Fluïdummechanica

Behoudsvergelijkingen langs stroomlijnen

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24 november 2016

Inhoud

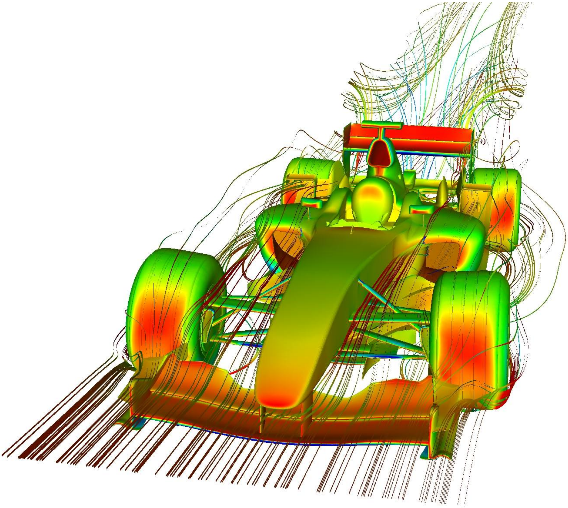
1 Inleiding

2 Bewegingsvergelijking

3 Bernoulli

4 Navier-Stokes

Voorbeeld

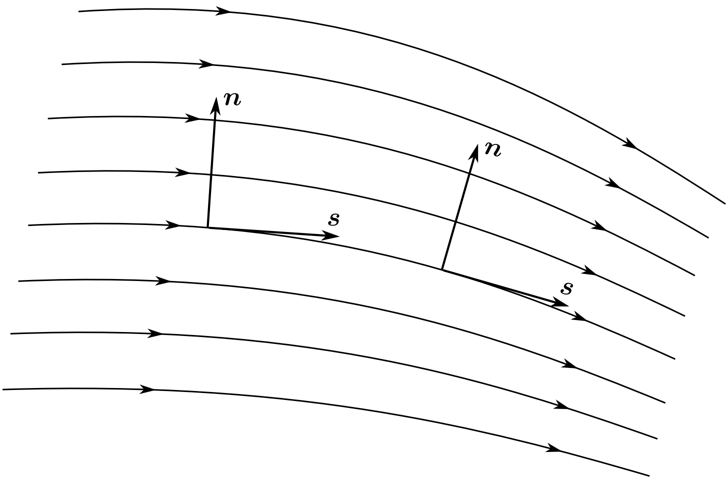


Bron: http://www.dalco.ch/

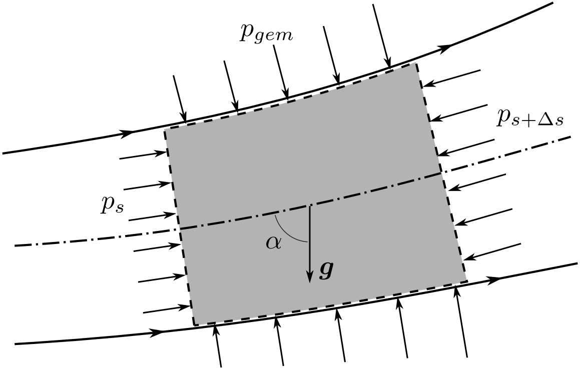
Inhoud

1. Inleiding
2. Bewegingsvergelijking
3. Bernoulli
4. Navier-Stokes

Stroomlijncoordinaten



Bewegingsvergelijking



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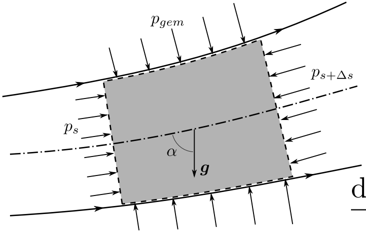

$$\frac{\mathrm{d}\boldsymbol{P}_{CV}}{\mathrm{d}t} + \dot{\boldsymbol{P}}_{\partial CV} = \boldsymbol{F}$$

