Tarea #1.1

$z_1 = a_1 + ib_1$, $z_2 = a_2 + i b_2$ y $z_3 = a_3 + i b_3$

Ley conmutativa para la suma

$$z_1 + z_2 = z_2 + z_1$$

Dem:

$$z_1 + z_2 = (a_1 + i b_1) + (a_2 + i b_2) = (a_1 + a_2) + i(b_1 + b_2)$$
$$= (a_2 + a_1) + i(b_2 + b_1) = (a_2 + i b_2) + (a_1 + i b_1)$$
$$= z_2 + z_1$$

por lo tanto

$\boxed{z_1 + z_2 = z_2 + z_1}$ q.e.d

Ley asociativa para la suma

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Dem:

$$z_1 + (z_2 + z_3) = (a_1 + i b_1) + ((a_2 + i b_2) + (a_3 + i b_3))$$
$$= (a_1 + i b_1) + ((a_2 + a_3) + i(b_2 + b_3))$$
$$= (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3)$$
$$= ((a_1 + a_2) + i(b_1 + b_2)) + (a_3 + i b_0)$$
$$= ((a_1 + i b_1) + (a_2 + i b_2)) + (a_3 + i b_3)$$
$$= (z_1 + z_2) + z_3$$

por lo tanto

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ q.e.d

Ley distributiva para la suma

$$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$$

Dem:

$$z_1(z_2+z_3) = z_1((a_2+ib_2)+(a_3+ib_3))$$

$$= z_1((a_2+a_3)+i(b_2+b_3))$$

$$= (a_1+ib_2)((a_2+a_3)+i(b_2+b_3))$$

$$= a_1\cdot(a_2+a_3) - b_1(b_2+b_3) + i(b_1(a_2+a_3) + a_1(b_2+b$$

$$= (a_1a_2 + a_1a_3 - (b_1b_2 + b_1b_3) + i((b_1a_2 + b_1a_3) + a_1b_2 + a_1$$

$$= [(a_1\cdot a_2 - b_1\cdot b_2) + (a_1a_3 - b_1b_3)] + i[(b_1a_2 + a_1b_2) + (b_1a_3+$$

$$= [(a_1\cdot a_2 - b_1\cdot b_2) + i(b_1a_2 + a_1b_2)] + [(a_1a_3 - b_1b_3) + i(b_1a_3 + a_1b$$

$$= [(a_1+ib_1)\cdot(a_2+ib_2)] + [(a_1+ib_1)\cdot(a_3+ib_3)]$$

$$= z_1\cdot z_2 + z_1\cdot z_3$$

por lo tanto

$$\boxed{z_1(z_2+z_3) = z_1\cdot z_2 + z_1 z_3}$$

qed

Ley Conmutativa para la multiplicación

$z_1 \cdot z_2 = z_2 \cdot z_1$

Dem:

$$z_1 \cdot z_2 = (a_1 + i b_1)\cdot(a_2 + i b_2)$$
$$= (a_1 \cdot a_2 - b_1 \cdot b_2) + i(b_1 a_2 + a_1 b_2)$$
$$= (a_2 \cdot a_1 - b_2 \cdot b_1) + i(a_2 b_1 + b_2 a_1)$$
$$= (a_2 + i b_2)\cdot(a_1 + i b_1)$$
$$= \boxed{z_2 \cdot z_1}$$

por lo tanto

$$\boxed{z_1 \cdot z_2 = z_2 \cdot z_1}$$

qed

Ley Asociativa para la multiplicación

$z_1(z_2 z_3) = (z_1 z_2) z_3$

Dem

$$z_1(z_2 z_3) = z_1[(a_2 + i b_2)\cdot(a_3 + i b_3)] = (a_1 + i b_1)\cdot[(a_2 a_3 - b_2 b_3) + i(b_2 a_3 +$$
$$= [a_1(a_2 a_3 - b_2 b_3) - b_1(b_2 a_3 + a_2 b_3)] + i[b_1(a_2 a_3 - b_2 b_3)$$
$$+ a_1(b_2 a_3 + a_2 b_3$$
$$= [(a_1 a_2 a_3 - a_1 b_2 b_3) - (b_1 b_2 a_3 + a_2 b_1 b_3)] + i[(b_1 a_2 a_3 - b_2 b_3 b_1)$$
$$+ (a_1 b_2 a_3 + a_1 a_2 b$$

= si [illegible] pano

$$= [(a_1 a_2 a_3 - b_1 b_2 a_3) - (a_1 b_2 b_3 + a_2 b_1 b_3)] + i[(a_1 a_2 b_3 - b_2 b_3 b_1) + (a_1 b_2 a_3 + b_1 a_2 a_3)]$$

$$= [a_3(a_1 a_2 - b_1 b_2) - b_3(a_1 b_2 + a_2 b_1)] + i[b_3(a_1 a_2 - b_2 b_1) + a_3(a_1 b_2 + b_1 a_2)]$$

$$= [(a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_2 b_1)] \cdot (a_3 + i b_3)$$

$$= [(a_1 + i b_1) \cdot (a_2 + i b_2)] \cdot z_3$$

$$= (z_1 \cdot z_2) \cdot z_3$$

por lo tanto

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

qed

z

Tarea 1.2

$$\frac{z+\bar{z}}{2} = Re(z)$$

donde $z = a + ib$

Dem:

$$\frac{z+\bar{z}}{2} = \frac{(a+ib)+(\overline{a+ib})}{2} = \frac{(a+ib)+(a-ib)}{2}$$

$$= \frac{(a+a)+i(b-b)}{2} = \frac{2a}{2} = \boxed{a}$$

por lo tanto

$$\boxed{\frac{z+\bar{z}}{2} = Re(z) \text{ qed}}$$

$$\frac{z-\bar{z}}{2i} = Im(z)$$

Dem:

$$\frac{z-\bar{z}}{2i} = \frac{(a+ib)-(\overline{a+ib})}{2i} = \frac{(a+ib)-(a-ib)}{2i}$$

$$= \frac{(a-a)+i(b+b)}{2i} = \frac{2bi}{2i} = \boxed{b}$$

por lo tanto

$$\boxed{\frac{z-\bar{z}}{2i} = Im(z)} \text{ qed}$$

Tarea 1.3

Demostrar la propiedad $|z_1 z_2| = |z_1||z_2|$

donde $z_1 = x_1 + iy_1$ y $z_2 = x_2 + iy_2$

Dem:

$$|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 \cdot z_2})$$
$$= z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2}$$
$$= z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2}$$
$$= |z_1|^2 \cdot |z_2|^2$$

