Introduction to Topology (Mendelson): Solutions to Select Exercises

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1 Indexed Families of Sets

Lemma 1.1. *Exercise 1*

Proof. Given $\{A_\alpha\}_{\alpha \in I}$ where $\forall_{\alpha \in I} A_\alpha \subset S$ or equivalently $\forall_{\alpha \in I} A_\alpha \in \mathcal{P}(S)$ and $\{B_\alpha\}_{\alpha \in I}$ where $\forall_{\alpha \in I} B_\alpha \subset S$ or equivalently $\forall_{\alpha \in I} B_\alpha \in \mathcal{P}(S)$.

(c)

$$x \in \bigcup_{\alpha \in I} (A_\alpha \cup B_\alpha) \Rightarrow \exists_{\gamma \in I} x \in A_\gamma \cup B_\gamma \Rightarrow x \in A_\gamma \vee x \in B_\gamma$$

and therefore since $A_\gamma \subset \bigcup_{\alpha \in I} A_\alpha$ and $B_\gamma \subset \bigcup_{\alpha \in I} B_\alpha$, we have that

$$x \in A_\gamma \vee x \in B_\gamma \Rightarrow x \in \bigcup_{\alpha \in I} A_\alpha \vee x \in \bigcup_{\alpha \in I} B_\alpha$$

$$\Rightarrow x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in I} B_\alpha).$$

and thus

$$\bigcup_{\alpha \in I} (A_\alpha \cup B_\alpha) \subset (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in I} B_\alpha).$$

$\square$

2 Metric Spaces

Lemma 2.1. *Exercise 3*

Proof. First, given $x, a \in \mathbb{R}$ where $x = \{x_1, x_2, ..., x_n\}$ and $a = \{a_1, a_2, ..., a_n\}$, let's define our metrics,

$$d(x,a) = \max_{1 \le i \le n} \{d_i(x_i, a_i)\} = \max_{1 \le i \le n} \{|x_i - a_i|\},$$

$$d'(x,a) = \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2},$$

$$d''(x,a) = \sum_{i=1}^{n} |x_i - a_i|.$$

Then, let's say that $|x_j - a_j| = d(x,a) = \max_{1 \le i \le n}\{|x_i - a_i|\}$ for some $j \in [1, n]$. Then we can say that

$$|x_j - a_j| \le |x_j - a_j| + \sqrt{\sum_{i=1, i \ne j}^{n} (x_i - a_i)^2} = \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2},$$

since $\sqrt{\sum_{i=1, i \ne j}^{n} (x_i - a_i)^2} > 0$. Therefore, we can say that

$$\max_{1 \le i \le n} \{|x_i - a_i|\} \le \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2}$$

and since $|x_j - a_j| \ge |x_i - a_i|, \forall_{i \in [1,n]} i \ne j$, then we can say that

$$\sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \le \sqrt{\sum_{i=1}^{n} (x_j - a_j)^2} = \sqrt{n(x_j - a_j)^2} = \sqrt{n}(x_j - a_j) \le \sqrt{n}|x_j - a_j|.$$

Putting it all together we have that

$$\max_{1 \le i \le n} \{|x_i - a_i|\} \le \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \le \sqrt{n} \max_{1 \le i \le n} \{|x_i - a_i|\},$$

or

$$d(x, a) \le d'(x, a) \le \sqrt{n}\ d(x, a).$$

This proved the first inequality. Now we need to prove the second. Assuming everything from part 1, we can say that

$$|x_j - a_j| \le |x_j - a_j| + \sum_{i=1, i \ne j}^{n} |x_i - a_i| = \sum_{i=1}^{n} |x_i - a_i|.$$

