AP Calculus AB
AP Classroom: 4.3 – 4.4

October 23, 2024

Brayden Price

All final solutions are $\boxed{\textit{boxed}}$

List of Questions

Question 1

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.
If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}$ at $x = 2$.

Solution: Given that $g(1) = 2$, the point of tangency on the graph of $y = g^{-1}$ is the point $(2, 1)$. (since g^{-1} is an inverse of g, $g(2) = 1$.)
We now want to find the derivative at the point of tangency:
To find $\left[g^{-1}\right]'(2)$, we must use the rule for derivatives of inverses:

$$\begin{aligned}\left[g^{-1}\right]'(x) &= \frac{1}{g'\left(g^{-1}(x)\right)} \\ \left[g^{-1}\right]'(2) &= \frac{1}{g'\left(g^{-1}(2)\right)} \\ &= \frac{1}{g'(1)} \\ &= \frac{1}{5}\end{aligned}$$

Now, we just need to use the point $(2, 1)$ and the derivative at that point to write the tangent line:

$$\boxed{y - 1 = {}^{1}/_{5}\,(x - 2)}$$

Question 2

A student attempted to confirm that the function f defined by $f(x) = \frac{x^2+x-6}{x^2-7x+10}$ is continuous at $x = 2$ In which step, if any, does an error first appear?

Step 1:

$$f(x) = \frac{x^2 + x - 6}{x^2 - 7x + 10} = \frac{(x-2)(x+3)}{(x-2)(x-5)}$$

Step 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x+3}{x-5} = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 3:

$$f(2) = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 4:

$$\lim_{x \to 2} f(x) = f(2), \text{ so } f \text{ is continuous at } x = 2.$$

Solution: During Step 3, we made a little bit of a mistake. While in the limit in Step 2, we can simplify the rational definition of $f(x)$, when finding the actual value (or lack thereof) of $f(2)$, we ***cannot*** do this.
Even though f is actually undefined at $x = 2$ (removable discontinuity / hole), we incorrectly said it was equal to $\frac{-5}{3}$, and therefore incorrectly stated f as continuous at $x = 2$.

Question 3

Two particles move along the x-axis with velocities given by $v_1(t) = -3t^2 + 8t + 5$ and $v_2(t) = \sin t$ for time $t \geq 0$. At what time t do the two particles have equal acceleration?

Solution:

To find acceleration from give velocity equations, we must take the derivative of the velocity functions to find two new acceleration equations

$$\begin{aligned} a_1(t) &= \frac{d}{dx}\left(v_1(t)\right) \\ &= \frac{d}{dx}\left(-3t^2 + 8t + 5\right) \\ &= -6t + 8 \end{aligned}$$

$$\begin{aligned} a_2(t) &= \frac{d}{dx}\left(v_2(t)\right) \\ &= \frac{d}{dx}\left(\sin t\right) \\ &= \cos t \end{aligned}$$

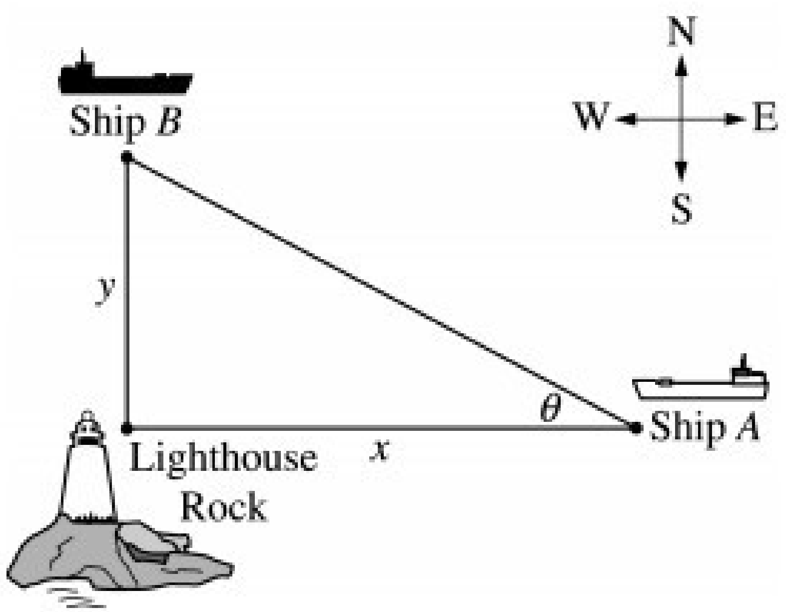
Now, we need to find the time t where $a_1(t) = a_2(t)$

$$\begin{aligned} a_1(t) &= a_2(t) \\ -6t + 8 &= \cos t \end{aligned}$$

Since this is a calculator required question, just use your calculator to find the intersection of the line and the cosine function. (**Make sure your calculator is in radian mode.**)

$$\boxed{t \approx 1.2866}$$

Question 4



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

Solution:

Equation (Pythagorean Theorem) of situation: $c^2 = y^2 + x^2$ Find: $\frac{dc}{dt}$
When: $x = 4$ km and $y = 3$ km Other Rates: $\frac{dx}{dt} = -15\frac{\text{km}}{\text{hr}}$ and $\frac{dy}{dt} = 10\frac{\text{km}}{\text{hr}}$
c is the hypotenuse / distance between Ship B and Ship A.
Using Pythagorean Theorem: $c = \sqrt{4^2 + 3^2} = 5$

Find derivative of equation:

$$\frac{dc}{dt} \cdot 2c = \frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x$$

Solve for $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{\frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x}{2c}$$

Plug in knowns and solve:

$$\frac{dc}{dt} = \frac{10 \cdot 2(3) - 15 \cdot 2(4)}{2(5)} = \frac{60 - 120}{10} = -\frac{60}{10} = \boxed{-6\frac{\text{km}}{\text{hr}}}$$

Question 5

Consider the curve give by $y^2 = 2 + xy$

Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $dy/dt = 6$. Find the value of dx/dt at time $t = 5$.

Solution: Find x at $y = 3$ (will be needed later):

$$(3)^2 = 2 + 3x$$
$$9 - 2 = 3x$$
$$\frac{7}{3} = x$$

Take derivative of given curve and then solve for dx/dt:

$$2y \cdot \frac{dy}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$
$$\frac{2y \cdot \frac{dy}{dt} - x \cdot \frac{dy}{dt}}{y} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(3) \cdot 6 - \frac{7}{3} \cdot 6}{3} = \frac{36 - 14}{3} = \boxed{\frac{22}{3} = \frac{dx}{dt}}$$

Question 6

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume V of a sphere with radius r is $V = {}^4/_3\pi r^3$.)

At the time when the radius, r, of the sphere is 10 centimeters, what is the rate of increase of its volume, dV/dt?

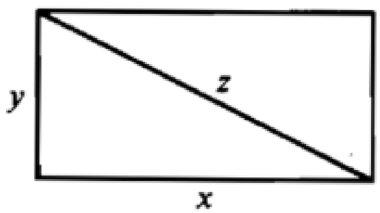
Solution: Differentiate given equation with respect to t (time):

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\left(\frac{dr}{dt}\right)$$

Plug in knowns and simplify:

$$\frac{4}{3}\pi(3(10)^2)(0.04) = \frac{4}{3}\pi(12) = \boxed{16\pi = \frac{dV}{dt}}$$

Question 7



The sides of the rectangle above increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant when $x = 4$ and $y = 3$, what is the value of $\frac{dx}{dt}$?

Solution: Write $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$ (will be needed later):

$$\frac{1}{3} \cdot \frac{dx}{dt} = \frac{dy}{dt}$$

Find z using Pythagorean Theorem (will be needed later):

$$z = \sqrt{4^2 + 3^2} = 5$$

Write derivative of Formula for side lengths of triangle (Pythagorean Theorem):

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Solve for $\frac{dx}{dt}$:

$$\frac{2z\frac{dz}{dt} - 2y\frac{dy}{dt}}{2x} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(5)(1) - 2(3)(\frac{1}{3} \cdot \frac{dx}{dt})}{2(4)} = \frac{10 - 6\left(\frac{1}{3} \cdot \frac{dx}{dt}\right)}{8} = \frac{10 - 2\frac{dx}{dt}}{8} = \frac{dx}{dt}$$

$$\frac{dx}{dt} + \frac{2}{8}\left(\frac{dx}{dt}\right) = \frac{10}{8}$$
$$\frac{10}{8}\left(\frac{dx}{dt}\right) = \frac{10}{8}$$
$$\boxed{\frac{dx}{dt} = 1}$$

Question 8

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \leq t \leq 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of the water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometer per day, with respect to t when $t = 30$ days.

