Problema 4.92

Si $F = \nabla\phi$ y $\nabla \cdot F = 0$ para una región R limitada por S, entonces

$$\iiint_R |F|^2 dV = \oiint \phi \, dF \cdot dS$$

como $F = \nabla\phi$ entonces $\nabla \cdot (\nabla\phi) = \nabla^2\phi = 0$ en R

si n es un vector unitario normal a S

$$F \cdot n = \nabla\phi \cdot n = \frac{\partial\phi}{\partial n} = 0 \quad \text{sobre } S$$

usando el vector identidad $\nabla \cdot (\phi F) = \phi \nabla \cdot F + F \cdot \nabla\phi$

$$\nabla \cdot (\phi\nabla\phi) = \phi\nabla\cdot\nabla\phi + \nabla\phi + \nabla\psi \cdot \nabla\phi = \phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi$$

$$\iiint_R |F|^2 dv = \iiint_R \nabla \cdot (\phi\nabla\phi)\, dv = \iiint_R (\phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi) dv$$

Aplicando el teorema de divergencia

$$\oiint \phi \frac{d\phi}{dn} \cdot ds$$

$$\Rightarrow \boxed{\oiint \phi \, dF \cdot ds}$$

Problema 4.93

$$\oiint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,ds$$

superficie elipsoide $ax^2+by^2+cz^2=1$

$$\oiint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\cdot ds = \oiint_S \frac{\sqrt{a^2x^2+b^2y^2+c^2z^2}\cdot ds}{a^2x^2+b^2y^2+c^2z^2}$$

$$= \oiint_S \frac{1}{\hat{n}}\,ds = \iiint_V dV$$

parametrizando la superficie

como $ax^2+by^2+cz^2=1$, entonces $\rho=1$

$\Rightarrow$ $x = a\,\text{sen}\,\theta\cos\phi$, $y = b\,\text{sen}\,\theta\,\text{sen}\,\phi$, $z = c\cos\theta$

$J\,d\rho\,d\theta\,d\phi = abc\,\text{sen}\,\theta$ $\quad 0\le\rho\le 1$

$0\le\theta\le\pi$

$$\Rightarrow \int_{\rho=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi} abc\,\text{sen}\,\theta\,d\theta\,d\phi\,d\rho \qquad 0\le\phi\le 2\pi$$

$$= abc\left[-\cos\theta\right]\Big|_0^{\pi}\cdot\left[\phi\right]\Big|_0^{2\pi}$$

$$= \boxed{abc\ 4\pi}$$

Problema 4.94

$F = ax\hat{i} + by\hat{j} + cz\hat{k}$

calcular $\oint\!\!\oint_S F \cdot ds$ cualquier superficie cerrada S que encierra una región de volumen V

$$\oint\!\!\oint_S F \cdot ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$\nabla \cdot F = \frac{\partial (ax)}{\partial x} + \frac{\partial (by)}{\partial y} + \frac{\partial (cz)}{\partial z} = (a+b+c)$$

$$= \iiint_V (a+b+c)\, dV$$

$$= (a+b+c) \iiint_V dV$$

$$= \boxed{(a+b+c)\, V}$$

Problema 4.95

Si $F = y\hat{i} + x\hat{j} + z^2\hat{k}$

calcular $\iiint_R \nabla \cdot F \, dV$, donde R es la región limitada por

$z = (1 - x^2 - y^2)^{\frac{1}{2}}$, $zy = 0$

$$\nabla \cdot F = \frac{\partial y}{dx} + \frac{\partial x}{dy} + \frac{\partial z^2}{dz} = 2z$$

$$z^2 = (1 - x^2 - y^2) \Rightarrow x^2 + y^2 + z^2 = 1$$

es una esfera de radio 1

$$\iiint_R \nabla \cdot F \, dV = \iiint_R 2z \, dV = 2\iiint_R r\cos\theta \, dv$$

$$= 2\int_{r=0}^{1} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} r\cos\theta \, r^2 \operatorname{sen}\theta \, dr\, d\phi\, d\theta$$

$$= 2\left(\frac{r^4}{4}\right)\Big|_{r=0}^{1} (2\pi)\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

problema 4.96

si $F = U(x,y)\hat{\imath} + V(x,y)\hat{\jmath}$

$$\oint_C F \times dr = K \iint_S \nabla \cdot F \, dx\, dy$$

C es una curva cerrada en el plano xy limita una región S sea $U = U(x,y)$ y $V = V(x,y)$

Dem

$$F \times dr = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ U & V & 0 \\ dx & dy & dz \end{vmatrix} = \hat{k} \oint_C (-V\,dx + U\,dy)$$

si $F \cdot dr = (-V\,dx + U\,dy)$

además $$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -V & U & 0 \end{vmatrix} = -\frac{\partial U}{\partial z}\hat{\imath} + \frac{\partial P}{\partial z}\hat{\jmath} + \left(\frac{\partial U}{\partial x} - \frac{\partial (-V)}{\partial y}\right.$$

entonces

$$\nabla \times F \cdot dS = \nabla \times F \cdot K\, dx\, dy = \left(\frac{\partial U}{\partial x} - \frac{\partial (-V)}{\partial y}\right) dx\, dy$$

por el teorema de stokes

$$\hat{k} \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\, dy = \hat{k} \iint_S (U\hat{\imath} + V\hat{\jmath}) \cdot \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) dx\, dy$$

$$= \boxed{\hat{k} \iint_S \nabla \cdot F \, dx\, dy}$$

qed

problema 4.97

$$V = \frac{1}{6} \oiint_S \nabla(r^2) \cdot ds$$

Si $f = \nabla(r^2)$

donde

$\vec{r}^2 = x^2 \hat{\imath} + y^2 \hat{\jmath} + z^2 \hat{k}$

entonces

$$\frac{1}{6} \oiint_S f \cdot ds$$

por el teorema de gauss

$$\frac{1}{6} \oiint f \cdot ds = \frac{1}{6} \iiint_V \nabla \cdot f \, dv = \frac{1}{6} \iiint_V \nabla \cdot (\nabla(r^2)) \, dV$$

como

$$\nabla \cdot (\nabla(r^2)) = \nabla^2(r^2) = \frac{\partial^2(x^2)}{dx^2} + \frac{\partial^2(y^2)}{dy^2} + \frac{\partial^2(z^2)}{dz^2} = 6$$

$$\Rightarrow \frac{1}{6} \iiint_V 6 \, dv = \iiint_V dv = \boxed{V}$$

$$\boxed{\therefore \frac{1}{6} \oiint_S \nabla(r^2) \cdot ds = V}$$

Problema 4.98

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

S es la superficie de la esfera $x^2+y^2+z^2=1$ y n es el vector unitario normal exterior

$$= \iiint_S \left(\frac{dx}{dx} + \frac{dy}{dy} + \frac{dz}{dz}\right) dx\,dy\,dz$$

$$3 \iiint_S dx\,dy\,dz$$

$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

en coordenadas esféricas

$$3 \int_{\rho=0}^{1} \rho^2\,d\rho \int_0^{\theta} d\theta \int_0^{\phi} \operatorname{sen} \phi \, d\phi$$

$$= 3\left(\frac{1^3}{3}\right)(2\pi)(2)$$

$$= \boxed{4\pi}$$