

AP Calculus AB  
AP Classroom: 5.5 – 5.6

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Brayden Price

All final solutions are *boxed*  
Only FRQs are included (for now).

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**Question 1**

Let  $f$  be a function defined by  $f(x) = \begin{cases} 2x - x^2 & \text{for } x \leq 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

- For what values of  $k$  and  $p$  will  $f$  be continuous and differentiable at  $x = 1$ ?
- For the values of  $k$  and  $p$  found in part (a), on what interval or intervals is  $f$  increasing?
- Using the values of  $k$  and  $p$  found in part (a), find all points of inflection of the graph of  $f$ . Support your conclusion.

**Solution:**

(a) Definition of continuity of  $f$  at  $x = 1$ :

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) \quad (1)$$

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 + 1k + p \quad (2)$$

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = 2(1) - (1)^2 = 2 - 1 = 1 \quad (3)$$

$$\text{Therefore, } 1 = 1 + k + p \implies \boxed{-k = p} \quad (4)$$

Differentiate  $f$  with respect to  $x$  (needed for next step):

$$f'(x) = \begin{cases} \frac{d}{dx}(2x - x^2) & \text{for } x \leq 1, \\ \frac{d}{dx}(x^2 + kx + p) & \text{for } x > 1. \end{cases} \quad (5)$$

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \leq 1, \\ 2x + k + 0 & \text{for } x > 1. \end{cases} \quad (6)$$

Definition of differentiability of  $f$  at  $x = 1$ :

$$\lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^-} f'(x) = f'(1) \quad (7)$$

$$\lim_{x \rightarrow 1^-} f'(x) = f'(1) = 2 - 2(1) = 0 \quad (8)$$

$$\lim_{x \rightarrow 1^+} f'(x) = 2(1) + k \quad (9)$$

$$\text{Therefore, } 2 + k = 0 \implies \boxed{-2 = k} \quad (10)$$

Conclusion: Using Equations 10 and 4:

$$-2 = k \quad -k = p \quad (11)$$

$$-(-2) = p \implies p = 2 \quad (12)$$

$$\boxed{\text{Therefore, } k = -2 \text{ and } p = 2} \quad (13)$$

(b)  $f'(x) > 0$  when  $f$  is increasing.

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \leq 1, \\ 2x - 2 & \text{for } x > 1. \end{cases}$$

On  $x \leq 1$ , set  $f'(x) = 0$ :

$$2 - 2x = 0 \implies 2 = 2x \implies 1 = x$$

Sign chart for  $2 - 2x$ : (Only applies to  $f'(x)$  on  $x > 1$ )

$$\begin{array}{c} 2 - 2x \quad + \quad | \quad - \\ \hline 1 \end{array}$$

On  $x > 1$ , set  $f'(x) = 0$ :

$$2x - 2 = 0 \implies 2x = 2 \implies x = 1$$

Sign chart for  $2x - 2$ : (Only applies to  $f'(x)$  on  $x > 1$ )

$$\begin{array}{c} 2x - 2 \quad - \quad | \quad + \\ \hline 1 \end{array}$$

Combined sign chart for  $f'(x)$ :

$$\begin{array}{c} f'(x) \quad + \quad | \quad + \\ \hline 1 \end{array}$$

Conclusion:

*Therefore,  $f(x)$  is increasing when  $x \neq 1$*

*because  $f'(x)$  is positive on those regions. (At  $x = 1$ ,  $f'(x) = 0$ .)*

*$f$  is continuous at  $x = 1$ , so  $f(x)$  is increasing on  $(-\infty, \infty)$ .*

(c) Inflection points are where  $f''$  changes signs.

$$f''(x) = \begin{cases} \frac{d}{dx}(2 - 2x) = -2 & \text{for } x \leq 1, \\ \frac{d}{dx}(2x - 2) = 2 & \text{for } x > 1. \end{cases}$$

Sign chart of  $f''(x)$ :

$$\begin{array}{c} f''(x) \quad - \quad | \quad + \\ \hline 1 \end{array}$$

*Therefore,  $f'(x)$  has an inflection point at  $x = 1$ .*

$$f(1) = 1 \quad \boxed{\text{Inflection Point: } (1, 1)}$$

**Question 2**

Consider the curve given by  $x^2 - xy + 2y^2 = 7$ .

- (a) Show that  $\frac{dy}{dx} = \frac{y-2x}{4y-x}$
- (b) Determine the  $y$ -coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.
- (c) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$ ,  $y$ , and  $\frac{dy}{dx}$ . The line tangent to the curve at the point  $(1, 2)$  is horizontal. Determine whether the curve is concave up or concave down at the point  $(1, 2)$ .