Therefore, we can say that

$$\max_{1 \le i \le n} \{|x_i - a_i|\} \le \sum_{i=1}^{n} |x_i - a_i|$$

and since we already stated that we know $|x_j - a_j| \ge |x_i - a_i|, \forall_{i \in [1,n]} i \ne j$ by the definition of the max function, then we can say that

$$\sum_{i=1}^{n} |x_i - a_i| \le \sum_{i=1}^{n} |x_j - a_j| = n|x_j - a_j|.$$

Putting it all together we have that

$$\max_{1 \le i \le n} \{|x_i - a_i|\} \le \sum_{i=1}^{n} |x_i - a_i| \le n \max_{1 \le i \le n} \{|x_i - a_i|\},$$

or

$$d(x, a) \le d'(x, a) \le n \cdot d(x, a).$$

□

Lemma 2.2. *Exercise 4*

Proof. We have that for $f, g \in X$,

$$d(f, g) = \int_a^b |f(t) - g(t)|dt.$$

In order to prove that (X, d) is a metric space, we will have to prove all four properties of a metric for d, starting with the first property.

$$\int_a^b |f(t) - g(t)|dt \geq |\int_a^b f(t) - g(t)dt| \geq 0$$

and therefore

$$d(f, g) \geq 0.$$

Now we need to prove the second property. First, assume that $d(f, g) = 0$. This means that

$$\int_a^b |f(t) - g(t)|dt = 0 \Rightarrow \frac{\partial}{\partial b}[\int_a^b |f(t) - g(t)|dt] = \frac{\partial}{\partial b}[0] \Rightarrow$$

$$f(b) - g(b) = 0 \Rightarrow f = g.$$

Now, let's assume that $f = g$. Let's call $c = f = g$. Therefore, we have that

$$\int_a^b |f(t) - g(t)|dt = \int_a^b |c(t) - c(t)|dt = \int_a^b 0dt = 0.$$

Therefore, we have proved that $d(f, g) = 0 \iff f = g$. Now we need to prove the third property. We have that

$$\int_a^b |f(t)-g(t)|dt = \int_a^b |-(-f(t)+g(t))|dt = \int_a^b |-f(t)+g(t)|dt = \int_a^b |g(t)-f(t)|dt.$$

Therefore, we have proved that $d(f, g) = d(g, f)$. Lastly, we'll prove the last property. Let $h \in X$, so we have that

$$d(f,h) = \int_a^b |f(t)-h(t)|dt = \int_a^b |f(t)-g(t)+g(t)-h(t)|dt \leq \int_a^b (|f(t)-g(t)|+|g(t)-h(t)|)dt$$

$$= \int_a^b |f(t)-g(t)|dt + \int_a^b |g(t)-h(t)|dt = d(f,g) + d(g,h).$$

Therefore, we have that

$$d(f,h) \leq d(f,g) + d(g,h).$$

Since $d(f,g)$ satisfies all conditions to be a metric for X, then (X,d) is a metric space. □

Lemma 2.3. *Exercise 7*

Proof. For $x, y \in X$, we have that

$$d(x,x) = 0,$$

$$d(x,y) = 1,$$

if $x \neq y$. In order to prove (X,d) is a metric space, we have to prove that d satisfies all of the conditions of a metric.

(1) $d(x,y) = 1 \geq 0$,

(2) $d(x,y) = 0$ only if $x = y$ by the given definition and thus $d(x,x) = 0$,

(3) $d(x,y) = 1$ and $d(y,x) = 1$, therefore $d(x,y) = d(y,x)$,

(4) Let $z \in X$ where $z \neq x$ and $z \neq y$, therefore $d(x,z) \leq d(x,y) + d(y,z) \Rightarrow 1 \leq 1+1 = 2$.

Therefore, (X,d) is a metric space. □

3 Continuity

Lemma 3.1. *Exercise 1*

Proof. We have that for $f, g \in X$

$$d^*(f,g) = \int_a^b |f(t) - g(t)|dt,$$

$$d(f,g) = |f(t) - g(t)|.$$

Given the function $I : (X, d^*) \to (\mathbb{R}, d)$ where

$$I(f) = \int_a^b f(t)dt.$$

Given $\epsilon > 0$, choose $\delta = \epsilon$. Suppose $d^*(f,g) < \delta$, or

$$\int_a^b |f(t) - g(t)|dt < \delta,$$

check that $d(I(f), I(g)) < \epsilon$.

