AP Calculus AB
AP Classroom: 4.3 – 4.4

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All final solutions are $\boxed{\textit{boxed}}$

List of Questions

Question 1

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.
If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}$ at $x = 2$.

Solution: Given that $g(1) = 2$, the point of tangency on the graph of $y = g^{-1}$ is the point $(2, 1)$. (since g^{-1} is an inverse of g, $g(2) = 1$.)
We now want to find the derivative at the point of tangency:
To find $\left[g^{-1}\right]'(2)$, we must use the rule for derivatives of inverses:

$$\begin{aligned}\left[g^{-1}\right]'(x) &= \frac{1}{g'\left(g^{-1}(x)\right)}\\ \left[g^{-1}\right]'(2) &= \frac{1}{g'\left(g^{-1}(2)\right)}\\ &= \frac{1}{g'(1)}\\ &= \frac{1}{5}\end{aligned}$$

Now, we just need to use the point $(2, 1)$ and the derivative at that point to write the tangent line:

$$\boxed{y - 1 = {}^{1}\!/\!{}_{5}\,(x-2)}$$

Question 2

A student attempted to confirm that the function f defined by $f(x) = \frac{x^2+x-6}{x^2-7x+10}$ is continuous at $x = 2$ In which step, if any, does an error first appear?

Step 1:

$$f(x) = \frac{x^2+x-6}{x^2-7x+10} = \frac{(x-2)(x+3)}{(x-2)(x-5)}$$

Step 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x+3}{x-5} = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 3:

$$f(2) = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 4:

$$\lim_{x \to 2} f(x) = f(2), \text{ so } f \text{ is continuous at } x = 2.$$

Solution: During Step 3, we made a little bit of a mistake. While in the limit in Step 2, we can simplify the rational definition of $f(x)$, when finding the actual value (or lack thereof) of $f(2)$, we ***cannot*** do this.
Even though f is actually undefined at $x = 2$ (removable discontinuity / hole), we incorrectly said it was equal to $\frac{-5}{3}$, and therefore incorrectly stated f as continuous at $x = 2$.

Question 3

Two particles move along the x-axis with velocities given by $v_1(t) = -3t^2 + 8t + 5$ and $v_2(t) = \sin t$ for time $t \geq 0$. At what time t do the two particles have equal acceleration?

Solution:

To find acceleration from give velocity equations, we must take the derivative of the velocity functions to find two new acceleration equations

$$\begin{aligned} a_1(t) &= \frac{d}{dx}\left(v_1(t)\right) \\ &= \frac{d}{dx}\left(-3t^2 + 8t + 5\right) \\ &= -6t + 8 \end{aligned}$$

$$\begin{aligned} a_2(t) &= \frac{d}{dx}\left(v_2(t)\right) \\ &= \frac{d}{dx}\left(\sin t\right) \\ &= \cos t \end{aligned}$$

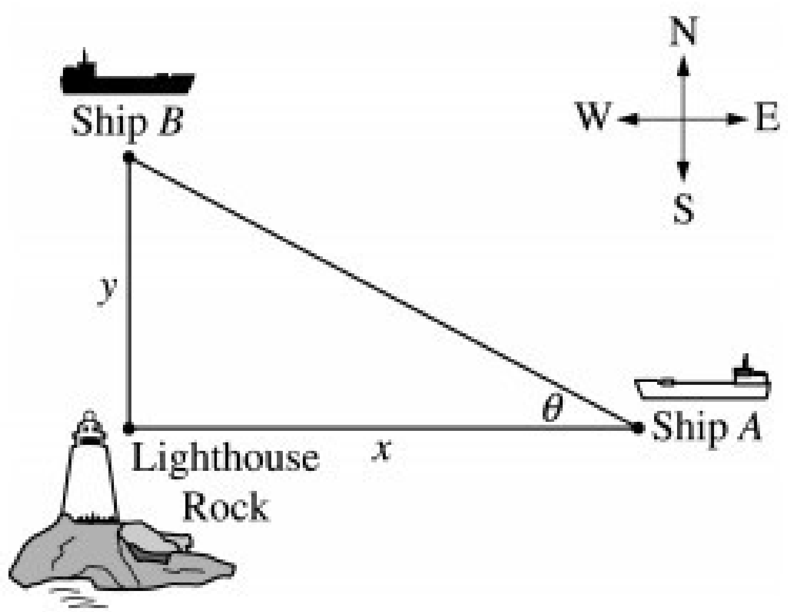
Now, we need to find the time t where $a_1(t) = a_2(t)$

$$\begin{aligned} a_1(t) &= a_2(t) \\ -6t + 8 &= \cos t \end{aligned}$$

Since this is a calculator required question, just use your calculator to find the intersection of the line and the cosine function. (**Make sure your calculator is in radian mode.**)

$$\boxed{t \approx 1.2866}$$

Question 4



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

Solution:

Equation (Pythagorean Theorem) of situation: $c^2 = y^2 + x^2$ Find: $\frac{dc}{dt}$
When: $x = 4$ km and $y = 3$ km Other Rates: $\frac{dx}{dt} = -15\frac{\text{km}}{\text{hr}}$ and $\frac{dy}{dt} = 10\frac{\text{km}}{\text{hr}}$
c is the hypotenuse / distance between Ship B and Ship A.
Using Pythagorean Theorem: $c = \sqrt{4^2 + 3^2} = 5$

Find derivative of equation:

$$\frac{dc}{dt} \cdot 2c = \frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x$$

Solve for $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{\frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x}{2c}$$

Plug in knowns and solve:

$$\frac{dc}{dt} = \frac{10 \cdot 2(3) - 15 \cdot 2(4)}{2(5)} = \frac{60 - 120}{10} = -\frac{60}{10} = \boxed{-6\frac{\text{km}}{\text{hr}}}$$

Question 5

Consider the curve give by $y^2 = 2 + xy$

Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $dy/dt = 6$. Find the value of dx/dt at time $t = 5$.

Solution: Find x at $y = 3$ (will be needed later):

$$\begin{aligned} (3)^2 &= 2 + 3x \\ 9 - 2 &= 3x \\ \frac{7}{3} &= x \end{aligned}$$

Take derivative of given curve and then solve for dx/dt:

$$\begin{aligned} 2y \cdot \frac{dy}{dt} &= x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} \\ \frac{2y \cdot \frac{dy}{dt} - x \cdot \frac{dy}{dt}}{y} &= \frac{dx}{dt} \end{aligned}$$

Plug in knowns and simplify:

$$\frac{2(3) \cdot 6 - \frac{7}{3} \cdot 6}{3} = \frac{36 - 14}{3} = \boxed{\frac{22}{3} = \frac{dx}{dt}}$$

Question 6

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume V of a sphere with radius r is $V = {}^4\!/\!_3\pi r^3$.)

At the time when the radius, r, of the sphere is 10 centimeters, what is the rate of increase of its volume, dV/dt?

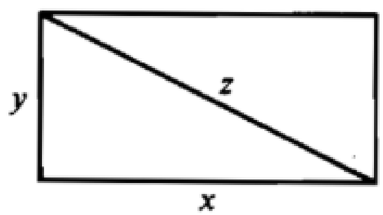
Solution: Differentiate given equation with respect to t (time):

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\left(\frac{dr}{dt}\right)$$

Plug in knowns and simplify:

$$\frac{4}{3}\pi(3(10)^2)(0.04) = \frac{4}{3}\pi(12) = \boxed{16\pi = \frac{dV}{dt}}$$

Question 7



The sides of the rectangle above increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant when $x = 4$ and $y = 3$, what is the value of $\frac{dx}{dt}$?

Solution: Write $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$ (will be needed later):

$$\frac{1}{3} \cdot \frac{dx}{dt} = \frac{dy}{dt}$$

Find z using Pythagorean Theorem (will be needed later):

$$z = \sqrt{4^2 + 3^2} = 5$$

Write derivative of Formula for side lengths of triangle (Pythagorean Theorem):

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Solve for $\frac{dx}{dt}$:

$$\frac{2z\frac{dz}{dt} - 2y\frac{dy}{dt}}{2x} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(5)(1) - 2(3)(\frac{1}{3} \cdot \frac{dx}{dt})}{2(4)} = \frac{10 - 6\left(\frac{1}{3} \cdot \frac{dx}{dt}\right)}{8} = \frac{10 - 2\frac{dx}{dt}}{8} = \frac{dx}{dt}$$

$$\frac{dx}{dt} + \frac{2}{8}\left(\frac{dx}{dt}\right) = \frac{10}{8}$$

$$\frac{10}{8}\left(\frac{dx}{dt}\right) = \frac{10}{8}$$

$$\boxed{\frac{dx}{dt} = 1}$$

Question 8

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \leq t \leq 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of the water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometer per day, with respect to t when $t = 30$ days.

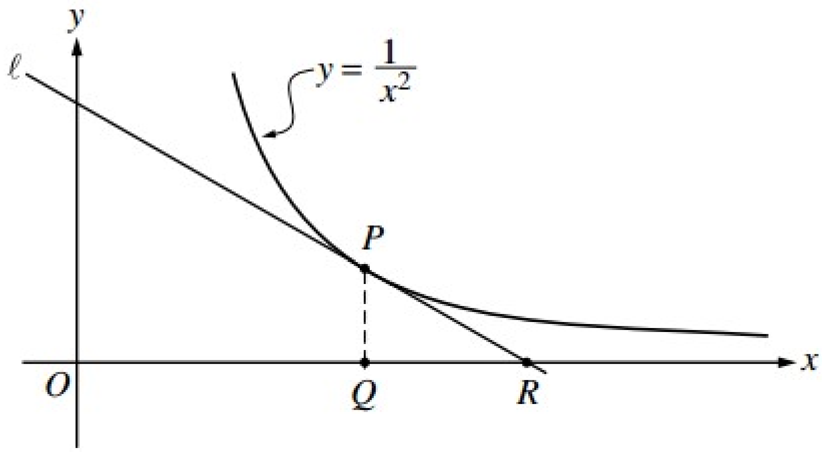
Solution: We are attempting to find $A'(t)$ when $t = 30$ days.
Knowns: $W'(30) = 0.5$, $W(30) = 125$ and $A = 7.5$ Differentiate given equation for A:

$$A' = 0.4(2/3W^{1/3})(W')$$

Plug in and simplify:

$$\left(\frac{3}{10}\right)\left(\frac{2}{3\sqrt[3]{125}}\right)\left(\frac{1}{2}\right) = \frac{6}{300} = \boxed{0.02 = A'}$$

Question 9



In the figure above, line ℓ is tangent to the graph of $y = \frac{1}{x^2}$ at point P, with coordinates $w, \frac{1}{w^2}$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x-axis at point R, with coordinates $(k, 0)$.

