2}{(r_2-r_1)^2} \tag{6.39}$$

that determines whether or not 'baroclinic instability' occurs; the system is baroclinically stable or unstable according as $\mathscr{F}$ is greater or less than a certain critical value $\mathscr{F}_c$, which tends to the value 2·4 as the ratio of the gap width $(r_2-r_1)$ to the mean radius $\frac{1}{2}(r_1+r_2)$ tends to zero.

The tendency for rotation to promote baroclinic instability can be understood in terms of elementary parcel-interchange arguments (see figure 15). The interchange of two parcels in a horizontal plane or in a surface of constant density neither absorbs nor releases energy. Because the vertical density gradient is negative the interchange of parcels in a plane inclined at a greater angle to the horizontal than a surface of constant density raises the potential energy of the system and therefore absorbs energy. However, the interchange of parcels in planes less steeply inclined than the surfaces of constant density lowers the centre of gravity and thus releases potential energy. Baroclinic instability arises when $\Omega$ is so large that, owing to the tendency for Coriolis forces to reduce $\partial w/\partial z$ to zero (see equation (6.9)), the slopes of typical trajectories of individual fluid elements are less than the slopes of equal density surfaces. According to this physical interpretation of baroclinic instability, in the presence of a magnetic field rotation would (in a system bounded by horizontal end walls) be less effective as a destabilizing agent.

This conclusion is consistent with the results of Gilman's (1967a,b, 1969) extensive study of baroclinic instability in various hydromagnetic systems, carried out largely with applications to solar physics in mind. Gilman found that the phase speeds of the waves generated on a 'beta plane' (see §4.5) by baroclinic instability are bounded in the hydromagnetic case exactly as in the nonhydromagnetic case (see Pedlosky 1964), indicating that at low values of the parameter $\mathscr{L}$ only the fast planetary waves (see equation (4.52)) are likely to be generated by this mechanism.

This difficulty of generating 'slow' hydromagnetic-planetary waves by baroclinic instability may admit a simple physical interpretation (cf §6.4) for, as we have seen, such waves require a delicate balance between Coriolis and Lorentz forces. Just as the comparatively weak net constraints on the fluid motion implied by such 'magnetostrophic' balance promote instability in the system considered in the previous section, so they *oppose* it in the present system, where the reduction of vertical motion by gyroscopic forces is an essential ingredient of the instability mechanism.

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## References

Acheson D J 1971 *PhD Thesis* University of East Anglia
—— 1972a *J. Fluid Mech.* **52** 529–41
—— 1972b to be published
—— 1972c *J. Fluid Mech.* **53** 401–15
Akasofu S-I and Chapman S 1972 *Solar-terrestrial Physics* (Oxford: Clarendon Press)
Alfvén H 1942a *Ark. Mat. Astr. Fys.* **29** 1–7
—— 1942b *Nature, Lond.* **150** 405–6
Alfvén H and Fälthammar C G 1963 *Cosmical Electrodynamics* (London: Oxford University Press)
Ariel P D 1970 *Proc. Indian Nat. Acad. Sci.* **2** 230–45
Backus G 1968 *Phil. Trans. R. Soc.* A **263** 239–66
—— 1971 *Lect. Appl. Math.* **14** 1–106 (American Mathematical Society)
Baldwin P and Roberts P H 1972 *Phil. Trans. R. Soc.* A **272** 303–30
Barcilon V 1970 *Phys. Fluids* **13** 537–44
Batchelor G K 1967 *An Introduction to Fluid Dynamics* (London: Cambridge University Press)
Benton E R and Loper D E 1969 *J. Fluid Mech.* **39** 561–86