Tomando raíces cuadradas en la igualdad obtenida

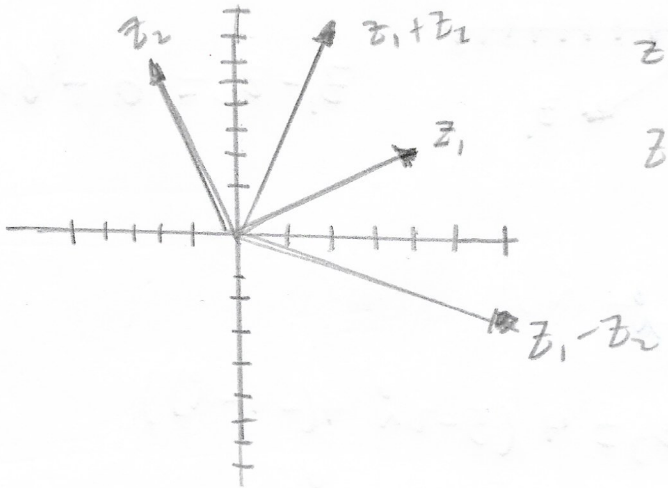
$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

qed

[illegible]

Exercises 1.2

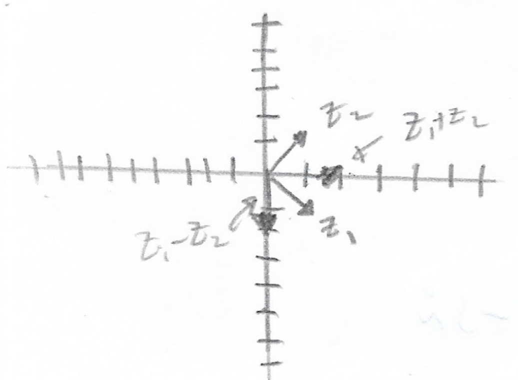
$z_1 = 4+2i$; $z_2 = -2+5i$, z_1+z_2, z_1-z_2



$z_1+z_2 = 2+7i$

$z_1-z_2 = 6-3i$

2. $z_1 = 1-i$, $z_2 = 1+i$; z_1+z_2, z_1-z_2

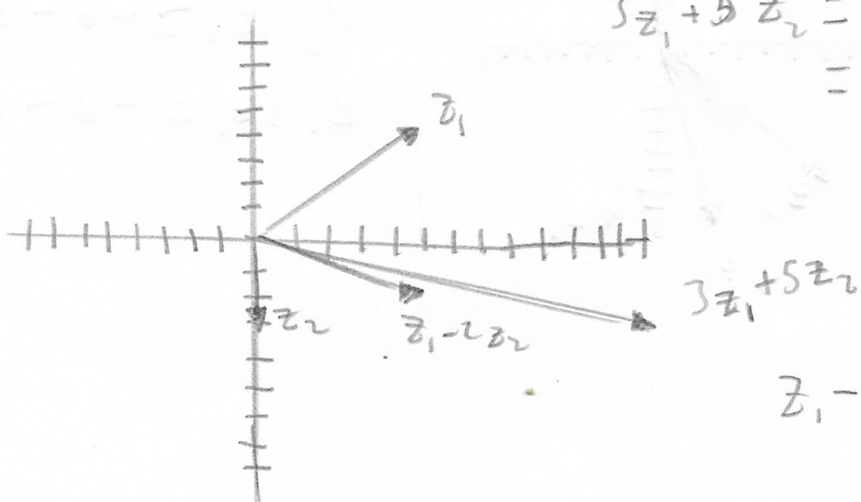


$z_1+z_2 = 2+0i = 2$

$z_1-z_2 = 0-2i = -2i$

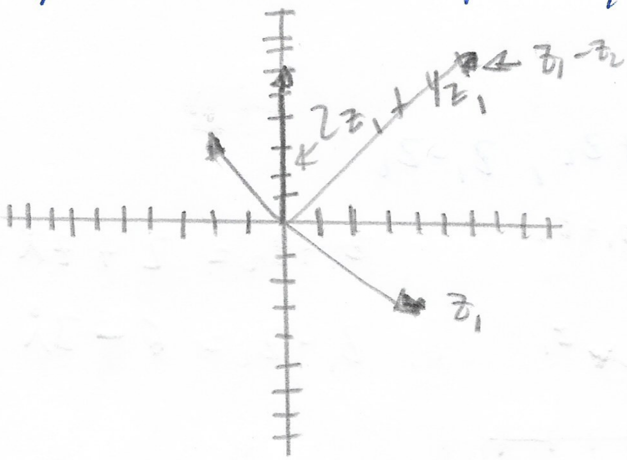
3. $z_1 = 5+4i$, $z_2 = -3i$, $3z_1+5z_2$, z_1-2z_2

