Diagnosing angular (directional) spectra by Tikhonov regularization (ridge regression, tapered least squares) in numerical studies of wave scattering (wave-topography interaction)

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Abstract

Realistic numerical simulations of wave propagation (both surface and internal) involve complex patterns of interference that conceals energy transfers. The elementary wave components can be deduced from angular spectrum. Here, the energy distribution is found by solving an inverse model based on plane wave dynamics. The solution method is a least square minimization augmented with penalizing term (Tikhonov regularization). This aids stability in presence of noise and provides better spectral resolution for under-sampled signal. Further, applicability of the method for energy estimates is tested against analytical solution of wave scattering by bottom topography. In the first problem a periodic shallow gravity wave attacks a circular island. The inverse model correctly identifies energy lobes and their amplitude, hence, allowing to examine scattering amplitude in more complex environments. In the second test, a low mode internal wave reflects from a continental slope represented as step topography. Estimated bulk reflection coefficient compares excellent with the analytical solution. But in case of three dimensional topography the method as well provides information on spatial variation of point-wise coefficient due to inhomogeneity of the prescribed incident field. The results lend confidence in robustness of the inverse technique, hence, suggesting its applicability to field of the real ocean.

1 Introduction

The oceanic waves interfere and create obscure character (change?) in currents and pressure. The staggered pattern makes difficult to answer the simplest questions about waves: from where are they coming and how strong they are. Hence, there will always be an uncertainty in estimate of energy transfers and budgets. In an example of a standing wave, no energy transfer is taken place, yet two waves are present but oppositely traveling. The problem of diagnosing elementary components inevitably emerges in studies of wave-topography interaction. Energy is transfered into scattered wave field that undoubtedly has large number of components. This might produce spatially (horizontally) tight beams of energy transport. Such phenomena was observed in numerical simulations of tsunami waves (Tang et al. (2012)) and internal waves of tidal frequency (internal tides) (Simmons et al. (2004), Arbic et al. (2010)(?)) (and in observations (Zhao et al. (2016))). To present a method to deduce dynamics of a complex wave field such as aforementioned ones is a chief reason for this note (chapter, paper). And it is application to problems of tsunami wave scattering by submarine island and reflection of low-mode internal tide.

A straightforward method is to obtain spatial frequencies of elementary waves (Barber (1963)) by means of two dimensional Fourier transform. It further can be mapped onto a circle leading to angular spectra. In the field studies this approach was extensively used in relation to surface wave phenomena, e.g. propagation (W. H. Munk et al. (1963), reflection (Dickson et al. (1995), Thomson et al. (2005), island diffraction (Pawka (1983)). The spectral description was also applied to tsunami trapping (Romano et al. (2013)) and internal tide propagation (Hendry (1977), Lozovatsky et al. (2003)). Estimation of spectra is a problem that one encounters and here is dealt by inverse modeling (Long (1986)). As discrete Fourier transform leads to ambiguity in elementary wave direction, additional inferences are necessary. Even though application of the Hilbert transform can resolve the issue (Mercier et al. (2008)) [1], it does not utilize all wave field characteristics available. Logically, constrains of wave dynamics are expected to lessen Gibbs phenomena, increase directional resolution and provide a way to super-resolution (Kay & Marple (1981), Sacchi et al. (1998)) so essential (viable, relevant, crucial) for resolving short-crested waves.

The inverse approach has to comprise dynamical statements of the wave field. So that angular spectrum will be related to (translated into) observed pressure and currents simultaneously. But such postulated problem leads to overdetermined and ill-posed system of equations, (least square) solution is unstable in relation to small changes or errors in data. For example, errors in numerical simulations might arise due to numerical and physical friction (reference?) leading to nonuniform wave amplitudes over sampling window of the numerical ocean. Hence, regularization is generally necessary to stabilize and improve inverse estimate (W. Munk et al. (2009), Snieder & Trampert (1999)). In studies of wind wave spectra this is a well known approach when penalizing terms are added to cost functions. These terms can comprise constrains on spectrum shape and smoothness (Long & Hasselmann (1979), Herbers & Guza (1990)), spectrum sparseness (Hashimoto & Kobune (1989), Sacchi & Ulrych (1996)[2], or statistical properties (Benoit et al. (1997)). Unfortunately, the mentioned methods cannot be utilized in studies of wave-topography interaction. The scattered field is "phase-locked" with incident one and thence, phase must be retained in the model equations (e.g. Thomson et al. (2005)) leading to deterministic wave decomposition. This is in contrary to wind waves of open sea that are mainly stochastic.

