

Acquaintance Graph Party Problem

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Math 479

Definition

The **floor function** is a function that gives us the greatest integer that is less than or equal to a real number x , denoted $\text{floor}(x)$ or $\lfloor x \rfloor$.

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Definition

The number of ways to choose k objects from a set of n objects is denoted as $\binom{n}{k}$ and is read as “ n choose k ”. The formula for **n choose k** is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Definition

A **bipartite** graph (or bigraph) is a graph where V can be partitioned into two disjoint sets such that no two vertices within the same set are adjacent.

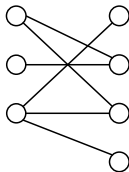


Figure: An example of a bipartite graph.

Definition

A **complete bipartite** graph is a bipartite graph where every vertex in one set is adjacent to every vertex in the other set.

Definition

A **regular** graph is a graph where each vertex has the same degree. A graph is called K -regular if the degree of each vertex in the graph is K .

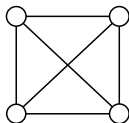


Figure:

Acquaintance
Graph Party
Problem

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Definitions
and Examples

Introduction

Theorem One

Theorem Two

Conclusion

A **party** yields a graph G where people are represented as vertices and two vertices are adjacent if those two people know each other, i.e. are acquaintances.

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- A **full triangle** is a subset of three people who all know each other. In other words, three adjacent vertices.

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- A **full triangle** is a subset of three people who all know each other. In other words, three adjacent vertices.
- An **empty triangle** is a subset of three people who are all strangers. In other words, three non-adjacent vertices.

Theorem

Let E and F be the number of empty and full triangles respectively. Then in every graph with p vertices

$$E + F \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor \quad (1)$$

and this lower bound is sharp for each positive integer p .

Proof.

Let P be the number of partial triangles in G , meaning the number of triangles containing exactly one or two edges. It's clear that

$$E + F + P = \binom{p}{3}. \quad (2)$$

Proof.

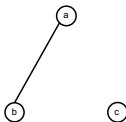
Let d_i be the degree of a vertex v_i , in other words, the number of people acquainted with the i^{th} person. For each vertex v_i , picking one of their d_i acquaintances and one of their $p - 1 - d_i$ nonacquaintances produces a partial triangle. Thus, each v_i produces $d_i(p - 1 - d_i)$ partial triangles.

Proof.

Furthermore, we note that every partial triangle is counted twice in this manner. To show this is true, let vertex a , b , and c represent v_i , one of their acquaintances, and one of their nonacquaintances respectively. Producing a partial triangle from v_i yields two cases.

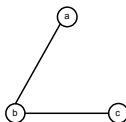
Proof.

Case 1: b and c are not acquainted. Then this partial triangle is counted when v_i is a and v_j is b .



Proof.

Case 2: b and c are acquainted. Then this partial triangle is counted when v_i is a and v_j is c .



Proof.

In both cases, the partial triangle is counted twice, which follows for every partial triangle. Consequently,

$$P = \frac{1}{2} \sum_{i=1}^p d_i(p-1-d_i). \quad (2)$$

Proof.

Looking back at Equation (2), we can minimize $E + F$ by maximizing P . To maximize P , we must maximize the sum in Equation (??). We can view each term of the sum as a quadratic function of d_i . After finding the derivative of the equation it's clear that we attain our maximum when $d_i = \frac{p-1}{2}$.

Proof.

However, this is a contradiction when p is even since d_i is an integer. Thus, if p is odd, we attain the maximum value of $\frac{(p-1)^2}{4}$ for each term when $d_i = \frac{p-1}{2}$. If p is even, we attain the maximum possible value of $\frac{p(p-2)}{4}$ for each term when $d_i = \frac{p}{2}$ or $d_i = \frac{p-2}{2}$.

Proof.

In either case, we can express the maximum value as

$$\left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor,$$

and so

$$P \leq \frac{1}{2} \sum_{i=1}^p \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor = \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor. \quad (2)$$

Proof.

But since P is an integer, we can strengthen this to read

$$P \leq \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \quad (2)$$

Proof.

Equations (2) and (??) now yield our desired bound:

$$E + F = \binom{p}{3} - P \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \quad (2)$$

Proof.

Next, for each p , we must find a graph G_p attaining this bound. But equality in Equation (1) is equivalent to equality in Equation (??) which occurs only when

$$P = \frac{1}{2} \sum_{i=1}^p d_i(p-1-d_i) = \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor. \quad (2)$$

Proof.

