Homework. October

✅Send 1 `.pdf` file to the `homework@merkulov.top`.

📅**Deadline:** 23:59, 29th of November

Conjugate sets

1. Find and sketch on the plane a conjugate set to a multi-faceted cone: $S = \mathbf{cone}\{(-3,1),(2,3),(4,5)\}$
2. Find the sets S^*, S^{**}, S^{***}, if
$$S = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0, \quad 2x_1 + x_2 \geq -4, \quad -2x_1 + x_2 \geq -4\}$$
3. Let $\mathbb{A}_n$ be the set of all n dimensional antisymmetric matrices. Show that $\left(\mathbb{A}_n\right)^* = \mathbb{S}_n$.
4. Find the conjugate cone for the exponential cone:
$$K = \{(x, y, z) \mid y > 0, ye^{x/y} \leq z\}$$
5. Find and sketch on the plane a conjugate set to a multifaced cone:
$$S = \mathbf{conv}\left\{(-4,-1),(-2,-1),(-2,1)\right\} + \mathbf{cone}\left\{(1,0),(2,1)\right\}$$
6. Prove, that if we define the conjugate set to S as follows:
$$S^* = \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 1 \quad \forall x \in S\},$$
, then unit ball with the zero point as the center is the only self conjugate set in $\mathbb{R}^n$.
7. Find the conjugate set to the ellipsoid:
$$S = \left\{ x \in \mathbb{R}^n \mid \sum\limits_{i=1}^n a_i^2 x_i^2 \leq \varepsilon^2 \right\}$$

Conjugate function

1. Find $f^*(y)$, if $f(x) = -\dfrac{1}{x}, \quad x \in \mathbb{R}_{++}$
2. Find $f^*(y)$, if $f(x) = -0,5 - \log x, \quad x > 0$
3. Find $f^*(y)$, if $f(x) = \log\left(\sum\limits_{i=1}^n e^{x_i} \right)$
4. Prove, that if $f(x) = \alpha g(x)$, then $f^*(y) = \alpha g^*(y/\alpha)$
5. Find $f^*(Y)$, if $f(X) = -\ln \det X, X \in \mathbb{S}^n_{++}$
6. Prove, that if $f(x) = g(Ax)$, then $f^*(y) = g^*(A^{-\top}y)$

Subgradient and subdifferential

1. Prove, that x_0 - is the minimum point of a convex function $f(x)$ if and only if $0 \in \partial f(x_0)$
2. Find $\partial f(x)$, if $f(x) = \text{ReLU}(x) = \max\{0, x\}$
3. Find $\partial f(x)$, if $f(x) = \|x\|_p$ при $p = 1, 2, \infty$
4. Find $\partial f(x)$, if $f(x) = \|Ax - b\|_1$
5. Find $\partial f(x)$, if $f(x) = e^{\|x\|}$
6. Find $\partial f(x)$, if $f(x) = \max\limits_i \left\{ \langle a_i, x\rangle + b_i \right\}, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}, i = 1, \ldots, m$