Problema 3.83

Hallar la divergencia y el rotacional de $f = (x-y)\hat{\imath} + (y-z)\hat{\jmath} + (z-x)\hat{k}$ y $g = (x^2+yz)\hat{\imath} + (y^2+zx)\hat{\jmath} + (z^2+xy)\hat{k}$

Para f

$$\nabla \cdot f = \frac{df_1}{dx} + \frac{df_2}{dy} + \frac{df_3}{dz} = \frac{d(x-y)}{dx} + \frac{d(y-z)}{dy} + \frac{d(z-x)}{dz}$$

$$= 1 + 1 + 1 = 3$$

$$\therefore \boxed{\nabla \cdot f = 3}$$

$$\nabla \times f = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{df_3}{dy} - \frac{df_2}{dz}\right)\hat{\imath} - \left(\frac{df_3}{dx} - \frac{df_1}{dz}\right)\hat{\jmath} + \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right)$$

$$= \left[\frac{d(z-x)}{dy} - \frac{d(y-z)}{dz}\right]\hat{\imath} - \left[\frac{d(z-x)}{dx} - \frac{d(x-y)}{dz}\right]\hat{\jmath} + \left[\frac{d(y-z)}{dx} - \frac{d(x-y)}{dy}\right]$$

$$= [0+1]\hat{\jmath} - [-1-0]\hat{\jmath} + [0+1]\hat{k} = \boxed{[1,1,1]}$$

Para g

$$\nabla \cdot g = \frac{dg_1}{dx} + \frac{dg_2}{dy} + \frac{dg_3}{dz} = \frac{d(x^2+yz)}{dx} + \frac{d(y^2+zx)}{dy} + \frac{d(z^2+xy)}{dz}$$

$$\nabla \cdot g = 2x + 2y + 2z = \boxed{2(x+y+z)}$$

$$\nabla \times g = \left[\frac{d(z^2+xy)}{dy} - \frac{d(y^2+zx)}{dz}\right]\hat{\imath} - \left[\frac{d(z^2+xy)}{dx} - \frac{d(x^2+yz)}{dz}\right] +$$

$$\left[\frac{d(y^2+zx)}{dx} - \frac{d(x^2+yz)}{dy}\right] = \boxed{0}$$

DIA	MES	AÑO	FOLIO

Problema 3.84 Si $\phi = 3x^2 - yz$

y $F = 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$

hallar en $(1, -1, 1)$

(a) $\nabla\phi$

$$\nabla\phi = \frac{d(3x^2-yz)}{dx}\hat{\imath} + \frac{d(3x^2-yz)}{dy}\hat{\jmath} + \frac{d(3x^2-yz)}{dz}\hat{k}$$

$$= 6x\hat{\imath} - z\hat{\jmath} - y\hat{k}$$

$$= 6(1) - (1) - (-1) = \boxed{[6, -1, 1]}$$

(b) $\nabla\cdot F$

$$\nabla\cdot F = \frac{d(3xyz^2)}{dx} + \frac{d(2xy^3)}{dy} + \frac{d(-x^2yz)}{dz}$$

$$= 3yz^2 + 6xy^2 - x^2y$$

$$= 3(-1)(1)^2 + 6(1)(-1)^2 - (1)^2(-1)$$

$$= -3 + 6 + 1 = \boxed{4}$$

(c) $\nabla\times F$

$$\nabla\times F = \left[\frac{d(-x^2yz)}{dy} - \frac{d(2xy^3)}{dz}\right]\hat{\imath} - \left[\frac{d(-x^2yz)}{dx} - \frac{d(3xyz^2)}{dz}\right]\hat{\jmath}$$

$$+ \left[\frac{d(2xy^3)}{dx} - \frac{d(3xyz^2)}{dy}\right]\hat{k} = (-x^2z)\hat{\imath} + (8xyz)\hat{\jmath} + (2y^3 - 3xz^2)\hat{k}$$