If $p = 2n$ for some integer n , let G_p be the complete bipartite graph $K_{n,n}$. Now G_p is regular of degree n , and we must check that Equation (??) is satisfied.

Proof.

$$\begin{aligned}
 P &= \frac{1}{2} \sum_{i=1}^p d_i(p-1-d_i) & P &= \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor \\
 &= \frac{1}{2} \sum_{i=1}^{2n} n(2n-1-n) & &= \left\lfloor \frac{2n}{2} \left\lfloor \left(\frac{2n-1}{2} \right)^2 \right\rfloor \right\rfloor \\
 &= \frac{1}{2} \sum_{i=1}^{2n} n^2 - n & &= \left\lfloor n \left\lfloor \frac{4n^2 - 4n + 1}{4} \right\rfloor \right\rfloor \\
 &= \frac{1}{2} (2n^3 - 2n^2) & &= \left\lfloor n \left\lfloor n^2 - n + \frac{1}{4} \right\rfloor \right\rfloor \\
 &= n^3 - n^2 & &= \left\lfloor n(n^2 - n) \right\rfloor
 \end{aligned}$$

Proof.

We can see that Equation (??) holds when $p = 2n$.

Proof.

If $p = 2n + 1$, the construction of G_p is a bit more involved. As depicted for $p = 9$ in Figure (??), we start with $K_{n,n}$ with its vertices labeled $u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_n$ and we subdivide edge $u_i v_i$ for $i \leq \frac{n}{2}$. We now obtain G_p by using these $\lfloor \frac{n}{2} \rfloor$ subdivision points to form a single vertex labeled w . We observe that G_p has $2n$ vertices of degree n and one vertex of degree $2\lfloor \frac{n}{2} \rfloor$. It is routine to check that Equation (??) is satisfied.

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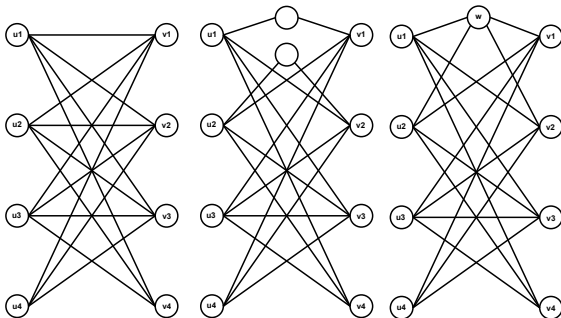
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Theorem One

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Proof.



Proof.

$$P = \frac{1}{2} \sum_{i=1}^p d_i(p-1-d_i)$$

$$= \frac{1}{2} \sum_{i=1}^{2n+1} d_i(2n-d_i)$$

$$= \frac{1}{2} \left(\sum_{i=1}^{2n} n(2n-n) + 2 \lfloor \frac{n}{2} \rfloor (2n - 2 \lfloor \frac{n}{2} \rfloor) \right)$$

$$= \frac{1}{2} \left(\sum_{i=1}^{2n} n^2 + 2 \lfloor \frac{n}{2} \rfloor (2n - 2 \lfloor \frac{n}{2} \rfloor) \right)$$

$$= \frac{1}{2} \left(2n^3 + 2 \lfloor \frac{n}{2} \rfloor (2n - 2 \lfloor \frac{n}{2} \rfloor) \right)$$

$$P = \left\lfloor \frac{p}{2} \right\rfloor \left(\frac{p-1}{2} \right)$$

$$= \left\lfloor \frac{2n+1}{2} \right\rfloor \left(\frac{2n}{2} \right)$$

$$= \left\lfloor \frac{2n+1}{2} \right\rfloor n^2$$

$$= \left\lfloor \left(\frac{2n+1}{2} \right) n^2 \right\rfloor$$

$$= \left\lfloor \frac{2n^3 + n^2}{2} \right\rfloor$$

Proof.

