Skoltech

Skolkovo Institute of Science and Technology

Experimental Data Processing
MA060238

Tracking of a moving object which trajectory is disturbed by random acceleration

Authors:

Luca Breggion
Irina Yareshko

October 16, 2022

1 Introduction

The aim of this work is to develop standard Kalman filter for tracking a moving object which trajectory is disturbed by random acceleration.

2 Development of Kalman filter algorithm and estimation of state vector X_i (Q1 - Q6)

At the first step, the actual trajectory x_i is determined (**Equation 1**) considering:

- Size of the trajectory 200 *points*
- Initial conditions: $x_1 = 5$, $V_1 = 1$, $T = 1$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2} \tag{1}$$

where V_i is evaluated as:

$$V_i = V_{i-1} + a_{i-1}T \tag{2}$$

and a_i is a random vector from normal distribution with zero mean and given variance $\sigma_a^2 = 0.2^2$. Similarly, to generate the measurements z_i of the process x_i, it is considered a variance of $\sigma_n^2 = 20^2$.

$$z_i = x_i + \eta_i \tag{3}$$

The standard Kalman filter recurrent algorithm is built accordingly to the following procedure: firstly, the prediction is made. At this step the prediction of state vector at time i using $i-1$ measurements is computed by **Equation 4** and the prediction error of covariance matrix is determined by **Equation 5**.

$$X_{i,i-1} = \Phi_{i,i-1}X_{i-1,i-1} \tag{4}$$

$$P_{i,i-1} = \Phi_{i,i-1}P_{i-1,i-1}\Phi_{i,i-1}^T + Q_i \tag{5}$$

The second step is filtration, that is adjustment of predicted estimate. The estimation is improved by incorporating a new measurements $X_{i,i}$ by **Equation 6**, the filter gain K_i by **Equation 7**, and filtration error covariance matrix P **Equation 8**:

$$X_{i,i} = X_{i,i-1} + K_i(z_i - HX_{i,i-1}) \tag{6}$$

$$K_i = P_{i,i-1}H_i^T(H_iP_{i,i-1}H_i^t + R_i)^{-1} \tag{7}$$

$$P_{i,i} = (I - K_iH_i)P_{i,i-1} \tag{8}$$

The initial conditions taken into account for the simulation are:

$$X_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{9}$$

$$P_{0,0}^{II} = \begin{bmatrix} 10000 & 0 \\ 0 & 10000 \end{bmatrix} \tag{10}$$

The result of Kalman filter development are presented in the figures below. The code used for the computation generates random numbers for each run, thus it allows to observe that the estimation results are different with every new trajectory after running the filter several times, as shown in **Figure 1**.

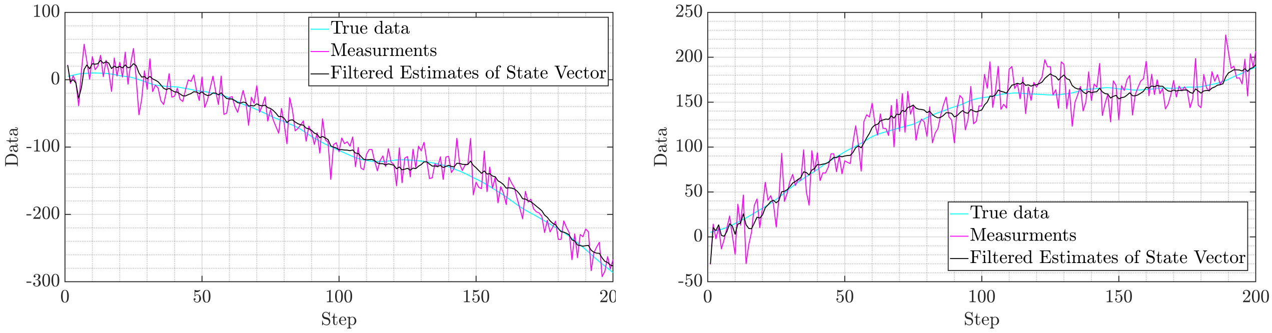

Figure 1: True trajectory, Measurements, Filtered Estimates of State Vector X_i for different trajectories

