Acquaintance Graph Party Problem

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Math 479

Definition

The **floor function** is a function that gives us the greatest integer that is less than or equal to a real number x, denoted $floor(x)$ or $\lfloor x \rfloor$.

Definition

The number of ways to choose k objects from a set of n objects is denoted as $\binom{n}{k}$ and is read as "n choose k". The formula for **n choose k** is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Definition

A **bipartite** graph (or bigraph) is a graph where V can be partitioned into two disjoint sets such that no two vertices within the same set are adjacent.

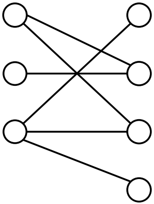

Figure: An example of a bipartite graph.

Definition

A **complete bipartite** graph is a bipartite graph where every vertex in one set is adjacent to every vertex in the other set. If two sets have m and n vertices, then we denote the complete bipartite graph by $K_{m,n}$.

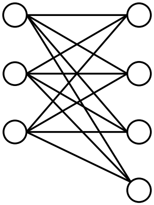

Figure: The graph $K_{3,4}$

Definition

A **regular** graph is a graph where each vertex has the same degree. A graph is called K-regular if the degree of each vertex in the graph is K.

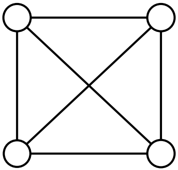

Figure: An example of a 3-regular graph.

- A **party** can be represented as a graph.

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  - An edge exists between two people if they are acquainted
- A **full triangle** is a subset of three adjacent vertices.
- An **empty triangle** is a subset of three non-adjacent vertices.

Theorem

Let E and F be the number of empty and full triangles respectively. Then in every graph with p vertices

$$E + F \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor \quad (1)$$

and this lower bound is sharp for each positive integer p.

Proof

- Let P be the number of partial triangles in G, meaning the number of triangles containing exactly one or two edges.

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- It's clear that

$$E + F + P = \binom{p}{3}. \tag{2}$$

Proof

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Proof

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- For each vertex v_i, picking one of their d_i acquaintances and one of their $p-1-d_i$ nonacquaintances produces a partial triangle.
- Thus, each v_i produces $d_i(p-1-d_i)$ partial triangles.

Proof

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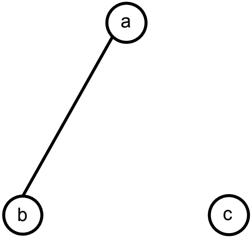
- To show this is true, let vertex a, b, and c represent v_i, one of their acquaintances, and one of their nonacquaintances respectively.
- Producing a partial triangle from v_i yields two cases.

Proof

- Case 1: b and c are not acquainted.



Proof

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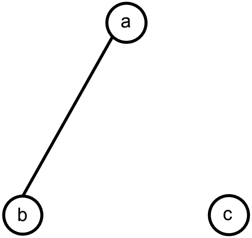
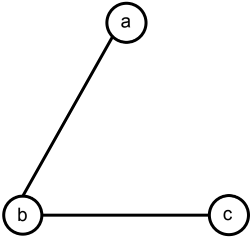


- Then this partial triangle is counted for a and b.

Proof

- Case 2: b and c are acquainted.



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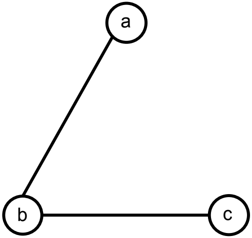

- Then this partial triangle is counted for a and c.

Proof

- In both cases, the partial triangle is counted twice, which follows for every partial triangle.

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- Consequently,

$$P = \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i). \tag{3}$$

Proof

- Looking back at Equation (2), we can minimize $E + F$ by maximizing P.

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- Looking back at Equation (2), we can minimize $E + F$ by maximizing P.
- To maximize P, we must maximize the sum in Equation (3).

$$E + F + P = \binom{p}{3} \qquad P = \frac{1}{2}\sum_{i=1}^{p} d_i(p - 1 - d_i)$$

Proof

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- We can view each term of the sum as a quadratic function of d_i.