Case 1: n is even. Let $n = 2t$ for some integer t .

$$\begin{aligned}
 P &= \frac{1}{2} \left(2n^3 + 2 \lfloor \frac{n}{2} \rfloor (2n - 2 \lfloor \frac{n}{2} \rfloor) \right) & P &= \left\lfloor n^3 + \frac{1}{2} n^2 \right\rfloor \\
 &= \frac{1}{2} \left(2(2t)^3 + 2 \lfloor \frac{2t}{2} \rfloor (2(2t) - 2 \lfloor \frac{2t}{2} \rfloor) \right) & &= \left\lfloor (2t)^3 + \frac{1}{2} (2t)^2 \right\rfloor \\
 &= \frac{1}{2} \left(16t^3 + 2t(4t - 2t) \right) & &= \left\lfloor 8t^3 + \frac{1}{2} 4t^2 \right\rfloor \\
 &= \frac{1}{2} \left(16t^3 + 4t^2 \right) & &= \left\lfloor 8t^3 + 2t^2 \right\rfloor \\
 &= 8t^3 + 2t^2 & &= 8t^3 + 2t^2
 \end{aligned}$$

Proof.

Case 2: n is odd. Let $n = 2t + 1$ for some integer t .

$$\begin{aligned}
 P &= \frac{1}{2} \left(2n^3 + 2 \left\lfloor \frac{n}{2} \right\rfloor (2n - 2 \left\lfloor \frac{n}{2} \right\rfloor) \right) \\
 &= \frac{1}{2} \left(2(2t+1)^3 + 2 \left\lfloor \frac{2t+1}{2} \right\rfloor (2(2t+1) - 2 \left\lfloor \frac{2t+1}{2} \right\rfloor) \right) \\
 &= \frac{1}{2} \left(2(8t^3 + 12t^2 + 6t + 1) + 2 \left\lfloor t + \frac{1}{2} \right\rfloor (4t + 2 - 2 \left\lfloor t + \frac{1}{2} \right\rfloor) \right) \\
 &= \frac{1}{2} \left(16t^3 + 24t^2 + 12t + 2 + 2t(4t + 2 - 2t) \right) \\
 &= \frac{1}{2} \left(16t^3 + 28t^2 + 16t + 2 \right)
 \end{aligned}$$

Proof.

Thus, in both cases, Equation (??) holds.

Theorem

In every graph attaining the minimum possible value for $E + F$,

$$F \geq \begin{cases} 0 & \text{if } p = 2n \\ n(n-1) & \text{if } p = 4n+1 \text{ or } 4n+3 \end{cases} \quad (2)$$

and this lower bound is sharp for each positive integer p .

Proof.

We first observe that this bound is attained by the graphs G_p constructed in the first proof. If $p = 2n$, then $G_p = K_{n,n}$ which obviously has no full triangles. If $p = 4n + 1$ or $p = 4n + 3$, we notice that $G_p - w$ is a bigraph, which has no full triangles. Thus, every full triangle of G_p has the form u_i, v_j, w . But recalling the construction of G_p , we see that this will be a full triangle if and only if $i \leq n, j \leq n$, and $i \neq j$. Consequently, G_p has $F = n(n - 1)$ full triangles as desired.

Proof.

It remains to be shown that the bound in Equation (2) cannot be violated. If p is even, then this is trivial, since we cannot have a negative number of full triangles. If p is odd and $n = 0$ or $n = 1$, then this is also trivial, since $n(n - 1) = 0$. Therefore, we will assume $n \geq 2$.

Proof.

Case 1: $p = 4n + 1$

Let H be a graph minimizing $E + F$ with the smallest value of F . We will assume $F \leq n(n - 1)$ and show that the bound cannot be violated. A vertex in H lies in an average of $\frac{3F}{4n+1}$ full triangles, that is, the number of vertices that lie in a full triangle over the total number of vertices. Thus, there exists a vertex v_0 that lies in t full triangles where

$$t \leq \frac{3F}{4n+1} \leq \frac{3n(n-1)}{4n+1} < n-1. \quad (3)$$

Proof.

Since t and n are integers, we can rewrite this to be $t \leq n - 2$.
Let V be the vertex set of H , let A be the set of vertices adjacent to v_0 and let B be the vertices not adjacent to v_0 .

Proof.

Recall the construction of G_p in the first proof, in order to minimize $E + F$, H must satisfy Equation (??) by being a regular graph of degree $2n$. Consequently, there are $2n$ vertices in both A and B . Since v_0 lies in t full triangles, there are t edges whose endpoints both lie within A .

Proof.

Now, we know the sum of the degrees of the vertices in A is $4n^2$, we know all $2n$ vertices in A are adjacent to v_0 , and we know $2t$ vertices in A are adjacent to another vertex within A . Thus, there must be $4n^2 - 2n - 2t$ edges joining sets A and B .

Proof.

