

AP Calculus AB  
AP Classroom: 4.3 – 4.4

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All final solutions are *boxed*

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**Question 1**

$x$	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions  $f$  and  $g$  are differentiable for all real numbers, and  $g$  is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of  $x$ . The function  $h$  is given by  $h(x) = f(g(x)) - 6$ .

If  $g^{-1}$  is the inverse function of  $g$ , write an equation for the line tangent to the graph of  $y = g^{-1}$  at  $x = 2$ .

**Solution:** Given that  $g(1) = 2$ , the point of tangency on the graph of  $y = g^{-1}$  is the point  $(2, 1)$ . (since  $g^{-1}$  is an inverse of  $g$ ,  $g(2) = 1$ .)

We now want to find the derivative at the point of tangency:

To find  $[g^{-1}]'(2)$ , we must use the rule for derivatives of inverses:

$$\begin{aligned} [g^{-1}]'(x) &= \frac{1}{g'(g^{-1}(x))} \\ [g^{-1}]'(2) &= \frac{1}{g'(g^{-1}(2))} \\ &= \frac{1}{g'(1)} \\ &= \frac{1}{5} \end{aligned}$$

Now, we just need to use the point  $(2, 1)$  and the derivative at that point to write the tangent line:

$$y - 1 = \frac{1}{5}(x - 2)$$

**Question 2**

A student attempted to confirm that the function  $f$  defined by  $f(x) = \frac{x^2+x-6}{x^2-7x+10}$  is continuous at  $x = 2$ . In which step, if any, does an error first appear?

Step 1:

$$f(x) = \frac{x^2 + x - 6}{x^2 - 7x + 10} = \frac{(x - 2)(x + 3)}{(x - 2)(x - 5)}$$

Step 2:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x + 3}{x - 5} = \frac{2 + 3}{2 - 5} = -\frac{5}{3}$$

Step 3:

$$f(2) = \frac{2+3}{2-5} = -\frac{5}{3}$$

Step 4:

$$\lim_{x \rightarrow 2} f(x) = f(2), \text{ so } f \text{ is continuous at } x = 2.$$

**Solution:** During Step 3, we made a little bit of a mistake. While in the limit in Step 2, we can simplify the rational definition of  $f(x)$ , when finding the actual value (or lack thereof) of  $f(2)$ , we **cannot** do this.

Even though  $f$  is actually undefined at  $x = 2$  (removable discontinuity / hole), we incorrectly said it was equal to  $-\frac{5}{3}$ , and therefore incorrectly stated  $f$  as continuous at  $x = 2$ .

### Question 3

Two particles move along the  $x$ -axis with velocities given by  $v_1(t) = -3t^2 + 8t + 5$  and  $v_2(t) = \sin t$  for time  $t \geq 0$ . At what time  $t$  do the two particles have equal acceleration?

#### Solution:

To find acceleration from give velocity equations, we must take the derivative of the velocity functions to find two new acceleration equations

$$\begin{aligned} a_1(t) &= \frac{d}{dt} (v_1(t)) \\ &= \frac{d}{dt} (-3t^2 + 8t + 5) \\ &= -6t + 8 \end{aligned}$$

$$\begin{aligned} a_2(t) &= \frac{d}{dt} (v_2(t)) \\ &= \frac{d}{dt} (\sin t) \\ &= \cos t \end{aligned}$$