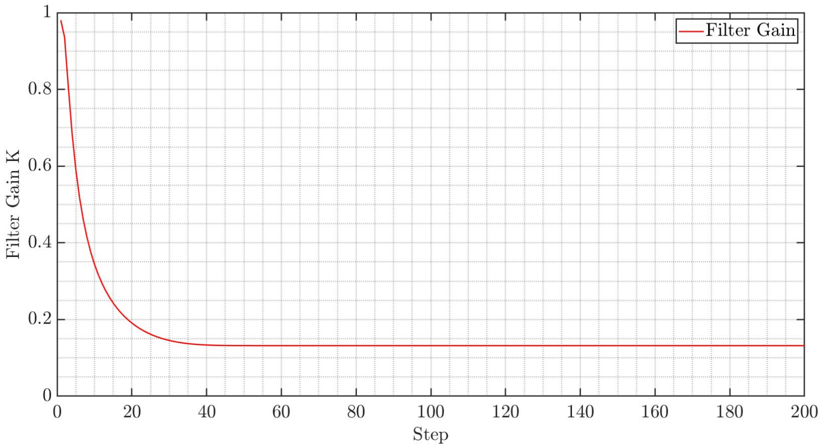

Figure 2: Filter Gain

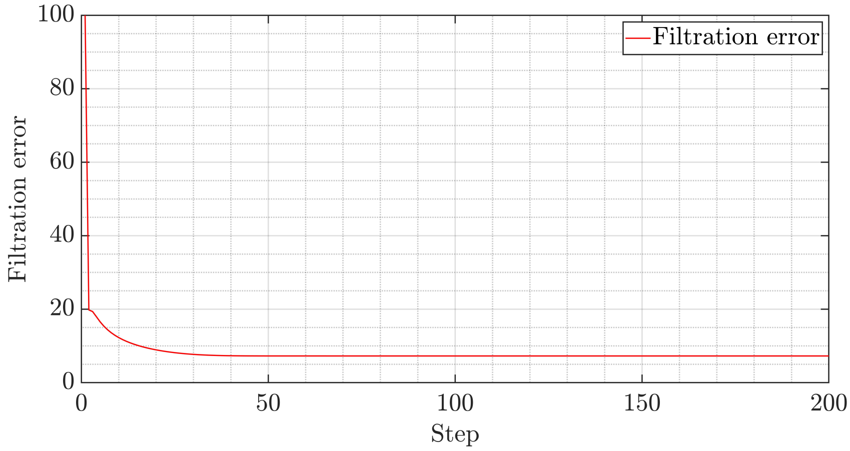

Figure 3: Standard Deviation of Estimation Error of Coordinate x_i

The filter gain K plot is presented in the **Figure 2**. To analyze filtration error covariance matrix $P_{i,i}$ over the observation period, another plot of square root of its first diagonal element corresponding to standard deviation of estimation error of coordinate x_i is shown in **Figure 3**. According to the plots, the standard deviation of estimation error of coordinate x_i and the filter gain K become constant after a few steps. This indicates that in presence of random noise, the estimation error cannot exceed the stated accuracy limit due to the uncertainty.

3 The 7-step extrapolation and dynamics of mean-squared error of estimation over observation interval (Q7 - Q11)

In this section the dynamics of mean-squared error is analyzed. Firstly, the extrapolation on 7 step ahead prediction was added as described by **Equation 11 - 14**

$$X_{i+m-1,i} = \Phi_{i+m-1,i} X_{i,i} \tag{11}$$

$$\Phi_{i+m-1,i} = \Phi_{i+m-1,i+m-2} \Phi_{i+m-2,i+m-3} ... \Phi_{i+1,i} \tag{12}$$

$$X_{7,1} = \Phi_{7,1} X_{1,1} \tag{13}$$

$$\Phi_{7,1} = \Phi_{7,6} \Phi_{6,5} \Phi_{5,4} \Phi_{4,3} \Phi_{3,2} \Phi_{2,1} \tag{14}$$

Squared deviation of true coordinate x_i from its estimation $\widehat{x}_{i,i}$ for every run:

$$Error(i) = (x_i - \widehat{x}_{i,i})^2 \tag{15}$$

$$FinalError(i) = \sqrt{\frac{1}{M-1} \sum_{k=1}^{M} Error(i)} \tag{16}$$

The results of calculation of final error are presented on the pictures below. It can be observed from **Figure 4** that the final error is essentially constant. As the filter becomes stationary, the algorithm can rely on his constant filter gain instead of computing it at each time step, providing smoother results than the true estimation error.