$$\boxed{\text{evaluando} = [-1, -8, -5]}$$

DIA	MES	AÑO	FOLIO

(d) $F \cdot \nabla \phi$

$$F \cdot \nabla \phi = (3xyz^2\,\hat{i} + 2xy^3\,\hat{j} - x^2yz\,\hat{k}) \cdot (6x\,\hat{i} - z\,\hat{j} - y\,\hat{k})$$

$$= 18x^2yz^2 - 2xy^3z + x^2y^2z$$

$$= 18(1)(-1)(1)^2 - 2(1)(-1)^3(1) + (1)^2(-1)^2(1)$$

$$= -18 + 2 + 1 = \boxed{-15}$$

(e) $\nabla \cdot (\phi F)$

$$\nabla \cdot (\phi F) = \phi \nabla \cdot F + F \cdot \nabla \phi$$

$$= (3x^2 - yz) \cdot (4) + (-15) \quad \text{inciso (d)(b)}$$

$$= (3(1)^2 - (-1)(1))(4) - 15$$

$$= (4 \cdot 4) - 15$$

$$= 16 - 15$$

$$\boxed{= 1}$$

DIA	MES	AÑO	FOLIO

(f) $\nabla \times (\phi f)$

$$\nabla \times (\phi f) = \phi \nabla \times f - f \times \nabla \phi$$

$$f \times \nabla \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_1 & f_2 & f_3 \\ \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{vmatrix} = (f_2 \nabla\phi_3 - f_3 \cdot \nabla\phi_2)\hat{i} - (f_1 \nabla\phi_3 - f_3 \nabla\phi_1)\hat{j} + (f_1 \nabla\phi_2 - f_2 \nabla\phi_1)\hat{k}$$

$$f \times \nabla \phi = [3xy^3 \cdot (-y) - (-x^2yz \cdot -z)]\hat{i}$$
$$- [3xyz^2 \cdot (-y) - (-x^2yz) \cdot 6x]\hat{j}$$
$$+ [3xyz^2 \cdot (-z) - (2xy^3) \cdot 6x]\hat{k}$$
$$= [-3xy^4 - x^2yz^2]\hat{i} + [3xy^2z^2 - 6x^3yz]\hat{j}$$
$$+ [-3xyz^3 - 12x^2y^3]\hat{k}$$
$$= [-2, 9, 15]$$

$$\phi \nabla \times f = [(3x^2 - yz) \cdot (-x^2z)]\hat{i} + [(3x^2 - yz) \cdot 8xyz]\hat{j}$$
$$+ [(3x^2 - yz) \cdot (2y^3 - xz^2)]\hat{k}$$
$$= [-4, -32, -20]$$

$$\nabla \times (\phi f) = [-4, -32, -20] - [-2, 9, 15]$$
$$\boxed{= [-2, -41, 35]}$$

FOLIO	AÑO	MES	DIA	DIA	MES	AÑO	FOLIO

(g) $\nabla^2 \phi$

$$\nabla^2 \phi = \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2}$$

$$= \frac{d^2(3x^2 - yz)}{dx^2} + \frac{d^2(3x^2 - yz)}{dy^2} + \frac{d^2(3x^2 - yz)}{dz^2}$$

$$= 6 + 0 + 0$$

$$\boxed{= 6}$$

Problema 3.85

Si $\nabla \times f = 0$, donde $f = (xyz)^m (x^n \hat{\imath} + y^n \hat{\jmath} + z^n \hat{k})$

mostrar que $m=0$ o bien $n=-1$

$$f = x^{n+m} \cdot y^m \cdot z^m \hat{\imath} + x^m \cdot y^{n+m} \cdot z^m \hat{\jmath} + x^m y^m z^{n+m}$$

$$\Rightarrow \nabla \times f = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{d(x^m y^m z^{n+m})}{dy} - \frac{d(x^m y^{n+m} z^m)}{dz} \right] \hat{\imath} - \left[\frac{d(x^m y^m z^{n+m})}{dx} - \frac{d(x^{n+m} y^m z}{dz} \right.$$