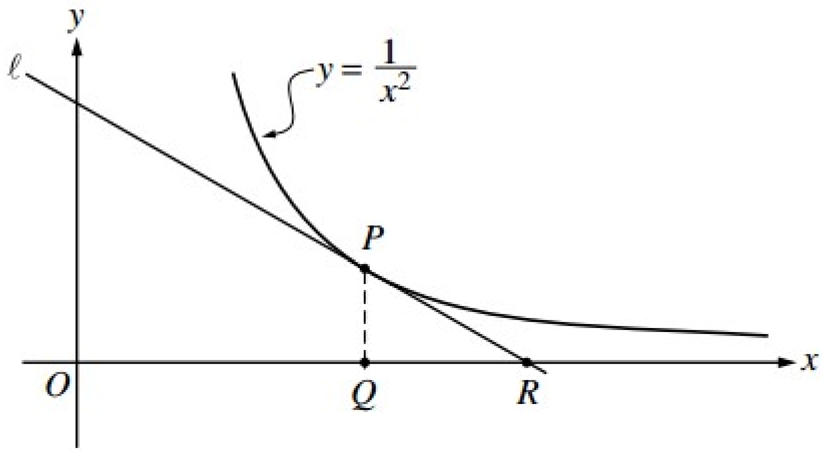
Solution: We are attempting to find $A'(t)$ when $t = 30$ days.
Knowns: $W'(30) = 0.5$, $W(30) = 125$ and $A = 7.5$ Differentiate given equation for A:

$$A' = 0.4({}^2\!/\!{}_3 W^{1/3})(W')$$

Plug in and simplify:

$$\left(\frac{3}{10}\right)\left(\frac{2}{3\sqrt[3]{125}}\right)\left(\frac{1}{2}\right) = \frac{6}{300} = \boxed{0.02 = A'}$$

Question 9



In the figure above, line ℓ is tangent to the graph of $y = \frac{1}{x^2}$ at point P, with coordinates $w, \frac{1}{w^2}$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x-axis at point R, with coordinates $(k, 0)$.

(a) Find the value of k when $w = 3$.

(b) For all $w > 0$, find k in terms of w.

(c) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

(d) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of PQR with respect to time? Determine whether the area is increasing or decreasing at this instant.

Solution: The slope of the tangent line ℓ is $\frac{dy}{dx}\big|_{x=w}$ (Since w is the x coordinate of point P.)
Derivative of y:

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$$

(a) In order to find Point R, where $y = 0$, we should write the equation of line ℓ where $x = w$ (the x-coordinate of point P) and then plug in $y = 0$:

$$y - \frac{1}{w^2} = \left(\frac{dy}{dx}\bigg|_{x=w} \right)(k - w)$$
$$-\frac{1}{w^2} = \left(\frac{dy}{dx}\bigg|_{x=w} \right)(k - w)$$

Now, we need to plug in $w = 3$ and solve for x:

$$
\begin{aligned}
-\frac{1}{3^2} &= \left(\left.\frac{dy}{dx}\right|_{x=3}\right)(k-3) \\
-\frac{1}{9} &= \left(-\frac{2}{3^3}\right)(k-3) \\
-\frac{1}{9} &= \left(-\frac{2}{27}\right)(k-3) \\
k-3 &= \frac{-\frac{1}{9}}{-\frac{2}{27}} \\
k &= \left(-\frac{2}{27}\cdot -\frac{9}{1}\right)+3 \\
k &= \boxed{\frac{18}{27}+3 = \frac{2}{3}+3 = \frac{11}{3} \approx 3.6667}
\end{aligned}
$$

(b)

$$
\begin{aligned}
-\frac{1}{w^2} &= \left(\left.\frac{dy}{dx}\right|_{x=w}\right)(k-w) \\
-\frac{1}{w^2} &= \left(\frac{-2}{w^3}\right)(k-w) \\
k-w &= -\frac{1}{w^2} \div \left(\frac{-2}{w^3}\right) \\
k &= -\frac{1}{w^2}\left(\frac{w^3}{-2}\right)+w \\
k &= \frac{w}{2}+w = \boxed{\frac{3w}{2}}
\end{aligned}
$$