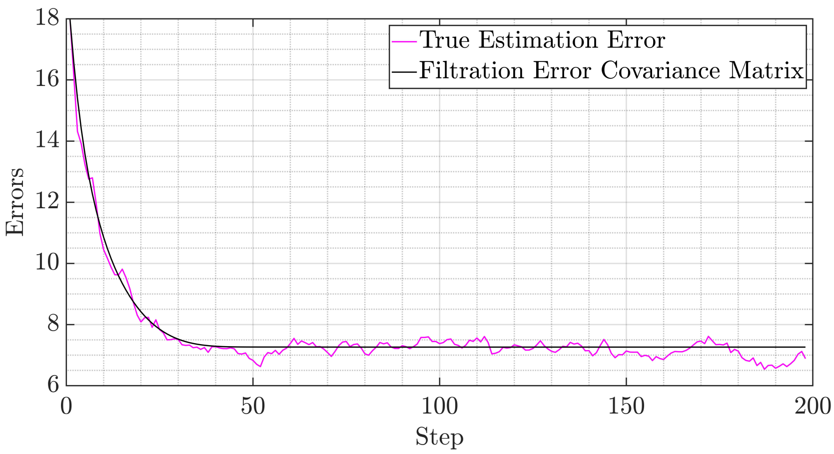

Figure 4: True and Filtration error with $P^I_{0,0}$

In addition, the comparison of estimation error is presented for 1-step ahead and 7-step ahead in **Figure 5**. It was observed that the general accuracy is lower for 7-step ahead prediction. Moreover, an overestimation problem is observed when 7-step prediction is applied, causing the abrupt jump in the beginning.

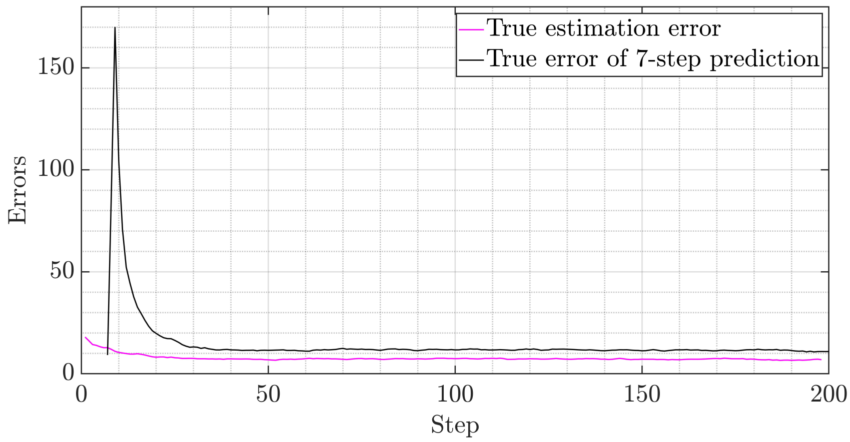

Figure 5: True and 7-steps prediction error with $P^{I}_{0,0}$

Finally, the analysis with a more accurate initial filtration error covariance matrix $P^{II}_{0,0}$ was performed and presented on the figures below.