(a) Find the value of k when $w = 3$.

(b) For all $w > 0$, find k in terms of w.

(c) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

(d) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of PQR with respect to time? Determine whether the area is increasing or decreasing at this instant.

Solution: The slope of the tangent line ℓ is $\frac{dy}{dx}\Big|_{x=w}$ (Since w is the x coordinate of point P.)
Derivative of y:

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$$

(a) In order to find Point R, where $y = 0$, we should write the equation of line ℓ where $x = w$ (the x-coordinate of point P) and then plug in $y = 0$:

$$y - \frac{1}{w^2} = \left(\frac{dy}{dx}\bigg|_{x=w}\right)(k - w)$$
$$-\frac{1}{w^2} = \left(\frac{dy}{dx}\bigg|_{x=w}\right)(k - w)$$

Now, we need to plug in $w = 3$ and solve for x:

$$-\frac{1}{3^2} = \left(\frac{dy}{dx}\bigg|_{x=3}\right)(k - 3)$$
$$-\frac{1}{9} = \left(-\frac{2}{3^3}\right)(k - 3)$$
$$-\frac{1}{9} = \left(-\frac{2}{27}\right)(k - 3)$$
$$k - 3 = \frac{-\frac{1}{27}}{-\frac{2}{9}}$$
$$k = \left(-\frac{1}{27} \cdot -\frac{9}{2}\right) + 3$$
$$k = \boxed{\frac{9}{54} + 3 = \frac{1}{6} + 3 = \frac{19}{6} \approx 3.1667}$$

(b)

$$-\frac{1}{w^2} = \left(\frac{dy}{dx}\bigg|_{x=w}\right)(k - w)$$
$$-\frac{1}{w^2} = \left(\frac{-2}{w^3}\right)(k - w)$$
$$k - w = -\frac{1}{w^2} \div \left(\frac{-2}{w^3}\right)$$
$$k = -\frac{1}{w^2}\left(\frac{w^3}{-2}\right) + w$$
$$k = \frac{w}{2} + w = \boxed{\frac{3w}{2}}$$

(c) Knowns:

$$\frac{dw}{dt} = 7\text{units p/ sec}$$
$$w = 5$$

Find $\frac{d}{dt}$ of k:

$$\frac{dk}{dt} = \frac{3}{2}\left(\frac{dw}{dt}\right)$$

Plug in Knowns:

$$\frac{dk}{dt} = \frac{3}{2}(7)$$
$$\frac{dk}{dt} = \boxed{\frac{21}{2} = 10.5 \text{ units p/ sec}}$$

(d)

$$\overline{\mathrm{PQ}} = \frac{1}{w^2} - 0$$
$$\overline{\mathrm{QR}} = k - w$$

$$\frac{dw}{dt} = 7(\text{given in problem})\frac{dk}{dt} = \frac{3}{2}\left(\frac{dw}{dt}\right)$$

$$A = \frac{1}{2}\left(\overline{\mathrm{QR}}\right)\left(\overline{\mathrm{PQ}}\right)$$
$$A = \frac{1}{2}\left(\frac{k-w}{w^2}\right)$$
$$A = \frac{k}{2w^2} - \frac{w}{2w^2}$$
$$\frac{dA}{dt} = \frac{2w^2\left(\frac{d}{dt}(k-w)\right)}{4w^4} - \frac{(k-w)\left(\frac{d}{dt}(2w^2)\right)}{4w^4}$$
$$\frac{dA}{dt} = \frac{2w^2\left(\frac{dk}{dt} - \frac{dw}{dt}\right)}{4w^4} - \frac{(k-w)\left(4w\frac{dw}{dt}\right)}{4w^4}$$

Find k and $\frac{dk}{dt}$ so we can plug them in:

$$\frac{dk}{dt} = \frac{3}{2}(7) = \frac{17}{2} = 8.5$$
$$k = \frac{3w}{2} = \frac{15}{2} = 7.5$$

Plug in $\frac{dw}{dt}, w$ and $\frac{dk}{dt}$:

$$\frac{dA}{dt} = \frac{(50)\,(8.5-7)}{2500} - \frac{(7.5-5)\,(20(7))}{2500}$$
$$\frac{dA}{dt} = \frac{75-350}{2500} = \boxed{-\frac{275}{2500} = -0.11 \text{ units}^2 \text{ p/ sec}}$$

Question 10

Question 11

Question 12

Question 13

Question 14