**Solution:**

(a)

$$\frac{d}{dx}(x^2 - xy + 2y^2) = 7 \quad (14)$$

$$2x - (xy' + y) + 4yy' = 0 \quad (15)$$

$$2x - xyy' - y + 4yy' = 0 \quad (16)$$

$$2x - y = y'(x - 4y) \quad (17)$$

$$\frac{2x - y}{x - 4y} = y' = \frac{dy}{dx} = \frac{y - 2x}{4y - x} \quad (18)$$

(b) Vertical tangent lines have undefined slope.

Therefore,  $\frac{dy}{dx}$  will be undefined  $\iff$  the tangent line at the point is vertical.

( $\iff$  means 'if and only if')

$$4y - x = 0 \text{ makes } \frac{dy}{dx} \text{ undefined.} \quad (19)$$

$$4y = x \quad (20)$$

So,

$$7 = (4y)^2 - (4y)y + 2y^2 \quad (21)$$

$$7 = 16y^2 - 4y^2 + 2y^2 \quad (22)$$

$$7 = 14y^2 \quad (23)$$

$$y^2 = \frac{1}{2} \quad (24)$$

$$y = \boxed{\pm \frac{1}{\sqrt{2}}} \quad (25)$$

(c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{y-2x}{4y-x} \right) \quad (26)$$

$$= \frac{\left( \frac{d}{dx}(y-2x) \right) (4y-x) - \left( \frac{d}{dx}(4y-x) \right) (y-2x)}{(4y-x)^2} \quad (27)$$

$$= \frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2} \quad (28)$$

$y' = 0$  at  $(1, 2)$  since the tangent to the curve is vertical.

$$\left. \frac{d^2y}{dx^2} \right|_{(1,2)} = \frac{(-2)(4(2)-1) - (-1)(2-2(1))}{(4(2)-1)^2} = \frac{-2(7)-0}{7^2} = -\frac{2}{7} \quad (29)$$

Since  $\frac{d^2y}{dx^2}$  is negative at  $(1, 2)$ , the curve is concave down at  $(1, 2)$ .

### Question 3

Particle  $P$  moves along the  $y$ -axis so that its position at time  $t$  is given by  $y(t) = 4t - \frac{2}{3}$  for all times  $t$ . A second particle, particle  $Q$ , moves along the  $x$ -axis so that its position at time  $t$  is given by  $x(t) = \frac{\sin(\pi t)}{2-t}$  for all times  $t \neq 2$ .

- As time  $t$  approaches 2, what is the limit of the position of particle  $Q$ ? Show the work that leads to your answer.
- Show that the velocity of particle  $Q$  is given by  $v_Q(t) = \frac{2\pi \cos(\pi t) - \pi t \cos(\pi t) + \sin(\pi t)}{(2-t)^2}$  for all times.
- Find the rate of change of the distance between particle  $P$  and particle  $Q$  at time  $t = \frac{1}{2}$ . Show the work that leads to your answer.

### Solution:

(a)

$$\lim_{t \rightarrow 2} (\sin(\pi t)) = \sin(2\pi) = 0 \quad (30)$$

$$\lim_{t \rightarrow 2} (2-t) = 0 \quad (31)$$

Since  $\frac{\lim_{t \rightarrow 2} (\sin(\pi t))}{\lim_{t \rightarrow 2} (2-t)}$  approaches an indeterminate form  $(0/0)$ , L'Hôpital's rule

can be applied.

$$\lim_{t \rightarrow 2} (d/dt (\sin(\pi t))) = \lim_{t \rightarrow 2} (\pi \cos(\pi t)) = \pi \cos(2\pi) = \pi \quad (32)$$

$$\lim_{t \rightarrow 2} (d/dt (2 - t)) = \lim_{t \rightarrow 2} (-1) = -1 \quad (33)$$

$$\therefore \lim_{t \rightarrow 2} (x(t)) = \frac{\pi}{-1} = \boxed{-\pi} \quad (34)$$