$$P^{II}_{0,0} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \tag{17}$$

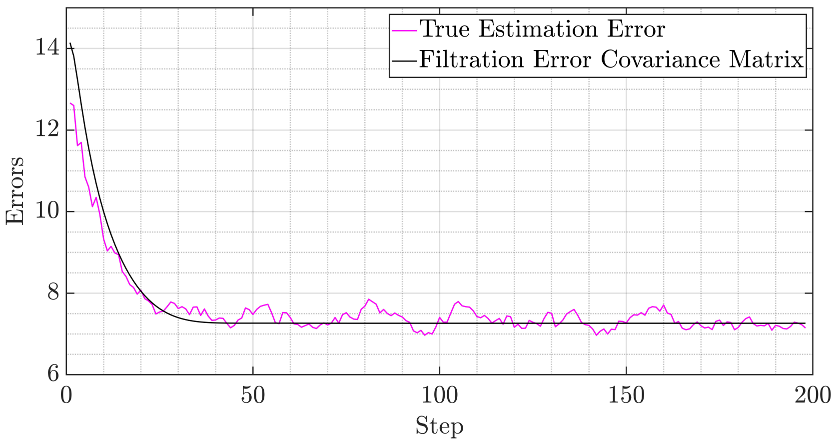

Figure 6: True and Filtration error with $P^{II}_{0,0}$

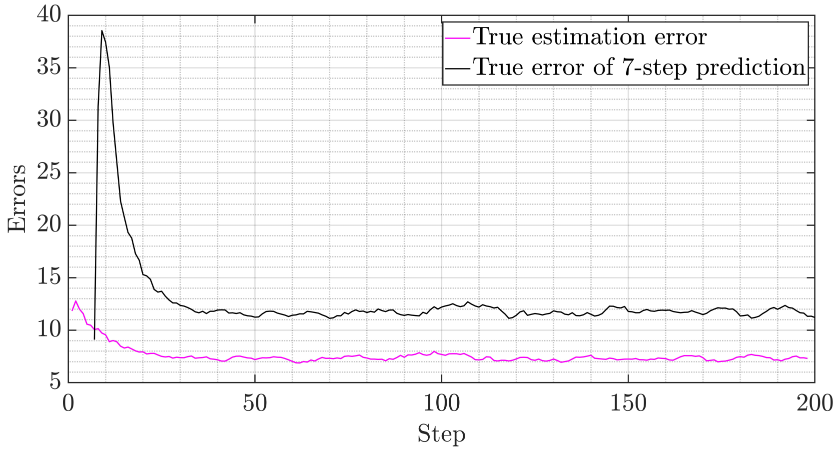

Figure 7: True and 7-steps prediction error with $P_{0,0}^{II}$

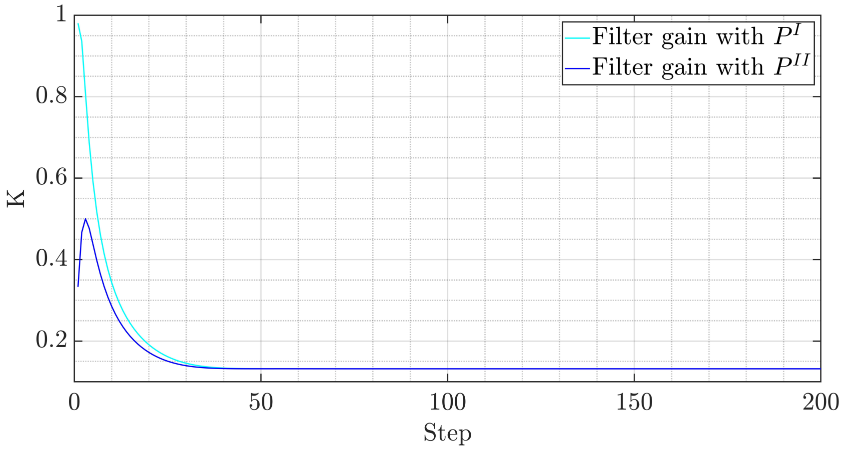

Figure 8: Gains comparison with $P_{0,0}^{I}$ $P_{0,0}^{II}$

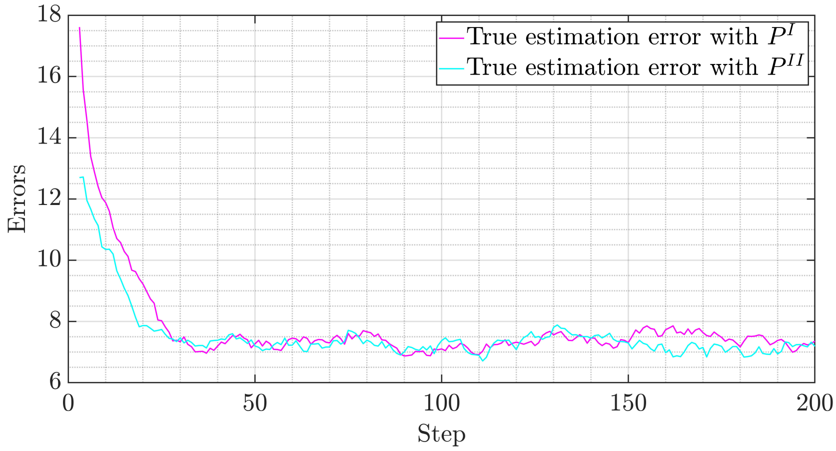

Figure 9: Estimation error comparison with $P_{0,0}^{I}$ $P_{0,0}^{II}$

A lower true estimation error is observed at the beginning of the experiment when the initial filtration error covariance matrix is more accurate, (**Figure 9**). However, with both values $P_{0,0}^{I}$ and $P_{0,0}^{II}$ almost the same level of final error is achieved. Moreover, both $P_{0,0}^{I}$ and $P_{0,0}^{II}$ allow convergence to this level within almost the same number of steps. It was also observed that the filter gain K for more accurate $P_{0,0}^{II}$ has an overestimation period at the beginning, compared to $P_{0,0}^{I1}$ (**Figure 8**). The influence of initial conditions dissipates more rapidly with an enlarged estimate.

4 Running the filter for a deterministic trajectory (Q12)

A deterministic trajectory was obtained by indicating the variance of the state noise equals to zero $\sigma_a^2 = 0$. The results of this experiments are presented in the following figures.

