Smoothing and Decay Estimates for Nonlinear Diffusion Equations

Equations of Porous Medium Type

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(rates), and also to obtain the best constants in the inequalities (in other words, we solve *optimal problems*).

In the case of the porous medium equation, there is a worst case with respect to the symmetrization-and-concentration comparison theorem of Section 1.3 in the class of solutions with the same initial mass $\|u_0\|_1 = M$. It is just the solution U with initial data a Dirac mass. This solution exists for $m > 1$ and is explicitly given by

$$U(x,t;M) = t^{-\alpha} F(x/t^{\alpha/n}), \quad F(\xi) = (C - k\xi^2)_+^{\frac{1}{m-1}} \tag{2.2}$$

where

$$\alpha = \frac{n}{n(m-1)+2}, \quad k = \frac{(m-1)\alpha}{2n}. \tag{2.3}$$

It is called the source solution because it has a Dirac delta as initial trace,

$$\lim_{t\to 0} u(x,t) = M\,\delta_0(x). \tag{2.4}$$

The remaining parameter $C > 0$ in formula (2.2) is in principle arbitrary; it can be uniquely determined by the mass condition $\int U dx = M$, which gives the following relation between the 'mass' M and C:

$$M = d\,C^{\gamma}, \quad d = n\,\omega_n \int_0^{\infty} (1 - k\,y^2)_+^{1/(m-1)}\, y^{n-1}\, dy, \quad \gamma = \frac{n}{2(m-1)\alpha} \tag{2.5}$$

(d and γ are functions of only m and n; the exact calculation of d will be performed later). This well-known solution is usually called the *source solution* or *Barenblatt solution*, since Barenblatt performed a complete study of these solutions in 1952. The name *ZKB solution* often used in this text honours the work of Zel'dovich and Kompaneyeets who supplied the first example. Using the mass as parameter, we denote it by $U(x,t;M)$ or $U_m(x,t;M)$. It is useful to write the complete expression as

$$\boxed{U_m(x,t;M)^{m-1} = \frac{(C\,t^{2\alpha/n} - k\,x^2)_+}{t}} \tag{2.6}$$

with $C = a(m,n)M^{2(m-1)\alpha/n}$. We can pass to the limit $m \to 1$ (with a fixed choice of the mass M) and obtain the fundamental solution of the heat equation,

$$\boxed{E(x,t) = M\,(4\pi t)^{-n/2} \exp(-x^2/4t)} \tag{2.7}$$

Therefore, $E(x,t;M) = U_1(x,t;M)$. Note the difference: U_m has compact support in the space variable for all $m > 1$, while E is positive everywhere with exponential tails at infinity. See Figure 2.1.

It was realized that the source solution also exists with many similar properties as long as $\alpha > 0$, i.e., it can be extended to the fast diffusion equation, $m < 1$, but only in the range $m_c < m < 1$, with $m_c = 0 \quad \text{for } n = 1, 2, \quad m_c = (n-2)/n \quad \text{for } n \geq 3$.

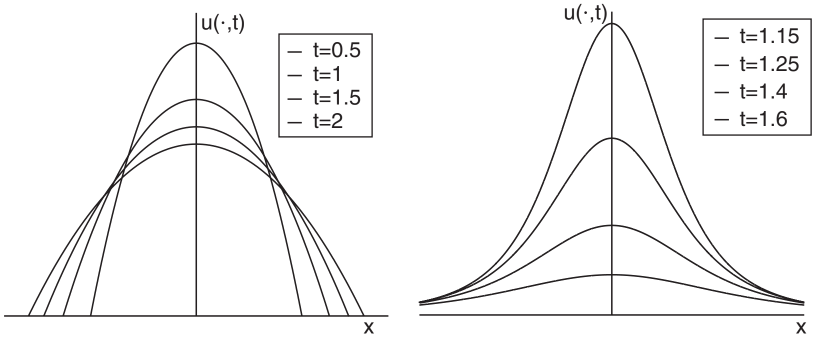

Figure 2.1 Left: evolution of the source solution for $m > 1$ with a free boundary. Right: the FDE source solution for $m < 1$ with a fat tail (polynomial decay). These profiles are the nonlinear alternative to the Gaussian profiles.

Formula (2.2) is basically the same, but now $m - 1$ and k are negative numbers, so that U_m is everywhere positive with a power-like tail at infinity. More precisely,

$$U_m(x, t; M) = t^{-\alpha} F(x/t^{\alpha/n}), \quad F(\xi) = (C + k_1 \xi^2)_+^{-\frac{1}{1-m}}. \tag{2.8}$$

with same value of α and $k_1 = -k = (1 - m)\alpha/2n$. The complete expression is

$$\boxed{U_m(x, t; M)^{1-m} = \frac{t}{C\, t^{2\alpha/n} + k_1 x^2}, \quad C = a(m, n)M^{-2(1-m)\alpha/n}} \tag{2.9}$$