$3z_1+5z_2 = 15+12i-15i$
$= 15-3i$



$z_1-2z_2 = 5+4i-6i$
$= 5-2i$

4. $z_1 = 4-3i$, $z_2 = -2+3i$, $2z_1 + 4z_2$, $z_1 - z_2$



$$2z_1 + 4z_2 = 8 - 6i - 8 + 12i$$
$$= 6i$$

$$z_1 - z_2 = 6 + 6i$$

5. $z_1 = 5-2i$ y $z_2 = -1-i$

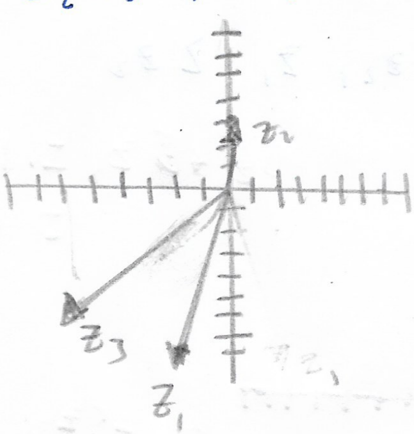
$$z_3 = 4(z_1 + z_2) \Rightarrow z_3 = 4(5-2i+(-1-i))$$
$$= 4[4-3i]$$
$$= 16-12i$$

por lo tanto

$$\boxed{z_3 = 16-12i}$$

6.

(a) $z_1 = -2-8i$, $z_2 = 3i$, $z_3 = -6-5i$



$$z_1 - z_2 = -2-11i$$
$$z_3 - z_2 = -6-8i$$
$$z_1 + z_2 = -8-13i$$

7. distancia z_1 a z_2

$$d_1 = \sqrt{(-2+0)^2 + (-8-3)^2} = \sqrt{4+121} = \sqrt{125} =$$

distancia z_2 a z_3

$$d_2 = \sqrt{(-6)^2 + (8)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

distancia z_3 a z_1

$$d_3 = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$(\sqrt{125})^2 = 10^2 + 5^2$$

$$125 = 100 + 25$$

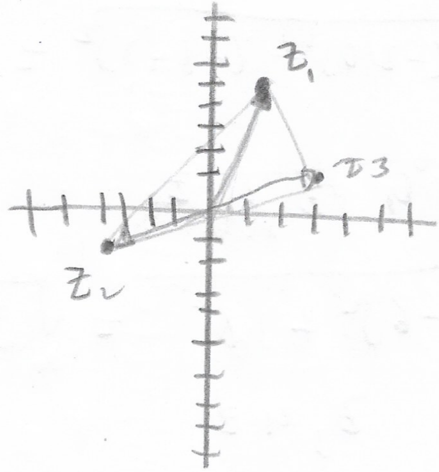
$$\boxed{128 = 125}$$

se satisface la igualdad aplicando el teorema de pitagoras, por lo tanto son vertice de un triangulo rectangulo

8. $z_1 = 1+5i$, $z_2 = -4-i$, $z_3 = 3+i$

$z_3 - z_2 = 7+2i$

$z_1(1,5)$ $\quad z_1 + (\frac{z}{2}(7+i)$



$$d = \sqrt{\left(\frac{5}{2}\right)^2 + (-4)^2}$$

$$= \sqrt{\frac{25}{4} + 16} = \sqrt{\frac{25}{4} + \frac{64}{4}}$$

$$= \sqrt{\frac{89}{4}} = \boxed{\frac{\sqrt{89}}{2}}$$

9. $(1-i)^2$

$= 1^2 - 2i + i^2 = 1 - 2i - 1$

$= -2i$

aplicando el modulo

$|-2i| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = \boxed{2}$

10. $i(2-i) - 4\left(1 + \frac{1}{4}i\right)$

$= (1 + 2i) - (4 + i) = -3 + i$

aplicando el modulo

$|-3+i| = \sqrt{(-3)^2 + 1^2} = \sqrt{9+1} = \boxed{\sqrt{10}}$

11. $\dfrac{2i}{3-4i}$

$= \dfrac{0 \cdot 3 - 8}{9+16} + i\,\dfrac{2 \cdot 3 - 0 \cdot -4}{9+16} = \dfrac{-8}{25} + \dfrac{6}{25}i$

aplicando el modulo

$\left|-\frac{8}{25} + \frac{6}{25}i\right| = \sqrt{\left(-\frac{8}{25}\right)^2 + \left(\frac{6}{25}\right)^2} = \sqrt{\frac{64}{625} + \frac{36}{625}} = \sqrt{\frac{100}{625}}$

$= \dfrac{10}{25} = \boxed{\dfrac{2}{5}}$

12. $\dfrac{1-2i}{1+i} + \dfrac{2-i}{1-i}$

$= \dfrac{1-2}{2} + i\dfrac{-2-1}{2} + \dfrac{2+1}{2} + i\dfrac{-1+2}{2}$

$= -\dfrac{1}{2} - \dfrac{3i}{2} + \dfrac{3}{2} + \dfrac{i}{2} = 1 - i$

aplicando el modulo

$|1-i| = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}$

12. $\cancel{10+8i}$, $\cancel{11}$

13. $|z-1-3i|^2$, $z = x+iy$

$|x+iy-1-3i|^2 = |(x-1)+(y-3)i|^2$

$= \left(\sqrt{(x-1)^2+(y-3)^2}\right)^2$

$= (x-1)^2+(y-3)^2$

$= x^2-2x+1+y^2-6y+9$

$= \boxed{x^2+y^2-2x-6y+10}$

14. $|z+5\bar{z}|$, $z = x+iy$

$|x+iy+5x-i5y| = |6x+i(y-5y)|$

$= |6x-i4y|$

$= \sqrt{(6x)^2+(4y)^2} = \boxed{\sqrt{36x^2+16y^2}}$

15. $10+8i \;;\; 11-6i$ origen y $1+i$

$|10+8i| = \sqrt{10^2+8^2} = \sqrt{100+64} = \sqrt{164}$

$|11-6i| = \sqrt{11^2+(-6)^2} = \sqrt{121+36} = \sqrt{157}$

como $\sqrt{164} > \sqrt{157}$ entonces $11-6i$ esta más cerca del origen

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

ademas también $11-6i$ esta más cerca de $1+i$

16. $\frac{1}{2} - \frac{1}{4}i \;,\; \frac{2}{3} + \frac{1}{6}i$

$|\frac{1}{2} - \frac{1}{4}i| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{4})^2} = \sqrt{\frac{2}{8} + \frac{1}{8}} = \sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{2\sqrt{2}}$