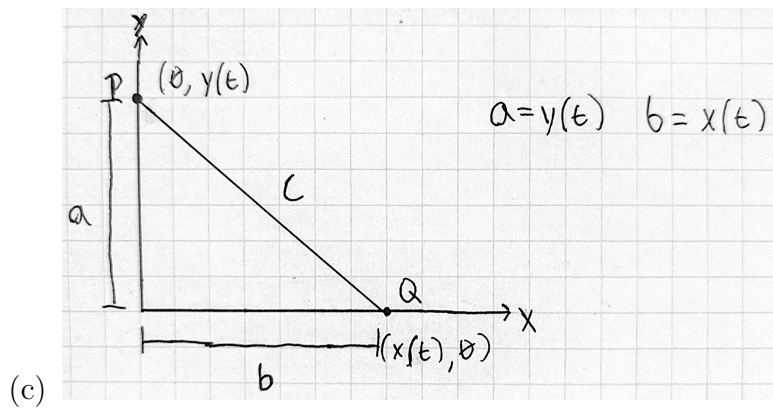
(b)

$$\frac{d}{dt} (x(t)) = v_Q(t) \quad (35)$$

$$\frac{d}{dt} \left( \frac{\sin(\pi t)}{2 - t} \right) = \frac{(d/dt(\sin(\pi t)))(2 - t) - (d/dt(2 - t))(\sin(\pi t))}{(2 - t)^2} \quad (36)$$

$$= \frac{\pi \cos(\pi t)(2 - t) - (-1)(\sin(\pi t))}{(2 - t)^2} \quad (37)$$

$$= \boxed{\frac{2\pi \cos(\pi t) - \pi t \cos(\pi t) + \sin(\pi t)}{(2 - t)^2}} \quad (38)$$



$$c^2 = a^2 + b^2 \quad (39)$$

$$c = \sqrt{a^2 + b^2} \quad (40)$$

$$(41)$$

$$2c \frac{dc}{dt} = 2a \frac{da}{dt} + 2b \frac{db}{dt} \quad (42)$$

$$\frac{dc}{dt} = \frac{a \frac{da}{dt} + b \frac{db}{dt}}{c} \quad (43)$$

$$\uparrow \text{Main rate equation} \uparrow \quad (44)$$

$$(45)$$

$$a = y(t) = 4t - \frac{2}{3} \quad (46)$$

$$\frac{da}{dt} = 4 \quad (47)$$

$$(48)$$

$$b = x(t) = \frac{\sin(\pi t)}{2 - t} \quad (49)$$

$$\frac{db}{dt} = \frac{(\pi \cos(\pi t))(2 - t) - (-1)(\sin(\pi t))}{(2 - t)^2} \quad (50)$$

Now, find all of those variables when  $t = \frac{1}{2}$

$$a|_{t=1/2} = 4(1/2) - 2/3 = 2 - 2/3 = 4/3 \quad (51)$$

$$b|_{t=1/2} = \frac{\sin(\pi/2)}{2 - 1/2} = \frac{1}{3/2} = \frac{2}{3} \quad (52)$$

$$\left. \frac{da}{dt} \right|_{t=1/2} = 4 \quad (53)$$

$$\left. \frac{db}{dt} \right|_{1/2} = \frac{(\pi \cos(\pi/2))(2 - 1/2) + (\sin(\pi/2))}{(2 - 1/2)^2} = \frac{4(\pi(0)(3/2) + 1)}{9} = \frac{4}{9} \quad (54)$$

$$c|_{t=1/2} = \sqrt{a^2 + b^2} = \sqrt{(4/3)^2 + (2/3)^2} = \sqrt{20/9} = \sqrt{20}/3 \quad (55)$$

Finally, plug all of those into our main rate equation:

$$\left. \frac{dc}{dt} \right|_{t=1/2} = \boxed{\frac{(4/3)(4) + (2/3)(4/9)}{\sqrt{20}/3}}$$

#### Question 4

Let  $f$  be the function given by  $f(x) = 2xe^{2x}$ .

Find the absolute minimum value of  $f$ . Justify that your answer is an absolute minimum.

**Solution:** All absolute minimums must also be relative minima.

At Relative minima of  $f$ , the derivative,  $f'$  changes from negative to positive.

Find  $f'(x)$

$$f'(x) = 2(1(e^{2x}) + x(2e^{2x})) \quad (56)$$

$$= 2e^{2x} + 4xe^{2x} \quad (57)$$

$$= 2e^{2x}(1 + 2x) \quad (58)$$

Set  $f'(x) = 0$  and solve:

$$2e^{2x}(1 + x)$$

$$2e^{2x} = 0 \quad | \quad (1 + 2x) = 0 \quad (59)$$

$$\ln(e^2 x) = 0 \quad | \quad 2x = -1 \quad (60)$$

$$\text{no solution} \quad | \quad x = -\frac{1}{2} \quad (61)$$



Sign chart of  $f'(x)$ :

$$\begin{array}{c} f'(x) \quad - \quad | \quad + \\ \hline -\frac{1}{2} \end{array}$$

