# AP Calculus AB
# AP Classroom: TOPIC1 – TOPIC2

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Brayden Price

All final solutions are $\boxed{boxed}$

Only FRQs are included (for now).

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## List of Questions

**Question 1**

Let $f$ be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of $k$ and $p$ will $f$ be continuous and differentiable at $x = 1$?

(b) For the values of $k$ and $p$ found in part (a), on what interval or intervals is $f$ increasing?

(c) Using the values of $k$ and $p$ found in part (a), find all points of inflection of the graph of $f$. Support your conclusion.

**Solution:**

(a) Definition of continuity of $f$ at $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) \tag{1}$$

$$\lim_{x\to 1^+} f(x) = 1^2 + 1k + p \tag{2}$$

$$\lim_{x\to 1^-} = f(1) = 2(1) - (1)^2 = 2 - 1 = 1 \tag{3}$$

$$\textit{Therefore, } 1 = 1 + k + p \implies \boxed{-k = p} \tag{4}$$

Differentiate $f$ with respect to $x$ (needed for next step):

$$f'(x) = \begin{cases} \frac{d}{dx}(2x - x^2) & \text{for } x \le 1, \\ \frac{d}{dx}(x^2 + kx + p) & \text{for } x > 1. \end{cases} \tag{5}$$

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \le 1, \\ 2x + k + 0 & \text{for } x > 1. \end{cases} \tag{6}$$

Definition of differentiability of $f$ at $x = 1$:

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^-} f'(x) = f'(1) \tag{7}$$

$$\lim_{x\to 1^-} f'(x) = f'(1) = 2 - 2(1) = 0 \tag{8}$$

$$\lim_{x\to 1^+} f'(x) = 2(1) + k \tag{9}$$

$$\textit{Therefore, } 2 + k = 0 \implies \boxed{-2 = k} \tag{10}$$

Conclusion: Using Equations 10 and 4:

$$-2 = k \quad -k = p \tag{11}$$

$$-(-2) = p \implies p = 2 \tag{12}$$

$$\boxed{\textit{Therefore, } k = -2 \text{ and } p = 2} \tag{13}$$

(b) $f'(x) > 0$ when $f$ is increasing.

$$f'(x) = \begin{cases} 2-2x & \text{for } x \le 1, \\ 2x-2 & \text{for } x > 1. \end{cases}$$

On $x \le 1$, set $f'(x) = 0$:

$$2 - 2x = 0 \implies 2 = 2x \implies 1 = x$$

Sign chart for $2-2x$: (Only applies to $f'(x)$ on $x > 1$)

| $2-2x$ | $+$ | 1 | $-$ |
|---|---|---|---|

On $x > 1$, set $f'(x) = 0$:

$$2x - 2 = 0 \implies 2x = 2 \implies x = 1$$

Sign chart for $2x-2$: (Only applies to $f'(x)$ on $x > 1$)

| $2x-2$ | $-$ | 1 | $+$ |
|---|---|---|---|

Combined sign chart for $f'(x)$:

| $f'(x)$ | $+$ | 1 | $+$ |
|---|---|---|---|

Conclusion:

*Therefore,* $f(x)$ is increasing when $x \neq 1$

because $f'(x)$ is positive on those regions. (At $x = 1, f'(x) = 0$.)

$f$ is continuous at $x = 1$, so $\boxed{f(x) \text{ is increasing on } (-\infty, \infty).}$

(c) Inflection points are where $f''$ changes signs.

$$f''(x) = \begin{cases} \frac{d}{dx}(2-2x) = -2 & \text{for } x \le 1, \\ \frac{d}{dx}(2x-2) = 2 & \text{for } x > 1. \end{cases}$$

Sign chart of $f''(x)$:

| $f(x)$ | $-$ | 1 | $+$ |
|---|---|---|---|

*Therefore,* $f'(x)$ has an inflection point at $x = 1$.

$f(1) = 1$ $\boxed{\text{Inflection Point: } (1,1)}$

**Question 2**

Consider the curve given by $x^2 - xy + 2y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{4y-x}$

(b) Determine the $y$-coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.

(c) Find $\frac{d^2y}{dx^2}$ in terms of $x$, $y$, and $\frac{dy}{dx}$. The line tangent to the curve at the point $(1, 2)$ is horizontal. Determine whether the curve is concave up or concave down at the point $(1, 2)$.

**Solution:**

(a)

$$\frac{d}{dx}(x^2 - xy + 2y^2) = 7 \tag{14}$$
$$2x - (xy' + y) + 4yy' = 0 \tag{15}$$
$$2x - xyy' - y + 4yy' = 0 \tag{16}$$
$$2x - y = y'(x - 4y) \tag{17}$$
$$\frac{2x - y}{x - 4y} = y' = \frac{dy}{dx} = \frac{y - 2x}{4y - x} \tag{18}$$

(b) Vertical tangent lines have undefined slope.
Therefore, $\frac{dy}{dx}$ will be undefined $\iff$ the tangent line at the point is vertical.
( $\iff$ means 'if and only if')

$$4y - x = 0 \text{ makes } \frac{dy}{dx} \text{ undefined.} \tag{19}$$
$$4y = x \tag{20}$$

