ACQUAINTANCE GRAPH PARTY PROBLEM

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Abstract. In this paper, I present a new, short solution to a problem that deals with the minimum number of empty and full triangles in a graph. Furthermore, I demonstrate a related result that deals with a minimum bound for the number of full triangles in a graph.

1. Introduction

Before we begin discussing the acquaintance graph party problem, let's introduce some preliminary definitions and corresponding examples.

Definition. *The* ***floor function*** *is a function that gives us the greatest integer that is less than or equal to a real number x, denoted $floor(x)$ or $\lfloor x \rfloor$.*

Definition. *The number of ways to choose k objects from a set of n objects is denoted as $\binom{n}{k}$ and is read as "n choose k". The formula for* ***n choose k*** *is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Definition. *A* ***graph*** *G consists of a vertex set $V(G)$ and an edge set $E(G)$, where each edge in $E(G)$ is a 2-element subset of $V(G)$.*

Definition. *Two vertices in a graph are* ***adjacent*** *if they are connected by an edge.*

Definition. *The* ***degree*** *of a vertex in a graph is the number of vertices adjacent to that vertex.*

Definition. *A* ***bipartite*** *graph (or bigraph) is a graph where V can be partitioned into two disjoint sets such that no two vertices within the same set are adjacent.*

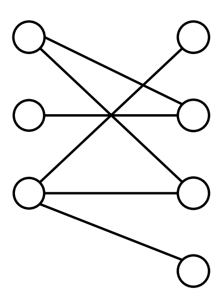

Figure 1. An example of a bipartite graph.

Definition. *A* ***complete bipartite*** *graph is a bipartite graph where every vertex in one set is adjacent to every vertex in the other set.*

Definition. *A* ***regular*** *graph is a graph where each vertex has the same degree. A graph is called K-regular if the degree of each vertex in the graph is K.*

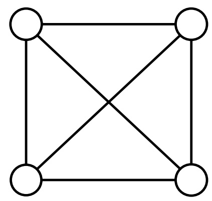

FIGURE 2. An example of a 3-regular graph.

2. Lower Bound For Empty and Full Triangles

In an article of *The American Mathematical Monthly* (REFERENCE), A. W. Goodman presents the following problem.

A **party** yields a graph G where people are represented as vertices and two vertices are adjacent if those two people know eachother, i.e. are acquaintances. A **full triangle** is a subset of three people who all know each other. In other words, three adjacent vertices. An **empty triangle** is a subset of three people who are all strangers. In other words, three non-adjacent vertices. Goodman proved the following theorem.

Theorem 1. *Let E and F be the number of empty and full triangles respectively. Then in every graph with p vertices*

$$E + F \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor \tag{1}$$

and this lower bound is sharp for each positive integer p.

Proof. Let P be the number of partial triangles in G, meaning the number of triangles containing exactly one or two edges. It's clear that

$$E + F + P = \binom{p}{3}. \tag{2}$$

Let d_i be the degree of a vertex v_i, in other words, the number of people acquainted with the i^{th} person. For each vertex v_i, picking one of their d_i acquaintances and one of their $p-1-d_i$ nonacquaintances produces a partial triangle. Thus, each v_i produces $d_i(p-1-d_i)$ partial triangles.

Furthermore, we note that every partial triangle is counted twice in this manner. To show this is true, let vertex a, b, and c represent v_i, one of their acquaintances, and one of their nonacquaintances respectively. Producing a partial triangle from v_i yields two cases.

Case 1: b and c are not acquainted. Then this partial triangle is counted when v_i is a and v_i is b.

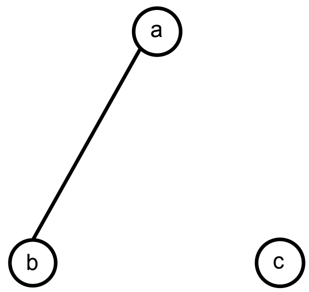

Figure 3. A visualization of Case 1.

Case 2: b and c are acquainted. Then this partial triangle is counted when v_i is a and v_i is c.

