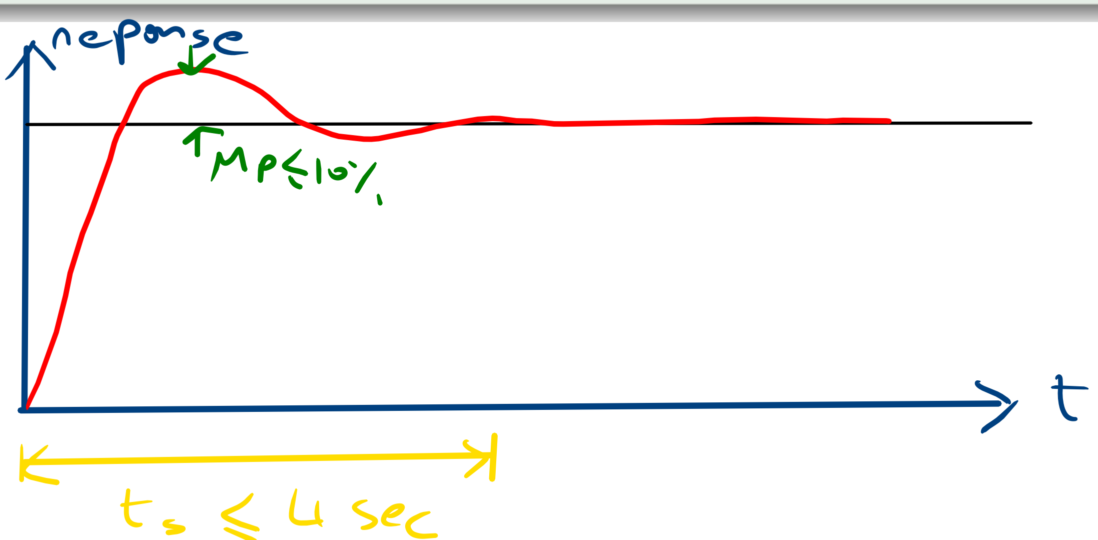
Example 1 (classwork)

Find the desired pole location to satisfy the following characteristics :

- Maximum overshoot : $M_p \leq 10\%$
- settling time : $t_s \leq 4\ sec$ (use 2% criterion)



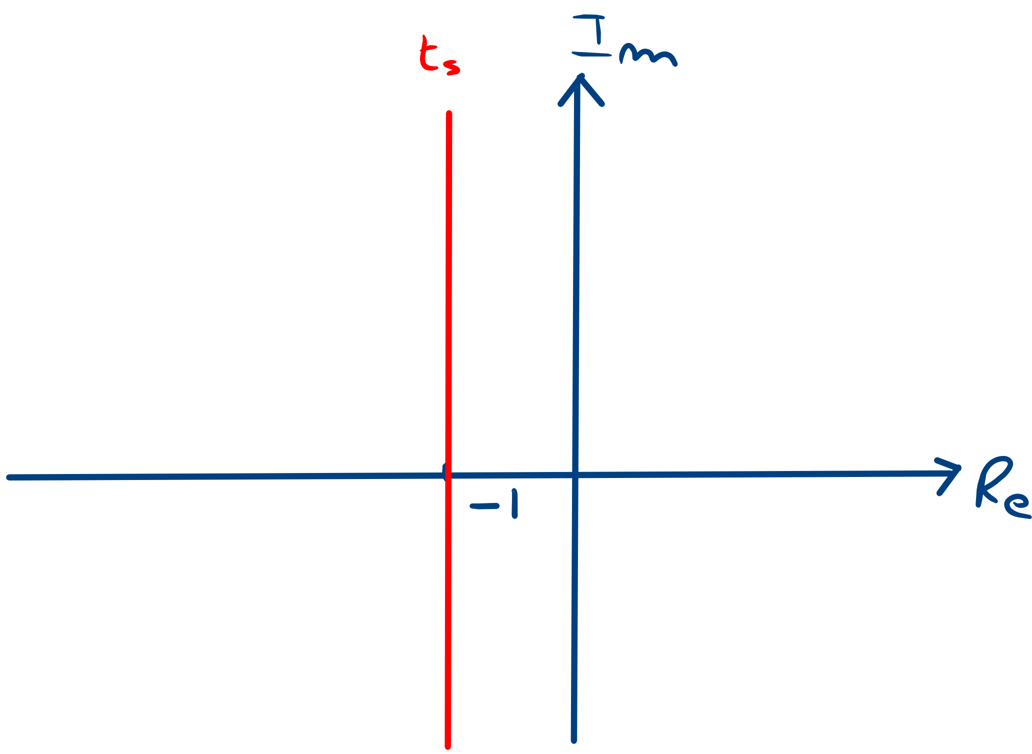
Trouvons ξ et ω_n qui correspondant à $M_p \leq 10\%$ et $t_s \leq 4$