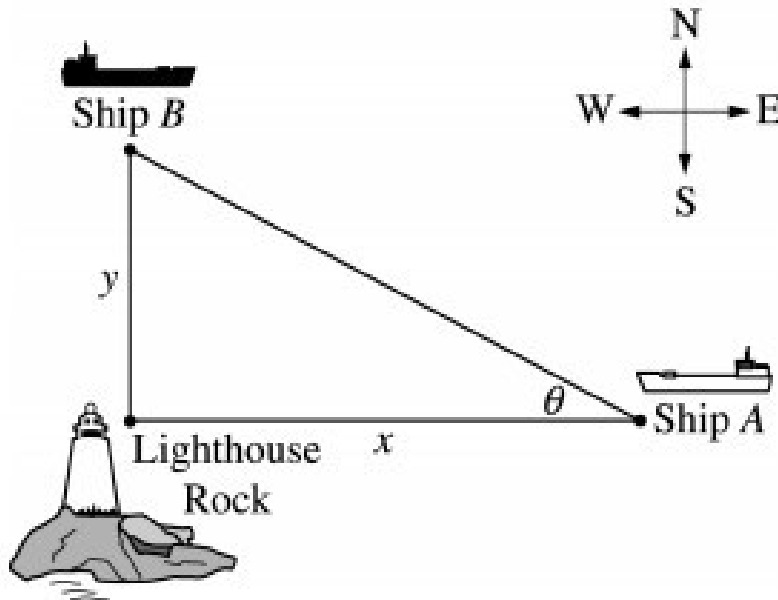
Now, we need to find the time  $t$  where  $a_1(t) = a_2(t)$

$$\begin{aligned} a_1(t) &= a_2(t) \\ -6t + 8 &= \cos t \end{aligned}$$

Since this is a calculator required question, just use your calculator to find the intersection of the line and the cosine function. (**Make sure your calculator is in radian mode.**)

$$t \approx 1.2866$$

#### Question 4



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse rock at a speed of 10 km/hr. Let  $x$  be the distance between Ship A and Lighthouse Rock at time  $t$ , and let  $y$  be the distance between Ship B and Lighthouse Rock at time  $t$ , as shown in the figure above.

#### Solution:

Equation (Pythagorean Theorem) of situation:  $c^2 = y^2 + x^2$  Find:  $\frac{dc}{dt}$

When:  $x = 4$  km and  $y = 3$  km Other Rates:  $\frac{dx}{dt} = -15 \frac{\text{km}}{\text{hr}}$  and  $\frac{dy}{dt} = 10 \frac{\text{km}}{\text{hr}}$

$c$  is the hypotenuse / distance between Ship B and Ship A.

Using Pythagorean Theorem:  $c = \sqrt{4^2 + 3^2} = 5$

Find derivative of equation:

$$\frac{dc}{dt} \cdot 2c = \frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x$$

Solve for  $\frac{dc}{dt}$ :

$$\frac{dc}{dt} = \frac{\frac{dy}{dt} \cdot 2y + \frac{dx}{dt} \cdot 2x}{2c}$$

Plug in knowns and solve:

$$\frac{dc}{dt} = \frac{10 \cdot 2(3) - 15 \cdot 2(4)}{2(5)} = \frac{60 - 120}{10} = -\frac{60}{10} = \boxed{-6 \frac{\text{km}}{\text{hr}}}$$

### Question 5

Consider the curve given by  $y^2 = 2 + xy$

Let  $x$  and  $y$  be functions of time  $t$  that are related by the equation  $y^2 = 2 + xy$ . At time  $t = 5$ , the value of  $y$  is 3 and  $dy/dt = 6$ . Find the value of  $dx/dt$  at time  $t = 5$ .

**Solution:** Find  $x$  at  $y = 3$  (will be needed later):

$$\begin{aligned}(3)^2 &= 2 + 3x \\ 9 - 2 &= 3x \\ \frac{7}{3} &= x\end{aligned}$$

Take derivative of given curve and then solve for  $dx/dt$ :

$$\begin{aligned}2y \cdot \frac{dy}{dt} &= x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} \\ \frac{2y \cdot \frac{dy}{dt} - x \cdot \frac{dy}{dt}}{y} &= \frac{dx}{dt}\end{aligned}$$

Plug in knowns and simplify:

$$\frac{2(3) \cdot 6 - \frac{7}{3} \cdot 6}{3} = \frac{36 - 14}{3} = \boxed{\frac{22}{3} = \frac{dx}{dt}}$$

### Question 6

The radius  $r$  of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume  $V$  of a sphere with radius  $r$  is  $V = \frac{4}{3}\pi r^3$ .)