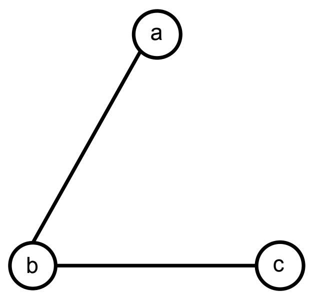

Figure 4. A visualization of Case 2.

In both cases, the partial triangle is counted twice, which follows for every partial triangle. Consequently,

$$P = \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i). \tag{3}$$

Looking back at Equation (2), we can minimize $E+F$ by maximizing P. To maximize P, we must maximize the sum in Equation (3). We can view each term of the sum as a quadratic function of d_i. After finding the derivative of the equation it's clear that we attain our maximum when $d_i = \frac{p-1}{2}$.

However, this is a contradiction when p is even since d_i is an integer. Thus, if p is odd, we attain the maximum value of $\frac{(p-1)^2}{4}$ for each term when $d_i = \frac{p-1}{2}$. If p is even, we attain the maximum possible value of $\frac{p(p-2)}{4}$ for each term when $d_i = \frac{p}{2}$ or $d_i = \frac{p-2}{2}$. In either case, we can express the maximum value as

$$\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor,$$

and so

$$P \leq \frac{1}{2}\sum_{i=1}^{p} \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor = \frac{p}{2}\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor. \tag{4}$$

But since P is an integer, we can strengthen this to read

$$P \leq \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \tag{5}$$

Equations (2) and (5) now yield our desired bound:

$$E + F = \binom{p}{3} - P \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \tag{6}$$

Next, for each p, we must find a graph G_p attaining this bound. But equality in Equation (1) is equivalent to equality in Equation (5) which occurs only when

$$P = \frac{1}{2} \sum_{i=1}^{p} d_i(p - 1 - d_i) = \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \tag{7}$$

If $p = 2n$ for some integer n, let G_p be the complete bipartite graph $K_{n,n}$. Now G_p is regular of degree n, and we must check that Equation (7) is satisfied.

$$\begin{aligned} P &= \frac{1}{2} \sum_{i=1}^{p} d_i(p - 1 - d_i) \\ &= \frac{1}{2} \sum_{i=1}^{2n} n(2n - 1 - n) \\ &= \frac{1}{2} \sum_{i=1}^{2n} n^2 - n \\ &= \frac{1}{2}(2n^3 - 2n^2) \\ &= n^3 - n^2 \end{aligned} \qquad \begin{aligned} P &= \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{2n}{2} \left\lfloor \left(\frac{2n-1}{2} \right)^2 \right\rfloor \right\rfloor \\ &= \left\lfloor n \left\lfloor \frac{4n^2 - 4n + 1}{4} \right\rfloor \right\rfloor \\ &= \left\lfloor n \left\lfloor n^2 - n + \frac{1}{4} \right\rfloor \right\rfloor \\ &= \left\lfloor n(n^2 - n) \right\rfloor \\ &= \left\lfloor n^3 - n^2 \right\rfloor \\ &= n^3 - n^2 \end{aligned}$$

We can see that Equation (7) holds when $p = 2n$.

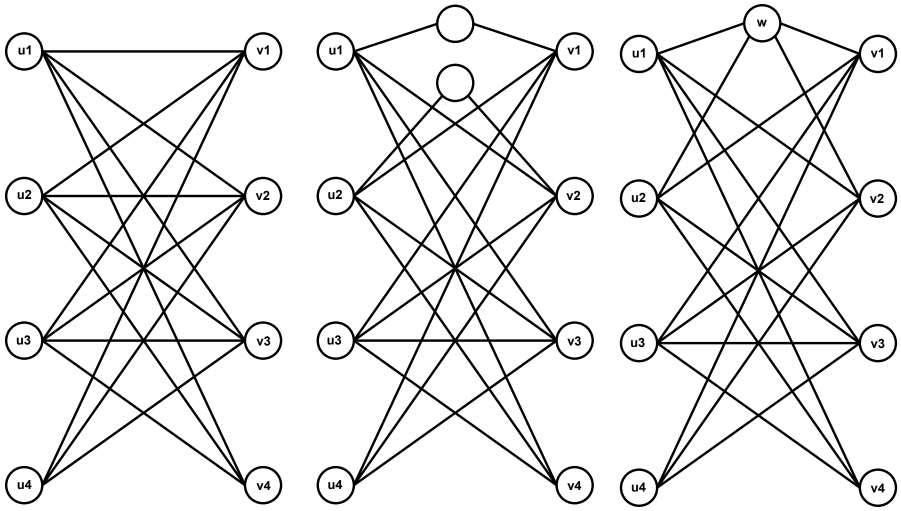

FIGURE 5. The construction of G_9.

