AP Calculus AB
AP Classroom: 5.1 – 5.3

November 5, 2024

Brayden Price

All final solutions are $\boxed{\textit{boxed}}$

Only FRQs are included (for now).

List of Questions

Question 1

$$f(t) = \begin{cases} 48t + t^2 - \frac{t^3}{12}, & \text{for } 0 \le t < 6 \\ g(t) & \text{for } 6 \le t \le 12 \end{cases}$$

t (hours)	6	8	10	12
$g(t)$ (cubic meters)	306	376	428	474

At an excavation site, the amount of dirt that has been removed, in cubic meters, is modeled by the function f defined above, where g is a differentiable function and t is measured in hours. Values of $g(t)$ at selected values of t are given in the table above.

(a) According to the model f, what is the average rate of change of the amount of dirt removed over the time interval $6 \le t \le 12$ hours?

(b) Use the data in the table to approximate $f'(9)$, the instantaneous rate of change in the amount of dirt removed, in cubic meters per hour, at time $t = 9$ hours. Show the computations that lead to your answer.

(c) Is f continuous for $0 \le t \le 12$? Justify your answer.

(d) Find $f'(t)$, the instantaneous rate of change in the amount of dirt removed, in cubic meters per hour, at time $t = 2$ hours.

Solution:

(a) Since $f(t)$ is defined by the $g(t)$ over $0 \le t < 6$, we can use values in the table of g to find the AROC.

$$\text{AROC} = \frac{g(12) - g(6)}{12 - 6} = \boxed{\frac{474 - 306}{6} \text{ m}^3 \text{ / min}}$$

(answer could be simplified but is not necessary to earn point)

(b) $f'(9)$ can be approximated using AROC over a neighboring interval, like $8 \le t \le 10$.
Since $f(t)$ is defined by $g(t)$ for $6 \le t \le 12$, we should use that to find our AROC.

$$\text{AROC} = \frac{g(10) - g(8)}{10 - 8} = \boxed{\frac{428 - 376}{2} \text{ m}^3 \text{ / min} \approx f'(9)}$$

(see above note about simplification)

(c) Since g is differentiable and $f(t)$ is defined by a polynomial over $0 \leq t < 6$, both individual intervals are continuous. We must check the continuity at $t = 6$, however.

$$\lim_{t \to 6^-} = 48(6) + 6^2 - \frac{6^3}{12} = 288 + 36 - 18 = 306$$

$$\lim_{t \to 6^+} = g(6) = 306$$

$$\therefore \lim_{t \to 6^-} = \lim_{t \to 6^+} = g(6)$$

$\therefore$ $\boxed{f(t) \text{ is continuous for } 0 \leq t \leq 12}$ since it is continuous at $t = 6$ and on $0 \leq t < 6$ and $6 < t \leq 12$.

(d)

$$f'(t) = 48 + 2t - \frac{t^2}{4} \text{ for } 0 \leq t \leq 6$$

$$f'(2) = 48 + 4 - \frac{4}{4} = \boxed{51}$$

Question 7

Just a note: this question is here in error because it uses integrals, which we haven't learned yet.

Question 10

Let f be the function given by the $f(x) = \frac{\ln x}{x}$ for all x > 0. The derivative of f is given by $f'(x) = \frac{1-\ln x}{x^2}$.

Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

Solution: $f'(x)$ must be undefined or zero to constitute a critical point. The only undefined point, $x = 0$, is excluded (not in the domain of $f(x)$). So, set numerator equal to zero and solve:

$$1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow \log_e x = 1 \Rightarrow e^1 = x$$

At point $x = e$, $f'(x) = 0$ and $f(x)$ is defined, so $\boxed{x = e \text{ is a critical point.}}$

$$\frac{1-\ln(e^2)}{(e^2)^2} = \frac{1-2}{e^4} = -\frac{1}{e^4}$$
$$\frac{1-\ln(e^0)}{(e^0)^2} = 1 - 0 = 1$$

$$\begin{array}{l|l} f'(x) \quad + & - \\ \hline \qquad\qquad\quad e & \end{array}$$

$f'(x) > 0$ when $0 < x < e$ and $f'(x) < 0$ when $x > e$.

Therefore, $\boxed{x = e \text{ must be a maximum on } f}$.

Question 12

Let f be defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

Solution:

(a)

$$f'(x) = 15x^4 - 15x^2 \tag{1}$$
$$= 15x^2(x^2 - 1) \tag{2}$$
$$x = \{0, \pm 1\} \tag{3}$$
$$\begin{array}{l|c|c|l} f'(x) \quad + & - & - & + \\ \hline \qquad\qquad -1 & 0 & +1 & \end{array} \tag{4}$$

$f'(x) > 0$ $\boxed{\text{on } (-\infty, 0) \text{ and } (1, +\infty)}$, therefore, f is increasing on these intervals.