$$d(I(f), I(g)) = |\int_a^b f(t)dt - \int_a^b g(t)dt| = |\int_a^b (f(t) - g(t))dt| \leq \int_a^b |f(t) - g(t)|dt < \delta = \epsilon.$$

Therefore, I is continuous. □

Lemma 3.2. *Exercise 3*

Proof. We have that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x_1, x_2) = x_1 + x_2$, where the metric on $\mathbb{R}^n$ is

$$\sqrt{\sum_{i=1}^{n} (x_i - a_i)^2}$$

and for $n = 2$ is

$$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}.$$

The metric on $\mathbb{R}$ is

$$|x - a|$$

Given $\epsilon > 0$, choose $\delta = \sqrt{\epsilon}$. Suppose

$$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta \Rightarrow (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2,$$

check that

$$|f(x) - f(a)| < \epsilon.$$

Let $x = \{x_1, x_2\}$ and $a = \{a_1, a_2\}$ where $x, a \in \mathbb{R}^2$. Then

$$|f(x) - f(a)| = |f(x_1, x_2) - f(a_1, a_2)| = |x_1 + x_2 - (a_1 + a_2)|$$

$$= |x_1 + x_2 - a_1 - a_2| = |x_1 - a_1 + x_2 - a_2| \leq |x_1 - a_1| + |x_2 - a_2|$$

$$\leq (|x_1 - a_1|)^2 + (|x_2 - a_2|)^2 = (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2 = \epsilon$$

Therefore, f is continuous. $\square$

4 Open Balls and Neighborhoods

Lemma 4.1. *Exercise 2*

Proof. Given $a \in \mathbb{R}$ with function $f : \mathbb{R} \to \mathbb{R}$ defined by $\forall_{x \leq a} f(x) = 0$ and $\forall_{x > a} f(x) = 1$. First, we'll prove that f is not continuous at a.

Assume that f is continuous at a. Therefore, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(B(a; \delta)) \subset B(f(a); \epsilon).$$

Since $B(a; \delta)$ will be the set of all $x \in \mathbb{R}$ such that $d(a, x) < \delta$, that means that $B(a; \delta)$ will consist of some elements that are less than or equal to a and some elements that are greater than a. Therefore, $f(B(a;\delta)) = \{0,1\}$, since for any $x \in B(a;\delta)$, $f(x)$ will be 0 if $x \leq a$ and $f(x)$ will be 1 if $x > a$. Therefore, we have that

$$\{0,1\} \subset B(f(a);\epsilon)$$

and since $f(a) = 0$ since $\forall_{x \leq a} f(x) = 0$ and $a = a$, we have that

$$\{0,1\} \subset B(0;\epsilon),$$

where $B(0;\epsilon)$ is the set of all elements that are less than ϵ and thus $B(0;\epsilon) \subset [0,\epsilon)$. Therefore, if $\epsilon \leq 1$, we have that

$$\{0,1\} \subset B(0;\epsilon),$$

but since $1 \notin [0,1)$, then

$$\{0,1\} \not\subset B(0;\epsilon) \Rightarrow f(B(a;\delta)) \not\subset B(f(a);\epsilon)$$

and thus we have a contradiction. Therefore, f is not continuous at a.

Now, we will first prove that f is continuous for all real numbers less than a, or the interval $(-\infty, a)$. Given $\epsilon > 0$, let $\delta > 0$. Therefore, we have that for any $x \in (-\infty, a)$, $f(B(x;\delta)) = \{0\}$ since $B(x;\delta)$ will be the set of all real numbers that are less than a and the metric is less than δ, and $\forall_{x \in \mathbb{R}} f(x) = 0$ where $x \leq a$ by the definition of f. We also have that $B(f(x);\epsilon) = \{\forall_{x \in \mathbb{R}} x : x < \epsilon\}$, since for any $x \in (-\infty, a)$, $x < a$ and thus $f(x) = 0$ and the open ball with center point 0 will just be a set of all the numbers less than ϵ. By default, $\{0\} \subset B(0;\epsilon)$ and thus we have that

$$f(B(x;\delta)) \subset B(f(x);\epsilon).$$

Therefore, f is continuous for all real numbers in the interval $(-\infty, a)$. The same logic can be applied to the interval (a, ∞) in order to prove f is continuous on that interval. $\square$