$$+ \left[\frac{d(x^m y^{n+m} z^m)}{dx} - \frac{d(x^{n+m} \cdot y^m z^m)}{dy} \right] \hat{k}$$

$$= \left[m y^{m-1} \cdot x^m z^{n+m} - m z^{m-1} x^m y^{n+m} \right] \hat{\imath}$$

$$+ \left[-m x^{m-1} \cdot y^m \cdot z^{n+m} + x^{m+n} \cdot y^m \cdot m z^{m-1} \right] \hat{\jmath}$$

$$+ \left[m x^{m-1} \cdot y^{m+n} \cdot z^m - x^{m+n} \cdot m y^{m-1} z^m \right] \hat{k}$$

$$\Rightarrow m y^{m-1} \cdot x^m z^{n+m} - m z^{m-1} x^m y^{n+m} = 0$$

$$m (y^{m-1} \cdot x^m \cdot z^{n+m} - z^{m-1} \cdot x^m y^{n+m}) = 0$$

$$\boxed{m = 0}$$

qed

Problema 3.86
Para un vector arbitrario constante a, y el vector de posición r, mostrar

(a) $\nabla \cdot (a \times r) = 0$

como $\nabla \cdot (a \times r) = r \cdot (\nabla \times a) - a \cdot (\nabla \times r)$,

además sabemos que $\nabla \times r$ es 0 puesto que es un vector de posición

$$\nabla \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ a_1 & a_2 & a_3 \end{vmatrix} = \left[\frac{da_3}{dx} - \frac{da_2}{dy}\right]\hat{i} - \left[\frac{da_3}{dx} - \frac{da_1}{dz}\right]\hat{j} + \left[\frac{da_2}{dx} - \frac{da_1}{dy}\right]\hat{k}$$

puesto que a es un vector constante, entonces
$\nabla \times a = 0$

$$\boxed{\therefore\ \nabla (a \times r) = r \cdot (0) - a \cdot (0) = 0}$$

(b) $\nabla \times (a \times r) = 2a$

$\nabla \times (a \times r) = a(\nabla \cdot r) - r(\nabla \cdot a) + (r \cdot \nabla)a - (a \cdot \nabla)r$

como r es un vector de posición, entonces
$(a \cdot \nabla)r = a$ y $a(\nabla \cdot r) = 3a$

además, dado que a es constante
$r(\nabla \cdot a) = 0$ $(r \cdot \nabla)a = 0$

$$\boxed{\Rightarrow \nabla \times (a \times r) = 3a - a = 2a}$$

Problema 3.87

Para cualquier función vectorial f diferenciable y el vector de posición r, mostrar que

dem:

$\nabla \cdot (f \times r) = r \cdot (\nabla \times f)$

$\nabla \cdot (f \times r) = r \cdot (\nabla \times f) - f \cdot (\nabla \times r)$

como r es un vector de posición,

$\nabla \cdot (\nabla \times r) = 0$

$\Rightarrow f \cdot (\nabla \times r) = f \cdot (0) = 0$

$\boxed{\nabla \cdot (f \times r) = r \cdot (\nabla \times f) \quad \text{q.e.d}}$

Problema 3.88

Mostrar que si ϕ y ψ son armónicas, entonces $\phi + \psi$ y $c\phi$, para una cte arbitraria c, son también armónicas.

dem:

$\beta = \phi + \psi$

β es armónica si y solo si $\nabla^2 \beta = 0$

$\nabla^2 \beta = \frac{d^2(\phi+\psi)}{dx^2} + \frac{d^2(\phi+\psi)}{dy^2} + \frac{d^2(\phi+\psi)}{dz^2}$

$= \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} + \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2}$

$= \nabla^2 \phi + \nabla^2 \psi$

como ϕ y ψ son armónicas,

$\Rightarrow \nabla^2\phi = 0$ y $\nabla^2\psi = 0$

entonces

$\nabla^2\beta = 0 + 0 = 0$

$\therefore$ β es armónica

$c\phi$ ¿es armónica?

