Problema 3.92

$\nabla \times f = 0$

$\nabla \times g = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left[\frac{df_3}{dy} - \frac{df_2}{dz}\right]\hat{i} - \left[\frac{df_3}{dx} - \frac{df_1}{dz}\right]\hat{j} + \left[\frac{df_2}{dx} - \frac{df_1}{dy}\right.$$

ademas

$$\left[\frac{dg_3}{dy} - \frac{dg_2}{dz}\right]\hat{i} - \left[\frac{dg_3}{dx} - \frac{dg_1}{dz}\right]\hat{j} + \left[\frac{dg_2}{dx} - \frac{dg_1}{dy}\right]\hat{k} = 0$$

$$\nabla \cdot (f \times g) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g)$$

$$= g \cdot (0) - f \cdot (0)$$

$$= 0 - 0$$

$$= 0$$

$\therefore$ $\nabla \cdot (f \times g) = 0$

$f \times g$ es solenoidal

Problema 3.93

Si $\nabla \cdot (\nabla \phi \times \nabla \psi)$

$$\nabla \cdot (\nabla \phi \times \nabla \psi) = \nabla \psi \cdot (\nabla \times \nabla \phi) - \nabla \phi \cdot (\nabla \times \nabla \psi)$$

pero
$$\nabla \times (\nabla \phi) = \left[\frac{d^2 \phi}{dy\,dz} - \frac{d^2 \phi}{dz\,dy}\right]\hat{\imath} + \left[\frac{d^2 \phi}{dz\,dx} - \frac{d^2 \phi}{dx\,dz}\right]\hat{\jmath} + \left[\frac{d^2 \phi}{dx\,dy} - \frac{d^2 \phi}{dy\,dx}\right]\hat{k}$$

$\Rightarrow \quad \nabla \times (\nabla \phi) = 0$

analogamente para $\nabla \times (\nabla \psi)$, entonces

$$\nabla \cdot (\nabla \phi \times \nabla \psi) = \nabla \psi \cdot (0) - \nabla \phi \cdot (0) = 0$$

$\boxed{\therefore\ \nabla \phi \times \nabla \psi \text{ es solenoidal}}$

Problema 7.94

ϕ es armonica

$\Rightarrow \nabla^2\phi = 0$ ademas $\nabla\phi = \frac{d\phi}{dx}\hat{\imath} + \frac{d\phi}{dy}\hat{\jmath} + \frac{d\phi}{dz}\hat{k}$

si $\nabla\cdot(\nabla\phi) = \frac{d}{dx}\left(\frac{d\phi}{dx}\right) + \frac{d}{dy}\left(\frac{d\phi}{dy}\right) + \frac{d}{dz}\left(\frac{d\phi}{dz}\right)$

$$= \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}$$

$$= \nabla^2\phi$$

$$= 0$$

$\boxed{\therefore \nabla\phi \text{ es solenoidal}}$

si $\nabla\times(\nabla\phi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \frac{d\phi}{dz} \end{vmatrix}$

$$= \left[\frac{d^2\phi}{dy\,dz} - \frac{d^2\phi}{dz\,dy}\right]\hat{\imath} + \left[\frac{d^2\phi}{dz\,dx} - \frac{d^2\phi}{dx\,dz}\right]\hat{\jmath} + \left[\frac{d^2\phi}{dx\,dy} - \frac{d^2\phi}{dy\,dx}\right]$$

$$= 0$$

$\boxed{\therefore \nabla\phi \text{ es irrotacional}}$

qed

Problema 3.95

si $cf = \nabla \emptyset$

$\Rightarrow f = \frac{\nabla \emptyset}{c}$

ahora si $f \cdot \nabla \times f = f \cdot (\nabla \times \frac{\nabla \emptyset}{c})$

como c es una cte podemos re acomodar la

$f \cdot \nabla \times f = \frac{f}{c} \cdot (\nabla \times \nabla \emptyset)$

pero $\nabla \times (\nabla \emptyset) = 0$

$\Rightarrow f \cdot \nabla \times f = \frac{f}{c} \cdot (0)$

$= 0$

$$\boxed{\therefore \quad f \cdot \nabla \times f = 0}$$

qed

Problema 3.96

Por la fórmula del triple producto vectorial

$$f \times (\nabla \times f) = (f \cdot f)\nabla - (f \cdot \nabla) f$$

despejando $(f \cdot \nabla) f$

$$f \times (\nabla \times f) + (f \cdot \nabla) f = (f \cdot f)\nabla$$

$$(f \cdot \nabla) f = (f \cdot f)\nabla - f \times (\nabla \times f)$$

por lo tanto

$$(f \cdot \nabla) f = (f \cdot f)\nabla - f \times (\nabla \times f)$$

qed

Problema 3.97

si $[\nabla u \ \nabla v \ \nabla w] = (\nabla u \times \nabla v) \cdot \nabla w$

