ТЕСТ 1

№1 $f: \mathbb{R}^n \to \mathbb{R}$ $\nabla f = \left(\frac{\partial f}{\partial x_i}\right)_{i=\overline{1,n}} \in \mathbb{R}^n$ — Gradient

$f: \mathbb{R}^n \to \mathbb{R}^m$ $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ — Jacobian $\in \mathbb{R}^{n\times m}$ 30

$\nabla^2 f = H_f = f'' = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ — Hessian $\in \mathbb{R}^{n\times n}$ $f: \mathbb{R}^n \to \mathbb{R}$

Гессиан — это якобиан градиента.

№2 $f(x) = -e^{-x^T x}$

$df = -e^{-x^T x} \cdot d(-x^T x) = e^{-x^T x} \cdot 2\langle x, dx\rangle \Longrightarrow \boxed{\nabla f(x) = 2e^{-x^T x} \cdot x}$ 20

$g = 2e^{-x^T x} \cdot x^T dx_1$

$dg = 2d\left(e^{-x^T x} \cdot x^T dx_1\right) = 2d\left(e^{-x^T x}\right) x^T dx_1 + 2e^{-x^T x} d(x^T dx_1) = 2 \cdot e^{-x^T x} d(-x^T x) x^T dx_1 + 2e^{-x^T x} dx_2^T dx_1 =$

$= 2e^{-x^T x}\left(-x^T dx_1 2x^T dx_2 + dx_1^T dx_2\right) = 2e^{-x^T x} \cdot \langle dx_1 - 2x x^T dx_1, dx_2\rangle = 2e^{-x^T x}\langle (I - 2xx^T) dx_1, dx_2\rangle$

$\boxed{\nabla^2 f = 2e^{-x^T x} \cdot (I - 2xx^T)}$ 20

№3 $f(x) = \operatorname{tr} AX = \langle A^T, X\rangle$

$df = d(\langle A^T, X\rangle) = \langle A^T, dX\rangle \Rightarrow \boxed{\nabla f = A^T}$ 30

$\langle A, B\rangle = \operatorname{tr} A^T B =$
$= \operatorname{tr} B^T A =$
$= \operatorname{tr}(AB^T) =$
$= \operatorname{tr}(BA^T)$

ТЕСТ 2 | **№1** S - convex set iff $\forall x_1, x_2 \in S$, $\forall \theta \in [0,1] \to \theta x_1 + (1-\theta) x_2 \in S$

возможны другие определения. 30

№2 $S = \{x \in \mathbb{R}^n \mid \|x - x_c\| \le r\}$ 30

$x, y: \|x - x_c\| \le r$, $\|y - x_c\| \le r$

$z = \theta x + (1-\theta) y \quad \Big| \quad \|\theta x + (1-\theta)y - \theta x_c + (1-\theta)x_c\| \le \theta \underbrace{\|x - x_c\|}_{\le r} + (1-\theta)\underbrace{\|y - x_c\|}_{\le r} \le r$

№3 If S - convex $\to S + S = 2S$ 40

1) $\underbrace{S+S}_{A} \subset \underbrace{2S}_{B}$ $\quad \forall x \in A \to x \in B$ $\quad x = s_1 + s_2$, $\quad s_1, s_2 \in S$

$\frac{x}{2} = \frac{s_1}{2} + \frac{s_2}{2} = s_3 \in S$ (т.к. $\theta = \frac{1}{2}$, $s_1, s_2 \in S$ - выпуклое)

$\Rightarrow x = 2 s_3 \in 2S$

2) $\underbrace{2S}_{A} \subset \underbrace{S+S}_{B}$ $\quad \forall x \in A \longrightarrow x \in B$

$x = 2 s_1$, $s_1 \in S$ $\quad \Big| \; x = \underset{\substack{\cap \\ S}}{s_1} + \underset{\substack{\cap \\ S}}{s_1} \in S + S$

3) Контрпример

$S = [0;1] \cup [2;3]$ $\quad 2S = [0;2] \cup [4;6]$ $\quad S + S = [0;6]$

Teem 3

1. $y = kx,\ k \geq 0$; $y = kx,\ k < 0$ $\left[\begin{matrix} x \geq 0 \\ x \leq 0 \end{matrix}\right.$ или; $2xy+1 \leq 0$

типичный неполный ответ: $\nabla^2 f = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2 e^{xy} \end{pmatrix} \geq 0$ 35

$\Rightarrow 2xy+1 \leq 0$.

2. $f(x) = \log\left(\sum_{i=1}^{n} e^{x_i}\right)$

I: $f(y) \geq f(x) + (\nabla f(x))^T (y-x)$ I

II: $\nabla^2 f \geq 0$

$\nabla f = \dfrac{e^x}{\sum_i e^{x_i}}$; $\dfrac{\partial f}{\partial x_k} = \dfrac{e^{x_k}}{\sum_i e^{x_i}}$

$\nabla^2 f = \dfrac{1}{\sum_i e^{x_i}} \cdot \mathrm{diag}(e^x) - (e^x)(e^x)^T \cdot \dfrac{1}{(\sum_i e^{x_i})^2}$

$df = \dfrac{\langle e^x, dx \rangle}{\sum_i e^{x_i}}$

$\dfrac{\partial^2 f}{\partial x_k \partial x_p} = \dfrac{\partial(e^{x_k})/\partial x_p \cdot (\sum_i e^{x_i}) - e^{x_k} \cdot e^{x_p}}{(\sum_i e^{x_i})^2}$

$= \dfrac{(\sum_i e^{x_i})\,\mathrm{diag}(e^x) - e^x \cdot e^{x^T}}{(\sum_i e^{x_i})^2}$

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3. convexity of $g(x)$: $\nabla^2 g(x) \geq 0$; $\nabla^2 f(x) = \underbrace{\nabla^2 g(x)}_{\geq 0} + 2\lambda I$

$t^T \nabla^2 f t \geq 0 \quad \forall t$:

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$\Rightarrow \nabla^2 f(x) \geq 2\lambda I$, $\boxed{2\lambda = \mu}$

This approach requires 2-differentiable functions BUT IT'S OK.

