Ejercicios (1.6.6)

2. Demuestre:
$$\begin{cases} a.\ \cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha) \\ b.\ 3\cos^2(\alpha)\sin(\alpha) - \sin^3(\alpha) = \sin(3\alpha). \end{cases}$$

→ Con la fórmula de De Moivre:

$$\cos(3\alpha) + i\sin(3\alpha) = [\cos\alpha + i\sin\alpha]^3$$
$$= \cos^3\alpha + 3\cos^2\alpha\, i\sin\alpha + 3\cos\alpha(i\sin\alpha)^2 + (i\sin\alpha)^3$$
$$= \cos^3\alpha + (3\cos^2\alpha\sin\alpha)i - 3\cos\alpha\sin^2\alpha - \sin^3\alpha\, i$$
$$= [\cos^3\alpha - 3\cos\alpha\sin^2\alpha] + i[3\cos^2\alpha\sin\alpha - \sin^3\alpha]$$

→ Igualamos las partes reales e imaginarias y obtenemos las identidades:

$$\cos(3\alpha) = \cos^3\alpha - 3\cos\alpha\sin^2\alpha \quad // \qquad \wedge \quad i\sin(3\alpha) = i[3\cos^2\alpha\sin\alpha - \sin^3\alpha]$$
$$\sin(3\alpha) = 3\cos^2\alpha\sin\alpha - \sin^3\alpha. \quad //$$

5. Encuentre las raíces de:

a) $(2i)^{1/2}$

$z = 2i = 2e^{i\pi/2}$

$z = 2e^{i\frac{\pi}{2}}e^{2\pi k i}$

$$\longrightarrow z^{1/2} = 2^{1/2} e^{i\frac{\pi}{4}} \cdot e^{\frac{i2\pi k}{2}} = 2^{1/2} e^{i[\frac{\pi}{4} + \pi k]}$$
$$z_0^{1/2} = 2^{1/2} e^{i\frac{\pi}{4}} = \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = (1+i) //$$
$$z_1^{1/2} = 2^{1/2} e^{i[\frac{\pi}{4}+\pi]} = \sqrt{2}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = (-1-i) //$$

b). $(1-\sqrt{3}i)^{1/2}$

$z = (1-\sqrt{3}i)e^{i2\pi k}$

$|z| = \sqrt{1+3} = 2$

$(1-\sqrt{3}i) = 2e^{-\frac{1}{3}\pi i}$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{1}{3}\pi$ →

$$z^{1/2} = 2^{1/2} e^{i\left(\frac{2\pi k - \frac{1}{3}\pi}{2}\right)} = 2^{1/2} e^{i\left(\pi k - \frac{1}{6}\pi\right)}$$
$$z_0^{1/2} = \sqrt{2}\, e^{-i\frac{1}{6}\pi} = \left(\sqrt{6}/2 - \sqrt{2}/2\, i\right) //$$
$$z_1^{1/2} = \sqrt{2}^{\frac{5}{6}\pi i} = \left(-\sqrt{6}/2 + \sqrt{2}/2\, i\right) //$$

Primavera

c) $(-1)^{1/3}$

$$z^{1/3} = e^{\left(\frac{\pi+2k\pi}{3}\right)i}$$

$z = -1 = e^{i\pi}$

$\theta = \pi$

$|z| = 1$

$$z_0^{1/3} = e^{i\frac{\pi}{3}} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) //$$

$$z_1^{1/3} = e^{i\pi} = (-1) //$$

$$z_2^{1/3} = e^{i\frac{5}{3}\pi} = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) //$$

d) $8^{1/6}$

$$z = 8e^{0i + 2\pi k i} \rightarrow z^{1/6} = 8^{1/6} \cdot e^{\frac{\pi k i}{3}}$$

$z = 8$

$|z| = 8$

$\theta = 0$

$$z_0 = 8^{1/6} \qquad z_1 = 8^{1/6} e^{\frac{\pi}{3}i}$$

$$z_2 = 8^{1/6} e^{\frac{2}{3}\pi i} \qquad z_3 = 8^{1/6} e^{\pi i}$$

$$z_4 = 8^{1/6} e^{\frac{4}{3}\pi i} \qquad z_5 = 8^{1/6} e^{5/3\pi i}$$

e) $(-8 - 8\sqrt{3}i)^{1/4}$

$$z = 16e^{i\frac{4}{3}\pi + 2\pi k}$$

$|z| = 16$

$$z^{\frac{1}{4}} = 2e^{i\left(\frac{\frac{4}{3}\pi + 2\pi k}{4}\right)}$$

$\theta = \pi + \tan\left(\frac{-8\sqrt{3}}{-8}\right) = \frac{4}{3}\pi$

$$z_0^{1/4} = 2e^{i\frac{\pi}{3}} \qquad z_2^{1/4} = 2e^{i\frac{4}{3}\pi}$$

$$z_1^{1/4} = 2e^{\frac{5}{6}\pi i} \qquad z_3^{1/4} = 2e^{\frac{11}{6}\pi i}$$

6) Demuestre que:

a) $\text{Log}(-ie) = 1 - \frac{\pi}{2} i$

$\text{Log}(-ie) = \text{Log}(e \cdot e^{i(-\frac{\pi}{2} + 2\pi k)}) = \ln e + i(-\frac{\pi}{2} + 2\pi k)$

$= 1 + i(-\frac{\pi}{2} + 2\pi k)$

Si $k = 0$: $= 1 - \frac{\pi}{2} i$ //

b) $\text{Log}(1-i) = \frac{1}{2}\ln(2) - \frac{\pi}{4} i$

$z = 1-i$

$|z| = \sqrt{2}$

$\theta = \tan^{-1}(-1) =$

$\text{Log}(1-i) = \ln|2^{1/2}| + i(-\frac{1}{4}\pi + 2\pi k)$

Si $k=0$: $= \frac{1}{2}\ln 2 - \frac{\pi}{4} i$

c) $\text{Log}(e) = 1 + 2n\pi i$

$z = e$

$|z| = e$

$\theta = 0$

$\text{Log}(e) = \ln|e| + i(0 + 2\pi n)$

$= 1 + 2n\pi i$ //

d) $\text{Log}(i) = (2n + \frac{1}{2})\pi i$

$z = i = e^{\pi/2 i}$

$|z| = 1$

$\theta = \pi/2$

$\text{Log}(i) = \ln(1) + i(\frac{\pi}{2} + 2n\pi)$

$= i\pi(\frac{1}{2} + 2n)$

$= (2n + \frac{1}{2})\pi i$ //