on sait que $t_s = \dfrac{4}{\xi\omega_n} \leq 4$

$\Rightarrow \xi\omega_n \geq 1$

on sait que la partie reelle du pole $= -\xi\omega_n$

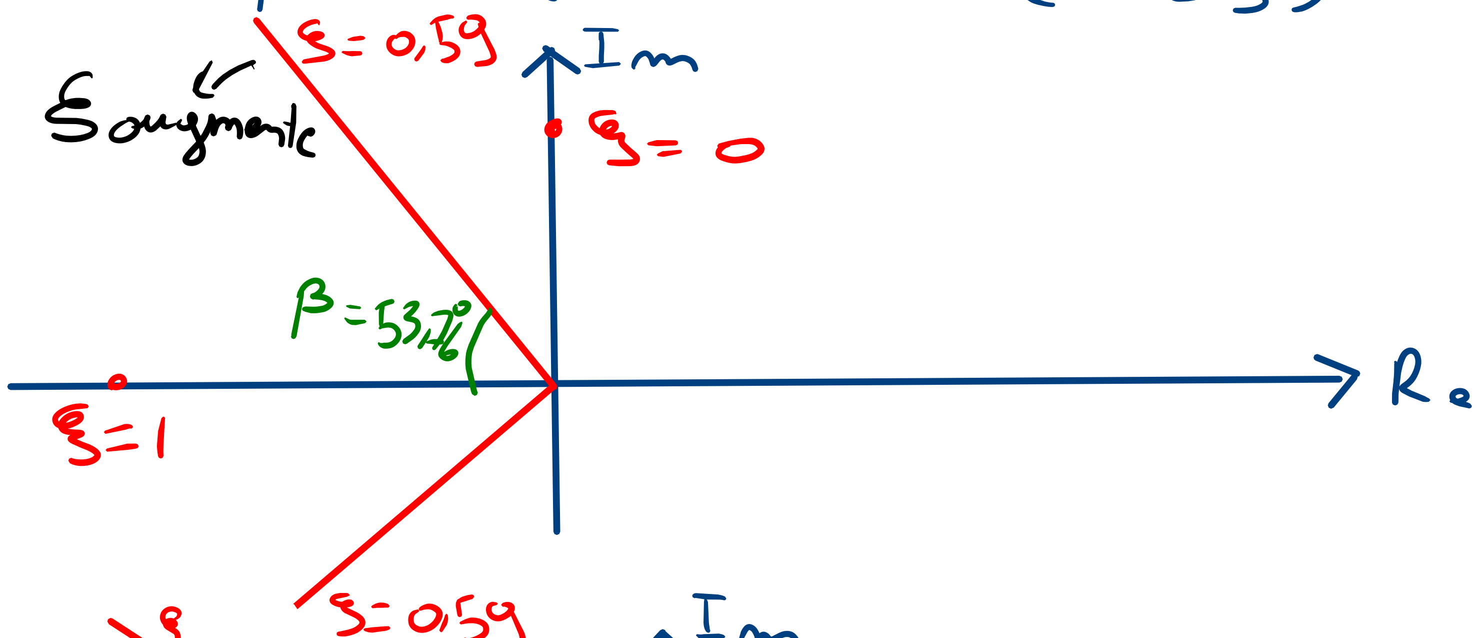
on $\xi\omega_n \geq 1 \Rightarrow \underbrace{-\xi\omega_n}_{\text{partie reelle du pole}} \leq -1$