$\left(\sum_i e^{x_i} t_i^2\right)\left(\sum_i e^{x_i}\right) - \left(\sum e^{x_i} t_i\right)^2 \geq 0$

КБШ: $\left(\sum_i t_i \cdot e^{x_i}\right)^2 \leq \left(\sum_i e^{x_i} t_i^2\right) \cdot \sum_i e^{x_i}$

ТЕСТ 4

1. $S^* = \{p \in \mathbb{R}^n \mid p^T x \geq -1, \forall x \in S\}$, $K = \{x \mid \forall x \in K \to \lambda x \in K, \lambda \geq 0\}$

$\boxed{20}$ $p^T \cdot \lambda x \geq -1 \to p^T x \geq -\frac{1}{\lambda}$ $\lambda \to \infty \to p^T x \geq 0$

2. $\boxed{30}$

3. $S = \{s \in \mathbb{R}^m \mid s = Ax, x \succeq 0\}$, $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ ← it is cone

$S^* = \{p \in \mathbb{R}^m \mid p^T s \geq 0 \ \forall s \in S\} \to p^T A x \geq 0$ $(A^T p)^T x \geq 0$

$\Rightarrow A^T p \succeq 0$

$\Rightarrow S^* = \{y \in \mathbb{R}^m \mid y = A^T p \succeq 0\}$

$\boxed{50}$

ТЕСТ 5 $\boxed{№1}$ $\boxed{30}$

$\boxed{№2}$ $f(x) = \frac{1}{x}$, $x \in R_{++}$; $f^*(y) = \sup\limits_x \underbrace{x^T y - f(x)}_{f(x,y)} = \frac{1}{\sqrt{-y}} \cdot y - \frac{1}{1/\sqrt{-y}} = 0$

$f'_x(x,y) = y + \frac{1}{x^2} = 0 \Rightarrow x = \frac{1}{\sqrt{-y}}$ $\boxed{35}$

$$\boxed{f^*(y) = \begin{cases} 0, & y < 0 \\ +\infty, & y \geq 0 \end{cases}}$$

$\boxed{35}$

$\boxed{№3}$ $f(x) = x^p$, $p > 1$, $x \in R_{++}$

$f^*(y) = \sup\limits_x \underbrace{xy - x^p}_{f(x,y)} = x(y - x^{p-1}) = x\left(y - \frac{1}{p}y\right) = xy\frac{p-1}{p} =$

$f'_x(x,y) = y - px^{p-1} = 0$

$x^{p-1} = \frac{1}{p}y$, $x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

$= y^{\frac{1}{p-1}+1} \cdot \underbrace{p^{-\frac{1}{p-1}} \cdot \frac{p-1}{p}}_{æ} =$

$= y^{\frac{p}{p-1}} \cdot æ$

$\boxed{f^*(y) = æ \cdot y^{\frac{p}{p-1}}}$, $æ = (p-1)\, p^{\frac{-p}{p-1}}$

ТЕСТ 6

№1 Пусть $f_i(x)$ - выпуклые функции на выпуклых множествах S_i

$f(x) = \sum_i a_i f_i(x)$ $\quad a_i > 0$

если $\bigcap_i \operatorname{ri} S_i \neq \varnothing$, то

$S = \bigcap_i S_i$

$$\partial_S f(x) = \sum_i a_i \partial_{S_i} f_i(x)$$

$\boxed{30}$

№2

$I(x_0) = \{ i \in [1, m] : f_i(x_0) = f(x_0) \}$

$$\partial_S f(x_0) = \operatorname{conv}\left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\}$$

$\boxed{30}$

№3 • $f(x_1, x_2) = |x_1 - x_2| = \max(x_1 - x_2, x_2 - x_1)$

$\boxed{40}$

$$\Rightarrow \partial f = \begin{cases} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & x_1 > x_2 \\ \theta \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (1-\theta) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \theta \in [0,1], & x_1 = x_2 \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, & x_1 < x_2 \end{cases}$$

• $f(x_1, x_2) = \max(e^{x_1}, x_2)$

$$\partial f = \begin{cases} \begin{pmatrix} e^{x_1} \\ 0 \end{pmatrix}, & e^{x_1} > x_2 \\ \theta \begin{pmatrix} e^{x_1} \\ 0 \end{pmatrix} + (1-\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \theta \in [0,1], & e^{x_1} = x_2 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x_2 > e^{x_1} \end{cases}$$

ТЕСТ 7) Теорема Каруша Куна Таккера

$$\min_{x \in \mathbb{R}^n} f(x) \quad (МП)$$
$$\text{s.t. } g_i(x) \le 0 \quad i = 1, \dots, m$$
$$h_j(x) = 0 \quad j = 1, \dots, p$$

Пусть x^* – решение (МП).

f, g_i, h_j – дифференцируемые функции

$\Rightarrow \exists \lambda_j^*, \mu_i^*$:

$$\begin{cases} \nabla_x L(x^*, \lambda^*, \mu^*) = 0 \\ \nabla_\lambda L(x^*, \lambda^*, \mu^*) = 0 \\ \mu_i^* \ge 0 \\ \mu_i^* \cdot g_i(x^*) = 0 \\ g_i(x^*) \le 0 \end{cases}$$

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