At the time when the radius,  $r$ , of the sphere is 10 centimeters, what is the rate of increase of its volume,  $dV/dt$ ?

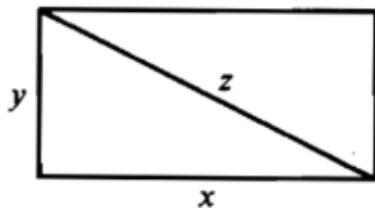
**Solution:** Differentiate given equation with respect to  $t$  (time):

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2) \left( \frac{dr}{dt} \right)$$

Plug in knowns and simplify:

$$\frac{4}{3}\pi(3(10)^2)(0.04) = \frac{4}{3}\pi(12) = \boxed{16\pi = \frac{dV}{dt}}$$

### Question 7



The sides of the rectangle above increase in such a way that  $\frac{dz}{dt} = 1$  and  $\frac{dx}{dt} = 3\frac{dy}{dt}$ . At the instant when  $x = 4$  and  $y = 3$ , what is the value of  $\frac{dx}{dt}$ ?

**Solution:** Write  $\frac{dy}{dt}$  in terms of  $\frac{dx}{dt}$  (will be needed later):

$$\frac{1}{3} \cdot \frac{dx}{dt} = \frac{dy}{dt}$$

Find  $z$  using Pythagorean Theorem (will be needed later):

$$z = \sqrt{4^2 + 3^2} = 5$$

Write derivative of Formula for side lengths of triangle (Pythagorean Theorem):

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

Solve for  $\frac{dx}{dt}$ :

$$\frac{2z \frac{dz}{dt} - 2y \frac{dy}{dt}}{2x} = \frac{dx}{dt}$$

Plug in knowns and simplify:

$$\frac{2(5)(1) - 2(3)\left(\frac{1}{3} \cdot \frac{dx}{dt}\right)}{2(4)} = \frac{10 - 6\left(\frac{1}{3} \cdot \frac{dx}{dt}\right)}{8} = \frac{10 - 2\frac{dx}{dt}}{8} = \frac{dx}{dt}$$

$$\frac{dx}{dt} + \frac{2}{8} \left( \frac{dx}{dt} \right) = \frac{10}{8}$$

$$\frac{10}{8} \left( \frac{dx}{dt} \right) = \frac{10}{8}$$

$$\boxed{\frac{dx}{dt} = 1}$$

**Question 8**

$t$ (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

The twice-differentiable function  $W$  models the volume of water in a reservoir at time  $t$ , where  $W(t)$  is measured in (GL) and  $t$  is measured in days. The table above gives values of  $W'(t)$  sampled at various times during the time interval  $0 \leq t \leq 30$  days. At time  $t = 30$ , the reservoir contains 125 gegaliters of water.

The equation  $A = 0.3W^{2/3}$  gives the relationship between the area  $A$ , in square kilometers, of the surface of the reservoir, and the volume of the water  $W(t)$ , in gegaliters, in the reservoir. Find the instantaneous rate of change of  $A$ , in square kilometer per day, with respect to  $t$  when  $t = 30$  days.