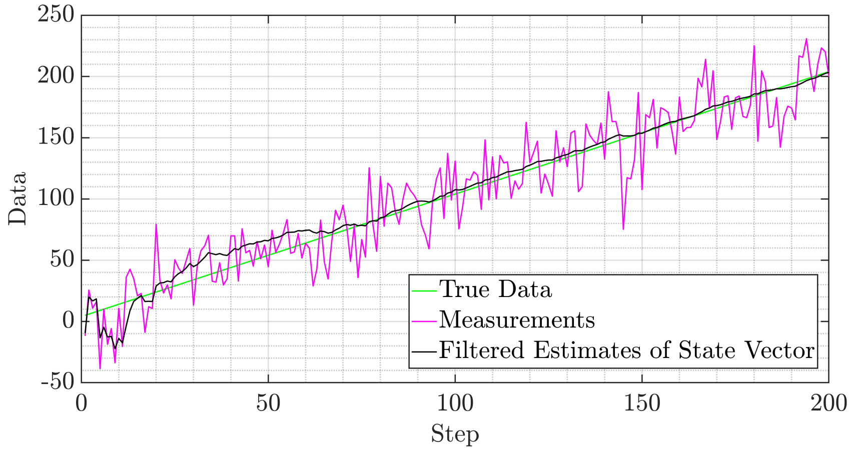

Figure 10: True trajectory, Measurements, Filtered Estimates for a deterministic trajectory

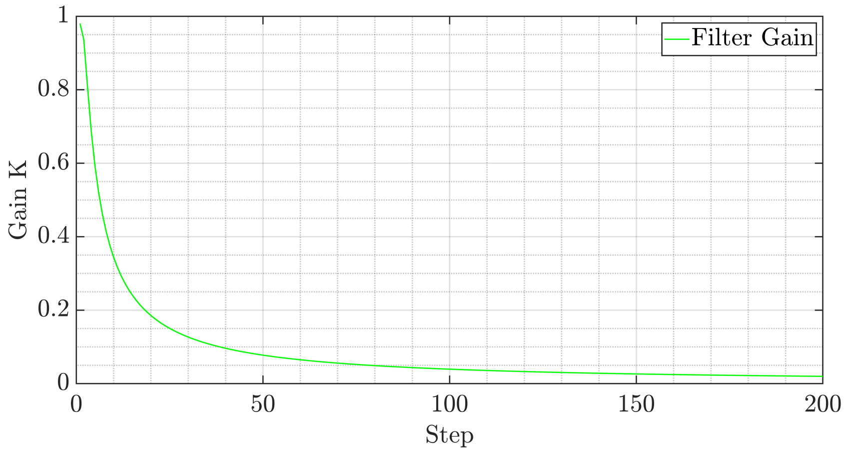

Figure 11: Filer gain for a deterministic trajectory

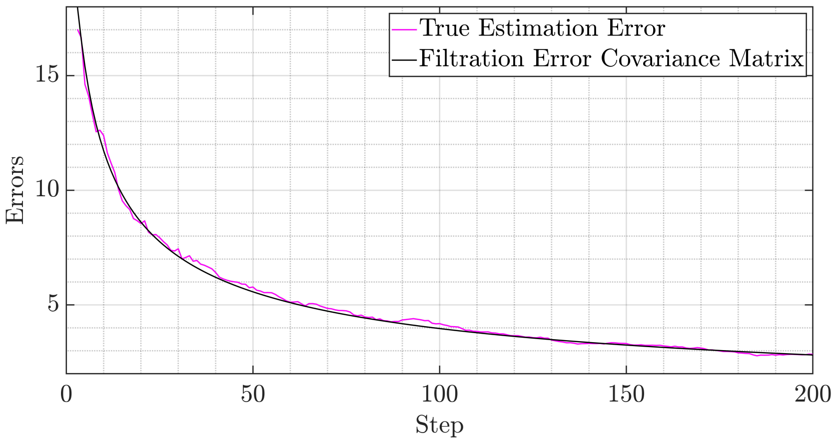

Figure 12: True and Filtration Errors for a deterministic trajectory

The obtained plot of filter gain presented in **Figure 11** show that the curve step by step approaches zero as expected. The same behaviour is observed for both true estimation and filtration errors, depicted in **Figure 12**.

5 Running the filter for a deterministic trajectory, disturbed by random acceleration (Q13)

A deterministic trajectory disturbed by random acceleration was obtained by indicating the variance of the state noise equals to zero $\sigma_a^2 = 0.2^2$ and neglecting the covariance matrix Q of state noise $w_i = Ga_i$ in assimilation algorithm. The results of this experiments are presented in the figures below.