$|\frac{2}{3} + \frac{1}{6}i| = \sqrt{(\frac{2}{3})^2 + (\frac{1}{6})^2} = \sqrt{\frac{4}{9} + \frac{1}{36}} = \sqrt{\frac{16}{36} + \frac{1}{36}} = \sqrt{\frac{17}{36}} = \frac{\sqrt{17}}{6}$

como $\frac{\sqrt{17}}{6} > \frac{\sqrt{3}}{2\sqrt{2}}$ entonces $\frac{1}{2} - \frac{1}{4}i$ esta mas cerca del origen

ademas $\frac{2}{3} + \frac{1}{6}i$ esta más cerca de $1+i$

17. $Re((1+i)z-1)=0$

donde $z=(a+bi)$

$Re(1+i)(a+bi)=1$

$a+i(a+b)-b=1$

$a-b=1$ ← ya que solo nos interesa la parte real

$a+b=0$

$Re(((b+1)+bi)(i+1))=1$

desarrollando

$Re(b+1+bi+bi+i-b=1)$ $Re(1+(2b+1)i)=1$

Así $b=\frac{1}{2}$ y $a=\frac{3}{2}$

sustituyendo

$$\boxed{Re\left(\frac{3}{2}+\frac{1}{2}i\right)(1+i)-1)=0}$$

18. $[Im(i\bar{z})]^2=2$

$z=a+bi$, $\bar{z}=a-bi$

Entonces

$[Im(i(a-bi)]^2=2$

$[Im(ai-bi^2)]^2=2 \Rightarrow [Im(ai+b)]^2=2$

tomamos la parte imaginaria y elevamos al cuadrado

$a^2=2 \Rightarrow \boxed{a=\sqrt{2}}$

19. $|z-i| = |z-1|$

$z = a + ib$

$|a+ib-i| = |a+ib-1|$

$|a+i(b-1)| = |(a-1)+ib|$

$\sqrt{a^2+(b-1)^2} = \sqrt{(a-1)^2+b^2}$

$a^2+b^2-2b+1 = a^2-2a+1+b^2$

De la ecuación anterior tenemos que $b=a$

una recta que pasa por el origen y forma un angulo de $\frac{\pi}{4}$ con el eje real

20. $\bar{z} = z^{-1}$

$z = a+bi$, $\bar{z} = a-bi$

$a-bi = (a+bi)^{-1}$

$(a-bi)(a+bi) = 1$

$a^2+b^2+i(-ba+ab) = 1$

$a^2+b^2 = 1$

el conjunto de puntos satisface la ecuación de una circunferencia

21. $\mathrm{Im}(z^2)=2$

$z=a+ib$

$\mathrm{Im}((a+ib)^2)=2$

$\mathrm{Im}(a^2+2abi-b^2)=2$

$2ab=2$

$ab=1$

El conjunto de puntos describe una hiperbola

22. $\mathrm{Re}(z^2)=|\sqrt{3}-i|$

$z=a+ib$

$\mathrm{Re}((a+ib)^2)=\sqrt{(\sqrt{3})^2+1^2}$

$\mathrm{Re}(a^2+2abi-b^2)=\sqrt{4}$

$a^2-b^2=2$

el conjunto de puntos describe una hiperbola

25. $|z-2| = Re(z)$

$z = a + ib$

$|a+ib-2| = Re(a+ib)$

$\sqrt{(a-2)^2+b^2} = a$

$a^2-2a+4+b^2 = a^2$

$b^2-2a+4=0$

el conjunto de puntos satisfacen la ecuación de una parabola

26. $|z| = Re(z)$

$z = a+ib$

$|a+ib| = Re(a+ib)$

$\sqrt{a^2+b^2} = a$

$a^2+b^2 = a^2$

$b^2 = 0$

el conjunto de puntos satisfacen la ecuación de una recta

27

si $|z|=2$, entonces $|z+6+8i| \leq 13$

Dem:

$$|z+(6+8i)| \leq |z|+|6+8i|$$
$$\leq 2+\sqrt{6^2+8^2}$$
$$\leq 2+\sqrt{100}$$
$$\leq 2+10$$
$$\leq 12$$

$\boxed{\therefore\ |z+6+8i| \leq 12}$ qed

28.

si $|z|=1$, then $1 \leq |z^2-3| \leq 4$

$$|z^2-3| \leq |z^2|+|-3|$$
$$\leq |z\cdot z|+|-3|$$
$$\leq |z|\cdot|z|+3$$
$$\leq 1+3=4 \quad \Rightarrow |z^2-3| \leq 4$$

como $|-3| \geq 0$ y $|z|=1$

$\boxed{\therefore\ 1 \leq |z^2-3| \leq 4}$ qed

29. $3z^2+2z+1$ y $|z|\leq 1$

Se tiene que

$3z^2+2z+1 \leq M$

$|3z^2+2z+1| \leq ||3z^2|+|2z+1||$

$\leq |3z^2|+|2z+1|$; $|2z|+|1| \geq 0$

$\leq |3z^2|+(|2z|+|1|)$,

Sustituyendo y simplificando

$|3z^2+2z+1| \leq |3(1)+2(1)+1|$

$\leq |6|$

Entonces $M = 6$ y $|z|\leq 1$

$\therefore |3z^2+2z+1| \leq 6$

Tarea 1.5

tan h w

Nota:

$\cos^2(iw) + \operatorname{sen}^2(iw) = \cosh^2 w - \operatorname{sen}h^2 w = 1$

Dem:

$$\frac{\operatorname{sen}h(w)}{\cos h(w)} = \frac{\frac{e^w - e^{-w}}{2}}{\frac{e^w + e^{-w}}{2}} = \frac{2(e^w - e^{-w})}{2(e^w + e^{-w})} = \frac{e^w - e^{-w}}{e^w + e^{-w}}$$

$$= \boxed{\tan h(w)}$$

qed

Dem:

Si

$\cosh^2 w - \operatorname{sen}h^2 w = 1 \quad \ldots\ldots (1)$

$$\frac{\cosh^2 w}{\cosh^2 w} - \frac{\operatorname{sen}h^2 w}{\cosh^2 w} = \frac{1}{\cosh^2 w}$$ Dividiendo por $\cosh^2 w$

$$1 - \tanh^2 w = \frac{1}{\cosh^2 w}$$

$$\boxed{1 - \tan h^2 w = \operatorname{sech}^2 w}$$

Tarea 1.6

1.- Encontrar las 2 raíces cuadradas de 1

Solución

$$|z| = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$z = e^{0i + 2k\pi i} = e^{2k\pi i}$$

$$\sqrt{z} = e^{\frac{2k\pi i}{2}} = e^{k\pi i}, \quad k = 0, 1$$

$$\Rightarrow k=0 \Rightarrow z_0 = e^{0i} = \cos(0) + i\,\mathrm{sen}(0) = 1$$

$$\Rightarrow k=1 \Rightarrow z_1 = e^{\pi i} = \cos(\pi) + i\,\mathrm{sen}(\pi) = -1$$

$$\therefore \boxed{\sqrt{1} = 1, -1}$$

2.- Encontrar las 3 raíces cúbicas de 1

$z=1$

$|z| = \sqrt{1} = 1$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$z = e^{0 + 2\pi k i} = e^{2\pi k i}$

$\sqrt[3]{z} = \left(e^{2\pi k i}\right)^{\frac{1}{3}} = e^{\frac{2}{3}\pi k i}, \quad k = 0, 1, 2$

Si

$k=0 \Rightarrow z_0 = e^{0i} = \cos(0) + i\,\text{sen}(0) = 1$

$k=1 \Rightarrow z_1 = e^{\frac{2}{3}\pi i} = \cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=2 \Rightarrow z_2 = e^{\frac{4}{3}\pi i} = \cos\left(\frac{4}{3}\pi\right) + i\,\text{sen}\left(\frac{4}{3}\pi\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore \boxed{\sqrt[3]{1} = 1, \ -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ -\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

3.- Encontrar todas las soluciones de la ecuación

$z^4 + 1 = 0$

Solución

$$z^4 + 1 = 0 \Rightarrow z^4 = -1 \Rightarrow z^4 = -1 = \cos\pi + i\,\mathrm{sen}\,\pi$$

$$z^4 = e^{i\pi}$$

$$z^4 = e^{i\pi + i2k\pi} \qquad k = 0, 1, 2, 3$$

$$z^4 = e^{i\pi(1+2k)}$$

$$z = \left[e^{i\pi(1+2k)}\right]^{\frac{1}{4}}$$

$$z = e^{i\frac{\pi}{4}(1+2k)} \qquad k = 0, 1, 2, 3$$

para

$$k = 0 \Rightarrow z_0 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\,\mathrm{sen}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 1 \Rightarrow z_1 = e^{\frac{3i\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 2 \Rightarrow z_2 = e^{i\frac{5}{4}\pi} = \cos\left(\frac{5\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k = 3 \Rightarrow z_3 = e^{i\frac{7}{4}\pi} = \cos\left(\frac{7}{4}\pi\right) + i\,\mathrm{sen}\left(\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

4.- Encontrar las raíces de $(-1+i\sqrt{3})^{\frac{1}{2}}$

$z = -1 + i\sqrt{3}$

$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\frac{\sqrt{3}}{-1} = -\frac{\pi}{3}$

$z = 2e^{-\frac{\pi}{3}i + 2k\pi i}$

$\sqrt{z} = \sqrt{2}\, e^{-\frac{\pi}{6}i + k\pi i}$ $\quad k = 0, 1$

$K=0 \Rightarrow z_0 = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\,\text{sen}\left(-\frac{\pi}{6}\right)\right) = \sqrt{\frac{3}{2}} - i\frac{\sqrt{2}}{2}$

$K=1 \Rightarrow z_0 = \sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\,\text{sen}\left(\frac{5\pi}{6}\right)\right) = -\sqrt{\frac{3}{2}} + i\frac{\sqrt{2}}{2}$