So,

$$7 = (4y)^2 - (4y)y + 2y^2 \tag{21}$$
$$7 = 16y^2 - 4y^2 + 2y^2 \tag{22}$$
$$7 = 14y^2 \tag{23}$$
$$y^2 = \frac{1}{2} \tag{24}$$
$$y = \boxed{\pm\frac{1}{\sqrt{2}}} \tag{25}$$

(c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y-2x}{4y-x}\right) \tag{26}$$

$$= \frac{\left({}^{d}\!/_{dx}\,(y-2x)\right)(4y-x) - \left({}^{d}\!/_{dx}\,(4y-x)\right)(y-2x)}{(4y-x)^2} \tag{27}$$

$$= \frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2} \tag{28}$$

$y' = 0$ at $(1, 2)$ since the tangent to the curve is vertical.

$$\left.\frac{d^2y}{dx^2}\right|_{(1,2)} = \frac{(-2)(4(2)-1) - (-1)(2-2(1))}{(4(2)-1)^2} = \frac{-2(7)-0}{7^2} = -\frac{2}{7} \tag{29}$$

$\boxed{\text{Since } \frac{d^2y}{dx^2} \text{ is negative at } (1,2)\text{, the curve is concave down at } (1,2).}$

**Question 3**

Particle $P$ moves along the $y$-axis so that its position at time $t$ is given by $y(t) = 4t - \frac{2}{3}$ for all times $t$. A second particle, particle, moves along the-axis so that its position at time $t$ is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

(a) As time $t$ approaches 2, what is the limit of the position of particle $Q$? Show the work that leads to your answer.

(b) Show that the velocity of particle $Q$ is given by $v_Q(t) = \frac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2}$ for all times.

(c) Find the rate of change of the distance between particle $P$ and particle $Q$ at time $t = \frac{1}{2}$. Show the work that leads to your answer.

**Solution:**

(a)

$$\lim_{t\to 2}\left(\sin(\pi t)\right) = \sin(2\pi) = 0 \tag{30}$$

$$\lim_{t\to 2}(2-t) = 0 \tag{31}$$

Since $\frac{\lim_{t\to 2}(\sin(\pi t))}{\lim_{t\to 2}(2-t)}$ approaches an indeterminate form (${}^{0}\!/_{0}$), L'Hôpital's rule can be applied.

$$\lim_{t\to 2}\left({}^{d}\!/\!_{dt}\left(\sin\left(\pi t\right)\right)\right) = \lim_{t\to 2}\left(\pi\cos\left(\pi t\right)\right) = \pi\cos(2\pi) = \pi \tag{32}$$

$$\lim_{t\to 2}\left({}^{d}\!/\!_{dt}\left(2-t\right)\right) = \lim_{t\to 2}\left(-1\right) = -1 \tag{33}$$

$$\therefore \lim_{t\to 2}(x(t)) = \frac{\pi}{-1} = \boxed{-\pi} \tag{34}$$

(b)

$$\frac{d}{dt}\left(x(t)\right) = v_Q(t) \tag{35}$$

$$\frac{d}{dt}\left(\frac{\sin(\pi t)}{2-t}\right) = \frac{({}^{d}\!/\!_{dt}(\sin(\pi t)))(2-t) - ({}^{d}\!/\!_{dt}(2-t))(\sin(\pi t))}{(2-t)^2} \tag{36}$$

$$= \frac{\pi\cos(\pi t)(2-t) - (-1)(\sin(\pi t))}{(2-t)^2} \tag{37}$$

$$= \frac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2} \tag{38}$$

(c) IDK yet... Work in Progress.

**Question 4**

Let $f$ be the function given by $f(x) = 2xe^{2x}$.
Find the absolute minimum value of $f$. Justify that your answer is an absolute minimum.

**Solution:** All absolute minimums must also be relative minima.
At Relative minima of $f$, the derivative, $f'$ changes from negative to positive. Find $f'(x)$

$$f'(x) = 2(1(e^{2x}) + x(2e^{2x})) \tag{39}$$

$$= 2e^{2x} + 2xe^{2x} \tag{40}$$

$$= 2e^{2x}(1+x) \tag{41}$$

Set $f'(x) = 0$ and solve:

$$2e^{2x}(1+x)$$

$$2e^{2x} = 0 \;\mid\; (1+x) = 0 \tag{42}$$

$$\ln(e^2x) = 0 \;\mid\; x = -1 \tag{43}$$

$$\text{no solution} \;\mid \tag{44}$$

Sign chart of $f'(x)$:

| | | |
|---|---|---|
| $f'(x)$ | $-$ | $+$ |
| | $-1$ | |

Find end behavior of $f(x)$:

$$\lim_{x\to\infty}\left(2xe^{2x}\right) = \lim_{x\to\infty}\left(e^{2x}\right) \quad = \infty \lim_{x\to-\infty}\left(2xe^{2x}\right) = \lim_{x\to-\infty}\left(e^{2x}\right) \quad = 0 \tag{45}$$

Conclusion:
Since $x = -1$ is the only relative minima and is less than the $\lim_{x\to\pm\infty}(f(x))$, it is the absolute minimum of $f(x)$.

$$f(-1) = 2(-1)e^{2(1)} = -2e^2$$

The absolute minimum of $f(x)$ is $\boxed{f(-1) = -2e^2}$