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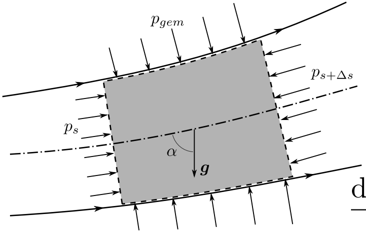

$$\frac{\mathrm{d}\boldsymbol{P}_{CV}}{\mathrm{d}t} + \dot{\boldsymbol{P}}_{\partial CV} = \boldsymbol{F}$$

$$\begin{aligned}\rho v v_\perp A\big|_{s+\Delta s} - \rho v v_\perp A\big|_{s} &= pA\big|_{s} - pA\big|_{s+\Delta s} \\ &\quad + p_{gem}\left(A\big|_{s+\Delta s} - A\big|_{s}\right) - \rho g A_{gem} \Delta s \cos\alpha\end{aligned}$$

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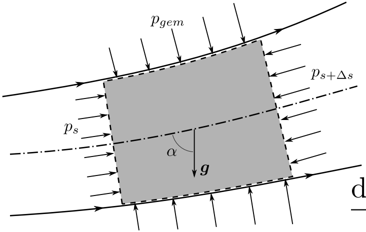

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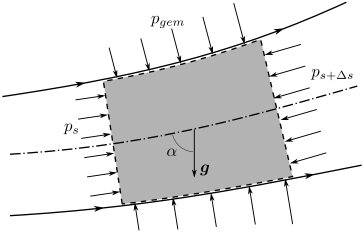

$$\frac{\mathrm{d}\rho v v A}{\mathrm{d}s} = -\frac{\mathrm{d}pA}{\mathrm{d}s} + p\frac{\mathrm{d}A}{\mathrm{d}s} - \rho g A \frac{\mathrm{d}z}{\mathrm{d}s}$$

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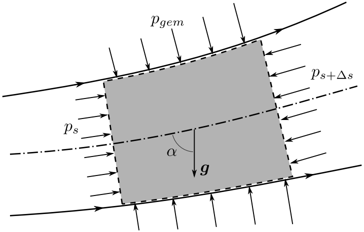

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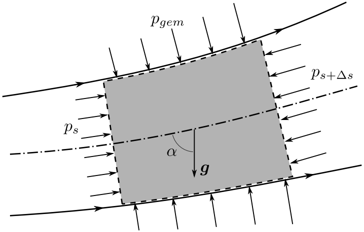

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$$\rho v\frac{\mathrm{d}v}{\mathrm{d}s} + \frac{\mathrm{d}p}{\mathrm{d}s} + \rho g\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \qquad (1)$$

Deeltjesversnelling

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \tag{2}$$

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$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \rho \boldsymbol{g} \tag{3}$$

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Integratie van de bewegingsvergelijking

$$\rho v \frac{\mathrm{d}v}{\mathrm{d}s} + \frac{\mathrm{d}p}{\mathrm{d}s} + \rho g \frac{\mathrm{d}z}{\mathrm{d}s} = 0$$

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$$\int \rho v \frac{\mathrm{d}v}{\mathrm{d}s} \mathrm{d}s + \int \frac{\mathrm{d}p}{\mathrm{d}s} \mathrm{d}s + \int \rho g \frac{\mathrm{d}z}{\mathrm{d}s} \mathrm{d}s = \mathrm{Cst}$$

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$$\Downarrow \quad \rho = \mathrm{Cst}$$

$$\frac{1}{2}\rho v^2 + p + \rho g z = \mathrm{Cst} \tag{4}$$

Bernoulli

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- Stationaire stroming

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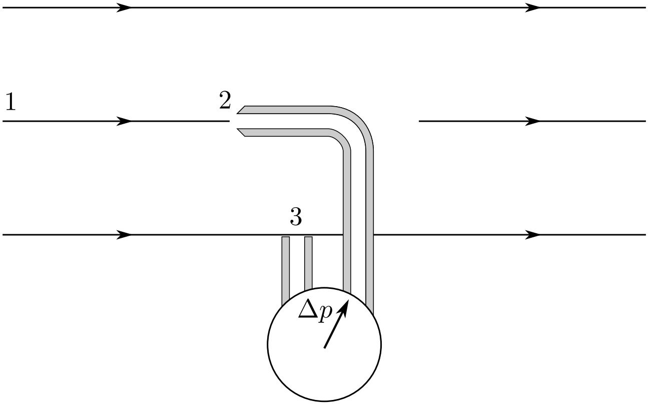