The simplest regularization technique, nevertheless powerful one seeks an inverse estimate of minimum variance. Initially developed by Tikhonov et al. (2013) for solution of

[1] (Will it work with many components? In presence of corrugated surfaces?)
[2] this is geophysics reference

Fredholm integral equations, Tikhonov regularization has a wide application starting from theoretical work in acoustics (Colton & Kirsch (1996)) and holography (Williams (2001)) to observational oceanography (W. Munk et al. (2009)) and geophysics (Snieder & Trampert (1999)) where it is named ”tapered” or “damped” least squares. But it has never been used [3] to oceanic wave problems. In the first section the inverse model will be formulated in order to find angular spectrum of waves. The developed approach than (Section 3) is illustrated on simple synthetic experiments to show its performance characteristics. And in later sections (4 and 5) is applied to numerical experiments of wave scattering where comparison with the well-known analytical solution is taken place. In section 6 concluding remarks are made and drawbacks are outlined.
[4]

2 Regularization technique of obtaining angular spectrum

2.1 Mathematical preliminary

Let pressure or sea level in complicated seas to be described by angular spectrum[5],

$$p(\vec{r},t) = \int_0^{2\pi} d\theta_k S(\theta_k) e^{i\vec{k}(\theta_k)\cdot\vec{r} + \phi(\theta_k) - i\omega t} \tag{1}$$

Here each elementary (monochromatic) sine wave of wavenumber k travels in direction θ with energy $S(\theta)^2 d\theta$ and temporal (spatial) lag of $\phi(\theta)$. Through out the note temporal dependence is omitted and complex notation is inferred. It is sought to estimate the angular spectrum $S(\theta)$ and phase distribution $\phi(\theta)$ from a finite set of observations. At first, this problem is reformulated in terms of polar[6] Fourier coefficients (Benoit et al. (1997), Rafaely (2004)) by Jacobi-Anger expansion [7],

$$p(r,\theta) = e^{i\vec{k}(\theta)\cdot\vec{r}} = \sum_{m=-\infty}^{m=\infty} i^m J_m(kr) e^{im(\theta-\theta_k)} \tag{2}$$

shows that a field at point (r, θ) produced by plane wave can be expanded in series of Bessel functions and circular functions. Than its substitution into (1) and reorganization lead to

$$p(r,\theta) = \sum_{m=-\infty}^{m=\infty} \Big[\int_0^{2\pi} d\theta_k S(\theta_k) e^{i\phi(\theta_k)} e^{-im\theta_k} \Big] i^m J_m(kr) e^{im\theta} \tag{3}$$

Term in brackets (square brackets) represent convolution integrals defining Fourier coefficients of order m, $A_m - iB_m$. Thence, series (3) state a model equation to find the unknown

[3] Dushaw used similar inverse
[4] Should I add plane-wave fit? “subtraction” method of Jody (and Zygmunt)?
[5] This is not spectrum per se, is term usage still possible?
[6] might not be correct term usage
[7] (should I cite anything?)

coefficients from the known, measured pressure field. Its spatial distribution can be sampled at a set of points (r_i, θ_i) and infinite series are truncated at order N both leading to matrix formulation[8],

$$p_i = \sum_{m=-N}^{m=N} J_m(kr_i)e^{im(\theta+\pi/2)}(A_m - iB_m) \tag{4}$$

Real and imaginary parts constitute two separate problems allowing deterministic definition of the spectrum. Angular spectrum and phase distribution will be recomposed as usual by

$$S(\theta_k)e^{i\phi(\theta_k)} = \frac{1}{\pi}A_0 + \frac{2}{\pi}\Big[\sum_{m=1}^{m=N} A_m \cos m\theta_k + iB_m \sin m\theta_k\Big]$$

The same steps are repeated but with current velocities instead, starting from (1) transfer functions are inserted invoking plane wave dynamics,

$$\begin{Bmatrix} u(\vec{r},t) \\ v(\vec{r},t) \end{Bmatrix} = \int_0^{2\pi} d\theta_k \frac{k}{\rho_0(\omega^2 - f^2)} \begin{Bmatrix} \omega k \cos\theta_k + if \sin\theta_k \\ \omega k \sin\theta_k - if \cos\theta_k \end{Bmatrix} S(\theta_k)e^{i\vec{k}(\theta_k)\cdot\vec{r}+\phi(\theta_k)} \tag{5}$$

Than dependence of currents on wave bearing causes splitting of Fourier coefficients and asymmetry via Coriolis effect,

$$\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \frac{1}{2}\sum_{m=-N}^{m=N} J_m(kr_i)e^{im(\theta+\pi/2)} \begin{Bmatrix} (\omega-f)A_{m+1} + (\omega+f)A_{m-1} - i[(\omega-f)B_{m+1} + (\omega+f)B_{m-1}] \\ (\omega-f)B_{m+1} - (\omega+f)B_{m-1} + i[(\omega-f)A_{m+1} - (\omega+f)A_{m-1}] \end{Bmatrix} \tag{6}$$