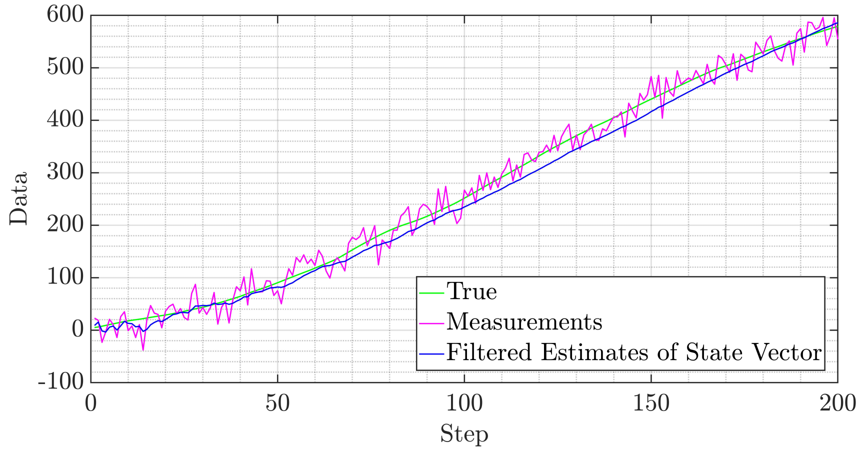

Figure 13: True trajectory, Measurements, Filtered Estimates for a deterministic trajectory, disturbed by random acceleration

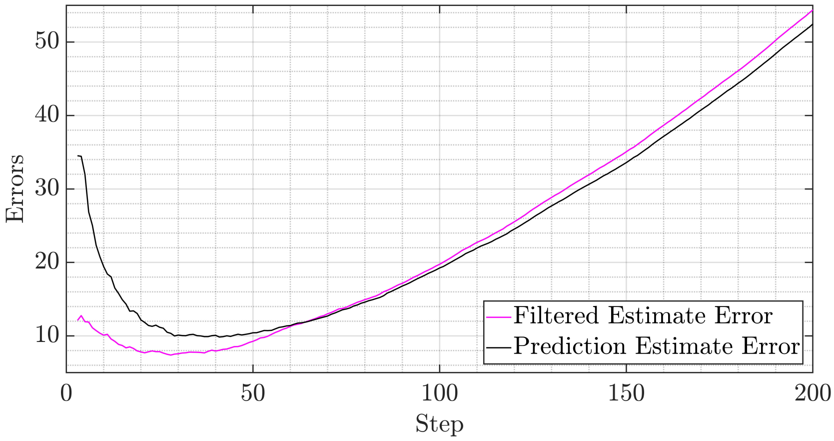

Figure 14: Filtered Estimate and Prediction Estimate Errors for a deterministic trajectory, disturbed by random acceleration

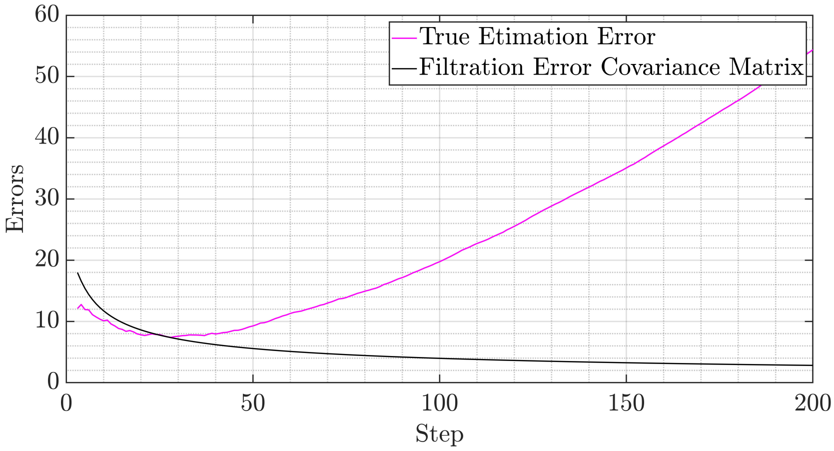

Figure 15: True Estimate and Filtration Errors for a deterministic trajectory, disturbed by random acceleration

In **Figure 13**, True trajectory, Measurements, Filtered Estimates are shown and it is seen that neglecting the covariance matrix of state noise Q leads to a delay in the filter. The level of accuracy is definitely lower than the one with the covariance matrix of state noise Q. In order to check the accuracy more deeply, the plot of filtered and prediction estimate errors is built **Figure 14**. Both filtered and forecast estimate errors show a significant growth in time. It is also shown in **Figure 15** that the calculation errors diverge from the true errors in time. This experiment demonstrates how important the covariance matrix of state noise is.