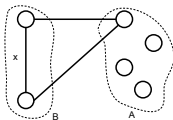
Furthermore, there must be $4n^2 - (4n^2 - 2n - 2t) = 2n + 2t$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0 . Hence, there are $n + t$ edges whose endpoints both lie within B .

Proof.

Consider an edge x within set B . Its two endpoints are incident with $4n - 2$ other edges. Of those edges, at most $n + t - 1$ of them can lie within B . Thus, at least $4n - 2 - (n + t - 1) = 3n - t - 1$ edges have an endpoint in A . Note that no vertex in A can lie on more than 2 of these edges because that would result in a double edge, and recall that A has $2n$ vertices.

Proof.

Therefore, we can conclude that at least $(3n - t - 1) - 2n = n - t - 1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x . Thus, each edge within B lies in at least $n - t - 1$ full ABB triangles, and so H should contain at least $(n + t)(n - t - 1)$ full ABB triangles.

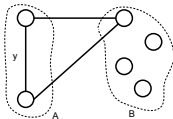


Proof.

Similarly, consider an edge y within set A . Its two endpoints are incident with $4n - 2$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t - 1$ of them lie within A . Thus, at least $4n - 2 - (2 + (t - 1)) = 4n - t - 3$ edges have an endpoint in B . Recall that no vertex in B can lie on more than two of these edges and B has $2n$ vertices.

Proof.

Consequently, at least $(4n - t - 3) - 2n = 2n - t - 3$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y . Thus, each edge within A lies in at least $2n - t - 3$ full AAB triangles, and so H should contain at least $t(2n - t - 3)$ full AAB triangles.



Proof.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t + (n+t)(n-t-1) + t(2n-t-3) = n^2 - n + 2t(n-t-\frac{3}{2}).$$

Finally, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2 - n$.

Proof.

Case 2: $p = 4n + 3$

As in Case 1, we let H be a graph minimizing $E + F$ with the smallest value of F . We will still assume $F \leq n(n - 1)$ and again show that the bound cannot be violated. Recall the construction of G_p in the first proof, in order to minimize $E + F$, H must satisfy Equation (??) by having $4n + 2$ vertices of degree $2n + 1$ and one vertex w of degree $2n$ or $2n + 2$.

Proof.

Note that a graph with an odd number of vertices cannot be regular with odd degree [?, Cor. 2.1 (a)]. We will go ahead and assume w has degree $2n$, for if it were $2n + 2$, we could just remove edges incident with w until we have a graph H' with the desired degrees and $F' \leq F$ full triangles.

Proof.

We continue to repeat the steps in Case 1 by selecting a vertex v_0 which does not lie in too many full triangles. However, the possibility that $v_0 = w$ is troublesome. Consequently, we select v_0 to be a point of degree $2n + 1$ lying in an average of $\frac{3F}{4n+3}$ full triangles, which is the number of vertices that lie in a full triangle over the total number of vertices. Recall that $n \geq 2$, so

$$t \leq \frac{3F}{4n+3} \leq \frac{3n(n-1)}{4n+3} < n-1. \quad (3)$$

Proof.

Since t and n are integers, we can again rewrite this to be $t \leq n - 2$. Furthermore, we will define sets A and B the same way that we defined them in Case 1, but now they both contain $2n + 1$ vertices. We will now repeat a similar counting process for the number of full triangles in H . However, we must consider two subcases depending on the location of w .

Proof.

Subcase i: w is in A

Then A has $2n$ vertices of degree $2n + 1$ and it has vertex w , which has a degree of $2n$. Additionally, v_0 lies in t full triangles, so there are t edges whose endpoints both lie within A . To add, B has $2n + 1$ vertices of degree $2n + 1$.

Proof.

Hence, the sum of the degrees of the vertices in A is $4n^2 + 4n$, all $2n + 1$ vertices in A are adjacent to v_0 , and $2t$ vertices in A are adjacent to another vertex within A . Thus, there must be $4n^2 + 4n - (2n + 1) - 2t = 4n^2 + 2n - 2t - 1$ edges joining sets A and B .

Proof.

Then there must be

$4n^2 + 4n + 1 - (4n^2 + 2n - 2t - 1) = 2n + 2t + 2$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0 . Hence, there are $n + t + 1$ edges whose endpoints both lie within B .

Proof.

Consider an edge x within set B . Its two endpoints are incident with $4n + 2 - 2$ other edges. Of those edges, at most $n + t$ of them can lie within B . Thus, at least $4n - (n + t) = 3n - t$ edges have an endpoint in A . Recall that no vertex in A can lie on more than two of these edges and A has $2n + 1$ vertices.