Lemma 4.2. *Exercise 6*

Proof. We are given neighborhoods N_a, N_b for points $a, b \in X$ respectively, therefore we have that $\exists \delta > 0, \theta > 0$ such that

$$B(a;\delta) \subset N_a,$$

$$B(b;\theta) \subset N_b.$$

Therefore, $B(a;\delta) \cap B(b;\theta) = \emptyset$ implies that there does not exist a point that is in both open balls. Therefore, since the balls do not overlap (intersect), the distance from a to b; $d(a, b)$ is greater than $\delta + \theta$. First, assume that a point $x \in X$ exists in both open balls, i.e.

$$x \in B(a;\delta) \wedge x \in B(b;\theta),$$

$$\Rightarrow d(a, x) < \delta \wedge d(b, x) < \theta.$$

Then we can say that

$$d(a, b) \leq d(a, x) + d(x, b) = d(a, x) + d(b, x) < \delta + \theta,$$

therefore we have that $d(a, b) < \delta + \theta$. However, this is a contradiction, since we already have $d(a, b) > \delta + \theta$. Therefore, there does not exists a point in X that is in both open balls, i.e.

$$B(a;\delta) \cap B(b;\theta) = \emptyset$$

and thus

$$N_a \cap N_b = \emptyset.$$

Another proof: Let $\delta < d(a,b) - \theta$ and $\theta > 0$. Let

$$x \in B(a;\delta) \wedge y \in B(b;\theta).$$

Therefore,

$$d(a,y) \leq d(a,b) + d(b,y) < d(a,b) + \theta,$$

in order for the two balls to have a non-empty intersection, $d(a,y) < \delta$ since $y \in B(b;\theta)$. However, since we have shown that

$$d(a,y) < d(a,b) + \theta,$$

$d(a,y) < \delta$ means that

$$d(a,y) < d(a,b) + \theta < \delta < d(a,b) - \theta$$

since $\delta < d(a,b) - \theta$. However, it cannot be true that $d(a,b) + \theta < d(a,b) - \theta$ since $\theta > 0$. Therefore, $d(a,y) \not< \delta$ and therefore

$$B(a;\delta) \cap B(b;\theta) = \emptyset$$

and thus

$$N_a \cap N_b = \emptyset.$$

□

Lemma 4.3. *Exercise 8*

Proof. To Do. □

5 Limits

Lemma 5.1. *Exercise 1*

Proof. Given that we have the metric space converted by

$$X = \prod_{i=1}^{k} X_i,$$

this implies that the metric for X is

$$d(a_n, c) = \max_{1 \leq i \leq k} \{d_i(a_i^n, c_i)\},$$

where $a_n = \{a_1^n, a_2^n, ..., a_k^n\}$ and $c = \{c_1, c_2, ..., c_k\}$ and $a_n, c \in X$. Assume that

$$\lim_n a_n = c,$$

this implies that for $N \in \mathbb{Z}$, $d(a_n, c) < \epsilon$ for $n > N$ and $\epsilon > 0$. Since $d(a_n, c) < \epsilon$, this implies that $\forall_{i \in [1,k]} d_i(a_i^n, c_i) < \epsilon$. Therefore, $\forall_{i \in [1,k]} \lim_n a_i^n = c_i$. Now assume that $\forall_{i \in [1,k]} \lim_n a_i^n = c_i$. This implies that $\forall_{i \in [1,k]} d_i(a_i^n, c_i) < \epsilon$ for $n > N$ and $\epsilon > 0$. Therefore $d(a_n, c) < \epsilon$ for all $n > N$. Therefore, by the definition of a limit, $\lim_n a_n = c$. □

Lemma 5.2. *Exercise 3*

Proof. Let $c = \{c_1, c_2, ..., c_k\}$ be a convergent sequence that converges to m. Let $\beta = \{c_a, c_{a+1}, ...c_b\}$ be a subset of c where $a, b \in [1, k]$ and $a < b$. Since c converges to m, this means that given $\epsilon > 0$ and $N \in \mathbb{Z}$ where $n > N$, $d(m, c_n) < \epsilon$.