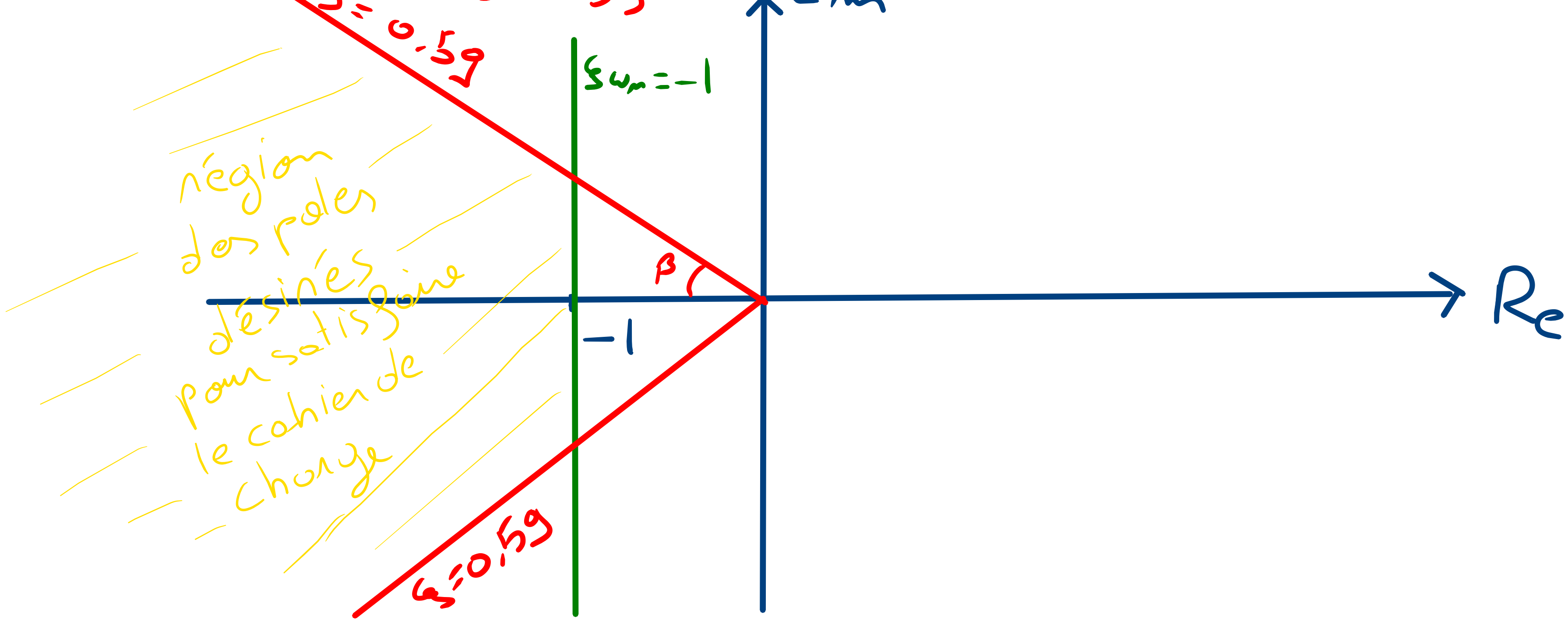


on sait que $\xi = -\dfrac{\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}}$

$M_p \leq 10\% = 0,1 \Rightarrow \xi \geq -\dfrac{\ln(0,1)}{\sqrt{\pi^2 + \ln^2(0,1)}} = 0,59$

on $\cos\beta = \xi \Rightarrow \beta = \cos^{-1}(\xi) = \cos^{-1}(0,59) = 53,76°$





Example 2 (classwork)

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0 \tag{13}$$

Routh table

s^4	1	3	5
s^3	2	4	0
s^2	$b_1 = 1$	$b_2 = 5$	
s^1	$c_1 = -6$	$c_2 = 0$	
s^0	$d_1 = 5$		

changement de signe

$$b_1 = \frac{-\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}{2} = 1 \qquad c_1 = \frac{-\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}}{1} = -6 \qquad d_1 = \frac{-\begin{vmatrix} 1 & 5 \\ -6 & 0 \end{vmatrix}}{-6} = 5$$

$$b_2 = \frac{-\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix}}{2} = 5 \qquad c_2 = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$$