Birch F 1972 *Geophys. J. R. Astr. Soc.* **29** 373–87
Booker J R and Bretherton F P 1967 *J. Fluid Mech.* **27** 513–39
Braginsky S I 1964 *Geomag. Aeronomy* **4** 698–712
—— 1967a *Geomag. Aeronomy* **7** 323–9
—— 1967b *Geomag. Aeronomy* **6** 851–9
—— 1970a *Geomag. Aeronomy* **10** 1–8
—— 1970b *Geomag. Aeronomy* **10** 172–81
Bullard E C 1949 *Proc. R. Soc.* A **197** 433–53
Bullard E C and Gellman H 1954 *Phil. Trans. R. Soc.* A **247** 213–78
Busse F H 1968a *J. Fluid Mech.* **33** 577–89
—— 1968b *J. Fluid Mech.* **33** 739–51
—— 1972 *J. Geophys. Res.* **77** 1589–90
Calkin M G 1963 *Can. J. Phys.* **41** 2241–51
Carstoiu J 1964 *C. R. Acad. Sci., Paris* **258** 2745–7
Causse R and Poirier Y 1960 *C. R. Acad. Sci., Paris* **251** 1056–8
Chandrasekhar S 1953 *Proc. R. Soc.* A **216** 293–309
—— 1954a *Astrophys. J.* **119** 7–9
—— 1954b *Proc. R. Soc.* A **225** 173–84
—— 1956 *Proc. R. Soc.* A **237** 476–84
—— 1960 *Proc. Nat. Acad. Sci., Wash.* **46** 253–7
—— 1961 *Hydrodynamic and Hydromagnetic Stability* (London: Oxford University Press)
Chandrasekhara B C and Rudraiah N 1971 *Appl. Sci. Res.* **25** 179–92
Chang T S and Sartory W K 1967 *Proc. R. Soc.* A **301** 451–71
Chawla S S 1972 *J. Fluid Mech.* **53** 545–55
Cowling T G 1955 *Proc. R. Soc.* A **233** 319–22
—— 1957 *Magnetohydrodynamics* (New York: Interscience)
—— 1962 *Rep. Prog. Phys.* **25** 244–86
Cox A 1968 *J. Geophys. Res.* **73** 3247–60
Cox A and Cain J C 1972 *Trans. Am. Geophys. Un.* **53** 591–623
Datta N 1964 *J. Phys. Soc. Japan* **19** 2219–23
Datta N and Biswas K K 1971 *J. Phys. Soc. Japan* **31** 1259–69
Donnelly R J and Ozima M 1960 *Phys. Rev. Lett.* **4** 497–8
Edmonds F N 1958 *Phys. Fluids* **1** 30–41
Ekman V W 1905 *Ark. Mat. Astr. Fys.* **2** 1–52
Elsasser W M 1946a *Phys. Rev.* **69** 106–16
—— 1946b *Phys. Rev.* **70** 202–12
—— 1947 *Phys. Rev.* **72** 821–33
—— 1956 *Rev. Mod. Phys.* **28** 135–63
Eltayeb I A 1972 *Proc. R. Soc.* A **326** 229–54
Eltayeb I A and Roberts P H 1970 *Astrophys. J.* **162** 699–701
Ertel H 1942 *Met. Z.* **59** 277–81
Ertel H and Rossby C G 1949 *Geofis. Pura Appl.* **14** 189–93
Ferraro V C A and Plumpton C 1966 *An Introduction to Magneto-fluid Mechanics* (Oxford: Clarendon Press)
Gans R F 1970 *J. Fluid Mech.* **45** 111–30
—— 1971 *J. Fluid Mech.* **50** 449–68
—— 1972 *J. Geophys. Res.* **77** 360–6
Gardiner R B and Stacey F D 1971 *Phys. Earth Planet. Interiors* **4** 406–10
Gilman P A 1967a *J. Atmos Sci.* **24** 101–43
—— 1967b *J. Atmos Sci.* **24** 333–6
—— 1968 *Science* **160** 760–3
—— 1969 *J. Atmos. Sci.* **26** 1003–9
—— 1971 *Phys. Fluids* **14** 7–12
Gilman P A and Benton E R 1968 *Phys. Fluids* **11** 2397–401
Govindarajulu T 1971 *Def. Sci. J. (India)* **21** 171–4
Greenspan H P 1968 *The Theory of Rotating Fluids* (London: Cambridge University Press)
Greenspan H P and Howard L N 1963 *J. Fluid Mech.* **17** 385–404
Gupta A S 1968 *J. Sci. Engng Res.* **12** 103–9

