Problema 3.70. Sean $A(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$ y $B(t) = \hat{\imath} + t\hat{\jmath} + (1-t)\hat{k}$. En $t=1$, calcular

(a) $A'(t)$

$$\frac{dA(t)}{dt} = \hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k} \Rightarrow A'(1) = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

(b) $\dfrac{d[A(t)\cdot B(t)]}{dt}$ $\qquad \dfrac{dB(t)}{dt} = \hat{\jmath} - \hat{k}$

$= A(t)\cdot B'(t) + A'(t)\cdot B(t)$

$= (t^2 - t^3) + 1 + 2t^2 + 3(1-t)$

$= t^2 - t^3 + 1 + 2t^2 + 3 - 3t$

$= -t^3 + 3t^2 - 3t + 4$

$= -1^3 + 3(1)^2 - 3(1) + 4$

$= 3$

(c) $\dfrac{d[A(t)\times B(t)]}{dt}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & t^2 & t^3 \\ 1 & t & (1-t) \end{vmatrix} = \hat{\imath}\begin{vmatrix} t^2 & t^3 \\ t & (1-t) \end{vmatrix} - \hat{\jmath}\begin{vmatrix} t & t^3 \\ 1 & 1-t \end{vmatrix} + \hat{k}\begin{vmatrix} t & t^2 \\ 1 & t \end{vmatrix}$$

$= (-t^4 - t^3 + t^2)\hat{\imath}$
$- (-t^3 - t^2 + t)\hat{\jmath}$

$= (-t^4 - t^3 + t^2)\hat{\imath}$
$+ (t^3 + t^2 - t)\hat{\jmath}$

$(-4t^3 - 3t^2 + 2t)\hat{\imath} + (3t^2 + 2t - 1)\hat{\jmath}$

$(-4 - 3 + 2)\hat{\imath} + (3 + 2 - 1)\hat{\jmath}$

$-5\hat{\imath} + 4\hat{\jmath}$

Problema 7.71

Mostrar que si las ~~vects~~ terceras derivadas de la función vectorial $f(t)$ existen, entonces

(a) $\dfrac{d}{dt}[f(t) \times f'(t)] = f(t) \times f''(t)$

(b) $\dfrac{d}{dt}[f(t) \cdot f'(t) \cdot f'''(t)] = [f(t) \cdot f'(t) \cdot f'''(t)]$

si $B(t) = f'(t)$, $C(t) = f''(t)$ y $D(t) = f'''(t)$, entonces

$$\frac{d}{dt}[f(t) \times B(t)] = f(t) \times B'(t) + f'(t) \times B(t)$$

$$= f(t) \times f''(t) + \cancelto{0}{f'(t) \times f'(t)}$$

Dado que el angulo será de $0°$, entonces

$$= \boxed{f(t) \times f''(t)}$$

qed

$$\frac{d}{dt}[f(t) \cdot f'(t) \times f''(t)] = \cancelto{0}{f'(t) \cdot f'(t) \times f''(t)} + \cancelto{0}{f(t) \cdot f''(t) \times f''(t)}$$

$$+ f(t) \cdot f'(t) \times f'''(t)$$

$$= \boxed{f(t) \cdot f'(t) \times f'''(t)}$$

problema 3.72

si $f(t) \times f'(t) = 0$, entonces mas fuar que $f(t)$ tiene una dirección fija,

si $f(t) \times f'(t) = 0$ es ortogonal la derivada

como, $f(t) = f(t) \cdot e(t)$, entonces $f'(t) = f(t) \cdot e'(t) + f'(t) \cdot e(t)$

$f(t) e(t) \times (f(t) e'(t) + f'(t) e(t)) = 0$

$f(t) e(t) \times f(t) e'(t) + \cancel{f(t) e(t) \times f'(t) e(t)}^{0} =$

$f(t) - f(t) \times e(t) \cdot e'(t) =$

$e(t) \cdot e'(t) \times f(t) \cdot f(t) =$

$\boxed{e(t) \cdot e'(t) = 0}$

qed

$- f(t) \cdot f(t) \times e(t) \cdot e'(t) =$

$e(t) \times f(t) f(t) \cdot e'(t) =$

$\boxed{e(t) \times e'(t) = 0}$

qed

Problema 3.73
Verificar la regla de la cadena (3-19) para derivadas

Si $t = g(s)$ es diferenciable, entonces

$$\frac{dA(t)}{ds} = \frac{dA[g(s)]}{ds} = \frac{dA(t)}{dt} \cdot \frac{dg(s)}{ds}$$