Finish □

Lemma 5.3. *Exercise 6*

Proof. We have from corollary 5.9 that for a metric space (X, d) and a subset $A \subset X$ and $a \in X$, then

$$\lim_n d(a, a_n) = d(a, A).$$

Therefore, for $x, y \in X$ we have that $d(x, A) = \lim_n d(x, a_n)$ and $d(y, A) = \lim_n d(y, a_n)$. We can say that

$$d(x, a_n) \leq d(x, y) + d(y, a_n) \Rightarrow \lim_n d(x, a_n) \leq \lim_n [d(x, y) + d(y, a_n)]$$

$$= \lim_n d(x, y) + \lim_n d(y, a_n) = d(x, y) + \lim_n d(y, a_n)$$

and thus we have that

$$\lim_n d(x, a_n) \leq \lim_n d(x, y) + \lim_n d(y, a_n) = d(x, y) + \lim_n d(y, a_n),$$

which is

$$d(x, A) \leq d(x, y) + d(y, A).$$

□

6 Open Sets and Closed Sets

Lemma 6.1. *Exercise 1*

Proof. Since we have the metric space

$$X = \prod_{i=1}^{n} X_i$$

for all of the metric spaces $\{X_i\}_{i \in [1,n]}$, we know that the metric on X is

$$d(x, a) = \max_{1 \leq i \leq n} \{d_i(x_i, a_i)\},$$

where $x = \{x_1, x_2, ..., x_n\}$ and $a = \{a_1, a_2, ..., a_n\}$. We are given that $\forall_{i \in [0,n]} O_i$ is an open subset of X_i. In order to prove that $\prod_{i=1}^{n} O_i$ is an open subset of X, we will use induction.

Base case: n=1:

$$\prod_{i=1}^{1} O_i = O_1,$$

where O_1 is open since $\forall_{i \in [0,n]} O_i$ is an open subset of X_i.

Inductive Hypothesis:

$$\prod_{i=1}^{k} O_i$$

is open for some k.

Inductive Step:

$$\prod_{i=1}^{k+1} O_i = O_{k+1} \times \prod_{i=1}^{k} O_i.$$

We have that O_{k+1} is open by definition and $\prod_{i=1}^{k} O_i$ is open by our IH. Now we need to prove that their Cartesian product is open.

For $a = (a_1, a_2)$, let

$$a \in O_{k+1} \times \prod_{i=1}^{k} O_i,$$

therefore $a_1 \in O_{k+1}$ and $a_2 \in \prod_{i=1}^{k} O_i$. Since O_{k+1} and $\prod_{i=1}^{k} O_i$ are both open,

$$B(a_1; \delta_1) \subset O_{k+1},$$

and

$$B(a_2; \delta_2) \subset \prod_{i=1}^{k} O_i.$$

Therefore,

$$B(a_1; \delta_1) \times B(a_2; \delta_2) \subset O_{k+1} \times \prod_{i=1}^{k} O_i.$$

We also have that

$$B((a_1,a_2);\min(\delta_1,\delta_2)) \subset B(a_1;\delta_1) \times B(a_2;\delta_2).$$

Therefore,

$$B((a_1,a_2);\min(\delta_1,\delta_2)) \subset O_{k+1} \times \prod_{i=1}^{k} O_i,$$

and thus,

$$\forall_{a\in O_{k+1}\times\prod_{i=1}^{k} O_i} B(a;\delta_3) \subset O_{k+1} \times \prod_{i=1}^{k} O_i.$$