HARTMANN J 1937 *K. Danske Vidensk. Selsk. Math.-fys. Medd.* **15** 6
HASIMOTO H 1959 *Phys. Fluids* **2** 337–8
HIDE R 1956 *Phys. Chem. Earth* **1** 94–137
—— 1966a *Phil. Trans. R. Soc.* A **259** 615–47
—— 1966b *Planet. Space Sci.* **14** 579–86
—— 1967 *Science* **157** 55–6
—— 1968 *J. Fluid Mech.* **32** 737–64
—— 1969a *J. Atmos Sci.* **26** 841–53
—— 1969b *Nature, Lond.* **222** 1055
—— 1969c *J. Fluid Mech.* **39** 283–8
—— 1969d *The Global Circulation of the Atmosphere* ed G A Corby (London: Royal Meteorological Society) pp196–221
—— 1970 *Q. J. R. Met. Soc.* **96** 579–90
—— 1971a *Nature, Lond.* **229** 114–5
—— 1971b *Nature Phys. Sci.* **233** 100–1
—— 1971c *J. Fluid Mech.* **49** 745–51
HIDE R and JONES M V 1972 *Scientific Papers, Meteorological Office, London* no33
HIDE R and MALIN S R C 1970 *Nature, Lond.* **225** 605–9
—— 1971a *Nature Phys. Sci.* **230** 63
—— 1971b *Nature Phys. Sci.* **232** 31–3
—— 1971c *Comments Earth Sci.: Geophys.* **2** 1–13
HIDE R and ROBERTS P H 1960 *Rev. Mod. Phys.* **32** 799–806
—— 1962 *Adv. Appl. Mech.* ed G Kuerti (New York: Academic Press) **7** pp215–316
HIDE R and STEWARTSON K 1972 *Rev. Geophys. Space Phys.* **10** 579–98
HIDE R and TITMAN C W 1967 *J. Fluid Mech.* **29** 39–60
HIGGINS G and KENNEDY G C 1971 *J. Geophys. Res.* **76** 1870–8
HOWARD L N and GUPTA A S 1962 *J. Fluid Mech.* **14** 463–76
HUNT J C R and LUDFORD G S S 1968 *J. Fluid Mech.* **33** 693–714
INGHAM D B 1969 *Phys. Fluids* **12** 389–96
IRVING E 1964 *Palaeomagnetism* (New York: Wiley)
JACOBS J A 1971 *Comments Earth Sci.: Geophys.* **2** 61–8
JACOBS J A, CHAN T and FRAZER M 1972 *Nature Phys. Sci.* **236** 24–5
KAKUTANI T 1962 *J. Phys. Soc. Japan* **17** 1496–506
KALRA G L 1969 *Publ. Astron. Soc. Japan* **21** 263–6
KALRA G L and TALWAR S P 1967 *Mon. Not. R. Astr. Soc.* **135** 391–400
KENDALL P C 1960 *Q. J. Mech. Appl. Math.* **13** 285–99
KHAN M A A and SRIVASTAVA R B 1970 *Indian J. Phys.* **44** 647–56
KIBBLE T W B 1966 *Classical Mechanics* (New York: McGraw-Hill)
KULIKOVSKY A G 1968 *Izv. Akad. Nauk SSSR Mekh. Zhidk. I Gaza* no2 p3
KURZWEG U H 1963 *J. Fluid Mech.* **17** 52–60
LAI W 1962 *Phys. Fluids* **5** 560–6
LEHNERT B 1954 *Astrophys. J.* **119** 647–54
—— 1955a *Astrophys J.* **121** 481–90
—— 1955b *Q. Appl. Math.* **12** 321–41
LIGHTHILL M J 1966 *J. Fluid Mech.* **26** 411–31
—— 1967 *J. Fluid Mech.* **27** 725–52
LONGUET-HIGGINS M S 1968 *Phil. Trans. R. Soc.* A **262** 511–607
LOPER D E 1970 *Phys. Fluids* **13** 2999–3002
—— 1971 *J. Fluid Mech.* **50** 609–23
LOPER D E and BENTON E R 1970 *J. Fluid Mech.* **43** 785–99
LORENZ E N 1967 *World Meteorological Organization Publication* no218
LUDWIEG H 1962 *Z. Flugwiss.* **10** 242–9
LUNDQUIST S 1952 *Ark. Fys.* **5** 297–347
LYNDEN-BELL D 1966 *Observatory* **86** 57–60
MALKUS W V R 1959 *Astrophys. J.* **130** 259–75
—— 1967a *J. Fluid Mech.* **28** 793–802
—— 1967b *Proc. Symp. Appl. Math.* **18** 64–77
—— 1968 *Science* **160** 259–64

