AP Calculus AB
AP Classroom: 5.5 – 5.6

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All final solutions are $\boxed{\textit{boxed}}$

Only FRQs are included (for now).

List of Questions

Question 1

Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of k and p will f be continuous and differentiable at $x = 1$?

(b) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

Solution:

(a) Definition of continuity of f at $x = 1$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) \tag{1}$$

$$\lim_{x \to 1^+} f(x) = 1^2 + 1k + p \tag{2}$$

$$\lim_{x \to 1^-} = f(1) = 2(1) - (1)^2 = 2 - 1 = 1 \tag{3}$$

$$\textit{Therefore, } 1 = 1 + k + p \implies \boxed{-k = p} \tag{4}$$

Differentiate f with respect to x (needed for next step):

$$f'(x) = \begin{cases} \frac{d}{dx}(2x - x^2) & \text{for } x \le 1, \\ \frac{d}{dx}(x^2 + kx + p) & \text{for } x > 1. \end{cases} \tag{5}$$

$$f'(x) = \begin{cases} 2 - 2x & \text{for } x \le 1, \\ 2x + k + 0 & \text{for } x > 1. \end{cases} \tag{6}$$

Definition of differentiability of f at $x = 1$:

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x) = f'(1) \tag{7}$$

$$\lim_{x \to 1^-} f'(x) = f'(1) = 2 - 2(1) = 0 \tag{8}$$

$$\lim_{x \to 1^+} f'(x) = 2(1) + k \tag{9}$$

$$\textit{Therefore, } 2 + k = 0 \implies \boxed{-2 = k} \tag{10}$$

Conclusion: Using Equations 10 and 4:

$$-2 = k \quad - k = p \tag{11}$$

$$-(-2) = p \implies p = 2 \tag{12}$$

$$\boxed{\textit{Therefore, } k = -2 \text{ and } p = 2} \tag{13}$$

(b) $f'(x) > 0$ when f is increasing.

$$f'(x) = \begin{cases} 2-2x & \text{for } x \le 1, \\ 2x-2 & \text{for } x > 1. \end{cases}$$

On $x \le 1$, set $f'(x) = 0$:

$$2 - 2x = 0 \implies 2 = 2x \implies 1 = x$$

Sign chart for $2 - 2x$: (Only applies to $f'(x)$ on $x > 1$)

	$x < 1$	$x = 1$	$x > 1$
$2-2x$	$+$	1 (●)	$-$

On $x > 1$, set $f'(x) = 0$:

$$2x - 2 = 0 \implies 2x = 2 \implies x = 1$$

Sign chart for $2x - 2$: (Only applies to $f'(x)$ on $x > 1$)

	$x < 1$	$x = 1$	$x > 1$
$2x-2$	$-$	1 (●)	$+$

Combined sign chart for $f'(x)$:

	$x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	1 (●)	$+$

Conclusion:

Therefore, $f(x)$ is increasing when $x \neq 1$

because $f'(x)$ is positive on those regions. (At $x = 1, f'(x) = 0$.)

f is continuous at $x = 1$, so $\boxed{f(x) \text{ is increasing on } (-\infty, \infty).}$

(c) Inflection points are where f'' changes signs.

$$f''(x) = \begin{cases} \frac{d}{dx}(2-2x) = -2 & \text{for } x \le 1, \\ \frac{d}{dx}(2x-2) = 2 & \text{for } x > 1. \end{cases}$$

Sign chart of $f''(x)$:

	$x < 1$	$x = 1$	$x > 1$
$f(x)$	$-$	1 (○)	$+$

Therefore, $f'(x)$ has an inflection point at $x = 1$.

$$f(1) = 1 \qquad \boxed{\text{Inflection Point: } (1, 1)}$$

Question 2

Consider the curve given by $x^2 - xy + 2y^2 = 7$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{4y-x}$

(b) Determine the y-coordinate of each point on the curve at which the line tangent to the curve at that point is vertical. Justify your answer.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x, y, and $\frac{dy}{dx}$. The line tangent to the curve at the point $(1, 2)$ is horizontal. Determine whether the curve is concave up or concave down at the point $(1, 2)$.