6 The relationship between state and measurement noise $\frac{\sigma_w^2}{\sigma_\eta^2}$ affect time when filter gain become almost constant (Q14)

The task was executed by indicating the variance of the state noise equals to zero $\sigma_a^2 = 1$. The results of this experiments are presented in the figures below.

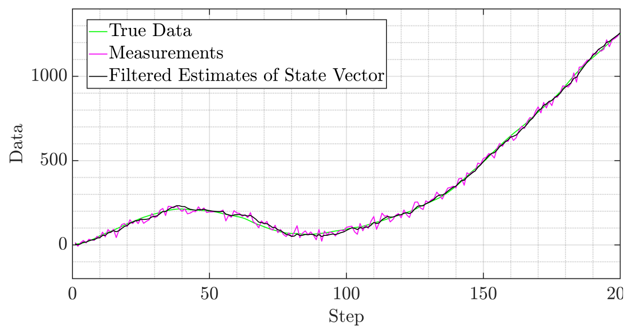

Figure 16: Data with $\sigma^2 = 0.2^2$

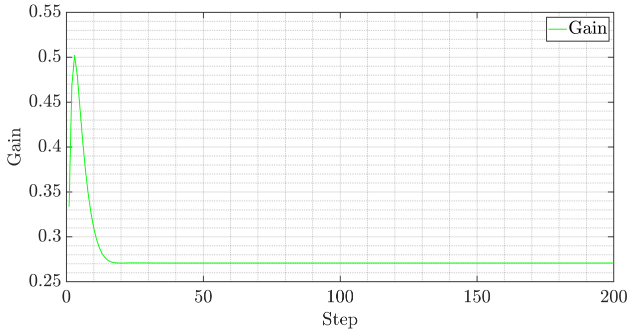

Figure 17: Filter gain with $\sigma^2 = 0.2^2$

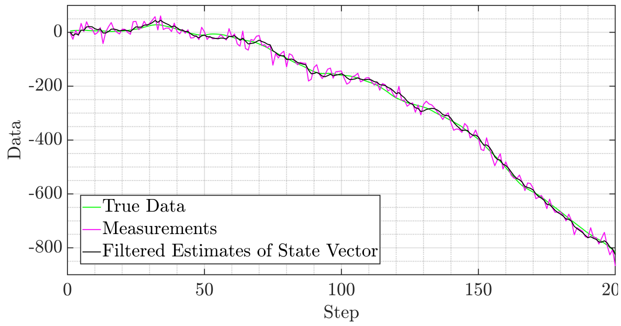

Figure 18: Data with $\sigma^2 = 1$

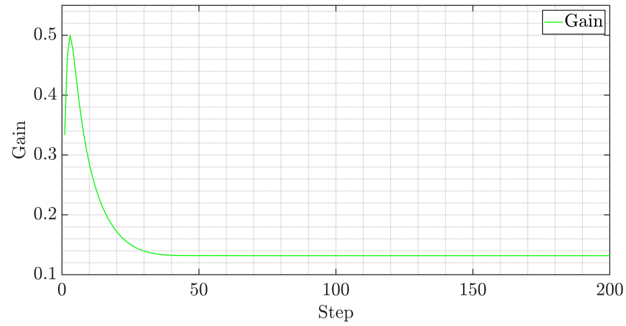

Figure 19: Filter gain with $\sigma^2 = 1$

In this part, two cases of variance of state noise were analysed, that is $\sigma_a^2 = 1$ and $\sigma_a^2 = 0.2^2$ respectively. Based on this numerical values, one can expect to see two inversely proportional trends, and that is shown in (**Figure 16** and **Figure 18**). Furthermore, both cases show that the filter gain remains constant over time, signalising a stagnation in estimation accuracy. However, for the case when $\sigma_a^2 = 1$ it happens almost two times more slowly, compared to the case when $\sigma_a^2 = 0.2^2$.