Malkus W V R 1971 *Lect. Appl. Math.* **14** 207–28 (American Mathematical Society)
Michael D H 1954 *Mathematika* **1** 45–50
Moffatt H K 1965 *J. Fluid Mech.* **22** 521–8
—— 1970 *J. Fluid Mech.* **44** 705–19
—— 1972 *J. Fluid Mech.* **53** 385–99
—— 1973 *J. Fluid Mech.* in press
Murphy J O and van de Borght R 1972 *Proc. Aust. Sci. Ass.* **2** 215–31
Murthy S N, Rudraiah N and Mariyappa B V 1972 *Appl. Sci. Res.* **26** 285–306
Nakagawa Y 1957 *Proc. R. Soc.* A **242** 81–8
—— 1959 *Proc. R. Soc.* A **249** 138–45
—— 1962 *Bull. Soc. R. Sci. Liége* no11–12 685–95
Namikawa T 1955 *J. Geomag. Geoelec.* **7** 97–104
—— 1961 *J. Inst. Polytechnics, Osaka City Univ.* ser G **4** 125–32
Nanda R S and Mohanty H K 1971 *Appl. Sci. Res.* **24** 65–78
Nigam P D 1965 *Appl. Sci. Res.* B **12** 139–50
Nigam S D and Nigam P D 1963 *Proc. R. Soc.* A **272**, 529–41
Pao H-P 1966 *Phys. Fluids* **9** 1254–5
Parker E N 1955 *Astrophys. J.* **122** 293–314
Pathak R S and Pathak L 1971 *Indian J. Phys.* **45** 49–58
Pedley T J 1968 *J. Fluid Mech.* **31** 603–7
Pedlosky J 1964 *J. Atmos. Sci.* **21** 201–19
—— 1971 *Lect. Appl. Math.* **13** 1–60 (American Mathematical Society)
Plumpton C and Ferraro V C A 1955 *Astrophys. J.* **121** 168–74
Poirier Y and Robert J 1960 *C. R. Acad. Sci., Paris* **251** 2880–2 (in French)
Pop I 1969 *Rev. Math. Pures Appl., Buc.* **14** 1173–86
Prandtl L 1952 *Essentials of Fluid Dynamics* (London: Blackie)
Proudman J 1916 *Proc. R. Soc.* A **92** 408–24
Rao A R 1971 *J. Fluid Mech.* **46** 357–63
Rayleigh Lord 1920 *Scientific Papers* **6** 447–53 (London: Cambridge University Press)
Rickard J A 1970 *PhD Thesis* University of London
Rikitake T 1956 *Bull. Earthq. Res. Inst. Tokyo Univ.* **34** 139–56
—— 1966 *Electromagnetism and the Earth's Interior* (Amsterdam: Elsevier)
Roberts B and Sozou C 1971 *J. Plasma Phys.* **6** 249–55
Roberts P H 1964 *Proc. Camb. Phil. Soc.* **60** 635–51
—— 1967 *An Introduction to Magnetohydrodynamics* (London: Longmans)
—— 1971 *Lect. Appl. Math.* **13** 129–206 (American Mathematical Society)
Roberts P H and Soward A M 1972 *Ann. Rev. Fluid Mech.* **4** 117–54
Roberts P H and Stix M 1971 *The Turbulent Dynamo* (a translation of a series of papers by F Krause, K H Rädler and M Steenbeck) *National Center for Atmospheric Research, Boulder, Colorado, Tech. Note* IA–60
Rochester M G 1970 *Earthquake Displacement Fields and the Rotation of the Earth* ed L Mansinha *et al* (Dordrecht: Reidel) pp136–48
Rudraiah N 1964 *Appl. Sci. Res.* B **11** 105–17
Rudraiah N and Murthy S N 1971 *Current Sci. (India)* **40** 343
Rudraiah N and Venkatachalappa M 1972 *J. Fluid Mech.* **52** 193–206
Sakurai K 1967 *Report of Ionosphere and Space Research in Japan* **21** 107–12
Schubert G 1968 *Astrophys. J.* **151** 1099–110
Shercliff J A 1965 *A Textbook of Magnetohydrodynamics* (Oxford: Pergamon Press)
Soundalgekar V M and Pop I 1970 *Bull. Math. Soc. Sci. Math. Phys. Répub. Pop. Roum.* **14** 375–81
Spiegel E A and Veronis G 1960 *Astrophys. J.* **131** 442–7
Spitzer L 1956 *Physics of Fully Ionized Gases* (New York: Interscience)
Squire H B 1933 *Proc. R. Soc.* A **142** 621–8
Stacey F D 1969 *Physics of the Earth* (London: Wiley)
Stephenson G 1961 *Mon. Not. R. Astr. Soc.* **122** 455–9
Stewartson K 1957 *Z. Angew. Math. Phys.* **8** 290–7
—— 1960 *J. Fluid Mech.* **8** 82–96
—— 1967 *Proc. R. Soc.* A **299** 173–87