$$\frac{dA[g(s)]}{ds} = \lim_{t \to s} \frac{A(g(t)) - A(g(s))}{t-s}$$

multiplicando por un 1 adecuado

$$= \lim_{t \to s} \frac{A(g(t)) - A(g(s))}{t-s} \cdot \frac{g(t)-g(s)}{g(t)-g(s)}$$

$$= \lim_{t \to s} \frac{A(g(t)) - A(g(s))}{g(t)-g(s)} \cdot \frac{g(t)-g(s)}{t-s}$$

$$= \left[\lim_{t \to s} \frac{A(g(t)) - A(g(s))}{g(t)-g(s)}\right] \cdot \left[\lim_{t \to s} \frac{g(t)-g(s)}{t-s}\right]$$

$$= A'(g(s)) \cdot g'(s)$$

$$= A'(t) \cdot g'(s)$$

qed

Problema 3.75

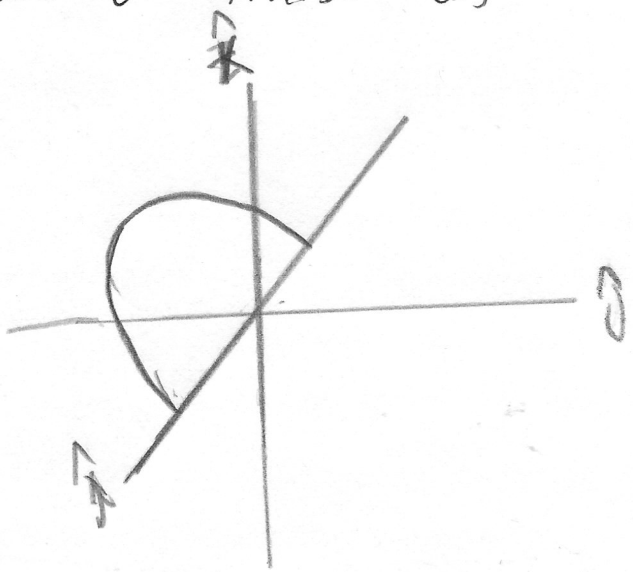
Trazar las curvas representadas por las siguientes funciones vectoriales

(a) $r(t) = \cos t\,\hat{i} + \operatorname{sen} t\,\hat{j} + \operatorname{sen} t\,\hat{k}$, $0 \leq t \leq 2\pi$

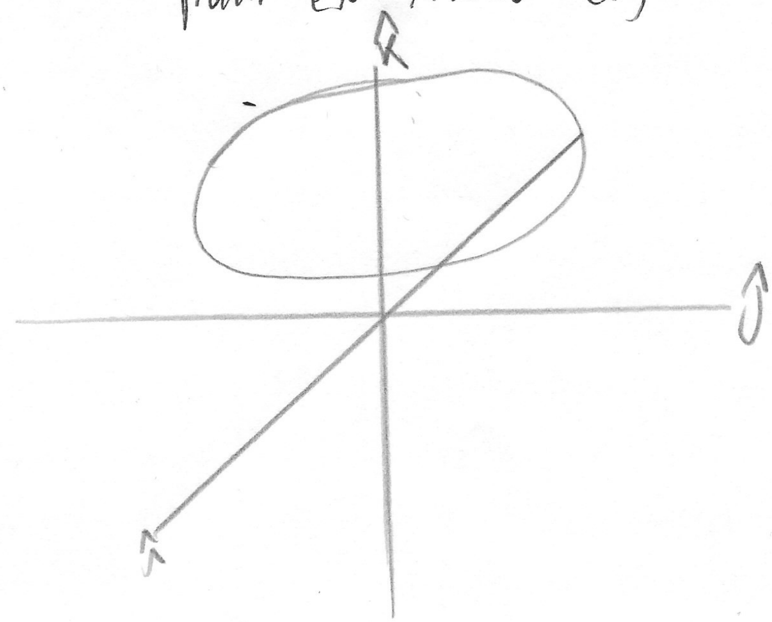
(b) $r(t) = a\cos t\,\hat{i} + a\operatorname{sen} t\,\hat{j} + bt\,\hat{k}$ $b > 0$ y $0 \leq t < \infty$

