AP Calculus AB
AP Classroom: 4.3 – 4.4

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All final solutions are $\boxed{\textit{boxed}}$

List of Questions

Question 1

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.
If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}$ at $x = 2$.

Solution: Given that $g(1) = 2$, the point of tangency on the graph of $y = g^{-1}$ is the point $(2, 1)$. (since g^{-1} is an inverse of g, $g(2) = 1$.)
We now want to find the derivative at the point of tangency:
To find $\left[g^{-1}\right]'(2)$, we must use the rule for derivatives of inverses:

$$\begin{aligned}
\left[g^{-1}\right]'(x) &= \frac{1}{g'\left(g^{-1}(x)\right)} \\
\left[g^{-1}\right]'(2) &= \frac{1}{g'\left(g^{-1}(2)\right)} \\
&= \frac{1}{g'(1)} \\
&= \frac{1}{5}
\end{aligned}$$

Now, we just need to use the point $(2, 1)$ and the derivative at that point to write the tangent line:

$$\boxed{y - 1 = {}^{1}\!/_{5}\,(x - 2)}$$

Question 2

A student attempted to confirm that the function f defined by $f(x) = \frac{x^2+x-6}{x^2-7x+10}$ is continuous at $x = 2$ In which step, if any, does an error first appear?

Step 1:

$$f(x) = \frac{x^2 + x - 6}{x^2 - 7x + 10} = \frac{(x-2)(x+3)}{(x-2)(x-5)}$$

Step 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x+3}{x-5} = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 3:

$$f(2) = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 4:

$$\lim_{x \to 2} f(x) = f(2), \text{ so } f \text{ is continuous at } x = 2.$$

Solution: During Step 3, we made a little bit of a mistake. While in the limit in Step 2, we can simplify the rational definition of $f(x)$, when finding the actual value (or lack thereof) of $f(2)$, we ***cannot*** do this.
Even though f is actually undefined at $x = 2$ (removable discontinuity / hole), we incorrectly said it was equal to $\frac{-5}{3}$, and therefore incorrectly stated f as continuous at $x = 2$.

Question 3

Two particles move along the x-axis with velocities given by $v_1(t) = -3t^2 + 8t + 5$ and $v_2(t) = \sin t$ for time $t \geq 0$. At what time t do the two particles have equal acceleration?

Solution:

To find acceleration from give velocity equations, we must take the derivative of the velocity functions to find two new acceleration equations

$$\begin{aligned} a_1(t) &= \frac{d}{dx}\left(v_1(t)\right) \\ &= \frac{d}{dx}\left(-3t^2 + 8t + 5\right) \\ &= -6t + 8 \end{aligned}$$

$$\begin{aligned} a_2(t) &= \frac{d}{dx}\left(v_2(t)\right) \\ &= \frac{d}{dx}\left(\sin t\right) \\ &= \cos t \end{aligned}$$

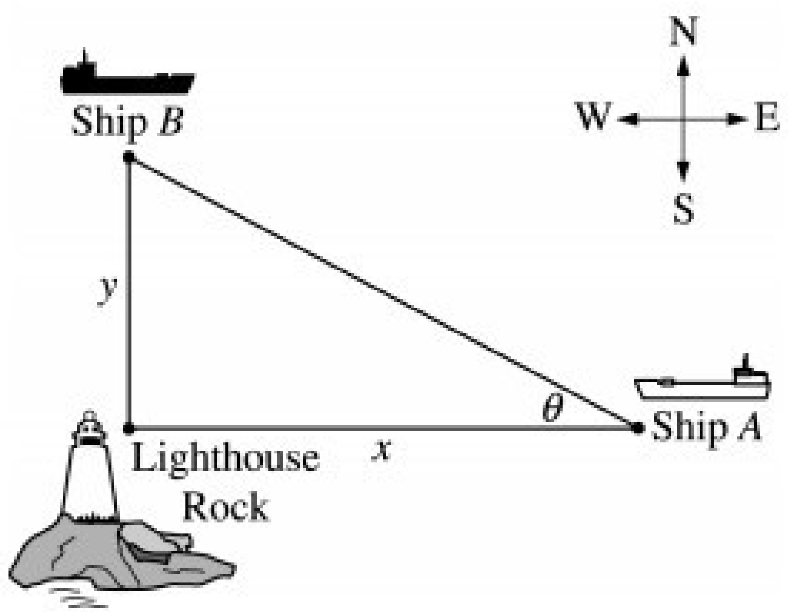
Now, we need to find the time t where $a_1(t) = a_2(t)$

$$\begin{aligned} a_1(t) &= a_2(t) \\ -6t + 8 &= \cos t \end{aligned}$$

Since this is a calculator required question, just use your calculator to find the intersection of the line and the cosine function. (**Make sure your calculator is in radian mode.**)

$$\boxed{t \approx 1.2866}$$

Question 4



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

Solution:

Equation (Pythagorean Theorem) of situation: $c^2 = y^2 + x^2$ Find: $\frac{dc}{dt}$
When: $x = 4$ km and $y = 3$ km Other Rates: $\frac{dx}{dt} = -15\frac{\text{km}}{\text{hr}}$ and $\frac{dy}{dt} = 10\frac{\text{km}}{\text{hr}}$
c is the hypotenuse / distance between Ship B and Ship A.
Using Pythagorean Theorem: $c = \sqrt{4^2 + 3^2} = 5$

Find derivative of equation:

$$\frac{dc}{dt} \cdot 2c = \frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x$$

Solve for $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{\frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x}{2c}$$

Plug in knowns and solve:

$$\frac{dc}{dt} = \frac{10 \cdot 2(3) - 15 \cdot 2(4)}{2(5)} = \frac{60 - 120}{10} = -\frac{60}{10} = \boxed{-6\frac{\text{km}}{\text{hr}}}$$

Question 5

Consider the curve give by $y^2 = 2 + xy$

Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $dy/dt = 6$. Find the value of dx/dt at time $t = 5$.

Solution: Find x at $y = 3$ (will be needed later):

$$(3)^2 = 2 + 3x$$
$$9 - 2 = 3x$$
$$\frac{7}{3} = x$$

Take derivative of given curve and then solve for dx/dt:

$$2y \cdot \frac{dy}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$
$$\frac{2y \cdot \frac{dy}{dt} - x \cdot \frac{dy}{dt}}{y} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(3) \cdot 6 - \frac{7}{3} \cdot 6}{3} = \frac{36 - 14}{3} = \boxed{\frac{22}{3} = \frac{dx}{dt}}$$

Question 6

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume V of a sphere with radius r is $V = {}^4\!/_3\pi r^3$.)

At the time when the radius, r, of the sphere is 10 centimeters, what is the rate of increase of its volume, dV/dt?

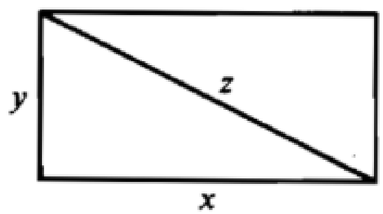
Solution: Differentiate given equation with respect to t (time):

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)(\frac{dr}{dt})$$

Plug in knowns and simplify:

$$\frac{4}{3}\pi(3(10)^2)(0.04) = \frac{4}{3}\pi(12) = \boxed{10\pi = \frac{dV}{dt}}$$

Question 7



The sides of the rectangle above increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant when $x = 4$ and $y = 3$, what is the value of $\frac{dx}{dt}$?

Solution: Write $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$ (will be needed later):

$$\frac{1}{3} \cdot \frac{dx}{dt} = \frac{dy}{dt}$$

Find z using Pythagorean Theorem (will be needed later):

$$z = \sqrt{4^2 + 3^2} = 5$$

Write derivative of Formula for side lengths of triangle (Pythagorean Theorem):

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Solve for $\frac{dx}{dt}$:

$$\frac{2z\frac{dz}{dt} - 2y\frac{dy}{dt}}{2x} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(5)(1) - 2(3)(\frac{1}{3} \cdot \frac{dx}{dt})}{2(4)} = \frac{10 - 6(\frac{1}{3} \cdot \frac{dx}{dt})}{8} = \frac{10 - 2\frac{dx}{dt}}{8} = \frac{dx}{dt}$$

$$\frac{dx}{dt} + \frac{2}{8}(\frac{dx}{dt}) = \frac{10}{8}$$
$$\frac{10}{8}(\frac{dx}{dt}) = \frac{10}{8}$$
$$\boxed{\frac{dx}{dt} = 1}$$

Question 8

Question 9

Question 10

Question 11

Question 12

Question 13

Question 14