Find end behavior of  $f(x)$ :

$$\lim_{x \rightarrow \infty} (2xe^{2x}) = \lim_{x \rightarrow \infty} (e^{2x}) = \infty \quad \lim_{x \rightarrow -\infty} (2xe^{2x}) = \lim_{x \rightarrow -\infty} (e^{2x}) = 0 \quad (62)$$

Conclusion:

Since  $x = -\frac{1}{2}$  is the only relative minima and is less than the  $\lim_{x \rightarrow \pm\infty} (f(x))$ , it is the absolute minimum of  $f(x)$ .

$$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)e^{2\left(-\frac{1}{2}\right)} = -e^{-1}$$

The absolute minimum of  $f(x)$  is  $f\left(-\frac{1}{2}\right) = -e^{-1}$

### Question 14

Unless otherwise specified, the domain of a function  $f$  is assumed to be the set of all real numbers  $x$  for which  $f(x)$  is a real number.

$t$ (hours)	0	5	15	30	35
$A(t)$ (gallons)	10	18	25	16	8

The number of gallons of olive oil in a tank at time  $t$  is given by the twice-differentiable function  $A$ , where  $t$  is measured in hours and  $0 \leq t \leq 35$ . Values of  $A(t)$  at selected times  $t$  are given in the table above.

- Use the data in the table to estimate the rate at which the number of gallons of olive oil in the tank is changing at time  $t = 10$  hours. Show the computations that lead to your answer. Indicate units of measure.
- For  $0 \leq t \leq 30$ , is there a time  $t$  at which  $A'(t) = \frac{1}{5}$ ? Justify your answer.
- The number of gallons of olive oil in the tank at time  $t$  is also modeled by the function  $G$  defined by  $G(t) = 5t - \frac{2}{3}(t+9)^{\frac{3}{2}} + 28$ , where  $t$  is measured in hours and  $0 \leq t \leq 35$ . Based on the model, at what time  $t$ , for  $0 \leq t \leq 35$ , is the number of gallons of olive oil in the tank an absolute maximum? Justify your answer.

Solution:

Use (5, 15), include units

(a)

$$A'(t) \approx \frac{A(30) - A(15)}{30 - 15} = \frac{16 - 25}{30 - 15} = -\frac{9}{15} = -\frac{3}{5}$$

(b) On  $0 \leq t \leq 30$ , the AROC can be found as follows:

$$\frac{A(30) - A(0)}{30 - 0} = \frac{16 - 10}{30} = \frac{6}{30} = \frac{1}{5}$$

Since  $A$  is twice-differentiable  $\implies A$  is differentiable  $\implies A$  is continuous.  
 Therefore, by MVT, there is a value  $c$  in  $(0, 30)$  where  $A'(c) = \frac{A(30) - A(0)}{30 - 0} = \frac{1}{5}$ .

(c)

$$G'(t) = 5 - \left(\frac{3}{2}\right) \left(\frac{2}{3}\right) (t+9)^{(3/2-1)}(1) \quad (63)$$

$$= 5 - (t+9)^{1/2} \quad (64)$$

$$= 5 - \sqrt{t+9} \quad (65)$$

Set  $G'(t) = 0$ :

$$5 - \sqrt{t+9} = 0 \quad (66)$$

$$5 = \sqrt{t+9} \quad (67)$$

$$25 = t+9 \quad (68)$$

$$16 = t \quad (69)$$

$$(70)$$

Sign Chart:

$$\begin{array}{c} G'(t) \quad + \quad | \quad - \\ \hline \quad \quad \quad 16 \end{array}$$

$t = 16$  must be the absolute maximum since it is the only relative max.  
 $(G(t))$  decreases continually on  $16 \leq t \leq 35$

$$(d) \quad G''(t) = -(t+9)^{1/2}(1) = \frac{-1}{\sqrt{t+9}}$$

Set  $G''(t) = 0$ :

$$\frac{-1}{\sqrt{t+9}} = 0 \implies \text{No Solution}$$

$$G''(t) < 0 \text{ on } 0 \leq t \leq 35.$$

$$\implies G'(t) \text{ is concave down on } 0 \leq t \leq 35.$$

Therefore,  $G(7)$  will provide an overestimate of  $G(8)$  if using a locally linear model. ( $G(t)$  is increasing at a decreasing rate, so if we assume a constant rate, it will provide an overestimate.)