Solution:

(a)

$$\frac{d}{dx}(x^2 - xy + 2y^2) = 7 \tag{14}$$
$$2x - (xy' + y) + 4yy' = 0 \tag{15}$$
$$2x - xyy' - y + 4yy' = 0 \tag{16}$$
$$2x - y = y'(x - 4y) \tag{17}$$
$$\frac{2x - y}{x - 4y} = y' = \frac{dy}{dx} = \frac{y - 2x}{4y - x} \tag{18}$$

(b) Vertical tangent lines have undefined slope.
Therefore, $\frac{dy}{dx}$ will be undefined $\iff$ the tangent line at the point is vertical.
($\iff$ means 'if and only if')

$$4y - x = 0 \text{ makes } \frac{dy}{dx} \text{ undefined.} \tag{19}$$
$$4y = x \tag{20}$$

So,

$$7 = (4y)^2 - (4y)y + 2y^2 \tag{21}$$
$$7 = 16y^2 - 4y^2 + 2y^2 \tag{22}$$
$$7 = 14y^2 \tag{23}$$
$$y^2 = \frac{1}{2} \tag{24}$$
$$y = \boxed{\pm\frac{1}{\sqrt{2}}} \tag{25}$$

(c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y-2x}{4y-x}\right) \tag{26}$$
$$= \frac{\left({}^{d}\!/\!{}_{dx}\,(y-2x)\right)(4y-x) - \left({}^{d}\!/\!{}_{dx}\,(4y-x)\right)(y-2x)}{(4y-x)^2} \tag{27}$$
$$= \frac{(y'-2)(4y-x) - (4y'-1)(y-2x)}{(4y-x)^2} \tag{28}$$

$y' = 0$ at $(1,2)$ since the tangent to the curve is vertical.

$$\left.\frac{d^2y}{dx^2}\right|_{(1,2)} = \frac{(-2)(4(2)-1)-(-1)(2-2(1))}{(4(2)-1)^2} = \frac{-2(7)-0}{7^2} = -\frac{2}{7} \tag{29}$$

$\boxed{\text{Since } \frac{d^2y}{dx^2} \text{ is negative at } (1,2)\text{, the curve is concave down at } (1,2).}$

Question 3

Particle P moves along the y-axis so that its position at time t is given by $y(t) = 4t-\frac{2}{3}$ for all times t. A second particle, particle, moves along the-axis so that its position at time t is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

(a) As time t approaches 2, what is the limit of the position of particle Q? Show the work that leads to your answer.

(b) Show that the velocity of particle Q is given by $v_Q(t) = \frac{2\pi\cos(\pi t)-\pi t\cos(\pi t)+\sin(\pi t)}{(2-t)^2}$ for all times.

(c) Find the rate of change of the distance between particle P and particle Q at time $t = \frac{1}{2}$. Show the work that leads to your answer.

Solution:

(a)

$$\lim_{t\to 2}(\sin(\pi t)) = \sin(2\pi) = 0 \tag{30}$$
$$\lim_{t\to 2}(2-t) = 0 \tag{31}$$

Since $\frac{\lim_{t\to 2}(\sin(\pi t))}{\lim_{t\to 2}(2-t)}$ approaches an indeterminate form (${}^{0}\!/\!{}_{0}$), L'Hôpital's rule can be applied.

$$\lim_{t\to 2}\left({}^{d}\!/\!_{dt}\left(\sin\left(\pi t\right)\right)\right) = \lim_{t\to 2}\left(\pi\cos\left(\pi t\right)\right) = \pi\cos(2\pi) = \pi \tag{32}$$

$$\lim_{t\to 2}\left({}^{d}\!/\!_{dt}\left(2-t\right)\right) = \lim_{t\to 2}\left(-1\right) = -1 \tag{33}$$

