

AP Calculus AB
AP Classroom: TOPIC1 – TOPIC2

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All final solutions are *boxed*
Only FRQs are included (for now).

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Question 1

Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \leq 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

- For what values of k and p will f be continuous and differentiable at $x = 1$?
- For the values of k and p found in part (a), on what interval or intervals is f increasing?
- Using the values of k and p found in part (a), find all points of inflection of the graph of f . Support your conclusion.

Solution:

(a) Definition of continuity of f at $x = 1$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) \quad (1)$$

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 + 1k + p \quad (2)$$

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = 2(1) - (1)^2 = 2 - 1 = 1 \quad (3)$$

$$\text{Therefore, } 1 = 1 + k + p \implies \boxed{-k = p} \quad (4)$$

Differentiate f with respect to x (needed for next step):

$$f'(x) = \begin{cases} \frac{d}{dx}(2x - x^2) & \text{for } x \leq 1, \\ \frac{d}{dx}(x^2 + kx + p) & \text{for } x > 1. \end{cases} \quad (5)$$

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \leq 1, \\ 2x + k + 0 & \text{for } x > 1. \end{cases} \quad (6)$$

Definition of differentiability of f at $x = 1$:

$$\lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^-} f'(x) = f'(1) \quad (7)$$

$$\lim_{x \rightarrow 1^-} f'(x) = f'(1) = 2 - 2(1) = 0 \quad (8)$$

$$\lim_{x \rightarrow 1^+} f'(x) = 2(1) + k \quad (9)$$

$$\text{Therefore, } 2 + k = 0 \implies \boxed{-2 = k} \quad (10)$$

Conclusion: Using Equations 10 and 4:

$$-2 = k \quad -k = p \quad (11)$$

$$-(-2) = p \implies p = 2 \quad (12)$$

$$\boxed{\text{Therefore, } k = -2 \text{ and } p = 2} \quad (13)$$

(b) $f'(x) > 0$ when f is increasing.

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \leq 1, \\ 2x - 2 & \text{for } x > 1. \end{cases}$$

On $x \leq 1$, set $f'(x) = 0$:

$$2 - 2x = 0 \implies 2 = 2x \implies 1 = x$$

Sign chart for $2 - 2x$: (Only applies to $f'(x)$ on $x > 1$)

$$\begin{array}{c} 2 - 2x \quad + \quad | \quad - \\ \hline 1 \end{array}$$

On $x > 1$, set $f'(x) = 0$:

$$2x - 2 = 0 \implies 2x = 2 \implies x = 1$$

Sign chart for $2x - 2$: (Only applies to $f'(x)$ on $x > 1$)

$$\begin{array}{c} 2x - 2 \quad - \quad | \quad + \\ \hline 1 \end{array}$$

Combined sign chart for $f'(x)$:

$$\begin{array}{c} f'(x) \quad + \quad | \quad + \\ \hline 1 \end{array}$$

Conclusion:

Therefore, $f(x)$ is increasing when $x \neq 1$

because $f'(x)$ is positive on those regions. (At $x = 1$, $f'(x) = 0$.)

f is continuous at $x = 1$, so $f(x)$ is increasing on $(-\infty, \infty)$.

(c) Inflection points are where f'' changes signs.

$$f''(x) = \begin{cases} \frac{d}{dx}(2 - 2x) = -2 & \text{for } x \leq 1, \\ \frac{d}{dx}(2x - 2) = 2 & \text{for } x > 1. \end{cases}$$

Sign chart of $f''(x)$:

$$\begin{array}{c} f''(x) \quad - \quad | \quad + \\ \hline 1 \end{array}$$

Therefore, $f'(x)$ has an inflection point at $x = 1$.

$$f(1) = 1 \quad \boxed{\text{Inflection Point: } (1, 1)}$$

Question 2

Consider the curve given by $x^2 - xy + 2y^2 = 7$.

- (a) Show that $\frac{dy}{dx} = \frac{y-2x}{4y-x}$
- (b) Determine the y -coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.
- (c) Find $\frac{d^2y}{dx^2}$ in terms of x , y , and $\frac{dy}{dx}$. The line tangent to the curve at the point $(1, 2)$ is horizontal. Determine whether the curve is concave up or concave down at the point $(1, 2)$.

