Problema 1

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \left(e^{\frac{1}{x^2}\ln \cos(x)}\right)$$ aplicando exponencial y ln

usando $f(x) = \ln \cos(x)$ y $g(x) = x^2$

$$= \lim_{x\to 0^+} \left(e^{\frac{-\tan x}{2x}}\right)$$ usando $f'(x) = -\tan x$ y $g'(x) = 2x$

$$= \lim_{x\to 0^+} \left(e^{\frac{-\sec^2 x}{2}}\right)$$ calculando el límite

$$= e^{\frac{-(-1)}{2}}$$

$$= \frac{1}{e^{\frac{1}{2}}}$$

$$= \boxed{\frac{1}{\sqrt{e}}}$$

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Operaciones

$f(x) = \ln(\cos(x))$

$f'(x) = \frac{1}{\cos(x)} \cdot (-\operatorname{sen} x) = -\tan x$

$g(x) = x^2$

$g'(x) = 2x$

si $f'(x) = -\tan x \Rightarrow f''(x) = -\sec^2 x$

$g'(x) = 2x \Rightarrow g''(x) = 2$

Problema 2

$$\int_0^\infty t e^{-st}\, dt$$

$u = -st \qquad du = -s\,dt \Rightarrow \frac{du}{-s} = dt$

además $\frac{u}{-s} = t$

$$\int \frac{u}{-s} \cdot e^u \frac{du}{-s} = \int \frac{ue^u}{2s} du = \frac{1}{2s}\int ue^u du$$

$v = u \Rightarrow dv = du$

$dw = e^u du \Rightarrow w = e^u$

$\Rightarrow \frac{1}{2s}\left(ue^u - \int e^u du\right)$

$= \frac{1}{2s}(ue^u - e^u)$

regresando el cambio

$$\lim_{b\to\infty} \frac{1}{2s}(-ste^{-st} - e^{-st})\Big|_0^b$$

$$\lim_{b\to\infty} \left(\frac{1}{2s}(-sbe^{-sb} - e^{-sb})\right) - \frac{1}{2s}(-s(0)e^0 - e^0)$$

$= \frac{1}{2s}(-s(0) - 0) - \frac{1}{2s}(-1)$

$= \boxed{\frac{1}{2s}}$

la integral converge y converge a $\frac{1}{2s}$

Problema 3

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

$u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow du\,2\sqrt{x} = dx$

$u^2 = x$ $\quad\quad\quad\quad\quad\quad\quad\quad du\,2u = dx$

$$\Rightarrow \int \frac{2u\,du}{u(1+u^2)} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u$$

regresando el cambio de variable

$$2\tan^{-1}\sqrt{x}\Big|_0^\infty$$

$$= \lim_{b\to\infty} 2\tan^{-1}\sqrt{b} - 2\tan^{-1}\sqrt{0}$$

$$= 2\left(\lim_{b\to\infty}\tan^{-1}\sqrt{b}\right) - 2(0)$$

$$= 2\left(\frac{\pi}{2}\right) - 0$$

$$= \boxed{\pi}$$

la integral converge en π