$$\therefore \lim_{t\to 2}(x(t)) = \frac{\pi}{-1} = \boxed{-\pi} \tag{34}$$

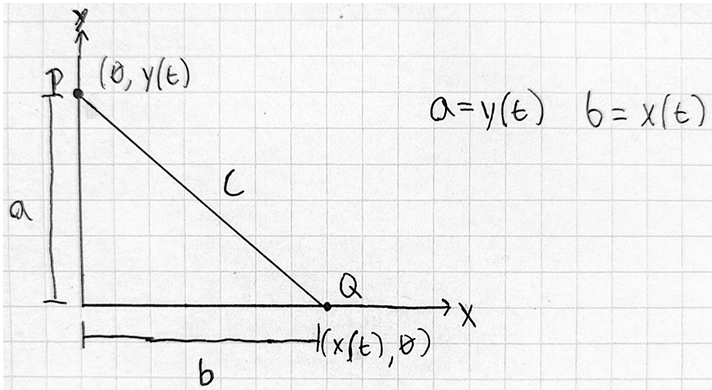
(b)

$$\frac{d}{dt}\left(x(t)\right) = v_Q(t) \tag{35}$$

$$\frac{d}{dt}\left(\frac{\sin(\pi t)}{2-t}\right) = \frac{({}^{d}\!/\!_{dt}(\sin(\pi t)))(2-t) - ({}^{d}\!/\!_{dt}(2-t))(\sin(\pi t))}{(2-t)^2} \tag{36}$$

$$= \frac{\pi\cos(\pi t)(2-t) - (-1)(\sin(\pi t))}{(2-t)^2} \tag{37}$$

$$= \boxed{\frac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2}} \tag{38}$$

(c) 

$$c^2 = a^2 + b^2 \tag{39}$$

$$c = \sqrt{a^2 + b^2} \tag{40}$$

$$\tag{41}$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} \tag{42}$$

$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c} \tag{43}$$

$$\uparrow \text{Main rate equation} \uparrow \tag{44}$$

$$\tag{45}$$

$$a = y(t) = 4t - \frac{2}{3} \tag{46}$$

$$\frac{da}{dt} = 4 \tag{47}$$

$$\tag{48}$$

$$b = x(t) = \frac{\sin(\pi t)}{2-t} \tag{49}$$

$$\frac{db}{dt} = \frac{(\pi\cos(\pi t))(2-t) - (-1)(sin(\pi t))}{(2-t)^2} \tag{50}$$

Now, find all of those variables when $t = \frac{1}{2}$

$$a\big|_{t=1/2} = 4(1/2) - 2/3 = 2 - 2/3 = 4/3 \tag{51}$$

$$b\big|_{t=1/2} = \frac{\sin(\pi/2)}{2 - 1/2} = \frac{1}{3/2} = \frac{2}{3} \tag{52}$$

$$\left.\frac{da}{dt}\right|_{t=1/2} = 4 \tag{53}$$

$$\left.\frac{db}{dt}\right|_{1/2} = \frac{(\pi\cos(\pi/2))(2-1/2) + (sin(\pi/2))}{(2-1/2)^2} = \frac{4(\pi(0)(3/2)+1)}{9} = \frac{4}{9} \tag{54}$$

$$c\big|_{t=1/2} = \sqrt{a^2+b^2} = \sqrt{(4/3)^2 + (2/3)^2} = \sqrt{20/9} = \sqrt{20}/3 \tag{55}$$

Finally, plug all of those into our main rate equation:

$$\left.\frac{dc}{dt}\right|_{t=1/2} = \boxed{\frac{(4/3)(4) + (2/3)(4/9)}{\sqrt{20}/3}}$$

Question 4

Let f be the function given by $f(x) = 2xe^{2x}$.
Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

