Homework. November - December.

✅ Send 1 `.pdf` file to the `homework@merkulov.top`

📅 **Deadline**: 23:59, 20th of December

General optimization problems

1. Give an explicit solution of the following LP.
$$
\begin{split}
&c^\top x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & Ax = b
\end{split}
$$
2. Give an explicit solution of the following LP.
$$
\begin{split}
&c^\top x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & 1^\top x = 1, \\
& x \succeq 0
\end{split}
$$
This problem can be considered as a simplest portfolio optimization problem.
3. Give an explicit solution of the following LP.
$$
\begin{split}
&c^\top x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & 1^\top x = \alpha, \\
& 0 \preceq x \preceq 1,
\end{split}
$$
where α is an integer between 0 and n. What happens if α is not an integer (but satisfies $0 \leq \alpha \leq n$)? What if we change the equality to an inequality $1^\top x \leq \alpha$?
4. Give an explicit solution of the following QP.
$$
\begin{split}
&c^\top x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & x^\top A x \leq 1,
\end{split}
$$
where $A \in \mathbb{S}^n_{++}, c \neq 0$. What is the solution if the problem is not convex $(A \notin \mathbb{S}^n_{++})$ (Hint: consider eigendecomposition of the matrix: $A = Q \mathbf{diag}(\lambda)Q^\top = \sum\limits_{i=1}^n \lambda_i q_i q_i^\top$) and different cases of $\lambda > 0, \lambda = 0, \lambda < 0$?
5. Give an explicit solution of the following QP.
$$
\begin{split}
&c^\top x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & (x - x_c)^\top A (x - x_c) \leq 1,
\end{split}
$$
where $A \in \mathbb{S}^n_{++}, c \neq 0, x_c \in \mathbb{R}^n$.
6. Give an explicit solution of the following QP.
$$
\begin{split}
&x^\top B x \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & x^\top A x \leq 1,
\end{split}
$$
where $A \in \mathbb{S}^n_{++}, B \in \mathbb{S}^n_{+}$.
7. Consider the equality constrained least-squares problem
$$
\begin{split}
&\|Ax - b\|_2^2 \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & Cx = d,
\end{split}
$$
where $A \in \mathbb{R}^{m \times n}$ with $\mathbf{rank} A = n$, and $C \in \mathbb{C}^{k \times n}$ with $\mathbf{rank} C = k$. Give the KKT conditions, and derive expressions for the primal solution x^* and the dual solution λ^*.

8. Derive the KKT conditions for the problem

$$\begin{split}& \mathbf{tr}\ X - \log\det X \to \min_{x \in \mathbb{S}^n_{++}} \\ \text{s.t. } & Xs = y,\end{split}$$

where $y \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ are given with $y^\top s = 1$. Verify that the optimal solution is given by

$$X^* = I + yy^\top - \frac{1}{s^\top s} ss^\top$$

9. **Supporting hyperplane interpretation of KKT conditions**. Consider a **convex** problem with no equality constraints

$$\begin{split}& f_0(x) \to \min_{x \in \mathbb{R}^n} \\ \text{s.t. } & f_i(x) \leq 0, \quad i = [1, m]\end{split}$$

Assume, that $\exists x^* \in \mathbb{R}^n, \mu^* \in \mathbb{R}^m$ satisfy the KKT conditions

$$\begin{split}& \nabla_x L(x^*, \mu^*) = \nabla f_0(x^*) + \sum_{i=1}^m \mu_i^* \nabla f_i(x^*) = 0 \\ & \mu_i^* \geq 0, \quad i = [1, m] \\ & \mu_i^* f_i(x^*) = 0, \quad i = [1, m] \\ & f_i(x^*) \leq 0, \quad i = [1, m]\end{split}$$

Show that

$$\nabla f_0(x^*)^\top (x - x^*) \geq 0$$

for all feasible x. In other words the KKT conditions imply the simple optimality criterion or $\nabla f_0(x^*)$ defines a supporting hyperplane to the feasible set at x^*

Duality

1. **Fenchel + Lagrange = ♥.** Express the dual problem of

$$\begin{split}& c^\top x \to \min_{x \in \mathbb{R}} \\ \text{s.t. } & f(x) \leq 0\end{split}$$

with $c \neq 0$, in terms of the conjugate function f^*. Explain why the problem you give is convex. We do not assume f is convex.

2. **Minimum volume covering ellipsoid.** Let we have the primal problem:

$$\begin{split}& \ln \det X^{-1} \to \min_{X \in \mathbb{S}^n_{++}} \\ \text{s.t. } & a_i^\top X a_i \leq 1, i = 1, \ldots, m\end{split}$$

   1. Find Lagrangian of the primal problem
   2. Find the dual function
   3. Write down the dual problem
   4. Check whether problem holds strong duality or not
   5. Write down the solution of the dual problem

3. **A penalty method for equality constraints.** We consider the problem minimize

$$\begin{split}& f_0(x) \to \min_{x \in \mathbb{R}^n} \\ \text{s.t. } & Ax = b,\end{split}$$

where $f_0(x) : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable, and $\alpha \in \mathbb{R}^{m \times n}$ with $\mathbf{rank} A = m$. In a quadratic penalty method, we form an auxiliary function

$$\phi(x) = f_0(x) + \alpha \|Ax - b\|_2^2,$$

where $\alpha > 0$ is a parameter. This auxiliary function consists of the objective plus the penalty term $\alpha\|Ax - b\|_2^2$. The idea is that a minimizer of the auxiliary function, $\tilde{x}$, should be an approximate solution of the original problem. Intuition suggests that the larger the penalty weight α, the better the approximation $\tilde{x}$ to a solution of the original problem. Suppose $\tilde{x}$ is a minimizer of $\phi(x)$. Show how to find, from $\tilde{x}$, a dual feasible point for the original problem. Find the corresponding lower bound on the optimal value of the original problem.

4. **Analytic centering.** Derive a dual problem for

$$-\sum_{i=1}^{m} \log(b_i - a_i^\top x) \to \min_{x \in \mathbb{R}^n}$$

with domain $\{x | a_i^\top x < b_i, i = [1, m]\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^\top x$. (The solution of this problem is called the analytic center of the linear inequalities $a_i^\top x \leq b_i, i = [1, m]$. Analytic centers have geometric applications, and play an important role in barrier methods.) with domain $\{x | a_i^\top x < b_i, i = [1, m]\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^\top x$. (The solution of this problem is called the analytic center of the linear inequalities $a_i^\top x \leq b_i, i = [1, m]$. Analytic centers have geometric applications, and play an important role in barrier methods.)