(c) $r(t) = t^3\,\hat{i} + t\,\hat{j}$ $-\infty < t < \infty$

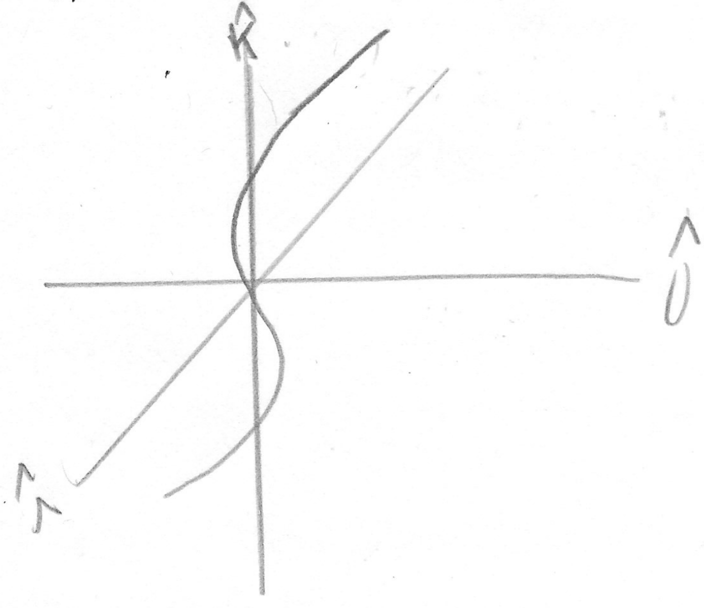
Para el inciso (a)



Para el inciso (b)



Para el inciso (c)



Problema 3.76 Hallar la longitud de las curvas representadas por las siguientes funciones vectoriales

(a) $r(t) = \cos t\,\hat{i} + \operatorname{sen} t\,\hat{j} + \operatorname{sen} t\,\hat{k}$, $0 \le t \le 2\pi$

(b) $r(t) = t\,\hat{i} + \operatorname{sen} 2\pi t\,\hat{j} + \cos 2\pi t\,\hat{k}$, $0 \le t \le 1$

(c) $r(t) = \cos 3t\,\hat{i} + \operatorname{sen} 3t\,\hat{j}$ $0 \le t \le 2\pi$

Para el inciso (a)

$$\int_a^b [f'(t)\cdot f'(t)]^{\frac{1}{2}}\,dt$$

$$r'(t) = -\operatorname{sen} t\,\hat{i} + \cos t\,\hat{j} + \cos t\,\hat{k}$$

$$[r'(t)\cdot r'(t)]^{\frac{1}{2}} = [\operatorname{sen}^2 t + \cos^2 t + \cos^2 t]^{\frac{1}{2}}$$

$$= [\operatorname{sen}^2 t + 2\cos^2 t]^{\frac{1}{2}}$$

$$\Rightarrow \int_0^{2\pi} \sqrt{\operatorname{sen}^2 t + 2\cos^2 t}\,dt$$

Para el inciso b

$$r'(t) = \hat{i} + 2\pi\cos 2\pi t\,\hat{j} + 2\pi \operatorname{sen} 2\pi t\,\hat{k}$$

$$[r'(t)\cdot r'(t)]^{\frac{1}{2}} = (1 + 4\pi^2[\cos^2 2\pi t + \operatorname{sen}^2 2\pi t])^{\frac{1}{2}}$$

$$= (1 + 4\pi^2)^{\frac{1}{2}}$$

$$\Rightarrow \int_0^1 \sqrt{1+4\pi^2}\,dt = [\sqrt{1+4\pi^2}\,t]\Big|_0^1$$

$$= \boxed{\sqrt{1+4\pi^2}}$$

para el inciso (c)

$$r'(t) = -3\,\text{sen}\,3t\,\hat{\imath} + 3\cos 3t\,\hat{\jmath}$$

$$[r'(t)\cdot r'(t)]^{\frac{1}{2}} = (9\,\text{sen}^2 3t + 9\cos^2 3t)^{\frac{1}{2}}$$

$$= \sqrt{9}\cdot\sqrt{\text{sen}^2 3t + \cos^2 3t}$$

$$= 3$$

$$\Rightarrow \int_0^{2\pi} 3\,dt = [3t]\Big|_0^{2\pi} = \boxed{6\pi}$$

Problema 3.37
Hallar el vector unitario T tangente a las curvas representadas por las siguientes funciones vectoriales en los puntos especificados

(a) $v(t) = \cos t\,\hat{i} + \operatorname{sen} t\,\hat{j} + t\,K$ en $t = \pi$

(b) $v(t) = \left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{K}$ en $t = 1$