Solution: All absolute minimums must also be relative minima.
At Relative minima of f, the derivative, f' changes from negative to positive. Find $f'(x)$

$$f'(x) = 2(1(e^{2x}) + x(2e^{2x})) \tag{56}$$

$$= 2e^{2x} + 2xe^{2x} \tag{57}$$

$$= 2e^{2x}(1+x) \tag{58}$$

Set $f'(x) = 0$ and solve:

$$2e^{2x}(1+x)$$

$$2e^{2x} = 0 \;\mid\; (1+x) = 0 \tag{59}$$

$$\ln(e^2x) = 0 \;\mid\; x = -1 \tag{60}$$

$$\text{no solution} \;\mid \tag{61}$$

Sign chart of $f'(x)$:

$$\begin{array}{c|c|c} f'(x) & - & + \\ \hline & -1 & \end{array}$$

Find end behavior of $f(x)$:

$$\lim_{x\to\infty}\left(2xe^{2x}\right) = \lim_{x\to\infty}\left(e^{2x}\right) \quad = \infty \quad \lim_{x\to-\infty}\left(2xe^{2x}\right) = \lim_{x\to-\infty}\left(e^{2x}\right) \quad = 0 \qquad (62)$$

Conclusion:
Since $x = -1$ is the only relative minima and is less than the $\lim_{x\to\pm\infty}(f(x))$, it is the absolute minimum of $f(x)$.

$$f(-1) = 2(-1)e^{2(1)} = -2e^2$$

The absolute minimum of $f(x)$ is $\boxed{f(-1) = -2e^2}$

Question 14

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

t (hours)	0	5	15	30	35
$A(t)$ (gallons)	10	18	25	16	8

The number of gallons of olive oil in a tank at time t is given by the twice-differentiable function A, where t is measured in hours and $0 \le t \le 35$. Values of $A(t)$ at selected times t are given in the table above.

(a) Use the data in the table to estimate the rate at which the number of gallons of olive oil in the tank is changing at time $t = 10$ hours. Show the computations that lead to your answer. Indicate units of measure.

(b) For $0 \le t \le 30$, is there a time t at which $A'(t) = \frac{1}{5}$? Justify your answer.

(c) The number of gallons of olive oil in the tank at time t is also modeled by the function G defined by $G(t) = 5t - \frac{2}{3}(t+9)^{\frac{3}{2}} + 28$, where t is measured in hours and $0 \le t \le 35$. Based on the model, at what time t, for $0 \le t \le 35$, is the number of gallons of olive oil in the tank an absolute maximum? Justify your answer.

Solution:

(a)

$$A'(t) \approx \frac{A(30) - A(15)}{30 - 15} = \frac{16 - 25}{30 - 15} = -\frac{9}{15} = \boxed{-\frac{3}{5}}$$

(b) On $0 \leq t \leq 30$, the AROC can be found as follows:

$$\frac{A(30) - A(0)}{30 - 0} = \frac{16 - 10}{30} = \frac{6}{30} = \frac{1}{5}$$

Since A is twice-differentiable $\implies$ A is differentiable $\implies$ A is continuous. Therefore, by MVT, there is a value c in $(0, 30)$ where $A'(c) = \frac{A(30)-A(0)}{30-0} = \frac{1}{5}$.

(c)

$$\begin{aligned} G'(t) &= 5 - \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)(t+9)^{(3/2-1)}(1) \qquad (63) \\ &= 5 - (t+9)^{1/2} \qquad (64) \\ &= 5 - \sqrt{t+9} \qquad (65) \end{aligned}$$

Set $G'(t) = 0$:

$$\begin{aligned} 5 - \sqrt{t+9} &= 0 \qquad (66) \\ 5 &= \sqrt{t+9} \qquad (67) \\ 25 &= t + 9 \qquad (68) \\ 16 &= t \qquad (69) \\ & \qquad (70) \end{aligned}$$

Sign Chart:

$G'(t)$ $+$ | $-$

16

$t = 16$ must be the absolute maximum since it is the only relative max. ($G(t)$ decreases continually on $16 \leq t \leq 35$)

(d) $G''(t) = -(t+9)^{1/2}(1) = \frac{-1}{\sqrt{t+9}}$
Set $G''(t) = 0$:

$$\frac{-1}{\sqrt{t+9}} = 0 \implies \text{No Solution}$$

$G''(t) < 0$ on $0 \leq t \leq 35$.
$\implies$ $G'(t)$ is concave down on $0 \leq t \leq 35$.

Therefore, $G(7)$ will provide an overestimate of $G(8)$ if using a locally linear model. ($G(t)$ is increasing at a decreasing rate, so if we assume a constant rate, it will provide an overestimate.)