Solution:

(a)

$$\frac{d}{dx}(x^2 - xy + 2y^2) = 7 \quad (14)$$

$$2x - (xy' + y) + 4yy' = 0 \quad (15)$$

$$2x - xyy' - y + 4yy' = 0 \quad (16)$$

$$2x - y = y'(x - 4y) \quad (17)$$

$$\frac{2x - y}{x - 4y} = y' = \frac{dy}{dx} = \frac{y - 2x}{4y - x} \quad (18)$$

(b) Vertical tangent lines have undefined slope.

Therefore, $\frac{dy}{dx}$ will be undefined \iff the tangent line at the point is vertical.

(\iff means 'if and only if')

$$4y - x = 0 \text{ makes } \frac{dy}{dx} \text{ undefined.} \quad (19)$$

$$4y = x \quad (20)$$

So,

$$7 = (4y)^2 - (4y)y + 2y^2 \quad (21)$$

$$7 = 16y^2 - 4y^2 + 2y^2 \quad (22)$$

$$7 = 14y^2 \quad (23)$$

$$y^2 = \frac{1}{2} \quad (24)$$

$$y = \boxed{\pm \frac{1}{\sqrt{2}}} \quad (25)$$

(c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{y-2x}{4y-x} \right) \quad (26)$$

$$= \frac{\left(\frac{d}{dx}(y-2x) \right) (4y-x) - \left(\frac{d}{dx}(4y-x) \right) (y-2x)}{(4y-x)^2} \quad (27)$$

$$= \frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2} \quad (28)$$

$y' = 0$ at $(1, 2)$ since the tangent to the curve is vertical.

$$\left. \frac{d^2y}{dx^2} \right|_{(1,2)} = \frac{(-2)(4(2)-1) - (-1)(2-2(1))}{(4(2)-1)^2} = \frac{-2(7)-0}{7^2} = -\frac{2}{7} \quad (29)$$

Since $\frac{d^2y}{dx^2}$ is negative at $(1, 2)$, the curve is concave down at $(1, 2)$.

Question 3

Particle P moves along the y -axis so that its position at time t is given by $y(t) = 4t - \frac{2}{3}$ for all times t . A second particle, particle Q , moves along the x -axis so that its position at time t is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

- As time t approaches 2, what is the limit of the position of particle Q ? Show the work that leads to your answer.
- Show that the velocity of particle Q is given by $v_Q(t) = \frac{2\pi \cos(\pi t) - \pi t \cos(\pi t) + \sin(\pi t)}{(2-t)^2}$ for all times.
- Find the rate of change of the distance between particle P and particle Q at time $t = \frac{1}{2}$. Show the work that leads to your answer.

Solution:

(a)

$$\lim_{t \rightarrow 2} (\sin(\pi t)) = \sin(2\pi) = 0 \quad (30)$$

$$\lim_{t \rightarrow 2} (2-t) = 0 \quad (31)$$

Since $\frac{\lim_{t \rightarrow 2} (\sin(\pi t))}{\lim_{t \rightarrow 2} (2-t)}$ approaches an indeterminate form $(0/0)$, L'Hôpital's rule

can be applied.

$$\lim_{t \rightarrow 2} (d/dt (\sin(\pi t))) = \lim_{t \rightarrow 2} (\pi \cos(\pi t)) = \pi \cos(2\pi) = \pi \quad (32)$$

$$\lim_{t \rightarrow 2} (d/dt (2 - t)) = \lim_{t \rightarrow 2} (-1) = -1 \quad (33)$$

$$\therefore \lim_{t \rightarrow 2} (x(t)) = \frac{\pi}{-1} = \boxed{-\pi} \quad (34)$$