dem: $\theta = c\phi$

θ será armónica sí y solo si $\nabla^2\theta = 0$

$$\nabla^2\theta = \frac{d^2 c\phi}{d^2x} + \frac{d^2 c\phi}{d^2y} + \frac{d^2 c\phi}{d^2z}$$

$$= c\left[\frac{d^2\phi}{d^2x} + \frac{d^2\phi}{d^2y} + \frac{d^2\phi}{d^2z}\right]$$

$$= c[\nabla^2\phi]$$

como ϕ es armónica

$\nabla^2\phi = 0$

, entonces

$$\nabla^2\theta = c[0]$$

$$= 0$$

$\therefore$ $\boxed{\theta \text{ es armónica} \quad \text{qed}}$

problema 3.89
Si las segundas derivadas de las funciones ϕ y ψ existen, entonces demostrar que

(a) $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$

dem:

como $\nabla(\phi\psi) = \phi\nabla\cdot\psi + \psi\cdot\nabla\phi$,

entonces

$$\nabla[\nabla(\phi\psi)] = \nabla[\phi\nabla\cdot\psi + \psi\cdot\nabla\phi]$$
$$= \nabla(\phi\nabla\cdot\psi) + \nabla(\psi\nabla\phi)$$
$$= \underline{\phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi}$$
$$= \boxed{\phi\nabla^2\psi + 2\nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi}$$

qed

(b) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$

dem:

$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{d\frac{\phi_1}{\psi_1}}{dx} + \frac{d\frac{\phi_2}{\psi_2}}{dy} + \frac{d\frac{\phi_3}{\psi_3}}{dz}$$

$$= \frac{\psi_1\frac{d\phi_1}{dx}}{\psi_1^2} - \frac{\phi_1\frac{d\psi_1}{dx}}{\psi_1^2} + \frac{\psi_2\frac{d\phi_2}{dx}}{\psi_2^2} - \frac{\phi_2\frac{d\psi_2}{dx}}{\psi_2} + \frac{\psi_3\frac{d\phi_3}{d}}{\psi_3}$$

$$= \frac{\psi_1\frac{d\phi_1}{dx} + \psi_2\frac{d\phi_2}{dy} + \psi_3\frac{d\phi_3}{dz} - \phi_1\frac{d\psi_1}{dx} - \phi_2\frac{d\psi_2}{dy} - \phi_3\frac{d\psi_3}{dz}}{\psi^2}$$

$$\boxed{= \frac{\psi \cdot \nabla \phi - \phi \nabla \psi}{\psi^2}}$$

qed

Problema 3.90

Si u es un vector unitario constante y f es una función vectorial diferenciable, mostrar que

$$u\cdot[\nabla(f\cdot u)-\nabla\times(f\times u)]=\nabla\cdot f$$

$$
\begin{aligned}
u\cdot[\nabla(f\cdot u)-\nabla\times(f\times u)] &\overset{u\cdot}{=}\Big[f(\cancelto{0}{\nabla\cdot u})-u(\nabla\cdot f)+(u\cdot\nabla)f \\
&\quad -\cancelto{0}{(f\cdot\nabla)u}-f\times(\cancelto{0}{\nabla\times u})-u\times(\nabla\times f) \\
&\quad -\cancelto{0}{(f\cdot\nabla)u}-(u\cdot\nabla)f\Big] \\
&=[-u(\nabla\cdot f)+(u\cdot\nabla)f-(u\cdot\nabla)f-u\times(\nabla\times f)]\cdot u \\
&=[-u(\nabla\cdot f)-u\times(\nabla\times f)]\cdot u \\
&=[-u(\nabla\cdot f)+\nabla\times(f\times u)]\cdot u \\
&=[-u(\nabla\cdot f)+u(\nabla\cdot f)-f(\cancelto{0}{\nabla\cdot u})+(\cancelto{0}{f\cdot\nabla})u+(u\cdot\nabla)f]\cdot u \\
&=[(u\cdot\nabla)f]\cdot u \\
&=\boxed{\nabla\cdot f}
\end{aligned}
$$

q.e.d