Toepassing

Bron: http://www.edwardsflighttest.com/

Mechanische arbeid van een deeltje

$$\rho v \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\mathrm{d}p}{\mathrm{d}s} - \rho g \frac{\mathrm{d}z}{\mathrm{d}s}$$

$$W = \int_1^2 F \mathrm{d}s$$

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$$\rho \frac{1}{2}(v_2^2 - v_1^2) = -(p_2 - p_1) - \rho g (z_2 - z_1)$$

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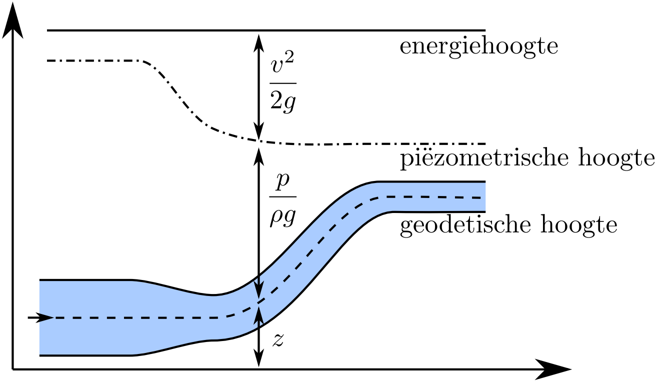
$$\rho \frac{1}{2}(v_2^2 - v_1^2) + (p_2 - p_1) + \rho g (z_2 - z_1) = 0 \tag{5}$$

Grafische voorstelling



Energiebeschouwingen en irreversibiliteit

$$\dot{m}(u_u + \frac{p_u}{\rho_u} + \frac{1}{2}v_u^2 + gz_u) - \dot{m}(u_i + \frac{p_i}{\rho_i} + \frac{1}{2}v_i^2 + gz_i) = \dot{Q} - \dot{W}_a$$

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$$\rho(u_u - u_i) + (p_u - p_i) + \frac{1}{2}\rho(v_u^2 - v_i^2) + \rho g(z_u - z_i) = 0$$

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vs

$$(p_2 - p_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(z_2 - z_1) = 0$$

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Navier-Stokes vergelijking

- Newtoniaanse vloeistof
- Niet-samendrukbare stroming

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$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{v} \tag{6}$$

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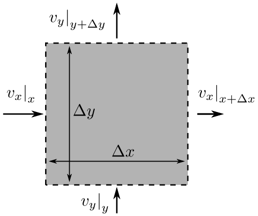
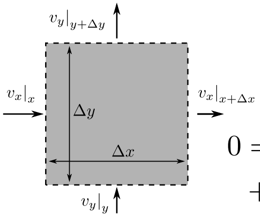
$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = \\ - \frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \end{aligned} \tag{7}$$

Continuiteitsvergelijking

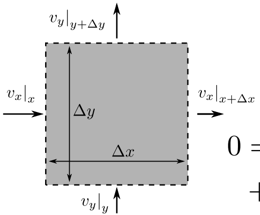


Continuiteitsvergelijking



$$
\begin{aligned}
0 = {} & (\rho v_x|_x - \rho v_x|_{x+\Delta x})\Delta y \Delta z \\
& + (\rho v_y|_y - \rho v_y|_{y+\Delta y})\Delta x \Delta z \\
& + (\rho v_z|_z - \rho v_z|_{z+\Delta z})\Delta x \Delta y
\end{aligned}
$$

Continuiteitsvergelijking



$$
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$$
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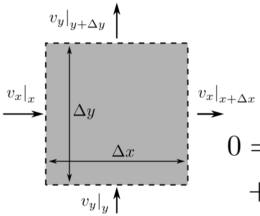
$$

Continuiteitsvergelijking



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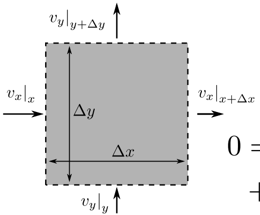

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$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0$$

$$\boldsymbol{\nabla} \boldsymbol{v} = 0 \tag{8}$$