This results points out that to describe velocity field higher circular harmonics have to be employed. Physically, velocity field has higher spatial variability. This is well known idea since spatial differentiation acts as a high-wavenumber filter (e.g. **?**). But in (**??**) additionally, the asymmetry is observed for clockwise and counterclockwise components.[9]

2.2 Inverse technique formulation

Inverse model sets its goal estimation of angular spectrum from observed field characteristics. This relation is posed by (4) and (6) and formally expressed as,

$$y = Kx \tag{7}$$

where y measured pressure and currents, x - Fourier coefficients and K - model coefficients. The statement does not need to be strictly satisfied as the model might not be fully correct such as in presence of trapped waves or presence of errors. Additionally, K does not have to be a square matrix, e.g. it is possible to have less measurements than unknowns. Hence, there will be a residue and measure of solution's success should be imposed, such as in least-square minimization, $\min\{||Kx - y||_2^2\}$ - stating to find a solution with minimum residue in L_2-norm

[8] should i drop infinite series here?
[9] This is bad, but discussion is necessary

sense. And the least-squares solution will be found as Moore-Penrose inverse (**?**). Generally, it will be unstable to small errors in data and hence, produce physically inconsistent results (ill-posedness). Addition of regularization parameter is to solve the issue. As in Tikhonov regularization, an inverse solution should minimize the following cost function (problem),

$$x_\alpha = \arg\min\{||Kx - y||_2^2 + \alpha||x||_2^2\} \tag{8}$$

The second term controls amount of solution's variance with a weight α called the regularization parameter. It can be shown that this parameter represents a low-pass filter over orthonormal vectors of $K^T K$ (Williams (2001)). This becomes clear as solution is explicitly expressed,

$$x_\alpha = (K^T K + \alpha I)^{-1} K^T y \tag{9}$$

So that factor $\frac{\lambda_k^2}{\lambda_k^2+\alpha}$ (λ_k - singular value) is introduced to damp small singular values. Unfortunately, this decreases solution and exerts dependence of x_α on the regularization parameter: for $\alpha = 0$ regular least-square estimate is obtained, while for large α minimizer is close to zero. Its choice cannot be made a priori and has to be data-driven. In field studies this is usually set by a signal-to-noise ratio (W. Munk et al. (2009)), since has to scale noise variance to actual signal's strength. Many techniques exist (Williams (2001)), but there is no silver bullet approach and everything depends on problem of interest. Here straightforward method will be used that is called L-curve (**?**).

As mentioned, in Tikhonov regularization amount of allowed error is competing with solution's norm. The simplest idea is to find an optimal parameter balancing both factors. This is seen as rapid change in behavior of curve associating residue with solution's norm as regularization varies (Figure 1). The two extremes for $\alpha \to 0$ and $\alpha \to \infty$ are observed as straight vertical lines. A sharp corner connects two regions and a point with maximum curvature is proposed to be used as a regularization parameter.

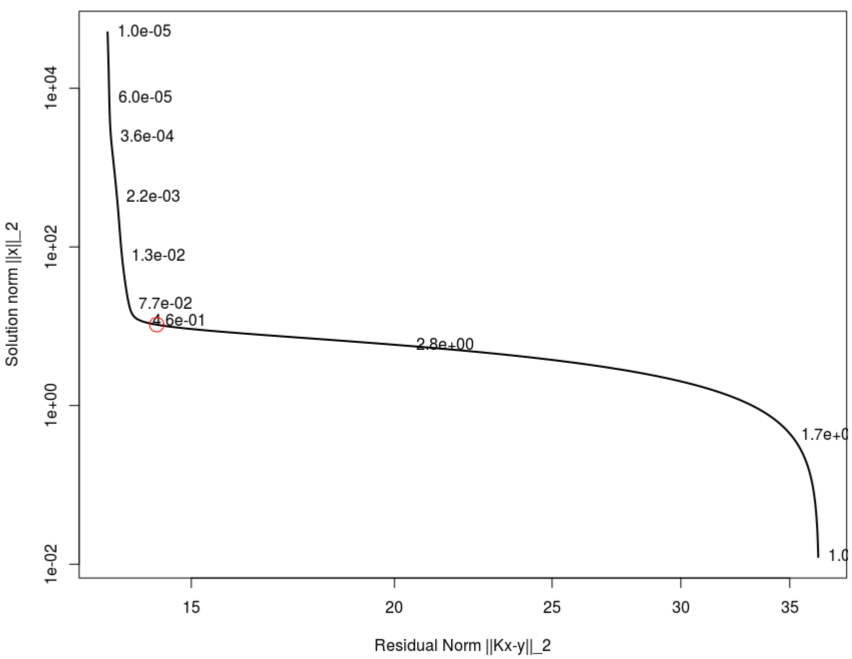

Figure 1: L-curve example