If $p = 2n+1$, the construction of G_p is a bit more involved. As depicted for $p = 9$ in Figure (5), we start with $K_{n,n}$ with its vertices labeled $u_1, u_2, ..., u_n$; $v_1, v_2, ..., v_n$ and we subdivide edge u_iv_i for $i \leq \frac{n}{2}$. We now obtain G_p by using these $\lfloor \frac{n}{2} \rfloor$ subdivision points to form a single vertex labeled w. We observe that G_p has $2n$ vertices of degree n and one vertex of degree $2\lfloor \frac{n}{2} \rfloor$. It is routine to check that Equation (7) is satisfied.

$$\begin{aligned}
P &= \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i) \\
&= \frac{1}{2}\sum_{i=1}^{2n+1} d_i(2n-d_i) \\
&= \frac{1}{2}\left(\sum_{i=1}^{2n} n(2n-n) + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{2n} n^2 + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right)
\end{aligned}$$

$$\begin{aligned}
P &= \left\lfloor \frac{p}{2}\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor \right\rfloor \\
&= \left\lfloor \frac{2n+1}{2}\left\lfloor \left(\frac{2n+1-1}{2}\right)^2 \right\rfloor \right\rfloor \\
&= \left\lfloor \frac{2n+1}{2}\left\lfloor n^2 \right\rfloor \right\rfloor \\
&= \left\lfloor \left(\frac{2n+1}{2}\right) n^2 \right\rfloor \\
&= \left\lfloor \frac{2n^3+n^2}{2} \right\rfloor \\
&= \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor
\end{aligned}$$

At this point, we run into two cases.

Case 1: n is even. Let $n = 2t$ for some integer t.

$$
\begin{aligned}
P &= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n - 2\lfloor\frac{n}{2}\rfloor)\right)\\
&= \frac{1}{2}\left(2(2t)^3 + 2\lfloor\frac{2t}{2}\rfloor(2(2t) - 2\lfloor\frac{2t}{2}\rfloor)\right)\\
&= \frac{1}{2}\left(16t^3 + 2t(4t - 2t)\right)\\
&= \frac{1}{2}\left(16t^3 + 4t^2\right)\\
&= 8t^3 + 2t^2
\end{aligned}
$$

$$
\begin{aligned}
P &= \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor\\
&= \left\lfloor (2t)^3 + \frac{1}{2}(2t)^2 \right\rfloor\\
&= \left\lfloor 8t^3 + \frac{1}{2}4t^2 \right\rfloor\\
&= \left\lfloor 8t^3 + 2t^2 \right\rfloor\\
&= 8t^3 + 2t^2
\end{aligned}
$$

Case 2: n is odd. Let $n = 2t + 1$ for some integer t.

$$
\begin{aligned}
P &= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n - 2\lfloor\frac{n}{2}\rfloor)\right)\\
&= \frac{1}{2}\left(2(2t+1)^3 + 2\left\lfloor\frac{2t+1}{2}\right\rfloor(2(2t+1) - 2\left\lfloor\frac{2t+1}{2}\right\rfloor)\right)\\
&= \frac{1}{2}\left(2(8t^3 + 12t^2 + 6t + 1) + 2\lfloor t + \frac{1}{2}\rfloor(4t + 2 - 2\lfloor t + \frac{1}{2}\rfloor)\right)\\
&= \frac{1}{2}\left(16t^3 + 24t^2 + 12t + 2 + 2t(4t + 2 - 2t)\right)\\
&= \frac{1}{2}\left(16t^3 + 28t^2 + 16t + 2\right)\\
&= 8t^3 + 14t^2 + 8t + 1
\end{aligned}
$$