(b)

$$f''(x) = 60x^3 - 30x \tag{5}$$
$$= 30x(2x^2 - 1) \tag{6}$$
$$x = \left\{0, \pm\frac{\sqrt{2}}{2}\right\} \tag{7}$$
$$\begin{array}{c|c|c|c|c} f''(x) & - & + & - & + \\ \hline & -\frac{\sqrt{2}}{2} & 0 & +\frac{\sqrt{2}}{2} & \end{array} \tag{8}$$

$f''(x) > 0$ $\boxed{\text{on } (-\frac{\sqrt{2}}{2}, 0) \text{ and } (+\frac{\sqrt{2}}{2}, +\infty)}$, therefore, f is increasing on these intervals

(c) Horizontal tangent lines have a slope of zero. Set $f'(x)$ to zero and solve:

$$f'(x) = 15x^4 - 15x^2 \tag{9}$$
$$15x^4 - 15x^2 = 0 \tag{10}$$
$$15x^2(x^2 - 1) = 0 \tag{11}$$
$$x = \{\pm 1, 0\} \tag{12}$$

Those are the x-coordinates of the points of tangency where the slope is zero.
Now we have to find the y-coordinates of those points (since horizontal lines have form $y =$ some number

$$f(-1) = 3(-1)^5 - 5(-1)^3 + 2 = -3 + 5 + 2 = 4 \tag{13}$$
$$f(0) = 2 \tag{14}$$
$$f(1) = 3(1)^5 - 5(1)^5 + 2 = 3 - 5 + 2 = 0 \tag{15}$$

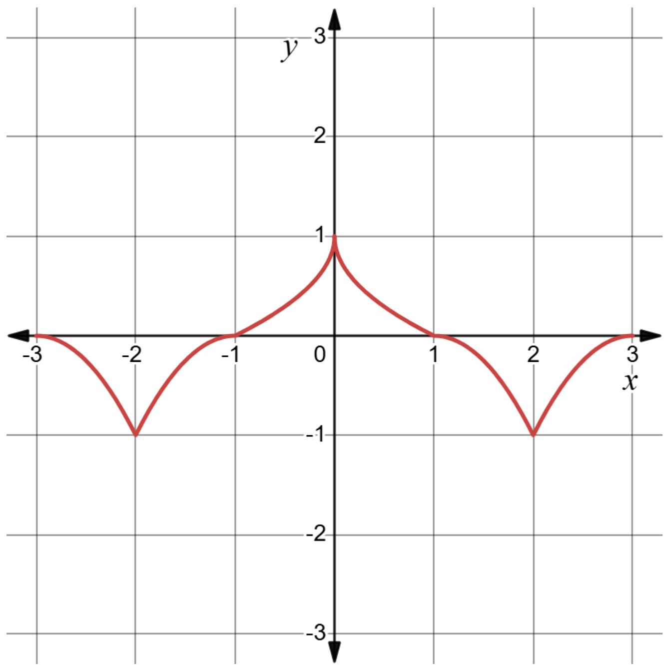
Points of tangency: $(-1, 4), (0, 2), (1, 0)$
$\boxed{y = 4, y = 0, \text{and } y = 2}$ are the horizontal tangent lines to the graph f.

Question 15

Let f be a function that is <u>even</u> and continuous on the closed interval $[3, 3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

Solution:



Question 20

Consider the curve given by $x^2 - xy + 2y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{4y-x}$

(b) Determine the y-coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x, y, and $\frac{dy}{dx}$. The line tangent to the curve at the point $(1, 2)$ is horizontal. Determine whether the curve is concave up or concave down at the point $(1, 2)$.

Solution:

(a)

$$\begin{aligned}\frac{d}{dx}\left(x^2 - xy + 2y^2\right) &= \frac{d}{dx}(7)\\ 2x - (y + xy') + 4yy' &= 0\\ 2x - y - xy' + 4yy'&\\ 2x - y &= y'(x - 4y)\\ y' &= \frac{2x-y}{x-4y} = \frac{y-2x}{4y-x}\end{aligned}$$

(b) Vertical tangent lines have undefined slope, meaning the denominator of the derivative of the curve must equal zero.
$4y - x = 0 \;\Rightarrow\; x = 4y$
Plug into curve:

$$\begin{aligned}7 &= (4y)^2 - (4y)y + 2y^2\\ 7 &= 16y^2 - 4y^2 + 2y^2\\ 7 &= 14y^2\\ y^2 &= \frac{1}{2}\\ y &= \pm\sqrt{\frac{1}{2}}\end{aligned}$$

(c)

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y-2x}{4y-x}\right)&\\ = \frac{(y-2x)'(4y-x) - (4y-x)'(y-2x)}{(4y-x)^2}&\\ = \boxed{\frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2}}&\end{aligned}$$

Find y' at $(1,2)$:

$$y'\Big|_{(1,2)} = \frac{2-2(1)}{4(2)-1} = \frac{0}{7} = 0$$

Plug in y' and $(1,2)$ to y'':

$$y''\Big|_{(1,2)} = \frac{(0-2)(4(2)-1) - (4(0)-1)((2)-2(1))}{(4(2)-1)^2} = \frac{(-2)(7)-(-1)(0)}{(7)^2} = \boxed{-\frac{14}{49}}$$

Therefore, y' is $\boxed{\text{concave down}}$ at $(1,2)$ because $y'' < 0$