AP Calculus AB
AP Classroom: 5.1 – 5.3

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All final solutions are $\boxed{boxed}$

Only FRQs are included (for now).

List of Questions

Question 1 . . . . page 2
Question 7 . . . . page 3
Question 10 . . . . page 4
Question 12 . . . . page 4
Question 15 . . . . page 6
Question 20 . . . . page 6

Question 1

$$f(t) = \begin{cases} 48t + t^2 - \frac{t^3}{12}, & \text{for } 0 \leq t < 6 \\ g(t) & \text{for } 6 \leq t \leq 12 \end{cases}$$

t (hours)	6	8	10	12
$g(t)$ (cubic meters)	306	376	428	474

At an excavation site, the amount of dirt that has been removed, in cubic meters, is modeled by the function f defined above, where g is a differentiable function and t is measured in hours. Values of $g(t)$ at selected values of t are given in the table above.

(a) According to the model f, what is the average rate of change of the amount of dirt removed over the time interval $6 \leq t \leq 12$ hours?

(b) Use the data in the table to approximate $f'(9)$, the instantaneous rate of change in the amount of dirt removed, in cubic meters per hour, at time $t = 9$ hours. Show the computations that lead to your answer.

(c) Is f continuous for $0 \leq t \leq 12$? Justify your answer.

(d) Find $f'(t)$, the instantaneous rate of change in the amount of dirt removed, in cubic meters per hour, at time $t = 2$ hours.

Solution:

(a) Since $f(t)$ is defined by the $g(t)$ over $0 \leq t < 6$, we can use values in the table of g to find the AROC.

$$\text{AROC} = \frac{g(12) - g(6)}{12 - 6} = \boxed{\frac{474 - 306}{6} \text{ m}^3 \text{ / min}}$$

(answer could be simplified but is not necessary to earn point)

(b) $f'(9)$ can be approximated using AROC over a neighboring interval, like $8 \leq t \leq 10$.
Since $f(t)$ is defined by $g(t)$ for $6 \leq t \leq 12$, we should use that to find our AROC.

$$\text{AROC} = \frac{g(10) - g(8)}{10 - 8} = \boxed{\frac{428 - 376}{2} \text{ m}^3 \text{ / min} \approx f'(9)}$$

(see above note about simplification)

(c) Since g is differentiable and $f(t)$ is defined by a polynomial over $0 \leq t < 6$, both individual intervals are continuous. We must check the continuity at $t = 6$, however.

$$\lim_{t \to 6^-} f(t) = 48(6) + 6^2 - \frac{6^3}{12} = 288 + 36 - 18 = 306$$

$$\lim_{t \to 6^+} f(t) = f(6) = g(6) = 306$$

$$\therefore \lim_{t \to 6^-} f(t) = \lim_{t \to 6^+} f(t) = f(6) = g(6)$$

$\therefore \boxed{f(t) \text{ is continuous for } 0 \leq t \leq 12}$ since it is continuous at $t = 6$ and on $0 \leq t < 6$ and $6 < t \leq 12$.

(d)

$$f'(t) = 48 + 2t - \frac{t^2}{4} \text{ for } 0 \leq t \leq 6$$

$$f'(2) = 48 + 4 - \frac{4}{4} = \boxed{51}$$

Question 7

The rate at which raw sewage enters a treatment tank is given by $E(t) = 850 + 715\cos\left(\frac{\pi t^2}{9}\right)$ gallons per hour for $0 \leq t \leq 4$ hours. Treated sewage is removed from the tank at the constant rate of 645 gallons per hour. The treatment tank is empty at time $t = 0$.

For $0 \leq t \leq 4$ at what time t is the amount of sewage in the treatment tank greatest? To the nearest gallon, what is the maximum amount of sewage in the tank? Justify your answers.

Solution: We only know enough calculus to solve the first part of this question. (We will learn integration, which have to be used to solve the second part of this question, later.)

When $E(t) = 645$, there is a critical point at that value t, and there could be a maximum there.

$$E(t) = 645 \rightarrow 850 + 715\cos\left(\frac{\pi t^2}{9}\right) = 645$$

In calculator, find intersection between $y = 615$ and $y = E(t)$.

$$t = \{2.3184, 3.5532\}$$

There could be a maximum at either one of critical values, and we would have to find values of the actual amount of sewage in order to test which point (or one of the endpoints) is the maximum.

Question 10

Let f be the function given by the $f(x) = \frac{\ln x}{x}$ for all x > 0. The derivative of f is given by $f'(x) = \frac{1-\ln x}{x^2}$.
Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

Solution: $f'(x)$ must be undefined or zero to constitute a critical point. The only undefined point, $x = 0$, is excluded (not in the domain of $f(x)$). So, set numerator equal to zero and solve:

$$1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow \log_e x = 1 \Rightarrow e^1 = x$$

At point $x = e$, $f'(x) = 0$ and $f(x)$ is defined, so $\boxed{x = e \text{ is a critical point.}}$

$$\frac{1-\ln(e^2)}{(e^2)^2} = \frac{1-2}{e^4} = -\frac{1}{e^4}$$
$$\frac{1-\ln(e^0)}{(e^0)^2} = 1 - 0 = 1$$

$f'(x)$ + | −

e

$f'(x) > 0$ when $0 < x < e$ and $f'(x) < 0$ when $x > e$.

Therefore, $\boxed{x = e \text{ must be a maximum on } f}$.