Therefore, $O_{k+1} \times \prod_{i=1}^{k} O_i$ is open and thus

$$\prod_{i=1}^{k+1} O_i$$

is open, which makes $\prod_{i=1}^{n} O_i$ open. □

Lemma 6.2. *Exercise 2*

Proof. □

Lemma 6.3. *Exercise 5*

Proof. Given that A is a closed, non-empty subset of the real numbers and A has a lower bound, A has a greatest lower bound. Lets say $c^* = g.l.b$ of A where $c^* \in \mathbb{R}$. Since c^* is the greatest lower bound of A, then by Corollary 5.7, there is a sequence of real numbers, $a_1, a_2, ..., a_n, ...$ such that $a_n \in A$ for each n and $\lim_n a_n = c^*$. Since A is closed, by Theorem 6.8, we have that for the sequence $a_1, a_2, ..., a_n, ...$ where $\lim_n a_n = c^*$, since the sequence converges to c^* we have that $c^* \in A$. Therefore, A contains its $g.l.b$. □

7 Topological Spaces

Lemma 7.1. *Exercise 2*

Proof. ☐

Lemma 7.2. *Exercise 4*

Proof. (X, τ) is a topological space. By definition, $X, \emptyset \in \tau$. $C(\emptyset) = X \in \tau$ therefore $C(\emptyset) \in \tau$ and thus $\emptyset$ is closed. But by definition, $\emptyset$ is open, therefore it is clopen. Same logic can be applied to X to show that it is clopen.

Let $\{S_i\}_{i\in I}$ for some index I where $\forall_{i\in I} C(S_i) \in \tau$. We want to prove that

$$C(\bigcup_{i=1}^{I} S_i) \in \tau.$$

$$C(\bigcup_{i=1}^{I} S_i) = \bigcap_{i=1}^{I} C(S_i),$$

where the compliment of a closed set is open, and the intersection of a finite number of open sets is open. Therefore,

$$\bigcap_{i=1}^{I} C(S_i) = C(\bigcup_{i=1}^{I} S_i) \in \tau.$$

Let $\{S_i\}_{i\in I}$ for some index I where $\forall_{i\in I} C(S_i) \in \tau$. We want to prove that

$$C(\bigcap_{i\in I} S_i) \in \tau.$$

$$C(\bigcap_{i\in I} S_i) = \bigcup_{i\in I} C(S_i),$$

where the compliment of a closed set is open, and the union of a finite number of open sets is open. Therefore,

$$\bigcup_{i \in I} C(S_i) = C(\bigcap_{i \in I} S_i) \in \tau.$$

$\square$

Lemma 7.3. *Exercise 6*

Proof. Let (X, τ) be a discrete topological space. Therefore, $\tau = 2^X$. Let S be a subset of X, therefore since $S \in 2^X$, $S \in \tau$. Therefore, S is open. Since $C(S)$ is another subset of X and thus $C(S) \in 2^X$, then $C(S) \in \tau$. Therefore, $C(S)$ is open. Therefore, S is clopen. $\square$

8 Closure, Interior, Boundary

Lemma 8.1. *Exercise 1*

Proof. First, we need to prove that $Int(A) \cap Bdry(A) = \emptyset$. $Int(A) = C(\overline{C(A)})$ and $Bdry(A) = \overline{A} \cap \overline{C(A)}$, therefore we need to prove that

$$C(\overline{C(A)}) \cap (\overline{A} \cap \overline{C(A)}) = \emptyset.$$

$$x \in C(\overline{C(A)}) \cap (\overline{A} \cap \overline{C(A)}),$$

therefore

$$x \in C(\overline{C(A)}) \wedge x \in \overline{A} \wedge x \in \overline{C(A)}.$$

Therefore,

$$x \notin \overline{C(A)} \wedge x \in \overline{A} \wedge x \in \overline{C(A)}.$$