$$
\begin{aligned}
P &= \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor\\
&= \left\lfloor (2t+1)^3 + \frac{1}{2}(2t+1)^2 \right\rfloor\\
&= \left\lfloor 8t^3 + 12t^2 + 6t + 1 + \frac{4t^2 + 4t + 1}{2} \right\rfloor\\
&= \left\lfloor 8t^3 + 12t^2 + 6t + 1 + 2t^2 + 2t + \frac{1}{2} \right\rfloor\\
&= \left\lfloor 8t^3 + 14t^2 + 8t + 1 + \frac{1}{2} \right\rfloor\\
&= 8t^3 + 14t^2 + 8t + 1
\end{aligned}
$$

Thus, in both cases, Equation (7) holds. □

3. Lower Bound For Full Triangles

In a different article of *The American Mathematical Monthly* (REFERENCE), L. Suave noted that a when p is even, $E+F$ can be minimized while keeping $F=0$. However, when p is odd and greater than seven, $F>0$ for all graphs attaining the minimum for $E+F$. We will now refine this result.

Theorem 2. *In every graph attaining the minimum possible value for $E+F$,*

$$F \geq \begin{cases} 0 & \text{if } p = 2n \\ n(n-1) & \text{if } p = 4n+1 \text{ or } 4n+3 \end{cases} \tag{8}$$

and this lower bound is sharp for each positive integer p.

Proof. We first observe that this bound is attained by the graphs G_p constructed in the first proof. If $p=2n$, then $G_p=K_{n,n}$ which obviously has no full triangles. If $p=4n+1$ or $p=4n+3$, we notice that G_p-w is a bigraph, which has no full triangles. Thus, every full triangle of G_p has the form u_i, v_j, w. But recalling the construction of G_p, we see that this will be a full triangle if and only if $i \leq n$, $j \leq n$, and $i \neq j$. Consequently, G_p has $F=n(n-1)$ full triangles as desired.

It remains to be shown that the bound in Equation (8) cannot be violated. If p is even, then this is trivial, since we cannot have a negative number of full triangles. If p is odd and $n=0$ or $n=1$, then this is also trivial, since $n(n-1)=0$. Therefore, we will assume $n \geq 2$.

Case 1: $p=4n+1$

Let H be a graph minimizing $E+F$ with the smallest value of F. We will assume $F \leq n(n-1)$ and show that the bound cannot be violated. A vertex in H lies in an average of $\frac{3F}{4n+1}$ full triangles, that is, the number of vertices that lie in a full triangle over the total number of vertices. Thus, there exists a vertex v_0 that lies in t full triangles where

$$t \leq \frac{3F}{4n+1} \leq \frac{3n(n-1)}{4n+1} < n-1. \tag{9}$$

Since t and n are integers, we can rewrite this to be $t \leq n-2$. Let V be the vertex set of H, let A be the set of vertices adjacent to v_0 and let B be the vertices not adjacent to v_0.

Recall the construction of G_p in the first proof, in order to minimize $E+F$, H must satisfy Equation (7) by being a regular graph of degree $2n$. Consequently, there are $2n$ vertices in both A and B. Since v_0 lies in t full triangles, there are t edges whose endpoints both lie within A.

Now, we know the sum of the degrees of the vertices in A is $4n^2$, we know all $2n$ vertices in A are adjacent to v_0, and we know $2t$ vertices in A are adjacent to another vertex within A. Thus, there must be $4n^2-2n-2t$ edges joining sets A and B.

Furthermore, there must be $4n^2-(4n^2-2n-2t)=2n+2t$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0. Hence, there are $n+t$ edges whose endpoints both lie within B.

Consider an edge x within set B. Its two endpoints are incident with $4n-2$ other edges. Of those edges, at most $n+t-1$ of them can lie within B. Thus, at least $4n-2-(n+t-1)=3n-t-1$ edges have an endpoint in A. Note that no vertex in A can lie on more than 2 of these edges because that would result in a double edge, and recall that A has $2n$ vertices. Therefore, we can conclude that at least $(3n-t-1)-2n=n-t-1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x.