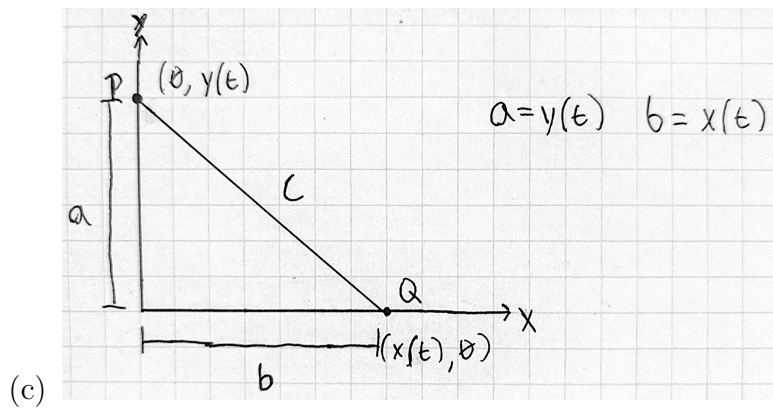
(b)

$$\frac{d}{dt} (x(t)) = v_Q(t) \quad (35)$$

$$\frac{d}{dt} \left(\frac{\sin(\pi t)}{2 - t} \right) = \frac{(d/dt(\sin(\pi t)))(2 - t) - (d/dt(2 - t))(\sin(\pi t))}{(2 - t)^2} \quad (36)$$

$$= \frac{\pi \cos(\pi t)(2 - t) - (-1)(\sin(\pi t))}{(2 - t)^2} \quad (37)$$

$$= \frac{2\pi \cos(\pi t) - \pi t \cos(\pi t) + \sin(\pi t)}{(2 - t)^2} \quad (38)$$



$$c^2 = a^2 + b^2 \quad (39)$$

$$c = \sqrt{a^2 + b^2} \quad (40)$$

$$(41)$$

$$2c \frac{dc}{dt} = 2a \frac{da}{dt} + 2b \frac{db}{dt} \quad (42)$$

$$\frac{dc}{dt} = \frac{a \frac{da}{dt} + b \frac{db}{dt}}{c} \quad (43)$$

$$\uparrow \text{Main rate equation} \uparrow \quad (44)$$

$$(45)$$

$$a = y(t) = 4t - \frac{2}{3} \quad (46)$$

$$\frac{da}{dt} = 4 \quad (47)$$

$$(48)$$

$$b = x(t) = \frac{\sin(\pi t)}{2 - t} \quad (49)$$

$$\frac{db}{dt} = \frac{(\pi \cos(\pi t))(2 - t) - (-1)(\sin(\pi t))}{(2 - t)^2} \quad (50)$$

Now, find all of those variables when $t = \frac{1}{2}$

$$a|_{t=1/2} = 4(1/2) - 2/3 = 2 - 2/3 = 4/3 \quad (51)$$

$$b|_{t=1/2} = \frac{\sin(\pi/2)}{2 - 1/2} = \frac{1}{3/2} = \frac{2}{3} \quad (52)$$

$$\left. \frac{da}{dt} \right|_{t=1/2} = 4 \quad (53)$$

$$\left. \frac{db}{dt} \right|_{1/2} = \frac{(\pi \cos(\pi/2))(2 - 1/2) + (\sin(\pi/2))}{(2 - 1/2)^2} = \frac{4(\pi(0)(3/2) + 1)}{9} = \frac{4}{9} \quad (54)$$

$$c|_{t=1/2} = \sqrt{a^2 + b^2} = \sqrt{(4/3)^2 + (2/3)^2} = \sqrt{20/9} = \sqrt{20}/3 \quad (55)$$

Finally, plug all of those into our main rate equation:

$$\left. \frac{dc}{dt} \right|_{t=1/2} = \boxed{\frac{(4/3)(4) + (2/3)(4/9)}{\sqrt{20}/3}}$$

Question 4

Let f be the function given by $f(x) = 2xe^{2x}$.

Find the absolute minimum value of f . Justify that your answer is an absolute minimum.

Solution: All absolute minimums must also be relative minima.

At Relative minima of f , the derivative, f' changes from negative to positive.