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- We can view each term of the sum as a quadratic function of d_i.
- After finding the derivative of the equation it's clear that we attain our maximum when $d_i = \frac{p-1}{2}$.

Proof

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Proof

- However, this is a contradiction when p is even since d_i is an integer.
- Thus, if p is odd, we attain the maximum value of $\frac{(p-1)^2}{4}$ for each term when $d_i = \frac{p-1}{2}$.
- If p is even, we attain the maximum possible value of $\frac{p(p-2)}{4}$ for each term when $d_i = \frac{p}{2}$ or $d_i = \frac{p-2}{2}$.

Proof

- In either case, we can express the maximum value as

$$\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor,$$

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- In either case, we can express the maximum value as

$$\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor,$$

- and so

$$P \leq \frac{1}{2}\sum_{i=1}^{p} \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor = \frac{p}{2}\left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor. \qquad (4)$$

Proof

- But since P is an integer, we can strengthen this to read

$$P \leq \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor . \tag{5}$$

Proof

- But since P is an integer, we can strengthen this to read

$$P \leq \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor . \tag{5}$$

- Equations (2) and (5) now yield our desired bound:

$$E + F = \binom{p}{3} - P \geq \binom{p}{3} - \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2} \right)^2 \right\rfloor \right\rfloor . \tag{6}$$

Proof

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- But equality in Equation (1) is equivalent to equality in Equation (5).
- Which occurs only when

$$P = \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i) = \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor \right\rfloor. \tag{7}$$

Proof

- If $p = 2n$ for some integer n, let G_p be the complete bipartite graph $K_{n,n}$.

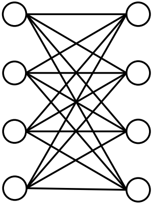

Figure: A visualization of G_8

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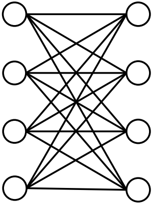

Figure: A visualization of G_8

- Now G_p is regular of degree n, and we must check that Equation (7) is satisfied.

Proof

$$\begin{aligned} P &= \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i) \\ &= \frac{1}{2}\sum_{i=1}^{2n} n(2n-1-n) \\ &= \frac{1}{2}\sum_{i=1}^{2n} n^2 - n \\ &= \frac{1}{2}(2n^3 - 2n^2) \\ &= n^3 - n^2 \end{aligned}$$

$$\begin{aligned} P &= \left\lfloor \frac{p}{2} \left\lfloor \left(\frac{p-1}{2}\right)^2 \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{2n}{2} \left\lfloor \left(\frac{2n-1}{2}\right)^2 \right\rfloor \right\rfloor \\ &= \left\lfloor n \left\lfloor \frac{4n^2-4n+1}{4} \right\rfloor \right\rfloor \\ &= \left\lfloor n \left\lfloor n^2 - n + \frac{1}{4} \right\rfloor \right\rfloor \\ &= \left\lfloor n(n^2-n) \right\rfloor \\ &= \left\lfloor n^3 - n^2 \right\rfloor \\ &= n^3 - n^2 \end{aligned}$$

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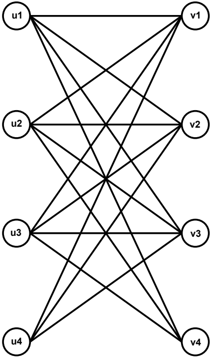
- If $p = 2n + 1$, the construction of G_p is a bit more involved.
- We will walk through the construction of G_9 as an example.

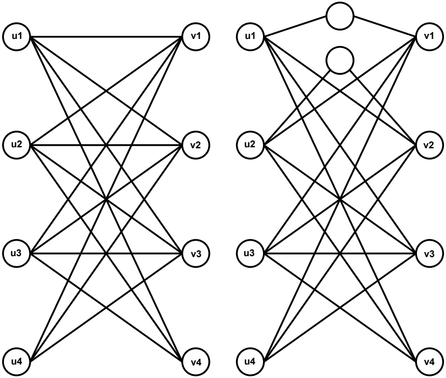
Proof

- We start with $K_{n,n}$ with its vertices labeled $u_1, u_2, ..., u_n$; $v_1, v_2, ..., v_n$.