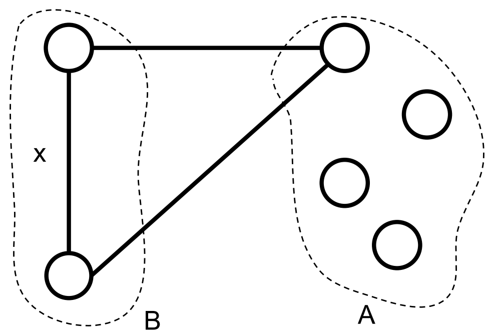

FIGURE 6. A visualization of an ABB triangle, other edges are omitted.

Thus, each edge within B lies in at least $n-t-1$ full ABB triangles, and so H should contain at least $(n+t)(n-t-1)$ full ABB triangles. Similarly, consider an edge y within set A. Its two endpoints are incident with $4n-2$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t-1$ of them lie within A. Thus, at least $4n-2-(2+(t-1))=4n-t-3$ edges have an endpoint in B. Recall that no vertex in B can lie on more than two of these edges and B has $2n$ vertices. Consequently, at least $(4n-t-3)-2n=2n-t-3$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y.

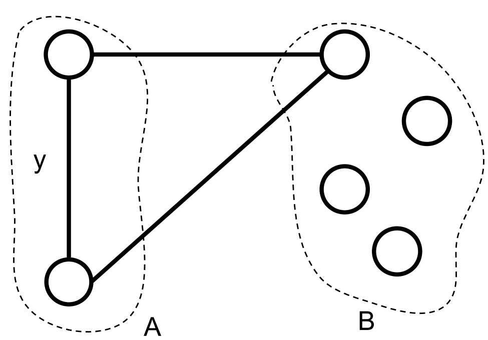

FIGURE 7. A visualization of an AAB triangle, other edges are omitted.

Thus, each edge within A lies in at least $2n-t-3$ full AAB triangles, and so H should contain at least $t(2n-t-3)$ full AAB triangles.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t + (n+t)(n-t-1) + t(2n-t-3) = n^2 - n + 2t(n-t-\frac{3}{2}).$$

Finally, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2 - n$.

Case 2: $p = 4n+3$

As in Case 1, we let H be a graph minimizing $E+F$ with the smallest value of F. We will still assume $F \leq n(n-1)$ and again show that the bound cannot be violated. Recall the construction of G_p in the first proof, in order to minimize $E+F$, H must satisfy Equation (7) by having $4n+2$ vertices of degree $2n+1$ and one vertex w of degree $2n$ or $2n+2$.

Note that a graph with an odd number of vertices cannot be regular with odd degree (REFERENCE). We will go ahead and assume w has degree $2n$, for if it were $2n+2$, we could just remove edges incident with w until we have a graph H' with the desired degrees and $F' \leq F$ full triangles.

We continue to repeat the steps in Case 1 by selecting a vertex v_0 which does not lie in too many full triangles. However, the posibility that $v_0 = w$ is troublesome. Consequently, we select v_0 to be a point of degree $2n+1$ lying in an average of $\frac{3F}{4n+3}$ full triangles, which is the number of vertices that lie in a full triangle over the total number of vertices. Recall that $n \geq 2$, so

$$t \leq \frac{3F}{4n+3} \leq \frac{3n(n-1)}{4n+3} < n-1. \tag{10}$$

Since t and n are integers, we can again rewrite this to be $t \leq n-2$. Furthermore, we will define sets A and B the same way that we defined them in Case 1, but now they both contain $2n+1$ vertices. We will now repeat a similar counting process for the number of full triangles in H. However, we must consider two subcases depending on the location of w.

Subcase i: w is in A

Then A has $2n$ vertices of degree $2n+1$ and it has vertex w, which has a degree of $2n$. Additionally, v_0 lies in t full triangles, so there are t edges whose endpoints both lie within A. To add, B has $2n+1$ vertices of degree $2n+1$.