Stewartson K 1971 *Lect. Appl. Math.* **14** 229–37 (American Mathematical Society)
Suffolk G C J and Allan D W 1969 *The Application of Modern Physics to the Earth and Planetary Interiors* ed S K Runcorn (London: Wiley-Interscience) pp653–6
Talwar S P 1960 *J. Fluid Mech.* **9** 581–92
—— 1962 *Indian J. Phys.* **36** 351–8
Tassoul J L 1967 *Mon. Not. R. Astr. Soc.* **137** 1–13
Taylor G I 1917 *Proc. R. Soc.* A **93** 99–113
—— 1921 *Proc. Camb. Phil. Soc.* **20** 326–9
Taylor J B 1963 *Proc. R. Soc.* A **274** 274–83
Trehan S K and Nakagawa Y 1969 *Astrophys. J.* **158** 1075–80
Trocheris M G 1949 *Phil. Mag.* **40** 1143–54
Velikhov E P 1959 *J. Exp. Theor. Phys. (USSR)* **36** 995–8
Veronis G 1970 *Ann. Rev. Fluid Mech.* **2** 37–67
Vidyanidhi V 1969 *J. Math. Phys. Sci. (India)* **3** 193–9
Vidyanidhi V and Rao V 1968 *J. Phys. Soc. Japan* **25** 1694–710
Weiss N O 1971 *Q. J. R. Ast. Soc.* **12** 432–46
Yih C-S 1958 *Q. Appl. Math.* **16** 409–15
—— 1959 *J. Fluid Mech.* **5** 436–44
—— 1965 *Dynamics of Non-homogeneous Fluids* (New York: Macmillan)
Yukutake T 1970 *Comments Earth Sci.: Geophys.* **1** 55–64