The file should be sent in the `.pdf` format created via $\LaTeX$ or [typora](https://typora.io) on the mail homework@merkulov.top with the subject `September. Fupm2021. Ivanova`, where `Ivanova` stands for your surname. **Deadline:** 16 October, 23:59.

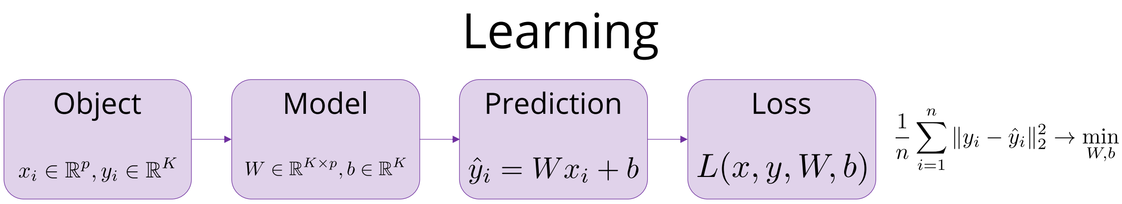
Matrix calculus

1. Find $\nabla f(x)$, if $f(x) = \|Ax\|_2, x \in \mathbb{R}^p \setminus \{0\}$.
2. Find $f''(X)$, if $f(X) = \log \det X$

   Note: here under $f''(X)$ assumes second order approximation of $f(X)$ using Taylor series:

$$
f(X + \Delta X) \approx f(X) + \mathbf{tr}(f'(X)^\top \Delta X) + \frac{1}{2}\mathbf{tr}(\Delta X^\top f''(X)\Delta X)
$$

3. Calculate the Frobenius norm derivative: $\dfrac{\partial}{\partial X}\|X\|_F^2$
4. Calculate the derivatives of the loss function with respect to parameters $\frac{\partial L}{\partial W}, \frac{\partial L}{\partial b}$ for the single object x_i (or, $n = 1$)



Convex sets

1. Prove that if the set is convex, its interior is also convex. Is the opposite true?
2. Prove that the set of square symmetric positive definite matrices is convex.
3. Show that the hyperbolic set of $\{x \in \mathbb{R}^n_+ \mid \prod\limits_{i=1}^n x_i \geq 1\}$ is convex. Hint: For $0 \leq \theta \leq 1$ it is valid, that $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$ with non-negative a, b.
4. Prove, that the set $S \subseteq \mathbb{R}^n$ is convex if and only if $(\alpha + \beta)S = \alpha S + \beta S$ for all non-negative α and β
5. Let $x \in \mathbb{R}$ is a random variable with a given probability distribution of $\mathbb{P}(x = a_i) = p_i$, where $i = 1, \ldots, n$, and $a_1 < \ldots < a_n$. It is said that the probability vector of outcomes of $p \in \mathbb{R}^n$ belongs to the probabilistic simplex, i.e.
   $P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \ldots + p_n = 1, p_i \geq 0\}$. Determine if the following sets of p are convex:
   1. $\mathbb{P}(x > \alpha) \leq \beta$
   2. $\mathbb{E}|x^{201}| \leq \alpha \mathbb{E}|x|$
   3. $\mathbb{E}|x^2| \geq \alpha$
   4. $\mathbb{V}x \geq \alpha$

Projection

1. Find the projection of the matrix X on a set of matrices of rank k, $\quad X \in \mathbb{R}^{m \times n}, k \leq n \leq m$. In Frobenius norm and spectral norm.

2. Prove that projection is a nonexpansive operator, i.e. prove, that if $S \in \mathbb{R}^n$ is nonempty, closed and convex set, then for any $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$

$$
\|\pi_S(x_2) - \pi_S(x_1)\|_2 \leq \|x_2 - x_1\|_2
$$

Convex functions

1. Prove, that function $f(X) = \mathbf{tr}(X^{-1}), X \in S^n_{++}$ is convex, while $g(X) = (\det X)^{1/n}, X \in S^n_{++}$ is concave.
2. Kullback–Leibler divergence between $p, q \in \mathbb{R}^n_{++}$ is:

$$
D(p, q) = \sum_{i=1}^n (p_i \log(p_i/q_i) - p_i + q_i)
$$

   Prove, that $D(p, q) \geq 0 \forall p, q \in \mathbb{R}^n_{++}$ и $D(p, q) = 0 \leftrightarrow p = q$

   Hint:

$$
D(p, q) = f(p) - f(q) - \nabla f(q)^T (p - q), \quad f(p) = \sum_{i=1}^n p_i \log p_i
$$

3. Let x be a real variable with the values $a_1 < a_2 < \ldots < a_n$ with probabilities $\mathbb{P}(x = a_i) = p_i$. Derive the convexity or concavity of the following functions from p on the set of $\left\{p \mid \sum\limits_{i=1}^n p_i = 1, p_i \geq 0\right\}$
   - $\mathbb{E}x$
   - $\mathbb{P}\{x \geq \alpha\}$
   - $\mathbb{P}\{\alpha \leq x \leq \beta\}$
   - $\sum\limits_{i=1}^n p_i \log p_i$
   - $\mathbb{V}x = \mathbb{E}(x - \mathbb{E}x)^2$
   - $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbb{P}\{x \leq \beta\} \geq 0.25\}$
4. Is the function returning the arithmetic mean of vector coordinates is a convex one: : $a(x) = \frac{1}{n}\sum\limits_{i=1}^n x_i$, what about geometric mean: $g(x) = \prod\limits_{i=1}^n (x_i)^{1/n}$?
5. Prove that spectral matrix norm $f(X) = \|X\|_2 = \sup\limits_{y \in \mathbb{R}^n} \dfrac{\|Xy\|_2}{\|y\|_2}$ is convex.
6. Is $f(x) = -x \ln x - (1 - x) \ln(1 - x)$ convex?