Hence, the sum of the degrees of the vertices in A is $4n^2+4n$, all $2n+1$ vertices in A are adjacent to v_0, and $2t$ vertices in A are adjacent to another vertex within A. Thus, there must be $4n^2+4n-(2n+1)-2t = 4n^2+2n-2t-1$ edges joining sets A and B.

Then there must be $4n^2+4n+1-(4n^2+2n-2t-1) = 2n+2t+2$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0. Hence, there are $n+t+1$ edges whose endpoints both lie within B.

Consider an edge x within set B. Its two endpoints are incident with $4n+2-2$ other edges. Of those edges, at most $n+t$ of them can lie within B. Thus, at least $4n-(n+t) = 3n-t$ edges have an endpoint in A. Recall that no vertex in A can lie on more than two of these edges and A has $2n+1$ vertices. Therefore, we can conclude that at least $(3n-t)-(2n+1)=n-t-1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x. Thus, each edge within B lies in at least $n-t-1$ full ABB triangles, and so H should contain at least $(n+t+1)(n-t-1)$ full ABB triangles.

Similarly, consider an edge y within set A. Its two endpoints are incident with at least $((2n+1)+2n)-2=4n-1$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t-1$ of them lie within A. Thus, at least $4n-1-(2+(t-1))=4n-t-2$ edges have an endpoint in B. Recall that no vertex in B can lie on more than two of these edges and B has $2n+1$ vertices. Consequently, at least $(4n-t-2)-(2n+1)=2n-t-3$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y. Thus, each edge within A lies in at least $2n-t-3$ full AAB triangles, and so H should contain at least $t(2n-t-3)$ full AAB triangles.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t+(n+t+1)(n-t-1)+t(2n-t-3)=n^2-1+2t(n-t-2).$$

As a result, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2-n$.

Subcase ii: w is in B

Then B has $2n$ vertices of degree $2n+1$ and it has vertex w, which has a degree of $2n$. It follows that A has $2n+1$ vertices of degree $2n+1$. Additionally, v_0 lies in t full triangles, so there are t edges whose endpoints both lie within A.

Hence, the sum of the degrees of the vertices in A is $4n^2+4n+1$, all $2n+1$ vertices in A are adjacent to v_0, and $2t$ vertices in A are adjacent to another vertex within A. Thus, there must be $4n^2+4n+1-(2n+1)-2t=4n^2+2n-2t$ edges joining sets A and B.

Then there must be $4n^2+4n-(4n^2+2n-2t)=2n+2t$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0. Hence, there are $n+t$ edges whose endpoints both lie within B.

Consider an edge x within set B. Its two endpoints are incident with at least $4n+1-2$ other edges. Of those edges, at most $n+t-1$ of them can lie within B. Thus, at least $4n-1-(n+t-1)=3n-t$ edges have an endpoint in A. Recall that no vertex in A can lie on more than two of these edges and A has $2n+1$ vertices. Therefore, we can conclude that at least $(3n-t)-(2n+1)=n-t-1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x. Thus, each edge within B lies in at least $n-t-1$ full ABB triangles, and so H should contain at least $(n+t)(n-t-1)$ full ABB triangles.

Similarly, consider an edge y within set A. Its two endpoints are incident with $4n+2-2=4n$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t-1$ of them lie within A. Thus, at least $4n-(2+(t-1))=4n-t-1$ edges have an endpoint in B. Recall that no vertex in B can lie on more than two of these edges and B has $2n+1$ vertices. Consequently, at least $(4n-t-1)-(2n+1)=2n-t-2$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y. Thus, each edge within A lies in at least $2n-t-2$ full AAB triangles, and so H should contain at least $t(2n-t-2)$ full AAB triangles.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t+(n+t)(n-t-1)+t(2n-t-2)=n^2-n+2t(n-t-1).$$

As a result, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2-n$. In all cases, the bound $F \geq n^2-n$ cannot be violated when p is odd. □

4. Conclusion

In conclusion, we observe that when p is even, many graphs attain the bound in Theorem 2. Conversely, not many graphs attain that bound when p is odd. In fact, it is even argued that G_p is the unique graph attaining that bound, but the proof is too long to include here.

NOTE: I didn't have time to add my references but I will be sure to add them in.

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