5.- $\left(\dfrac{1+i}{\sqrt{3}+i}\right)^{\frac{1}{6}}$

$$Z=\left(\frac{1+i}{\sqrt{3}+i}\right)=\frac{\sqrt{3}+1}{4}+\frac{\sqrt{3}-1}{4}i$$

$$|Z|=\sqrt{\left(\frac{\sqrt{3}+1}{4}\right)^2+\left(\frac{\sqrt{3}-1}{4}\right)^2}=\sqrt{\frac{1}{2}}$$

$$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}-1}{4}}{\frac{\sqrt{3}+1}{4}}\right)=\tan^{-1}(2+\sqrt{3})=\frac{5\pi}{12}$$

$$Z=\sqrt{\tfrac{1}{2}}\,e^{\frac{5\pi}{12}i+2k\pi i}$$

$$\sqrt[6]{Z}=\sqrt[12]{\tfrac{1}{2}}\cdot e^{\frac{5\pi}{72}i+\frac{k\pi i}{3}},\quad k=0,1,2,3,4,5$$

$$k=0 \Rightarrow Z_0=\sqrt[12]{\tfrac{1}{2}}\cdot e^{\frac{5\pi}{72}i}=\sqrt[12]{\tfrac{1}{2}}\left[\cos\left(\frac{5\pi}{72}\right)+i\,\mathrm{sen}\left(\frac{5\pi}{72}\right)\right]$$

$$k=1 \Rightarrow Z_1=\sqrt[12]{\tfrac{1}{2}}\,e^{\frac{5\pi}{72}i+\frac{\pi i}{3}}=\sqrt[12]{\tfrac{1}{2}}\left[\cos\left(\frac{29}{72}\pi\right)+i\,\mathrm{sen}\left(\frac{29\pi}{72}\right)\right]$$

$$k=2 \Rightarrow Z_2=\sqrt[12]{\tfrac{1}{2}}\,e^{\frac{5\pi}{72}i+\frac{2}{3}\pi i}=\sqrt[12]{\tfrac{1}{2}}\left[\cos\left(\frac{53}{72}\pi\right)+i\,\mathrm{sen}\left(\frac{53}{72}\pi\right)\right]$$

$$k=3 \Rightarrow Z_3=\sqrt[12]{\tfrac{1}{2}}\,e^{\frac{5\pi}{72}i+\pi i}=\sqrt[12]{\tfrac{1}{2}}\left[\cos\left(\frac{77}{72}\pi\right)+i\,\mathrm{sen}\left(\frac{77}{72}\pi\right)\right]$$

$$K=4 \Rightarrow z_4 = \sqrt[12]{\frac{1}{2}}\, e^{\frac{5\pi}{72} i + \frac{4}{3}\pi i} = \sqrt[12]{\frac{1}{2}}\left[\cos\left(\frac{101}{72}\pi\right) + i\sin\left(\frac{101}{72}\pi\right)\right]$$

$$K=5 \Rightarrow z_5 = \sqrt[12]{\frac{1}{2}}\, e^{\frac{5\pi}{72} i + \frac{5}{3}\pi i} = \sqrt[12]{\frac{1}{2}}\left[\cos\left(\frac{125}{72}\pi\right) + i\sin\left(\frac{120}{72}\pi\right)\right]$$

Tarea 1.8

(a) i^{i}

solución:

Utilizando la expresión $z^{a} = e^{a\ln z}$

$z = i$ y $a = i$

$\Rightarrow i^{i} = e^{i \ln i}$

donde $\ln(i) = \ln|i| + i\arg(z)$

$= \cancel{\ln 1}^{0} + i\left(\frac{\pi}{2} + 2K\pi\right) \quad K = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \quad e^{i\ln i} = e^{i\left(i\frac{\pi}{2} + 2K\pi\right)} = e^{-\left(\frac{1+4K}{2}\right)\pi}$

por tanto

$\boxed{i^{i} = e^{-\left(\frac{1+4K}{2}\right)\pi}} \quad K = 0, \pm 1, \pm 2, \ldots$

(b) $i^{\frac{1}{i}}$

solución:

Utilizando la expresión $z^{a} = e^{a\ln z}$

$z = i$ y $a = \frac{1}{i}$

$\Rightarrow i^{\frac{1}{i}} = e^{\frac{1}{i}\ln i}$

pero $\ln i = i\left(\frac{\pi}{2} + 2K\pi\right)$

Entonces

$i^{\frac{1}{i}} = e^{\frac{1}{i}\left(i\left(\frac{\pi}{2} + 2K\pi\right)\right)} = \boxed{e^{\left(\frac{1+4K}{2}\right)\pi}}$

para $K = 0, \pm 1, \pm 2, \ldots$

c) 1^{i}

solución

si $z=1$ y $a=i$

entonces

$$1^{i} = e^{i \ln 1}$$

$$\Rightarrow \ln 1 = \ln|1| + i \arg(1)$$
$$= i(2k\pi)$$

por lo tanto

$$1^{i} = e^{i 2k\pi} \quad ; \quad k = 0, \pm 1, \pm 2, \ldots$$

d) $(-1)^{\sqrt{2}}$

solución

si $z=(-1)$ y $a=\sqrt{2}$

entonces

$$(-1)^{\sqrt{2}} = e^{\sqrt{2} \ln(-1)}$$

$$\Rightarrow \ln(-1) = \ln|-1| + i \arg(-1)$$
$$= 2k\pi i$$

por lo tanto

$$(-1)^{\sqrt{2}} = e^{\sqrt{2} \cdot 2k\pi i} = e^{\sqrt{8}\, k\pi i}$$

$$(-1)^{\sqrt{2}} = e^{\sqrt{8}\, k\pi i} \quad ; \quad k = 0, \pm 1, \pm 2$$

(e) $\left(\frac{1+i}{\sqrt{2}}\right)^{2i}$

Solución:

si $z=\left(\frac{1+i}{\sqrt{2}}\right)$ y $a=2i$

$$\Rightarrow \left(\frac{1+i}{\sqrt{2}}\right) = e^{2i\,\mathrm{Ln}\left(\frac{1+i}{\sqrt{2}}\right)}$$

donde

$$\mathrm{Ln}\left(\frac{1+i}{\sqrt{2}}\right) = \mathrm{Ln}\left(\left|\frac{1+i}{\sqrt{2}}\right|\right) + i\arg\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= \mathrm{Ln}\left(\left|\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right|\right) + i\arctan\left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)$$

$$= \cancel{\mathrm{Ln}(1)}^{0} + i\left(\frac{\pi}{4}+2k\pi\right) = i\left(\frac{\pi}{4}+2k\pi\right)$$

$$\Rightarrow 2i\,\mathrm{ln}\left(\frac{1+i}{\sqrt{2}}\right) = 2i\left(i\left(\frac{\pi}{4}+2k\pi\right)\right)$$

$$= -\frac{\pi}{2}-4k\pi = -\left(\frac{1+8k}{2}\right)\pi$$

$$\Rightarrow \left(\frac{1+i}{\sqrt{2}}\right)^{2i} = e^{-\left(\frac{1+8k}{2}\right)\pi}\ ; \quad k=0,\pm1,\pm2,\ldots$$