Proof.

Therefore, we can conclude that at least $(3n - t) - (2n + 1) = n - t - 1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x . Thus, each edge within B lies in at least $n - t - 1$ full ABB triangles, and so H should contain at least $(n + t + 1)(n - t - 1)$ full ABB triangles.

Proof.

Similarly, consider an edge y within set A . Its two endpoints are incident with at least $((2n + 1) + 2n) - 2 = 4n - 1$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t - 1$ of them lie within A . Thus, at least $4n - 1 - (2 + (t - 1)) = 4n - t - 2$ edges have an endpoint in B . Recall that no vertex in B can lie on more than two of these edges and B has $2n + 1$ vertices.

Proof.

Consequently, at least $(4n - t - 2) - (2n + 1) = 2n - t - 3$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y . Thus, each edge within A lies in at least $2n - t - 3$ full AAB triangles, and so H should contain at least $t(2n - t - 3)$ full AAB triangles.

Proof.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t + (n+t+1)(n-t-1) + t(2n-t-3) = n^2 - 1 + 2t(n-t-2).$$

As a result, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2 - n$.

Proof.

Subcase ii: w is in B

Then B has $2n$ vertices of degree $2n + 1$ and it has vertex w , which has a degree of $2n$. It follows that A has $2n + 1$ vertices of degree $2n + 1$. Additionally, v_0 lies in t full triangles, so there are t edges whose endpoints both lie within A .

Proof.

Hence, the sum of the degrees of the vertices in A is $4n^2 + 4n + 1$, all $2n + 1$ vertices in A are adjacent to v_0 , and $2t$ vertices in A are adjacent to another vertex within A . Thus, there must be $4n^2 + 4n + 1 - (2n + 1) - 2t = 4n^2 + 2n - 2t$ edges joining sets A and B .

Proof.

Then there must be $4n^2 + 4n - (4n^2 + 2n - 2t) = 2n + 2t$ vertices in B that are adjacent to another vertex in B since no vertex in B is adjacent to v_0 . Hence, there are $n + t$ edges whose endpoints both lie within B .

Proof.

Consider an edge x within set B . Its two endpoints are incident with at least $4n + 1 - 2$ other edges. Of those edges, at most $n + t - 1$ of them can lie within B . Thus, at least $4n - 1 - (n + t - 1) = 3n - t$ edges have an endpoint in A . Recall that no vertex in A can lie on more than two of these edges and A has $2n + 1$ vertices.

Proof.

Therefore, we can conclude that at least $(3n - t) - (2n + 1) = n - t - 1$ vertices in A lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge x . Thus, each edge within B lies in at least $n - t - 1$ full ABB triangles, and so H should contain at least $(n + t)(n - t - 1)$ full ABB triangles.

Proof.

Similarly, consider an edge y within set A . Its two endpoints are incident with $4n + 2 - 2 = 4n$ other edges. Of those edges, we know that two of them have v_0 as an endpoint and at most $t - 1$ of them lie within A . Thus, at least $4n - (2 + (t - 1)) = 4n - t - 1$ edges have an endpoint in B . Recall that no vertex in B can lie on more than two of these edges and B has $2n + 1$ vertices.

Proof.

Consequently, at least $(4n - t - 1) - (2n + 1) = 2n - t - 2$ vertices in B lie on exactly two of these edges. Each of these vertices forms a full triangle containing the edge y . Thus, each edge within A lies in at least $2n - t - 2$ full AAB triangles, and so H should contain at least $t(2n - t - 2)$ full AAB triangles.

Proof.

Recall that v_0 lies on t full triangles. We can add these to the sum of the two bounds that we found above to conclude that

$$F \geq t + (n+t)(n-t-1) + t(2n-t-2) = n^2 - n + 2t(n-t-1).$$

As a result, since $0 \leq t \leq n-2$ and $n \geq 2$, we conclude that $F \geq n^2 - n$. In all cases, the bound $F \geq n^2 - n$ cannot be violated when p is odd.

In conclusion, we observe that when p is even, many graphs attain the bound in Theorem 2. Conversely, not many graphs attain that bound when p is odd. In fact, it is even argued that G_p is the unique graph attaining that bound, but the proof is too long to include here.

References

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- [1] M. Aigner and G. Ziegler, "Proofs From THE BOOK," 3rd edition, Springer-Verlag, 2004. 203-205