**Solution:** We are attempting to find  $A'(t)$  when  $t = 30$  days.

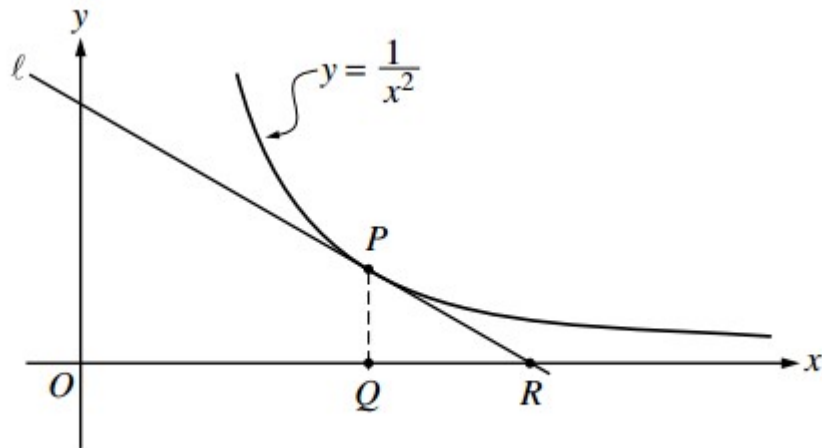
Knowns:  $W'(30) = 0.5$ ,  $W(30) = 125$  and  $A = 7.5$  Differentiate given equation for  $A$ :

$$A' = 0.4(2/3 W^{1/3})(W')$$

Plug in and simplify:

$$\left( \frac{3}{10} \right) \left( \frac{2}{3\sqrt[3]{125}} \right) \left( \frac{1}{2} \right) = \frac{6}{300} = \boxed{0.02 = A'}$$

**Question 9**



In the figure above, line  $\ell$  is tangent to the graph of  $y = \frac{1}{x^2}$  at point  $P$ , with coordinates  $w, \frac{1}{w^2}$ , where  $w > 0$ . Point  $Q$  has coordinates  $(w, 0)$ . Line  $\ell$  crosses the  $x$ -axis at point  $R$ , with coordinates  $(k, 0)$ .

- Find the value of  $k$  when  $w = 3$ .
- For all  $w > 0$ , find  $k$  in terms of  $w$ .
- Suppose that  $w$  is increasing at the constant rate of 7 units per second. When  $w = 5$ , what is the rate of change of  $k$  with respect to time?
- Suppose that  $w$  is increasing at the constant rate of 7 units per second. When  $w = 5$ , what is the rate of change of the area of  $PQR$  with respect to time? Determine whether the area is increasing or decreasing at this instant.

**Solution:** The slope of the tangent line  $\ell$  is  $\left. \frac{dy}{dx} \right|_{x=w}$  (Since  $w$  is the  $x$  coordinate of point  $P$ .)

Derivative of  $y$ :

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$$

- In order to find Point  $R$ , where  $y = 0$ , we should write the equation of line  $\ell$  where  $x = w$  (the  $x$ -coordinate of point  $P$ ) and then plug in  $y = 0$ :

$$\begin{aligned} y - \frac{1}{w^2} &= \left( \left. \frac{dy}{dx} \right|_{x=w} \right) (k - w) \\ -\frac{1}{w^2} &= \left( \left. \frac{dy}{dx} \right|_{x=w} \right) (k - w) \end{aligned}$$



Now, we need to plug in  $w = 3$  and solve for  $x$ :

$$-\frac{1}{3^2} = \left( \frac{dy}{dx} \Big|_{x=3} \right) (k - 3)$$

$$-\frac{1}{9} = \left( -\frac{2}{3^3} \right) (k - 3)$$

$$-\frac{1}{9} = \left( -\frac{2}{27} \right) (k - 3)$$

$$k - 3 = \frac{-\frac{2}{27}}{-\frac{1}{9}}$$

$$k = \left( -\frac{2}{27} \cdot -\frac{9}{1} \right) + 3$$

$$k = \frac{18}{27} + 3 = \frac{2}{3} + 3 = \frac{11}{3} \approx 3.6667$$

(b)

$$-\frac{1}{w^2} = \left( \frac{dy}{dx} \Big|_{x=w} \right) (k - w)$$

$$-\frac{1}{w^2} = \left( \frac{-2}{w^3} \right) (k - w)$$

$$k - w = -\frac{1}{w^2} \div \left( \frac{-2}{w^3} \right)$$

$$k = -\frac{1}{w^2} \left( \frac{w^3}{-2} \right) + w$$

$$k = \frac{w}{2} + w = \frac{3w}{2}$$

(c) Knowns:

$$\frac{dw}{dt} = 7 \text{ units p/ sec}$$

$$w = 5$$

Find  $\frac{d}{dt}$  of  $k$ :

$$\frac{dk}{dt} = \frac{3}{2} \left( \frac{dw}{dt} \right)$$

Plug in Knowns:

$$\frac{dk}{dt} = \frac{3}{2} (7)$$

$$\frac{dk}{dt} = \boxed{\frac{21}{2} = 10.5 \text{ units p/ sec}}$$

(d) Find side lengths and derivatives of  $w$  and  $k$ :

$$\overline{PQ} = \frac{1}{w^2} - 0$$

$$\overline{QR} = k - w$$

$$\frac{dw}{dt} = 7 \text{ (given in problem)}$$

$$\frac{dk}{dt} = \frac{3}{2} \left( \frac{dw}{dt} \right)$$

Solve for A, then find the derivative of A with respect to time:

$$A = \frac{1}{2} (\overline{QR}) (\overline{PQ})$$

$$A = \frac{1}{2} \left( \frac{k - w}{w^2} \right)$$

$$A = \frac{k}{2w^2} - \frac{w}{2w^2}$$

$$\frac{dA}{dt} = \frac{2w^2 \left( \frac{d}{dt} (k - w) \right)}{4w^4} - \frac{(k - w) \left( \frac{d}{dt} (2w^2) \right)}{4w^4}$$

$$\frac{dA}{dt} = \frac{2w^2 \left( \frac{dk}{dt} - \frac{dw}{dt} \right)}{4w^4} - \frac{(k - w) \left( 4w \frac{dw}{dt} \right)}{4w^4}$$

Find  $k$  and  $\frac{dk}{dt}$  so we can plug them in:

$$\frac{dk}{dt} = \frac{3}{2} (7) = \frac{17}{2} = 8.5$$

$$k = \frac{3w}{2} = \frac{15}{2} = 7.5$$

Plug in  $\frac{dw}{dt}$ ,  $w$  and  $\frac{dk}{dt}$  :

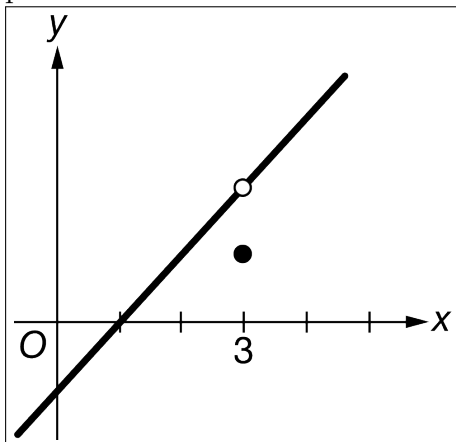
$$\frac{dA}{dt} = \frac{(50)(8.5 - 7)}{2500} - \frac{(7.5 - 5)(20(7))}{2500}$$

$$\frac{dA}{dt} = \frac{75 - 350}{2500} = \boxed{-\frac{275}{2500} = -0.11 \text{ units}^2 \text{ p/ sec}}$$

### Question 10

If  $f$  is a function that has a removable discontinuity at  $x = 3$ , which of the following could be the graph of  $f$ ?

**Solution:** This would indicate that the function  $f$  is continuous around  $x = 3$ , even if it is not continuous *at*  $x = 3$ . The only graph that fulfills this is The one pictured below:



### Question 11

The volume of a certain cone for which the sum of its radius,  $r$ , and height is constant is given by  $V = \frac{1}{3}\pi r^2(10 - r)$ . The rate of change of the radius of the cone with respect to time is 6. In terms of  $r$ , what is the rate of change of the volume of the cone with respect to time?

**Solution:**

$$\frac{d}{dt}(r + h) = 0$$

$$h = 0$$

$$\frac{dv}{dt} =$$

**Question 12**

**Question 13**

**Question 14**