Para $k=0$

$$\left(\frac{1+i}{\sqrt{2}}\right)^{2i} = e^{-\frac{\pi}{2}}$$

para $k=1$

$$\left(\frac{1+i}{\sqrt{2}}\right)^{2i} = e^{-5\pi}$$

para $k=-1$

$$\left(\frac{1+i}{\sqrt{2}}\right)^{2i} = e^{\frac{7}{2}\pi}$$

F) $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{1+i}$

solución:

si $z=\frac{\sqrt{3}}{2}+\frac{i}{2}$ y $a=1+i$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{1+i} = e^{(1+i)\ln\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)}$$

donde

$$\mathcal{L}n\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = \mathcal{L}n\left(\left|\frac{\sqrt{3}}{2}+\frac{i}{2}\right|\right) + i\arg\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$

$$= \cancel{\mathcal{L}n(1)}^{0} + i\arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= i\left(\frac{\pi}{6}+2K\pi\right) = i\left(\frac{1+12K}{6}\right)\pi$$

para $K=0,\pm1,\pm2,\dots$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{1+i} = e^{(1+i)\left(i\left(\frac{1+12K}{6}\right)\pi\right)}$$

$$= e^{i\left(\frac{1+12K}{6}\right)\pi}\cdot e^{-\left(\frac{1+12K}{6}\right)\pi}$$

por lo tanto

$$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{1+i} = e^{-\left(\frac{1+12K}{6}\right)\pi + i\left(\frac{1+12K}{6}\right)\pi}$$ para $K=0,\pm1,\pm2,\dots$

g) $(1-i)^{3-3i}$

solución:

si $\quad z = 1-i \quad$ y $\quad a = 3-3i$

$$\Rightarrow (1-i)^{3-3i} = e^{(3-3i)\operatorname{Ln}(1-i)}$$

donde

$$\begin{aligned} \operatorname{Ln}(1-i) &= \operatorname{Ln}(|1-i|) + i\arg(1-i) \\ &= \operatorname{Ln}(\sqrt{2}) + i\left(-\frac{\pi}{4} + 2k\pi\right) \\ &= \frac{1}{2}\operatorname{Ln}(2) + i\left(2K - \frac{1}{4}\right)\pi \end{aligned}$$

$$\begin{aligned} \Rightarrow (1-i)^{3-3i} &= e^{(3-3i)\left(\operatorname{Ln}(2) + i\left(2K-\frac{1}{4}\right)\pi\right)} \\ &= e^{\left(3\operatorname{Ln}(2) + 3i\left(2K-\frac{1}{4}\right)\pi - 3i\operatorname{Ln}(2) + 3\left(2K-\frac{1}{4}\right)\pi\right)} \\ &= e^{\left[\left(\operatorname{Ln} 8 + \left(6K-\frac{3}{4}\right)\pi\right) + 3i\left(\left(2K-\frac{1}{4}\right)\pi + \operatorname{Ln}(2)\right)\right]} \end{aligned}$$

por lo tanto

$$(1-i)^{3-3i} = e^{\left[\left(\operatorname{Ln} 8 + \left(6K-\frac{3}{4}\right)\pi\right) + i\left(\left(6K-\frac{3}{4}\right)\pi + \operatorname{Ln} 8\right)\right]}$$

para $K = 0, \pm 1, \pm 2, \ldots$

Tarea 1.9

(a) $\lim_{z \to 2i} (iz^4 + 3z^2 - 10i)$

$$\lim_{z \to 2i} (iz^4 + 3z^2 - 10i) = i(2i)^4 + 3(2i)^2 - 10i$$
$$= i\,16i^4 + 3(4i^2) - 10i$$
$$= i16 - 12 - 10i$$
$$= 6i - 12$$
$$=$$

por lo tanto

$$\boxed{\lim_{z \to 2i} (iz^4 + 3z^2 - 10i) = 6i - 12}$$

(b) $\lim_{z \to e^{i\pi/4}} \frac{z^2}{z^4 + z + 1}$

$$\lim_{z \to e^{i\frac{\pi}{4}}} \left(\frac{z^2}{z^4+z+1}\right) = \frac{\lim_{z \to e^{i\frac{\pi}{4}}} z^2}{\lim_{z \to e^{i\frac{\pi}{4}}} z^4+z+1} = \frac{\left(e^{\frac{\pi i}{4}}\right)^2}{\left(e^{\frac{\pi i}{4}}\right)^4 + e^{\frac{i\pi}{4}} + 1}$$

$$= \frac{e^{i\frac{\pi}{2}}}{e^{\frac{\pi i}{4}} + 1 - 1} = \frac{i}{e^{\frac{\pi}{4}i}} = i e^{-\frac{\pi}{4}i}$$

$$= i\left(\cos\frac{\pi}{4} - i \operatorname{sen}\frac{\pi}{4}\right) = i\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= \boxed{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}$$