The main difference in the ZKB profiles in the different ranges is probably the shape at infinity, which reflects the propagation mode. While for $m > 1$ the ZKB solutions are compactly supported, for $m = 1$ the Gaussian kernel has quadratic exponential decay, and for the fast diffusion range $m_c < m < 1$ we have profiles with an algebraic space decay, $u(x, t) \approx C(t)|x|^{-2/(1-m)}$, which are called in the statistical literature *fat tails*, *robust tails*, and also *overpopulated tails*. The difference between the tails for $m = 1$ and $m < 1$ is not only a question of decay rate. Thus, the tails for $m < 1$ behave in a universal way

$$U_m(x, t) \sim \left(\frac{t}{k_1 x^2}\right)^{\frac{1}{1-m}} \quad \text{as } |x| \to \infty, \tag{2.10}$$

whereas in the heat equation case the behaviour is $E(x, t) \sim M\,(4\pi t)^{-n/2} e^{-x^2/4t}$, an expression that depends on the total mass of the solution.[1] A related observation that will have a consequence in Chapter 5 is the following: in the fast diffusion case we can

[1] But note that this behaviour would be universal in logarithmic variables, $\log E(x, t) \sim -x^2/4t$. In any case, the difference has consequences for the physics and the theory.

put the constant $C = 0$ and still get a non-trivial solution. The 'solution' that we get corresponds to mass $M = \infty$ and reads

$$U_m(x, t; \infty) = \left(\frac{t}{k_1 x^2}\right)^{\frac{1}{1-m}}. \tag{2.11}$$

This singular solution has a standing non-integrable singularity at $x = 0$ and does not decay at all. In fact, it has a positive time derivative everywhere away from the singularity. It does not take part in the description of the asymptotic analysis of weak solutions as $t \to \infty$ though it is exact to describe the behaviour as $|x| \to \infty$ as (2.10) indicates. This solution is called infinite point-source solution, briefly IPSS, in [ChV02], where it is used to describe the behaviour of the class of continuous extended solutions for $m_c < m < 1$.

2.2 Smoothing effect and decay with L^1 functions or measures as data. Best constants

Using the existence and properties of the source solutions and the comparison theorems, we get the following result.

Theorem 2.1 *Let u be the solution of equation* (2.1) *in the range $m > m_c$ with initial datum $u_0 \in L^1(\mathbf{R}^n)$. Then, for every $t > 0$ we have $u(t) \in L^\infty(\mathbf{R}^n)$ and moreover there is a constant $c(m, n) > 0$ such that*

$$|u(x, t)| \le c(m, n)\, \|u_0\|_1^\sigma\, t^{-\alpha}, \tag{2.12}$$

with α given in (2.3) *and $\sigma = 2\alpha/n$. The best constant is attained by the ZKB solution and is given by formulas* (2.13), (2.14), and (2.16) *below.*

The same result holds when u_0 belongs to the space $\mathcal{M}(\mathbf{R}^n)$ of bounded and non-negative Radon measures if $\|u_0\|_1$ is replaced by $\|u_0\|_{\mathcal{M}(\mathbf{R}^n)}$.

Proof (i) It is clear that the worst case with respect to the symmetrization-and-concentration comparison in the class of solutions with the same initial mass M is just the ZKB solution U with initial data a Dirac mass, $u_0(x) = M\delta(x)$. We are thus reduced to perform the computation of the best constant for the ZKB solution. We have

$$\|U(t)\|_\infty = C^{1/(m-1)} t^{-\alpha} = d^{-2\alpha/n} M^{2\alpha/n} t^{-\alpha}.$$

Computing d is an exercise involving Euler's beta and gamma functions, see Appendix AI.1 for the basic formulas. For $m > 1$ we obtain

$$d = k^{-n/2} n\omega_n \int_0^1 (1 - s^2)^{1/(m-1)} s^{n-1} ds = \frac{1}{2} k^{-n/2} n\omega_n B\left(\frac{n}{2}, \frac{m}{m-1}\right),$$

and for $m < 1$

$$d = k^{-n/2} n\omega_n \int_0^\infty (1 + s^2)^{-1/(1-m)} s^{n-1} ds = \frac{1}{2} k^{-n/2} n\omega_n B\left(\frac{n}{2}, \frac{1}{m-1} - \frac{n}{2}\right).$$

Appendix I. Some analysis topics

AI.1 Some integrals and constants

We list some of the integrals that enter the calculation of the best constants in the smoothing effect.

(i) EULER'S GAMMA FUNCTION is defined as

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt, \qquad p > 0.$$

We have $\Gamma(p) = (p-1)\,\Gamma(p-1)$, and $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$. As $p \to \infty$ we have

$$\Gamma(p) \sim (p/e)^p (2\pi p)^{1/2}.$$

(ii) EULER'S BETA FUNCTION is defined for $p, q > 0$ as

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = 2\int_0^1 s^{2p-1}(1-s^2)^{q-1}\, ds.$$

We have $B(p,q) = B(q,p)$ and the basic relation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

as well as the equivalent expressions with parameter $r > 0$

$$B(p,q) = r\int_0^1 s^{rp-1}(1-s^r)^{q-1}\, ds = r\int_0^\infty \frac{x^{rq-1}}{(1+x^r)^{p+q}}\, dx.$$

These expressions are usually found for the value $r = 2$.

(iii) VOLUME OF THE UNIT SPHERE: It is well known that

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

This formula easily follows from the properties of Euler's integrals by induction.