Since we have that $x \notin \overline{C(A)}$ and $x \in \overline{C(A)}$, this means that x is an element in $C(Int(A))$ and $Int(A)$. This is a contradiction and therefore there is no intersection and thus

$$C(\overline{C(A)}) \cap (\overline{A} \cap \overline{C(A)}) = \emptyset.$$

Now we need to prove that $Bdry(A) \cap Int(C(A)) = \emptyset$. $Int(C(A)) = C(\overline{A})$ and $Bdry(A) = \overline{A} \cap \overline{C(A)}$. Therefore, we need to prove that

$$C(\overline{A}) \cap (\overline{A} \cap \overline{C(A)}) = \emptyset.$$

$$x \in C(\overline{A}) \cap (\overline{A} \cap \overline{C(A)}),$$

therefore

$$x \in C(\overline{A}) \wedge x \in \overline{A} \wedge x \in \overline{C(A)}.$$

Therefore,

$$x \notin \overline{A} \wedge x \in \overline{A} \wedge x \in \overline{C(A)}.$$

Since $x \notin \overline{A}$ and $x \in \overline{A}$, this is a contradiction and therefore there is no intersection and thus

$$C(\overline{A}) \cap (\overline{A} \cap \overline{C(A)}) = \emptyset.$$

Lastly we need to prove that $Int(A) \cap Int(C(A)) = \emptyset$. $Int(C(A)) = C(\overline{A})$ and $Int(A) = C(\overline{C(A)})$. Therefore, we need to prove that

$$C(\overline{A}) \cap C(\overline{C(A)}) = \emptyset.$$

$$x \in C(\overline{A}) \cap C(\overline{C(A)}),$$

therefore

$$x \in C(\overline{A}) \wedge x \in C(\overline{C(A)}),$$

therefore

$$x \notin \overline{A} \wedge x \notin \overline{C(A)}.$$

Therefore

$$x \in Int(C(A)) \wedge x \in Int(A)$$

which is a contradiction. Therefore,

$$C(\overline{A}) \cap C(\overline{C(A)}) = \emptyset.$$

Therefore, the three sets are mutually disjoint.

Now, we have to prove that

$$Int(A) \cup Bdry(A) \cup Int(C(A)) = X.$$

Written alternatively,

$$C(\overline{C(A)}) \cup (\overline{A} \cap \overline{C(A)}) \cup C(\overline{A}) = X.$$

We can rearrange the left side and get

$$C(\overline{C(A)}) \cup C(\overline{A}) \cup (\overline{A} \cap \overline{C(A)}) = X,$$

where

$$C(\overline{C(A)} \cap \overline{A}) \cup (\overline{A} \cap \overline{C(A)}) = X,$$

which is true since the union of a set and its compliment make up the whole set space, in this case, the topological space X. $\square$

Lemma 8.2. *Exercise 2*

Proof. (a) First, assume $x \in \overline{A}$. Therefore, by definition, for each neighborhood N of x, $N \cap A \neq \emptyset$. Therefore, each neighborhood of x contains a point of A. Therefore, by lemma 4.1, $d(x, A) = 0$. Conversely, assume that $d(x, A) = 0$ for some point x. Therefore, every neighborhood N or x contains a point from A. This can also be expressed as $N \cap A \neq \emptyset$. This is the definition of the closure of A and thus $x \in \overline{A}$. Therefore, $x \in \overline{A} \iff d(x, A) = 0$.

(b) ☐

Lemma 8.3. *Exercise 5*

Proof. ☐

Lemma 8.4. *Exercise 13*

Proof. First, assume A is closed. Therefore $A = \overline{A}$. We need to prove that

$$Bdry(A) \subset A \Rightarrow \overline{A} \cap \overline{C(A)} \subset A,$$

therefore

$$x \in \overline{A} \cap \overline{C(A)}$$

implies that

$$x \in \overline{A} \wedge x \in \overline{C(A)}.$$

Since we have that $A = \overline{A}$, we can say that

$$x \in A \wedge x \in \overline{C(A)}.$$

Therefore $x \in A$ and thus

$$\overline{A} \cap \overline{C(A)} \subset A.$$

Conversely, assume that

$$Bdry(A) \subset A \Rightarrow \overline{A} \cap \overline{C(A)} \subset A.$$

Then by definition $A = \overline{A}$ since an element x is in both according to the above equation. Therefore A is closed.