Find $f'(x)$

$$f'(x) = 2(1(e^{2x}) + x(2e^{2x})) \quad (56)$$

$$= 2e^{2x} + 2xe^{2x} \quad (57)$$

$$= 2e^{2x}(1 + x) \quad (58)$$

Set $f'(x) = 0$ and solve:

$$2e^{2x}(1 + x)$$

$$2e^{2x} = 0 \quad | \quad (1 + x) = 0 \quad (59)$$

$$\ln(e^{2x}) = 0 \quad | \quad x = -1 \quad (60)$$

$$\text{no solution} \quad | \quad (61)$$

Sign chart of $f'(x)$:

$$\begin{array}{c} f'(x) \quad - \quad | \quad + \\ \hline \quad \quad \quad \bullet \\ \quad \quad \quad -1 \end{array}$$

Find end behavior of $f(x)$:

$$\lim_{x \rightarrow \infty} (2xe^{2x}) = \lim_{x \rightarrow \infty} (e^{2x}) = \infty \quad \lim_{x \rightarrow -\infty} (2xe^{2x}) = \lim_{x \rightarrow -\infty} (e^{2x}) = 0 \quad (62)$$

Conclusion:

Since $x = -1$ is the only relative minima and is less than the $\lim_{x \rightarrow \pm\infty} (f(x))$, it is the absolute minimum of $f(x)$.

$$f(-1) = 2(-1)e^{2(-1)} = -2e^2$$

The absolute minimum of $f(x)$ is $f(-1) = -2e^2$

Question 14

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

t (hours)	0	5	15	30	35
$A(t)$ (gallons)	10	18	25	16	8

The number of gallons of olive oil in a tank at time t is given by the twice-differentiable function A , where t is measured in hours and $0 \leq t \leq 35$. Values of $A(t)$ at selected times t are given in the table above.

- Use the data in the table to estimate the rate at which the number of gallons of olive oil in the tank is changing at time $t = 10$ hours. Show the computations that lead to your answer. Indicate units of measure.
- For $0 \leq t \leq 30$, is there a time t at which $A'(t) = \frac{1}{5}$? Justify your answer.
- The number of gallons of olive oil in the tank at time t is also modeled by the function G defined by $G(t) = 5t - \frac{2}{3}(t+9)^{\frac{3}{2}} + 28$, where t is measured in hours and $0 \leq t \leq 35$. Based on the model, at what time t , for $0 \leq t \leq 35$, is the number of gallons of olive oil in the tank an absolute maximum? Justify your answer.

Solution:

(a)

$$A'(t) \approx \frac{A(30) - A(15)}{30 - 15} = \frac{16 - 25}{30 - 15} = -\frac{9}{15} = -\frac{3}{5}$$

(b) On $0 \leq t \leq 30$, the AROC can be found as follows:

$$\frac{A(30) - A(0)}{30 - 0} = \frac{16 - 10}{30} = \frac{6}{30} = \frac{1}{5}$$

Since A is twice-differentiable $\implies A$ is differentiable $\implies A$ is continuous.
Therefore, by MVT, there is a value c in $(0, 30)$ where $A'(c) = \frac{A(30) - A(0)}{30 - 0} = \frac{1}{5}$.

(c)

$$G'(t) = 5 - \left(\frac{3}{2}\right) \left(\frac{2}{3}\right) (t + 9)^{(3/2-1)}(1) \quad (63)$$

$$= 5 - (t + 9)^{1/2} \quad (64)$$

$$= 5 - \sqrt{t + 9} \quad (65)$$

Set $G'(t) = 0$:

$$5 - \sqrt{t + 9} = 0 \quad (66)$$

$$5 = \sqrt{t + 9} \quad (67)$$

$$25 = t + 9 \quad (68)$$

$$16 = t \quad (69)$$

$$(70)$$

Sign Chart:

$$\begin{array}{c} G'(t) \quad + \quad | \quad - \\ \hline \quad \quad \quad \bullet \\ \quad \quad \quad 16 \end{array}$$

$t = 16$ must be the absolute maximum since it is the only relative max.
($G(t)$ decreases continually on $16 \leq t \leq 35$)

(d) $G''(t) = -(t + 9)^{1/2}(1) = \frac{-1}{\sqrt{t+9}}$

Set $G''(t) = 0$:

$$\frac{-1}{\sqrt{t+9}} = 0 \implies \text{No Solution}$$

$G''(t) < 0$ on $0 \leq t \leq 35$.

$\implies G'(t)$ is concave down on $0 \leq t \leq 35$.

Therefore, $G(7)$ will provide an overestimate of $G(8)$ if using a locally linear model. ($G(t)$ is increasing at a decreasing rate, so if we assume a constant rate, it will provide an overestimate.)