7 Sensitivity of filter to underestimated non-optimal filter gain K (Q15)

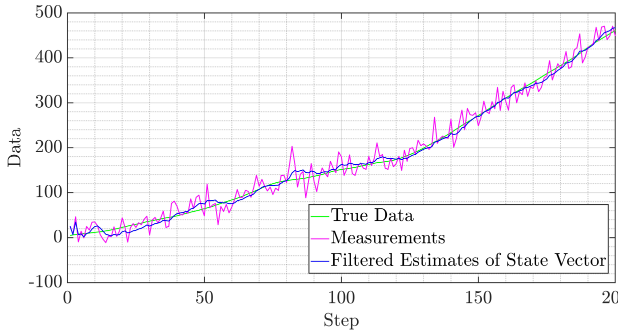

Figure 20: Data

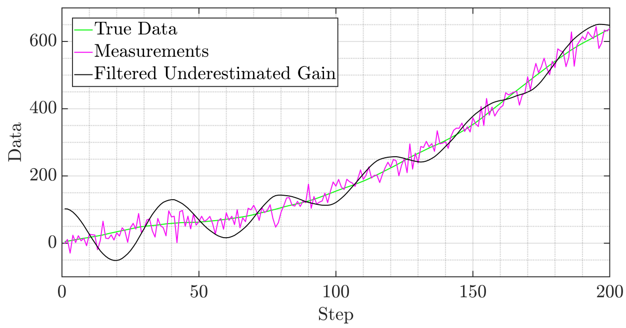

Figure 21: Data (Underestimated)

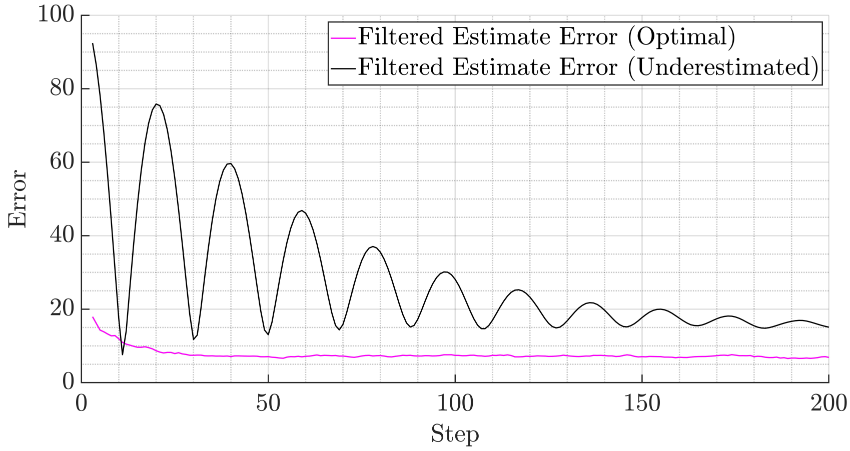

Figure 22: Comparison of Filtered Estimate Errors for Optimal and Underestimated filter gains

In this part, the results of experiment with underestimated filter gain are presented on the figures below. The optimal filter gain is calculated according to the equations of the Kalman filter, and the filtered estimation is performed with both the optimal and the underestimated gain. The errors in the **Figure 22** are compared, and it is noticeable that the underestimated gain yields lower values than the optimal gain. Further, the value of the optimal K was calculated: $K = 0.1319$.

8 Conclusion

By the end of this assignment, we developed a standard Kalman filter for tracking a moving object which trajectory is disturbed by random acceleration. One of the most important take away knowledge for us became the construction of Kalman filter. The standard Kalman filter recurrent algorithm is built accordingly to the following procedure: firstly, the prediction is made, then filtration. At the prediction step, the prediction of state vector at time i using $i-1$ measurements and the prediction error of covariance matrix are computed. At the stage of filtration, adjustment of predicted estimate is performed. The estimation is improved by incorporating a new measurements $X_{i,i}$, the filter gain K_i, and filtration error covariance matrix. After the construction and running the filtered several times we saw that estimation results are different with every new trajectory.
We analysed the filter gain K and filtration error covariance matrix $P_{i,i}$ over the observation period. We found out that the standard deviation of estimation error of coordinate x_i and the filter gain K become constant after a few steps. This gave us a knowledge of the fact that in presence of random noise, the estimation error cannot exceed the stated accuracy limit due to the uncertainty.
After that, we analysed the 7-step extrapolation and dynamics of mean-squared error of estimation over observation interval. The final error is essentially constant. As the filter becomes stationary, the algorithm can rely on its constant filter gain instead of computing it at each time step, providing smoother results than the true estimation error. We also learned by experience that the general accuracy is lower for 7-step ahead prediction. Moreover, an overestimation problem appears when multi-step prediction is applied, causing the abrupt jump in the beginning.
Finally, the analysis with a more accurate initial filtration error covariance matrix $P_{0,0}^{II}$ was performed. We learnt that a more accurate initial filtration error covariance matrix provides lower true estimation error in the beginning, though leads to almost the same level of accuracy with almost the same number of steps.
We learnt what a deterministic model is and which result to expect from application of Kalman filter. In details, filter gain approaches zero over time, and both true estimation and filtration errors do so. The experiment with a deterministic trajectory, disturbed by random acceleration, we learnt how important the covariance matrix of state noise Q is.

Individual contribution:

- Luca Breggion: Matlab code, report
- Irina Yareshko: Matlab code, report