First, assume that A is closed. Then, we need to prove that $\square$

9 Functions, Continuity, Homeomorphism

Lemma 9.1. *Exercise 1*

Proof. We have the function $f : (X, 2^X) \to (Y, \tau')$. Let O be any arbitrary open subset of Y, so $O \in \tau'$. Therefore, $f^{-1}(O) \subset X$ and thus $f^{-1}(O) \in \mathcal{P}(X) = 2^X$ by the definition of a discrete topology and a powerset. Therefore $f^{-1}(O)$ is an open set of X. Therefore, by theorem 5.3, f is always continuous for any topology τ' on Y.

We have the function $f : (X, \tau) \to \{Y, (\emptyset, Y\})$. Considering our first open subset of Y, Y, we have that $f^{-1}(Y) = X$ by our function. Since $X \in \tau$ by the definition of a topology, $f^{-1}(Y)$ is open in X. Now considering our second open subset of Y, $\emptyset$. We have that $f^{-1}(\emptyset) = \emptyset$ by default. Since $\emptyset \in \tau$ by the definition of a topology, $f^{-1}(\emptyset)$ is open in X. Therefore, f is always continuous. $\square$

Lemma 9.2. *Exercise 3*

Proof.

$$f : (X, \tau) \to (Y, \tau'),$$

$$h : (Y, \tau') \to (Z, \tau'').$$

Since f is homeomorphic, $f : X \to Y$ and inverse $g : Y \to X$ are continuous. First, assume h is continuous. Let f be continuous at $a \in X$ and h be continuous at $f(a) \in Y$ by the definition of continuous functions. By theorem 5.6, hf is continuous at a and since a is arbitrary, hf is continuous. Now assume that hf is continuous. Therefore, hf is continuous at $a \in X$. By theorem 5.6, f

is continuous at $f(a)$ and therefore h is continuous. Therefore h is continuous $\iff$ hf is continuous. $\square$

10 Subspaces

Lemma 10.1. *Exercise 1*

Proof. Y is a subspace of X and Z is a subspace of Y. Therefore, $Y \subset X$ and $Z \subset Y$, therefore $Z \subset X$. Let $\tau'' = \{\forall_{O \in \tau} O \cap Z\}$, where τ is the topology on X. Therefore, the topological space (Z, τ'') is a subspace of $(X, \tau) by proposition 6.2$. $\square$

Lemma 10.2. *Exercise 2*

Proof. Given (X, τ), $O \in 2^X$ and $O \in \tau$. Let $A \subset O$. Assume that A is relatively open. Therefore, for some set A' that is open in X, $A = A' \cap O$. Since $A' \in \tau$ and $O \in \tau$, then by the definition of a topology their intersection is in τ and thus $A \in \tau$. Now assume that A is an open subset of X. Therefore, $A \in \tau$. For some subset of X, O, let (O, τ'), where $A \cap O \in \tau'$. Therefore, by proposition 6.2 A is relatively open in O. $\square$

Lemma 10.3. *Exercise 4*

Proof. Hausdroff space (X, τ) with $x, y \in X$:

$$A \in \tau : x \in A,$$

$$B \in \tau : y \in B,$$

$$A \cap B = \emptyset.$$

Let (Y, τ') be a subspace of the Hausdroff space (X, τ) with distinct points $a, b \in Y$ with $a \in A \cap Y$ and $b \in B \cap Y$ where $A \cap Y, B \cap Y \in 2^Y$. Therefore,

$Y \subset X$ and by the definition of a subspace and proposition 6.2, $A \cap Y \in \tau'$ and $B \cap Y \in \tau'$, since $A, B \in \tau$. Therefore,

$$(A \cap Y) \cap (B \cap Y) = (A \cap B) \cap (Y \cap Y) = (A \cap B) \cap Y.$$

Since X is a Hausdroff space and $A, B \in \tau$, $A \cap B = \emptyset$. Thus

$$(A \cap Y) \cap (B \cap Y) = (A \cap B) \cap Y = \emptyset \cap Y = \emptyset.$$

Which shows that (Y, τ') is a Hausdroff space. Thus, any subspace of a Hausdroff space is a Hausdroff space. $\square$