Deux changements de signe $\Rightarrow$ Deux poles à partie reelles positives $\Rightarrow$ système instable

Example 3 (classwork)

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = s^3 + 2s^2 + s + 2 = 0 \tag{14}$$

Routh table:

s^3	1	1
s^2	2	2
s^1	$b_1 = 0 \approx \varepsilon$	
s^0	$c_1 = 2$	

$$b_1 = \frac{-\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}{2} = 0$$

$$c_1 = \frac{-\begin{vmatrix} 2 & 2 \\ \varepsilon & 0 \end{vmatrix}}{\varepsilon} = 2$$

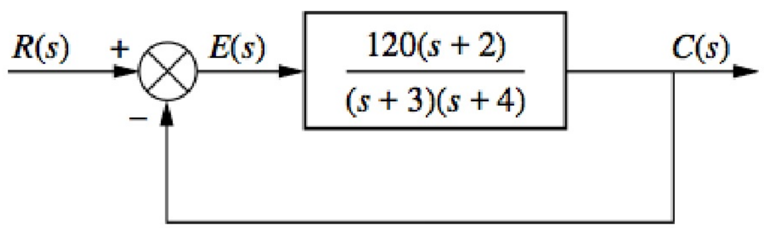
Comme on ne peut pas diviser par 0, on remplace b_1 par ε, un paramètre qui tend vers zéro

Pas de changement de signe $\Rightarrow$ pas de pole à partie réelle positive

La présence de $\varepsilon \approx 0$ dans les coefficients de la 1ère colonne, sans changement de signe, indique la présence d'un pole complexe (avec son conjugué) sur l'axe $j\omega$, donc le système est critiquement stable.

Example 4 (classwork)

Find the steady-state errors for inputs of $u(t)$, $tu(t)$, and $\frac{1}{2}t^2u(t)$ to the system shown beneath. The function $u(t)$ is the unit step.



$*$ si $r(t) = u(t) \Rightarrow e_{ss} = \frac{1}{1+K_p}$ avec $K_p = \lim\limits_{s \to 0} G(s) = \lim\limits_{s \to 0} \frac{120(s+2)}{(s+3)(s+4)} = 20$

$\Rightarrow e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+20} \boxed{= \frac{1}{21}}$

$*$ si $r(t) = t\,u(t) \Rightarrow e_{ss} = \frac{1}{K_v}$ avec $K_v = \lim\limits_{s \to 0} s\,G(s) = \lim\limits_{s \to 0} s \cdot \frac{120(s+2)}{(s+3)(s+4)} = 0$

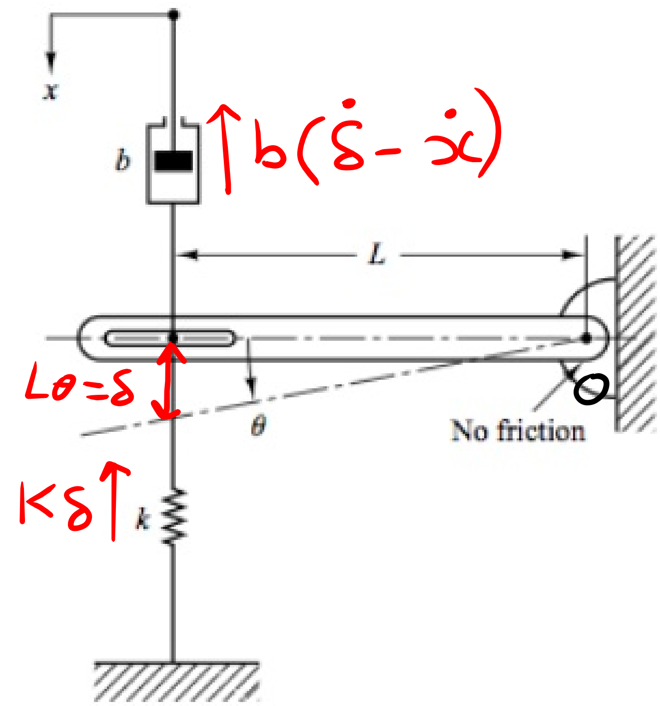
$\Rightarrow e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$