(c) $\lim_{z \to \frac{i}{2}} \dfrac{(2z-3)(4z+i)}{(iz-1)^2}$

$$\lim_{z \to \frac{i}{2}} \frac{(2z-3)(4z+i)}{(iz-1)^2} = \frac{(2(\frac{i}{2})-3)(4(\frac{i}{2})+i)}{(i(\frac{i}{2})-1)^2} = \frac{(i-3)(2i+i)}{(-\frac{1}{2}-1)^2}$$

$$= \frac{(i-3)(3i)}{(-\frac{3}{2})^2} = \frac{(-3-9i)}{\frac{9}{4}}$$

$$= \boxed{-\frac{4}{3} - 4i}$$

(d) $\lim_{z \to i} \dfrac{z^2+1}{z^6+1}$

$$\lim_{z \to i} \frac{z^2+1}{z^6+1} = \frac{(i)^2+1}{(i)^6+1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

aplicando l'Hopital

$$\lim_{z \to i} \left(\frac{2z}{6z^5}\right) = \lim_{z \to i} \frac{1}{3z^4} = \frac{1}{3(i)^4}$$

$$= \boxed{\frac{1}{3}}$$

(e) $\lim_{z \to 1+i} \left\{ \frac{z-1-i}{z^2-2z+2} \right\}^2$

$$\lim_{z\to 1+i} \left\{ \frac{z-1-i}{z^2-2z+2} \right\}^2 = \lim_{z\to 1+i} \left\{ \frac{z-1-i}{z^2-2z+2} \right\} \cdot \lim_{z\to 1+i} \left(\frac{z-1-i}{z^2-2z+2} \right)$$

Aplicando l'Hopital

$$\lim_{z\to 1+i} \left\{ \frac{1}{2z-2} \right\} = \frac{1}{2(1+i)-2} = \frac{1}{2+2i-2} = \frac{1}{2i} = -\frac{i}{2}$$

$$\Rightarrow \lim_{z\to 1+x} \left(\frac{z-1-i}{z^2-2z+2} \right) \lim_{z\to 1+i} \left(\frac{z-1-i}{z^2-2z+2} \right) = \left(-\frac{i}{2}\right)\left(-\frac{i}{2}\right) = \boxed{-\frac{1}{4}}$$

(f) $\lim_{z \to e^{i\frac{\pi}{3}}} \left(z - e^{i\frac{\pi}{3}} \right) \left(\frac{z}{z^3+1} \right)$

$$\lim_{z\to e^{i\frac{\pi}{3}}} \left(z - e^{i\frac{\pi}{3}} \right) \left(\frac{z}{z^3+1} \right) = \lim_{z\to e^{i\frac{\pi}{3}}} \left(z - e^{i\frac{\pi}{3}} \right) \lim_{z\to e^{i\frac{\pi}{3}}} \left(\frac{z}{z^3+1} \right)$$

para

$$\lim_{z\to e^{i\frac{\pi}{3}}} \left(z - e^{i\frac{\pi}{3}} \right) = \lim_{z\to e^{i\frac{\pi}{3}}} 1 = 1$$

para

$$\lim_{z\to e^{i\frac{\pi}{3}}} \left(\frac{z}{z^3+1} \right) = \lim_{z\to e^{i\frac{\pi}{3}}} \left(\frac{1}{3z^2} \right) = \frac{1}{3\left(e^{i\frac{\pi}{3}}\right)^2} = \frac{1}{3e^{\frac{2\pi i}{3}}} = -\frac{1}{6} - \frac{i\sqrt{3}}{6}$$

por lo tanto

$$\lim_{z\to e^{i\frac{\pi}{3}}} \left(z - e^{i\frac{\pi}{3}} \right) \left(\frac{z}{z^3+1} \right) = (1)\left(-\frac{1}{6} - \frac{i\sqrt{3}}{6} \right) = \boxed{-\frac{1}{6} - i\frac{\sqrt{3}}{6}}$$

(a) $f(z) = \dfrac{2z-3}{z^2+2z+2}$

solución

$$f(z) = \frac{2z-3}{z^2+2z+2} = \frac{2z-3}{(z+1+i)(z+1-i)}$$

$\Rightarrow$ observamos que el denominador se hace cero cuando

$z = -1-i$ y $z = -1+i$

$\therefore$ $f(z)$ va a ser continua para todos los valores de z excepto cuando $z=-1-i$ y $z=-1+i$

(b) $f(z) = \dfrac{3z^2+4}{z^2-16}$

solución

$$\frac{3z^2+4}{z^2-16} = \frac{3z^2+4}{(z-4)(z+4)}$$

$\Rightarrow$ observamos que el denominador se hace cero cuando

$z = -4+0i$ y $z = 4+0i$

$\therefore$ $f(z)$ será continua para todo z excepto cuando $z = \pm 4$

(c) $f(z) = \cot z$

Solución

Como $f(z) = \cot z = \dfrac{1}{\tan z}$

$\Rightarrow \tan z = 0$

esto se cumpliría cuando $z = 0, \pm\pi, \dots$

por lo tanto

$f(z)$ será continua para todo z excepto

para $z = 0, \pm\pi, 2\pi, \dots$

(d) $f(z) = \dfrac{1}{z} - \sec z$

Solución

$f(z) = \dfrac{1}{z} - \sec z = \dfrac{1 - z\sec z}{z}$

$\Rightarrow z = 0$

observemos que el denominador se hace

0 cuando $z = 0$

por lo tanto

$f(z)$ será continua para todo z excepto para

$z = 0$

(e) $f(z) = \dfrac{\tan h z}{z^2+1}$

Solución

$f(z) = \dfrac{\tan h z}{z^2+1}$

$\Rightarrow$ observamos que el denominador es cero cuando $z = i$ y $z = -i$

por lo tanto

$f(z)$ será continua para todo z excepto para $z \pm i$.