Question 12

Let f be defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

Solution:

(a)

$$f'(x) = 15x^4 - 15x^2 \tag{1}$$
$$= 15x^2(x^2 - 1) \tag{2}$$
$$x = \{0, \pm 1\} \tag{3}$$
$$\begin{array}{c|cccc} f'(x) & + & - & - & + \\ \hline & & -1 & 0 & +1 \end{array} \tag{4}$$

$f'(x) > 0$ $\boxed{\text{on } (-\infty, 0) \text{ and } (1, +\infty)}$, therefore, f is increasing on these intervals.

(b)

$$f''(x) = 60x^3 - 30x \tag{5}$$
$$= 30x(2x^2 - 1) \tag{6}$$
$$x = \left\{0, \pm\frac{\sqrt{2}}{2}\right\} \tag{7}$$
$$\begin{array}{c|cccc} f''(x) & - & + & - & + \\ \hline & & -\frac{\sqrt{2}}{2} & 0 & +\frac{\sqrt{2}}{2} \end{array} \tag{8}$$

$f''(x) > 0$ $\boxed{\text{on } (-\frac{\sqrt{2}}{2}, 0) \text{ and } (+\frac{\sqrt{2}}{2}, +\infty)}$, therefore, f is increasing on these intervals

(c) Horizontal tangent lines have a slope of zero. Set $f'(x)$ to zero and solve:

$$f'(x) = 15x^4 - 15x^2 \tag{9}$$
$$15x^4 - 15x^2 = 0 \tag{10}$$
$$15x^2(x^2 - 1) = 0 \tag{11}$$
$$x = \{\pm 1, 0\} \tag{12}$$

Those are the x-coordinates of the points of tangency where the slope is zero.

Now we have to find the y-coordinates of those points (since horizontal lines have form $y =$ some number

$$f(-1) = 3(-1)^5 - 5(-1)^3 + 2 = -3 + 5 + 2 = 4 \quad (13)$$
$$f(0) = 2 \quad (14)$$
$$f(1) = 3(1)^5 - 5(1)^5 + 2 = 3 - 5 + 2 = 0 \quad (15)$$

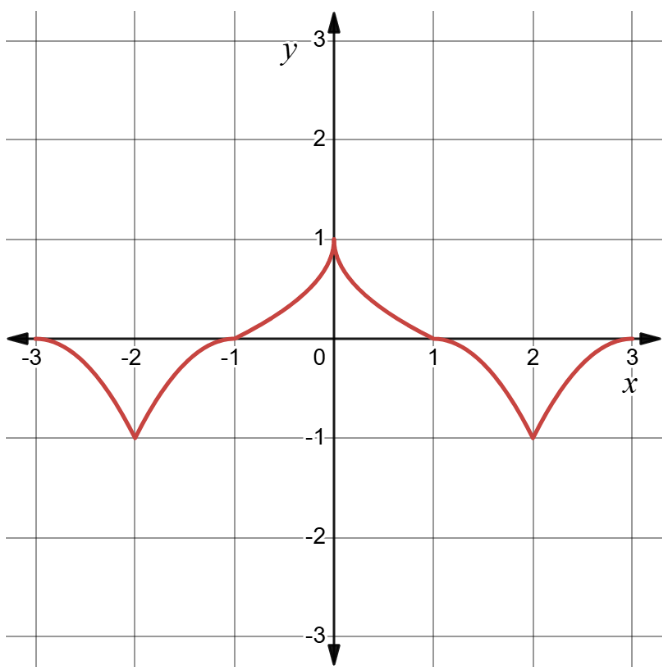
Points of tangency: $(-1, 4), (0, 2), (1, 0)$
$\boxed{y = 4, y = 0, \text{and } y = 2}$ are the horizontal tangent lines to the graph f.

Question 15

Let f be a function that is even and continuous on the closed interval $[3, 3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

Solution:



Question 20

Consider the curve given by $x^2 - xy + 2y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{4y-x}$

(b) Determine the y-coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x, y, and $\frac{dy}{dx}$. The line tangent to the curve at the point $(1, 2)$ is horizontal. Determine whether the curve is concave up or concave down at the point $(1, 2)$.

Solution:

(a)

$$\begin{aligned}
\frac{d}{dx}\left(x^2 - xy + 2y^2\right) &= \frac{d}{dx}(7) \\
2x - (y + xy') + 4yy' &= 0 \\
2x - y - xy' + 4yy' &= 0 \\
2x - y &= y'(x - 4y) \\
y' &= \frac{2x - y}{x - 4y} = \frac{y - 2x}{4y - x}
\end{aligned}$$

(b) Vertical tangent lines have undefined slope, meaning the denominator of the derivative of the curve must equal zero.
$4y - x = 0 \ \Rightarrow\ x = 4y$
Plug into curve:

$$\begin{aligned}
7 &= (4y)^2 - (4y)y + 2y^2 \\
7 &= 16y^2 - 4y^2 + 2y^2 \\
7 &= 14y^2 \\
y^2 &= \frac{1}{2} \\
y &= \pm\sqrt{\frac{1}{2}}
\end{aligned}$$

(c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y-2x}{4y-x}\right)$$
$$= \frac{(y-2x)'(4y-x) - (4y-x)'(y-2x)}{(4y-x)^2}$$
$$= \boxed{\frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2}}$$

Find y' at $(1, 2)$:

$$y'\Big|_{(1,2)} = \frac{2-2(1)}{4(2)-1} = \frac{0}{7} = 0$$

Plug in y' and $(1, 2)$ to y'':

$$y''\Big|_{(1,2)} = \frac{(0-2)(4(2)-1) - (4(0)-1)((2)-2(1))}{(4(2)-1)^2} = \frac{(-2)(7)-(-1)(0)}{(7)^2} = \boxed{-\frac{14}{49}}$$

Therefore, y' is $\boxed{\text{concave down}}$ at $(1, 2)$ because $y'' < 0$