$*$ si $r(t) = \frac{1}{2}t^2u(t) \Rightarrow e_{ss} = \frac{1}{K_a}$ avec $K_a = \lim\limits_{s \to 0} s^2 G(s) = \lim\limits_{s \to 0} s^2 \frac{120(s+2)}{(s+3)(s+4)} = 0$

$\Rightarrow e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$

Problem 1

In the system of Figure (1), $x(t)$ is the input displacement and $u(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for x and θ are zeros, or $x(0) = 0$ and $\theta(0) = 0$. Find the transfer function of the system. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.



$x(t)$: input

$\theta(t)$: output

$\sum_i M_o = J\ddot{\theta}$ J = moment d'inertie de la tige $= \frac{mL^2}{3} \approx 0$ car la masse est négligeable.

$\Rightarrow -k\delta . L - b(\dot{\delta} - \dot{x})L = 0$

$k\delta + b(\dot{\delta} - \dot{x}) = 0$

$kL\theta + b(L\dot{\theta} - \dot{x}) = 0$

$kL\theta + bL\dot{\theta} = b\dot{x}$

la transformée de Laplace donne :

$kL\,\Theta(s) + bLs\,\Theta(s) = bs\,X(s)$

$\Theta(s)\left(kL + bLs\right) = bs\,X(s)$

la fonction de transfert $\frac{\Theta(s)}{X(s)} = \frac{bs}{kL + bLs}$ (1)

réponse de $\theta(t)$ si l'entrée $x(t)$ = échelon unitaire $\Rightarrow X(s) = \frac{1}{s}$

(1): $\Theta(s) = \frac{bs}{kL + bLs} \cdot X(s)$

$= \frac{b\cancel{s}}{kL + bLs} \cdot \frac{1}{\cancel{s}} = \frac{b}{kL + bLs}$

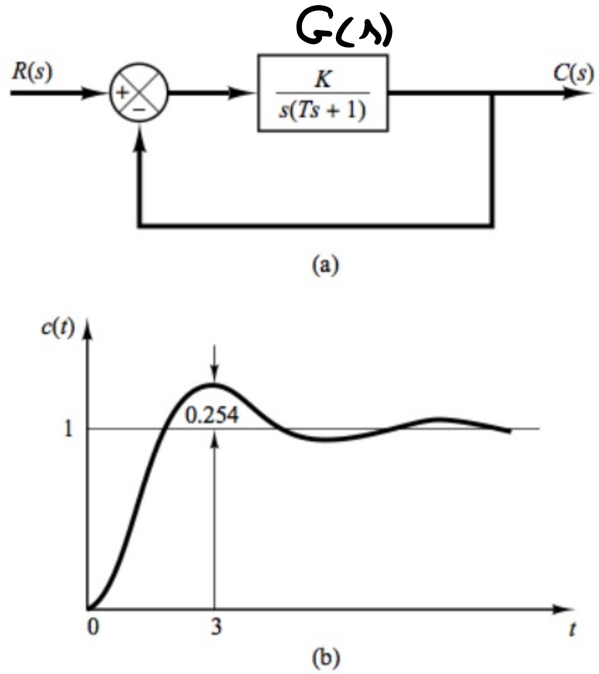
$= \frac{\cancel{b}}{\cancel{b}L\left(s + \frac{k\cancel{L}}{b\cancel{L}}\right)} = \frac{1}{L\left(s + \frac{k}{b}\right)}$

$\Theta(s) = \frac{1/L}{s + k/b}$

$\Rightarrow \boxed{\theta(t) = \frac{1}{L}e^{-\frac{k}{b}t}}$

Problem 2

When the system shown in Figure (2)(a) is subjected to a unit-step input, the system output responds as shown in Figure (2)(b). Determine the values of K and T from the response curve.