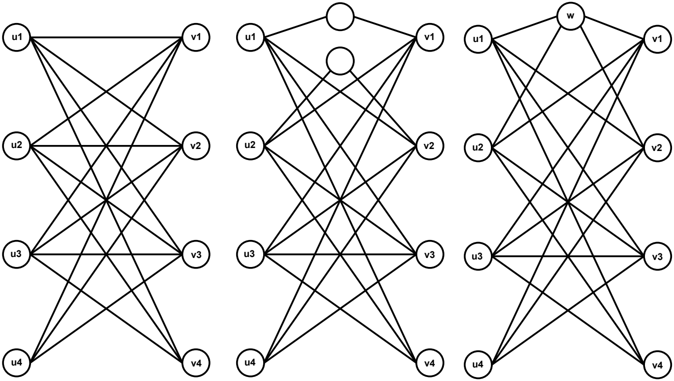
Proof

- We subdivide edge $u_i v_i$ for $i \leq \frac{n}{2}$.



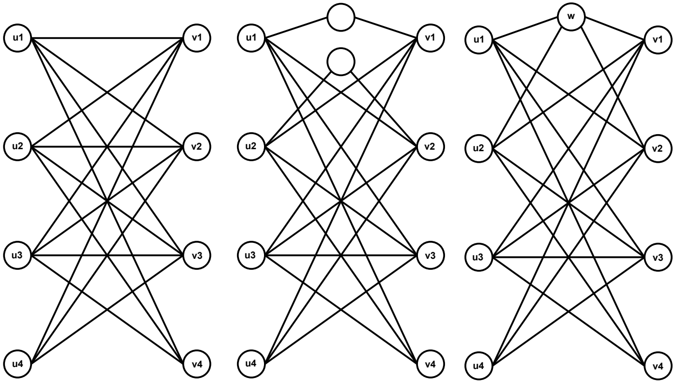
Proof

- We now obtain G_p by using these $\lfloor \frac{n}{2} \rfloor$ subdivision points to form a single vertex labeled w.



Proof

- We observe that G_p has $2n$ vertices of degree n and one vertex of degree $2\lfloor \frac{n}{2} \rfloor$.



Proof

- We observe that G_p has $2n$ vertices of degree n and one vertex of degree $2\lfloor \frac{n}{2} \rfloor$.
- It is routine to check that Equation (7) is satisfied.

Proof

$$
\begin{aligned}
P &= \frac{1}{2}\sum_{i=1}^{p} d_i(p-1-d_i) \\
&= \frac{1}{2}\sum_{i=1}^{2n+1} d_i(2n-d_i) \\
&= \frac{1}{2}\left(\sum_{i=1}^{2n} n(2n-n) + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{2n} n^2 + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n-2\lfloor\frac{n}{2}\rfloor)\right)
\end{aligned}
$$

$$
\begin{aligned}
P &= \left\lfloor \frac{p}{2}\left\lfloor\left(\frac{p-1}{2}\right)^2\right\rfloor\right\rfloor \\
&= \left\lfloor \frac{2n+1}{2}\left\lfloor\left(\frac{2n+1-1}{2}\right)^2\right\rfloor\right\rfloor \\
&= \left\lfloor \frac{2n+1}{2}\left\lfloor n^2\right\rfloor\right\rfloor \\
&= \left\lfloor \left(\frac{2n+1}{2}\right) n^2\right\rfloor \\
&= \left\lfloor \frac{2n^3+n^2}{2}\right\rfloor \\
&= \left\lfloor n^3+\frac{1}{2}n^2\right\rfloor
\end{aligned}
$$

Proof

- At this point, we run into two cases.

$$P = \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n - 2\lfloor\frac{n}{2}\rfloor)\right) \quad P = \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor$$