(c) $v(t) = a(t - \operatorname{sen})t\,\hat{i} + a(1-\cos t)\hat{j}$ en cualquier punto t

$$s = \int_a^t |f'(t)|\,dt \qquad \frac{d[g(s)]}{dt} = T$$

Para el inciso (a)

$$v'(t) = -\operatorname{sen} t\,\hat{i} + \cos t\,\hat{j} + \hat{K}_1$$

$$[v'(t)\cdot v'(t)]^{\frac{1}{2}} = (1 + (\operatorname{sen}^2 t + \cos^2 t))^{\frac{1}{2}}$$

$$= \sqrt{2}$$

$$\Rightarrow s = \int_0^t \sqrt{2}\,dt = [\sqrt{2}\,t]\Big|_0^t = \sqrt{2}\,t$$

$$\Rightarrow t = \frac{s}{\sqrt{2}} \qquad s = \sqrt{2}\pi$$

$$g(s) = -\cos\frac{s}{\sqrt{2}}\hat{i} + \operatorname{sen}\frac{s}{\sqrt{2}}\hat{j} + \frac{s}{\sqrt{2}}\hat{K}$$

$$g'(s) = -\frac{s}{\sqrt{2}}\operatorname{sen}\frac{s}{\sqrt{2}}\hat{i} + \frac{s}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{K}$$

$$g'(\sqrt{2}\pi) = -\frac{\sqrt{2}\pi}{\sqrt{2}}\cancel{\operatorname{sen}\frac{\sqrt{2}\pi}{\sqrt{2}}}\hat{i} + \frac{\sqrt{2}\pi}{\sqrt{2}}\cos\frac{\sqrt{2}\pi}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{K}$$

$$= \boxed{-\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{K}}$$

para el inciso (b)

$$r'(t) = (1-t^2)\hat{\imath} + 2t\hat{\jmath} + (1+t^2)\hat{k}$$

$$[r'(t)\cdot r'(t)] = \left[(1-\cancel{2t^2}+t^4) + 4t^2 + 1 + \cancel{2t^2} + t^4\right]^{\frac{1}{2}}$$

$$= \left[2t^4 + 4t^2 + 2\right]^{\frac{1}{2}}$$

$$= \sqrt{2}\cdot\sqrt{t^4+2t^2+1}$$

$$= \sqrt{2}\sqrt{(t^2+1)^2}$$

$$= \sqrt{2}(t^2+1)$$

como $\dfrac{r'(t)}{|r'(t)|} = T$, entonces

$$\frac{(1-t^2)}{\sqrt{2}(t^2+1)}\hat{\imath} + \frac{2t}{\sqrt{2}(t^2+1)}\hat{\jmath} + \frac{(\cancel{1+t^2})}{\sqrt{2}(\cancel{1+t^2})}\hat{k} = T$$

evaluando en 1

$$\frac{2}{\sqrt{2}(2)}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k} = \boxed{\frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}}$$

para el inciso (c)

$$r'(t) = a(1-\cos t)\hat{\imath} + a(\operatorname{sen} t)\hat{\jmath}$$

$$|r'(t)| = \left[a^2(1-\cos t)^2 + a^2(\operatorname{sen}^2 t)\right]^{\frac{1}{2}}$$

$$= a\left[1-2\cos t+\cos^2 t + \operatorname{sen}^2 t\right]^{\frac{1}{2}}$$

$$= a\sqrt{2}\left[1-\cos t\right]^{\frac{1}{2}}$$

$$= 2a \operatorname{sen} \frac{t}{2}$$

$$T = \frac{a(1-\cos t)\hat{\imath}}{2a\operatorname{sen}\frac{t}{2}} + \frac{a \operatorname{sen} t}{2a\operatorname{sen}\frac{t}{2}}\hat{\jmath}$$

$$\frac{\operatorname{sen} 2\frac{t}{2}}{2\operatorname{sen}\frac{t}{2}} = \cos\frac{t}{2}$$

$$= \frac{2\operatorname{sen}^2\frac{t}{2}}{2\operatorname{sen}\frac{t}{2}}\hat{\imath} + \cos\frac{t}{2}\hat{\jmath}$$

$$\frac{\operatorname{sen} t}{2\operatorname{sen}\frac{t}{2}} = \cos\frac{t}{2}$$

$$\boxed{= \operatorname{sen}\frac{t}{2}\hat{\imath} + \cos\frac{t}{2}\hat{\jmath}}$$