Problema 4.87

$F = x\hat{i} + y\hat{j} + z\hat{k}$, calcular $\iint_S F \cdot dS$

$r = \cos u\,\hat{i} + \operatorname{sen} u\,\hat{j} + v\hat{k}$

$x = \cos u$
$y = \operatorname{sen} u$
$z = v$

$0 \leq u \leq 2\pi$, $0 \leq v \leq 1$

$dS = n\,dS$

n es el vector unitario normal exterior

$r_u = -\operatorname{sen} u\,\hat{i} + \cos u\,\hat{j} + 0\hat{k}$

$r_v = 0\hat{i} + 0\hat{j} + \hat{k}$

$$r_u \times r_v = \begin{vmatrix} \cos u & \operatorname{sen} u & v \\ -\operatorname{sen} u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \cos u & \operatorname{sen} u & v \\ -\operatorname{sen} u & \cos u & 0 \end{matrix}$$

$\operatorname{sen}^2 u$

$\cos^2 u$

$= \operatorname{sen}^2 u + \cos^2 u = 1$

$$\Rightarrow \iint F \cdot dS = \int_0^{2\pi}\int_0^1 du\,dv \Rightarrow u\Big|_0^{2\pi} \cdot v\Big|_0^1$$

$= [2\pi - 0] \cdot [1 - 0]$

$= 2\pi \cdot 1$

$= \boxed{2\pi}$

Problem 4.88

$F = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

$$\iint_S F \cdot dS$$

$z^2 = x^2 + y^2,\ z = 0,\ z = 4 \quad dS = n\,dS$

$r = (x, y, x^2 + y^2)$

$r_x = (1, 0, 2x)$

$r_y = (0, 1, 2y)$

$$r_x \times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \end{matrix} \quad \begin{matrix} \to 0 \\ -2x\hat{i} \\ -2y\hat{j} \\ \to 0 \\ \to \hat{k} \\ \to 0 \end{matrix}$$

$0^2 = x^2 + y^2 \quad 4^2 = x^2 + y^2$

$0 = x^2 + y^2 \quad 16 = x^2 + y^2$

$\Rightarrow r_x \times r_y = (-2x, -2y, 1)$

$F = 4x(x^2+y^2)\hat{i} + xy(x^2+y^2)\hat{j} + 3(x^2+y^2)\hat{k}$

$$\int_0^2 \int_0^2 \left(-8x^2(x^2+y^2) - 2xy^2(x^2+y^2) + 3(x^2+y^2)\right) dx\, dy$$

$$= \int_0^2 \int_0^2 -8x^4 - 8x^2y^2 - 2x^3y^2 - 2xy^4 + 3x^2 + 3y^2 \, dx\, dy$$

$$= \int_0^2 \left(-\frac{8}{5}x^5 - \frac{8}{3}x^3y^2 - \frac{2x^4}{4}y^2 - \frac{2x^2}{2}y^4 + \frac{3x^3}{3} + 3y^2x\right)\Bigg|_0^2 dy$$

$$= \int_0^2 \left(-\frac{256}{5} - \frac{64}{3}y^2 - 8y^2 - 4y^4 + 8 + 6y^2\right) dy$$

$$= -\frac{256}{5}y - \frac{64}{9}y^3 - \frac{8y^3}{3} - \frac{4y^5}{5} + 8y + \frac{6y^3}{3}\Bigg|_0^2 = -\frac{7060\,48}{45}$$

Problema 4.89

si $F = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ y S es la superficie del cubo limitado por

$x=0 \quad x=1$
$y=0 \quad y=1$
$z=0 \quad z=1$

calcular

(a) $\iint_S F\,ds$

Dividimos el cubo en areas

$S_1 \quad z=1$

$$\int_0^1\int_0^1 (y+1)\hat{i} + (1+x)\hat{j} + (x+y)\hat{k}\,dx\,dy$$

$$= \int_0^1 \left(\tfrac{3}{2}\right)\hat{i} + (1+x)\hat{j} + \left(x+\tfrac{1}{2}\right)\hat{k}\,dx = \left(\tfrac{3}{2}, \tfrac{3}{2}, 1\right)$$

$S_2 \quad y=1$

$$\int_0^1\int_0^1 (z+1)\hat{i} + (z+x)\hat{j} + (x+1)\hat{k}\,dx\,dz$$

$$= \int_0^1\int_0^1 \left(\tfrac{3}{2}\right)\hat{i} + \left(\tfrac{1}{2}+x\right)\hat{j} + (x+1)\hat{k}\,dx = \left(\tfrac{3}{2}, 1, \tfrac{3}{2}\right)$$

$S_3 \Rightarrow \quad x=1$

$$\int_0^1\int_0^1 (z+y)\hat{i} + (z+1)\hat{j} + (1+y)\hat{k}\,dy\,dz = \left(1, \tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$\iint_S F\,ds = (4, 4, 4)$$

b)

$$\iint_S F\cdot ds$$

para

S_1 $z=0$, $n=-\hat{k}$

$F\cdot n = (x+y) =$

$S_2 = y=0$, $n=-\hat{j}$

$F\cdot n = -(z+x)$

$S_3 = x=0 \Rightarrow n=-\hat{i}$

$F\cdot n = -(y+z)$.

$S_4 \Rightarrow z=1$ $n=\hat{k}$

$F\cdot n = (x+y)$

$S_5 \Rightarrow y=1$ $n=\hat{j}$

$F\cdot n = (z+x)$

$S_6 \Rightarrow x=1$ $n=\hat{i}$

$F\cdot n = (y+z)$

$$\Rightarrow \iint_S F\cdot ds = -\int_0^1\int_0^1 (x+y)\,dx\,dy - \int_0^1\int_0^1 (z+x)\,dz\,dx$$

$$-\int_0^1\int_0^1 (y+z)\,dy\,dz + \int_0^1\int_0^1 (x+y)\,dx\,dy + \int_0^1\int_0^1 (z+x)\,dz\,dx$$

$$+\int_0^1\int_0^1 (y+z)\,dy\,dz$$

$$\Rightarrow \boxed{\iint_S F\cdot ds = 0}$$

(c)

$$\iint_S F \times ds$$

dividiendo por caras tal que

$S_1 \quad z=0, \Rightarrow n = -\hat{k}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ 0 & 0 & -1 \end{vmatrix} = f_1 \hat{\jmath} - f_2 \hat{\imath}$$

$S_4 \Rightarrow z = 1 \Rightarrow n = \hat{k}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ 0 & 0 & 1 \end{vmatrix} = f_2 \hat{\imath} - f_1 \hat{\jmath}$$

$S_2 \quad y=0, \Rightarrow n = -\hat{\jmath}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ 0 & -1 & 0 \end{vmatrix} f_3 \hat{\imath} - f_1 \hat{k}$$

$S_5 - \quad y = 1 \Rightarrow n \; \hat{\jmath}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ 0 & 1 & 0 \end{vmatrix} = f_1 \hat{k} - f_3 \hat{\imath}$$

$S_3 = \quad x=0 \Rightarrow n = -\hat{\imath}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ -1 & 0 & 0 \end{vmatrix} = f_2 \hat{k} - f_3 \hat{\jmath}$$

$S_6 \Rightarrow \quad x = 1 \quad \Rightarrow n = \hat{\imath}$

$$F \times n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ f_1 & f_2 & f_3 \\ 1 & 0 & 0 \end{vmatrix} = f_3 \hat{\jmath} - f_2 \hat{k}$$

dado que $\iint_S F \times ds$

es la suma de cada integral de superficie de cada cara, entonces

$$\boxed{\iint_S F \times ds = 0}$$