Proof

- Case 1: n is even. Let $n = 2t$ for some integer t.

$$\begin{aligned} P &= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n - 2\lfloor\frac{n}{2}\rfloor)\right) \\ &= \frac{1}{2}\left(2(2t)^3 + 2\lfloor\frac{2t}{2}\rfloor(2(2t) - 2\lfloor\frac{2t}{2}\rfloor)\right) \\ &= \frac{1}{2}\left(16t^3 + 2t(4t - 2t)\right) \\ &= \frac{1}{2}\left(16t^3 + 4t^2\right) \\ &= 8t^3 + 2t^2 \end{aligned}$$

$$\begin{aligned} P &= \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor \\ &= \left\lfloor (2t)^3 + \frac{1}{2}(2t)^2 \right\rfloor \\ &= \left\lfloor 8t^3 + \frac{1}{2}4t^2 \right\rfloor \\ &= \left\lfloor 8t^3 + 2t^2 \right\rfloor \\ &= 8t^3 + 2t^2 \end{aligned}$$

Proof

- Case 2: n is odd. Let $n = 2t + 1$ for some integer t.

$$
\begin{aligned}
P &= \frac{1}{2}\left(2n^3 + 2\lfloor\frac{n}{2}\rfloor(2n - 2\lfloor\frac{n}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(2(2t+1)^3 + 2\left\lfloor\frac{2t+1}{2}\right\rfloor(2(2t+1) - 2\left\lfloor\frac{2t+1}{2}\right\rfloor)\right) \\
&= \frac{1}{2}\left(2(8t^3 + 12t^2 + 6t + 1) + 2\lfloor t + \frac{1}{2}\rfloor(4t + 2 - 2\lfloor t + \frac{1}{2}\rfloor)\right) \\
&= \frac{1}{2}\left(16t^3 + 24t^2 + 12t + 2 + 2t(4t + 2 - 2t)\right) \\
&= \frac{1}{2}\left(16t^3 + 28t^2 + 16t + 2\right) \\
&= 8t^3 + 14t^2 + 8t + 1
\end{aligned}
$$

Proof

- Case 2: n is odd. Let $n = 2t + 1$ for some integer t.

$$
\begin{aligned}
P &= \left\lfloor n^3 + \frac{1}{2}n^2 \right\rfloor \\
&= \left\lfloor (2t+1)^3 + \frac{1}{2}(2t+1)^2 \right\rfloor \\
&= \left\lfloor 8t^3 + 12t^2 + 6t + 1 + \frac{4t^2 + 4t + 1}{2} \right\rfloor \\
&= \left\lfloor 8t^3 + 12t^2 + 6t + 1 + 2t^2 + 2t + \frac{1}{2} \right\rfloor \\
&= \left\lfloor 8t^3 + 14t^2 + 8t + 1 + \frac{1}{2} \right\rfloor \\
&= 8t^3 + 14t^2 + 8t + 1
\end{aligned}
$$

Proof

- Thus, in both cases, Equation (7) holds. $\square$

Theorem

In every graph attaining the minimum possible value for $E + F$,

$$F \geq \begin{cases} 0 & \text{if } p = 2n \\ n(n-1) & \text{if } p = 4n+1 \text{ or } 4n+3 \end{cases} \tag{8}$$

and this lower bound is sharp for each positive integer p.

- There isn't enough time to go over the proof for Theorem 2.

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- Essentially, we count an arbitrary number of full triangles to show that the bound can't be violated.

References

[1] A. W. Goodman, *On sets of acquaintances and strangers at any party*, The American Mathematical Monthly **66** (1959), 778-783.

[2] F. Harary, "Graph Theory," Addison-Wesley Pub. Co, Reading, Mass., 1969.

[3] L. Sauvé, *On chromatic graphs*, The American Mathematical Monthly **68** (1961), 107-111.

[4] A. J. Schwenk, *Acquaintance Graph Party Problem*, The American Mathematical Monthly **79** (1972), 1113-1117.