(c) Knowns:

$$
\begin{aligned}
\frac{dw}{dt} &= 7\text{units p/ sec} \\
w &= 5
\end{aligned}
$$

Find $\frac{d}{dt}$ of k:

$$
\frac{dk}{dt} = \frac{3}{2}\left(\frac{dw}{dt}\right)
$$

Plug in Knowns:

$$\frac{dk}{dt} = \frac{3}{2}(7)$$
$$\frac{dk}{dt} = \boxed{\frac{21}{2} = 10.5 \text{ units p/ sec}}$$

(d) Find side lengths and derivatives of w and k:

$$\overline{\text{PQ}} = \frac{1}{w^2} - 0$$
$$\overline{\text{QR}} = k - w$$

$$\frac{dw}{dt} = 7 \text{ (given in problem)}$$
$$\frac{dk}{dt} = \frac{3}{2}\left(\frac{dw}{dt}\right)$$

Solve for A, then find the derivative of A with respect to time:

$$A = \frac{1}{2}\left(\overline{\text{QR}}\right)\left(\overline{\text{PQ}}\right)$$
$$A = \frac{1}{2}\left(\frac{k-w}{w^2}\right)$$
$$A = \frac{k}{2w^2} - \frac{w}{2w^2}$$
$$\frac{dA}{dt} = \frac{2w^2\left(\frac{d}{dt}(k-w)\right)}{4w^4} - \frac{(k-w)\left(\frac{d}{dt}(2w^2)\right)}{4w^4}$$
$$\frac{dA}{dt} = \frac{2w^2\left(\frac{dk}{dt} - \frac{dw}{dt}\right)}{4w^4} - \frac{(k-w)\left(4w\frac{dw}{dt}\right)}{4w^4}$$

Find k and $\frac{dk}{dt}$ so we can plug them in:

$$\frac{dk}{dt} = \frac{3}{2}(7) = \frac{17}{2} = 8.5$$
$$k = \frac{3w}{2} = \frac{15}{2} = 7.5$$

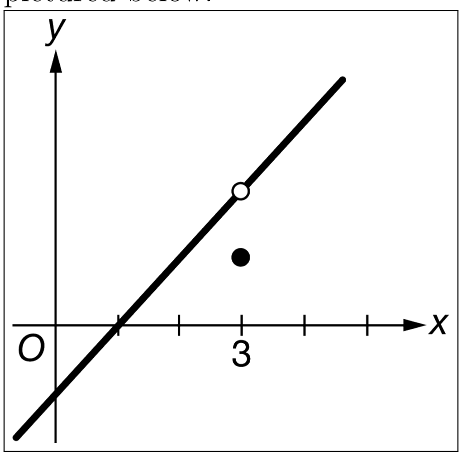
Plug in $\frac{dw}{dt}$, w and $\frac{dk}{dt}$:

$$\frac{dA}{dt} = \frac{(50)(8.5-7)}{2500} - \frac{(7.5-5)(20(7))}{2500}$$
$$\frac{dA}{dt} = \frac{75-350}{2500} = \boxed{-\frac{275}{2500} = -0.11 \text{ units}^2 \text{ p/ sec}}$$

Question 10

If f is a function that has a removable discontinuity at $x = 3$, which of the following could be the graph of f?

Solution: This would indicate that the function f is continuous around $x = 3$, even if it is not continuous *at* $x = 3$. The only graph that fulfills this is The one pictured below:



Question 11

The volume of a certain cone for which the sum of its radius, r, and height is constant is given by $V = \frac{1}{3}\pi r^2(10 - r)$. The rate of change of the radius of the cone with respect to time is 6. In terms of r, what is the rate of change of the volume of the cone with respect to time?

Solution:

$$\frac{d}{dt}(r+h) = 0h = 0$$

$$\boxed{\begin{aligned} V &= \frac{10\pi}{3}r^2 - \frac{\pi}{3}r^3 \\ V &= r^2\left(\frac{10\pi}{3} - \frac{\pi}{3}r\right) \\ \frac{dV}{dt} &= \left(\frac{d}{dt}(r^2)\right)\left(\frac{10\pi}{3} - \frac{\pi}{3}r\right) + \frac{d}{dt}\left(\frac{10\pi}{3} - \frac{\pi}{3}r\right)(r^2) \end{aligned}}$$

Question 12

Question 13

Question 14