Pero $\nabla u \times \nabla v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \end{vmatrix}$

entonces

$$\left[\frac{dw}{dx}\hat{\imath} + \frac{dw}{dy}\hat{\jmath} + \frac{dw}{dz}\hat{k}\right] \cdot \left[\left(\frac{dv}{dz}\frac{du}{dy} - \frac{du}{dz}\frac{dv}{dy}\right)\hat{\imath} - \left(\frac{du}{dx}\cdot\frac{dv}{dz} - \frac{du}{dz}\cdot\frac{dv}{dx}\right)\hat{\jmath} + \left(\frac{du}{dy}\cdot\frac{dv}{dx} - \frac{du}{dy}\cdot\frac{dv}{dx}\right)\hat{k}\right]$$

es igual a

$$\left[\left(\frac{dv}{dz}\cdot\frac{du}{dy} - \frac{du}{dz}\cdot\frac{dv}{dy}\right)\frac{dw}{dx} - \left(\frac{du}{dx}\frac{dv}{dz} - \frac{du}{dz}\cdot\frac{dv}{dx}\right)\frac{dw}{dy} + \left(\frac{du}{dy}\frac{dv}{dx} - \frac{du}{dx}\cdot\frac{dv}{dy}\right)\right]$$

ademas por la definición del triple producto escalar

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix}$$

Problema 3.98

Si
$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix}$$

entonces

$$\left[\left(\frac{dv}{dy}\cdot\frac{dw}{dz}-\frac{dv}{dz}\cdot\frac{dw}{dy}\right)\frac{du}{dx}-\left(\frac{dv\cdot dw}{dx\,dz}-\frac{dv\,dw}{dz\cdot dx}\right)\frac{du}{dy}+\left(\frac{dv\,dw}{dx\,dy}-\frac{dv\,dw}{dy\,dx}\right)\frac{du}{dz}\right.$$

ahora ordenando tenemos que

$$\left[\left(\frac{dv}{dy}\frac{dw}{dz}-\frac{dv}{dy}\frac{dw}{dz}\right)\frac{du}{dx}-\left(\frac{dv\,dw}{dx\,dz}-\frac{dv\,dw}{dx\,dz}\right)\frac{du}{dy}+\left(\frac{dv\,dw}{dx\,dy}-\frac{dv\,dw}{dx\,y}\right)\frac{du}{dz}\right]$$

Pero $\frac{dv\,dw}{dy\,dz}-\frac{dv\,dw}{dy\,dz}=0$, analogamente para las demas.

$$\Rightarrow \left[(0)\frac{du}{dx}-(0)\frac{du}{dy}+(0)\frac{du}{dz}\right]=0$$

Por lo tanto $F(u,v,w)=0$, es funcionalmente dependiente.

qed

Problema 3.99

del problema 3.98 sabemos que

$$\left[\left(\frac{du}{dy}\frac{dw}{dz}-\frac{dv}{dz}\frac{dw}{dy}\right)\frac{du}{dx}-\left(\frac{dv}{dx}\frac{dw}{dz}-\frac{dv}{dz}\cdot\frac{dw}{dx}\right)\frac{du}{dy}+\left(\frac{dv}{dx}\frac{dw}{dy}-\frac{dv}{dy}\frac{du}{dx}\right)\frac{dv}{dz}\right]$$

sustituyendo u, v y w

$$\left[\frac{d(x-y+z)}{dy}\cdot\frac{d((2x+z)^2+(2y-z)^2)}{dz}-\frac{d(x-y+z)}{dz}\frac{d((2x+z)^2+(2y-z)^2)}{dy}\right]\frac{d(x+}{dx}$$

$$=\left[(-1)(4x+2z-4y+2z)-(1)(8y-4z\right](1) \quad \ldots \ (1)$$

$$-\left[\frac{d(x-y+z)}{dx}\frac{d((2x+z)^2+(2y-z)^2)}{dz}-\frac{d(x-y+z)}{dz}\cdot\frac{d((2x+z)^2+(2y-z)^2)}{dx}\right]\frac{d(x+y)}{dy}$$

$$=\left[(1)(4x+2z-4y+2z)-(1)(8x+4z)\right](1) \quad \ldots \ (2)$$

$$\left[\frac{d(x-y+z)}{dx}\cdot\frac{d((2x+z)^2+(2y-z)^2)}{dy}-\frac{d(x-y+z)}{dy}\cdot\frac{d((2x+z)^2+(2y-z)^2)}{dx}\right]\frac{d(x}{d}$$

$$=0 \quad \ldots \ (3)$$

sumando 1, 2 y 3

$$-\cancel{4x}-\cancel{4z}+\cancel{4y}-\cancel{8y}+\cancel{4z}-\cancel{4x}+\cancel{4y}-\cancel{4z}+\cancel{8x}+\cancel{4z}+0$$

$$=0$$

$\boxed{\therefore\ u, v \text{ y } w \text{ son funcionalmente dependient}}$