on a $M_p = 0,254$ et $t_p = 3$ secondes

on $\xi = -\dfrac{\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}}$

$$\xi = -\frac{\ln(0,254)}{\sqrt{\pi^2 + \ln^2(0,254)}}$$

$$\boxed{\xi \simeq 0,4}$$

on sais que $t_p = \dfrac{\pi}{\omega_d} = 3$

$\Rightarrow \omega_d = \dfrac{3}{\pi}$

$\Rightarrow \omega_n \sqrt{1-\xi^2} = \dfrac{3}{\pi}$

$$\Rightarrow \omega_n = \frac{3}{\pi\sqrt{1-\xi^2}} = \frac{3}{\pi\sqrt{1-0,4^2}}$$

$$\boxed{\omega_n = 1,14}$$

on la fonction de transfert est donnée par:

$$\frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(Ts+1)}}{1 + \dfrac{K}{s(Ts+1)}}$$

$$= \frac{\dfrac{K}{\cancel{s(Ts+1)}}}{\dfrac{s(Ts+1)+K}{\cancel{s(Ts+1)}}}$$

$$= \frac{K}{s(Ts+1)+K}$$

$$= \frac{K}{Ts^2 + s + K}$$

Pour identifier k et T, il faut écrire la fonction de transfert sous la forme canonique d'un système du second ordre, c'est à dire, sous la forme de

$$\frac{\omega_m^2}{s^2 + 2\xi\omega_m s + \omega_m^2} \quad (1)$$

or on a $$\frac{k}{Ts^2 + s + k} = \frac{k}{T\left(s^2 + \frac{1}{T}s + \frac{k}{T}\right)} = \frac{k/T}{s^2 + \frac{1}{T}s + \frac{k}{T}} \quad (2)$$

En comparant les eq (1) et (2) : $\omega_m^2 = \frac{k}{T}$ (3) et $2\xi\omega_m = \frac{1}{T}$ (4)

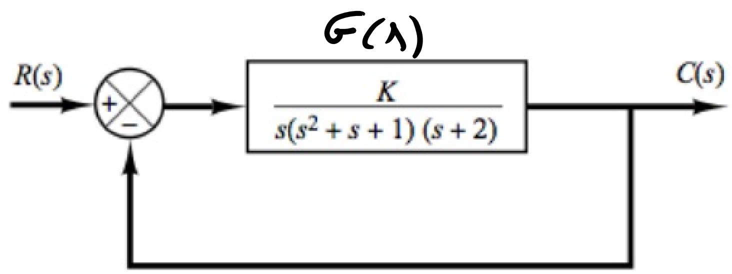
(4) $$T = \frac{1}{2\xi\omega_m} = \frac{1}{2 \times 0{,}4 \times 1{,}14} = 1{,}09$$

(3) $$k = T\omega_m^2 = 1{,}09 \times 1{,}14^2 = 1{,}42$$

$\boxed{T = 1{,}09}$ et $\boxed{k = 1{,}42}$

Problem 3

Determine the range of K for stability of the system shown in figure (3)



L'eq. caractéristique du système
(= denominateur de la fonction de transfert $\frac{G}{1+G}$)

$$1 + G(s) = 0$$

$$1 + \frac{K}{s(s^2+s+1)(s+2)} = 0$$

$$\frac{s(s^2+s+1)(s+2) + K}{s(s^2+s+1)(s+2)} = 0$$

$$s(s^2+s+1)(s+2) + K = 0$$

$$(s^3+s^2+s)(s+2) + K = 0$$

$$s^4 + 2s^3 + s^3 + 2s^2 + s^2 + 2s + K = 0$$

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

table de Routh :

s^4	1	3	K
s^3	3	2	
s^2	$\frac{7}{3}$	K	
s^1	$\left(\frac{14}{3} - 3K\right) / \frac{7}{3}$	0	
s^0	K		

Le système est stable si on a pas de changement de signe des coefficients de la 1ère colonne

$$\Rightarrow \begin{cases} \frac{14}{3} - 3K > 0 \Rightarrow K < \frac{14}{9} \\ K > 0 \end{cases}$$

Le système est stable si :

$$\boxed{0 < K < \frac{14}{9}}$$