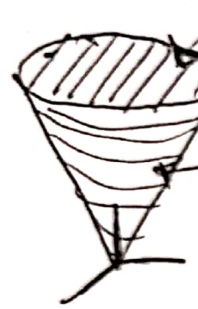
Problema 4.88) $\iint_S \vec{f} \cdot d\vec{s}$

donde: $\vec{f} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

& S: Superficie limitada por $z^2 = x^2 + y^2$, $z=0$, $z=4$



Dada la Simetria de la Superficie S

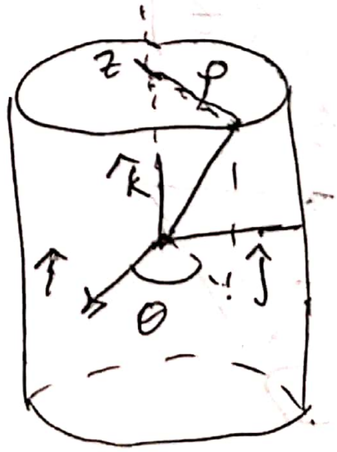
Se utilizara una Transformacion de Coordenadas Cilindrica:

En "general" para un cilindro de radio ρ

$x = \rho\cos\theta \qquad 0 \leq \rho < \infty$

$y = \rho\,\mathrm{sen}\,\theta \qquad 0 \leq \theta < 2\pi$

$z = z \qquad z \in \mathbb{R}$



$\vec{r}(\rho, \theta, z) = \rho\cos\theta\,\hat{i} + \rho\,\mathrm{sen}\,\theta\,\hat{j} + z\hat{k}$

La Superficie $S = S_1 \cup S_2$

donde S_1 es la Superficie Conica y S_2 es la tapa del Cono, entonces.

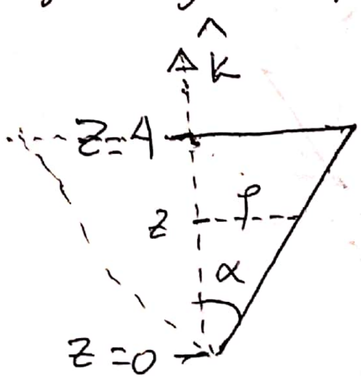
$$\iint_S \vec{f} \cdot d\vec{S} = \iint_{S_1 \cup S_2} \vec{f} \cdot d\vec{S} = \iint_{S_1} \vec{f} \cdot d\vec{S} + \iint_{S_2} \vec{f} \cdot d\vec{S}$$

Resolvemos por Separado.

$$\iint_{S_1} \vec{f} \cdot d\vec{S}$$

observemos la restriccion de $\vec{r}$ para describir S_1

$$\rho = \rho(z)$$



$$tg\alpha = \frac{\rho}{z}$$

así $\rho(z) = z\, tg\alpha$

recordemos que

$$z^2 = x^2 + y^2$$

si $z = 0 \Rightarrow \rho = 0$

si $z = 4 \Rightarrow \rho = 4$

$\therefore$ $tg\alpha = \frac{4}{4} = 1$

pues. $x^2 + y^2 = \rho^2 \cos^2\theta + \rho^2 \operatorname{sen}^2\theta = \rho^2$

i.e. $\boxed{z^2 = \rho^2}$

$$\vec{r} = \rho\cos\theta\,\hat{i} + \rho\,\text{Sen}\,\theta\,\hat{j} + \rho\,\hat{k}$$

$$\vec{r}_\rho = \cos\theta\,\hat{i} + \text{Sen}\,\theta\,\hat{j} + \hat{k}$$

$$\vec{r}_\theta = -\rho\,\text{Sen}\,\theta\,\hat{i} + \rho\cos\theta\,\hat{j}$$

$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \text{Sen}\,\theta & 1 \\ -\rho\,\text{Sen}\,\theta & \rho\cos\theta & 0 \end{vmatrix} = \hat{i}(-\rho\cos\theta) - \hat{j}(\rho\,\text{Sen}\,\theta) + \hat{k}(\rho)$$

$$\vec{r}_\theta \times \vec{r}_\rho = \rho\cos\theta\,\hat{i} + \rho\,\text{Sen}\,\theta\,\hat{j} - \hat{k}\rho$$

$$\vec{F}\cdot d\vec{S} = (4\rho^3\cos^2\theta + \rho^5\cos\theta\,\text{Sen}^2\theta - 3\rho^2)\,d\rho\,d\theta$$

$$\iint \vec{F}\cdot d\vec{S} = \int_{\rho=0}^{4}\int_{\theta=0}^{2\pi} (4\rho^3\cos^2\theta + \rho^5\cos\theta\,\text{Sen}^2\theta - 3\rho^2)\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(4^4\cos^2\theta\,d\theta + \frac{4^6}{6}\cos\theta\,\text{Sen}^2\theta - 4^3\right)d\theta$$

$u = \text{Sen}\,\theta$
$du = \cos\theta\,d\theta$

$$256\left[\frac{\theta}{2} + \frac{\text{Sen}\,2\theta}{4}\right]_0^{2\pi} + \frac{4^6}{6}\,\frac{\text{Sen}^3\theta}{3}\Bigg|_0^{2\pi} - 128\pi$$

$$= 256\pi - 128\pi = 128\pi$$

Tapa: S_2 es una superficie definida por: ④

$$\vec{r} = \rho\cos\theta\,\hat{i} + \rho\,\text{Sen}\,\theta\,\hat{j} + 4\hat{k}$$

$$\vec{f} = 4\rho^2\cos\theta\,\hat{i} + 4\rho^2\cos\theta\,\text{Sen}\,\theta\,\hat{j} + 12\hat{k}$$

$$\vec{r}_\theta \times \vec{r}_\rho = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\rho\,\text{Sen}\,\theta & \rho\cos\theta & 0 \\ \cos\theta & \text{Sen}\,\theta & 0 \end{vmatrix} = \hat{k}(-\rho) \quad ;\; \vec{r}_\rho \times \vec{r}_\theta = \rho\hat{k}$$

$$\vec{f}\cdot d\vec{S} = 12\rho\, d\rho\, d\theta$$

$$\int_{\rho=0}^{4}\int_{\theta=0}^{2\pi} 12\rho\, d\rho\, d\theta = \frac{12}{2}\rho^2\Big|_0^4 2\pi = 12\pi\cdot 16 = 192\pi$$

$$\iint_S \vec{f}\cdot d\vec{S} = 128\pi + 192\pi = 320\pi$$

finalmente S_1 es una Superficie definida por.

$$\vec{r} = \rho\cos\theta\,\hat{i} + \rho\,\text{Sen}\theta\,\hat{j} + \rho\,\hat{k}$$

note que: $\vec{r} = \vec{r}(\rho,\theta)$ con: $0 \leq \rho \leq 4$

$0 \leq \theta \leq 2\pi$

Utilizando esta parametrización de S_1 en

$$\vec{f} = 4\rho^2\cos\theta\,\hat{i} + \rho^4\cos\theta\,\text{Sen}\theta\,\hat{j} + 3\rho\,\hat{k}$$

y $d\vec{S} = \vec{r}_\theta \times \vec{r}_\rho \, d\rho\, d\theta$