Control engineering

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Contents

Chapter 2 : Mathematical Preliminaries (A Review)

Laplace Transform Review

Laplace Transform Review

The Laplace transform is defined as :

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. (1) exists, we can find a function, $F(s)$, that is called the Laplace transform of $f(t)$.

Example (classwork)

Example 1 (classwork)

Find the Laplace transform of $f(t) = Ke^{-at}$

Solution of example 1

Solution of example 1

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$F(s) = \int_0^\infty Ke^{-at}e^{-st}dt$$

$$F(s) = K\int_0^\infty e^{-(a+s)t}dt$$

$$F(s) = -\frac{K}{s+a}\left[e^{-(a+s)t}\right]_0^\infty$$

$$F(s) = \frac{K}{s+a}$$

Laplace Transform Review

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform Review

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Inverse Laplace Transformation

Partial Fraction Expansion

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a partial-fraction expansion.
Let

$$F(s) = \frac{P(s)}{Q(s)} = \frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \quad n > m \tag{2}$$

Expand into a sum of simple partial fractions as :

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + ... + \frac{a_n}{s-p_n} \tag{3}$$

where a_j is computed from the following relation :

$$a_j = (s-p_j)\left[\frac{P(s)}{Q(s)}\right]_{s=p_j} \tag{4}$$

Partial Fraction Expansion with repeated poles

Partial Fraction Expansion with repeated poles

Suppose, for the polynomial $F(s)$ there is pole located at $-r$ repeated q times so that $F(s)$ may be expressed as :

$$F(s) = \frac{N(s)}{(s-r)^q (s-p_2)(s-p_3)\dots(s-p_n)} \tag{5}$$

It can be expanded in partial fraction as

$$F(s) = \frac{a_1}{(s-r)^q} + \frac{a_2}{(s-r)^{q-1}} + \frac{a_3}{(s-r)^{q-2}} + \dots + \frac{a_q}{(s-r)} + \frac{a_{q+1}}{(s-p_2)} + \dots + \frac{a_{n+r-1}}{(s-p_n)} \tag{6}$$

it can be shown that the coefficients a_i of the repeated pole r, $(i = 1, \dots, q)$ are computed as follow :

$$a_i = \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[(s-r)^q F(s)\right]_{s=r} \tag{7}$$

for $i = 1, \dots, q$

Example (classwork)

Example 2

Find the time function from the following Laplace transform :

$$F(s) = \frac{s+4}{(s+1)(s+2)}$$

Solution of example (classwork)

Solution of example 2

$$F(s) = \frac{s+4}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

$s_1 = -1$ is a single pole, therefore

$$a_1 = \lim_{s \to -1} (s+1)F(s)$$

$$a_1 = \lim_{s \to -1} \frac{s+4}{s+2} = 3$$

$s_2 = -2$ is a single pole, therefore

$$a_2 = \lim_{s \to -2} (s+2)F(s)$$

$$a_2 = \lim_{s \to -2} \frac{s+4}{s+1} = -2$$

Solution of example (classwork)

Solution of example 2

Therefore

$$F(s) = \frac{s+4}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$$

The time domain function is

$$3e^{-t} - 2e^{-2t}$$

Example (classwork)

Example 3

Find the time function from the following Laplace transform :

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

Solution of example (classwork)

Solution of example 3

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2} + \frac{a_3}{s+1}$$

$s_1 = -1$ is a single pole, therefore

$$a_3 = \lim_{s \to -1} (s+1)F(s)$$

$$a_3 = \lim_{s \to -1} \frac{2}{(s+2)^2} = 2$$

$s_1 = -2$ is a repeated pole of order 2 ($q = 2$), therefore

$$a_1 = \frac{1}{(i-1)!} \lim_{s \to -2} \frac{d^{i-1}}{ds} (s+2)^q F(s)$$

$$a_1 = \lim_{s \to -2} \frac{2}{s+1} = -2$$

Solution of example 3

$s_2 = -2$ is a repeated pole of order 2 ($q = 2$), therefore

$$a_2 = \frac{1}{(i-1)!} \lim_{s \to -2} \frac{d^{i-1}}{ds} (s+2)^q F(s) = \lim_{s \to -2} \frac{d}{ds} \frac{2}{s+1}$$

$$a_1 = \lim_{s \to -2} \frac{-2}{(s+1)^2} = -2$$

Therefore,

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2} + \frac{a_3}{s+1} = \frac{-2}{(s+2)^2} + \frac{-2}{s+2} + \frac{2}{s+1}$$

and the inverse laplace transform is :

$$f(t) = -2te^{-2t} - 2e^{-2t} + 2e^{-t}$$

Transfer Function

The transfer function of a linear, time-invariant system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input with all initial conditions in the system set equal to zero.

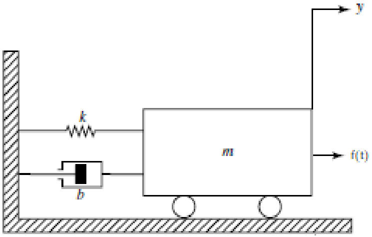
$$Transfer\ function = G(s) = \left[\frac{\mathcal{L}[output]}{\mathcal{L}[input]} \right]_{\text{zero initial conditions}} \tag{8}$$

The transfer function of a given system is unique and it is the property of the system itself and does not depend on the input and output and initial conditions. The concept is applicable to linear system only. Nonlinear systems cannot be represented by transfer function concept.

EXAMPLES (CLASSWORK)

EXAMPLE 3

Consider the mass-spring-damper system as shown below, find the transfer function of the system :



Solution of example 3

Solution example 3

Applying Newton's second law, we can find that the differential equation is given by :

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = f(t)$$

Take Laplace transform

$$ms^2Y(s) + bsY(s) + kY(s) = F(s)$$

$$(ms^2 + bs + k)Y(s) = F(s)$$

The transfer function is defined as the output/input

$$TF = G(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Example 4

Find the closed loop transfer function as shown in the figure

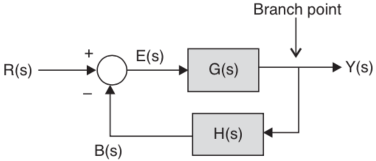


Figure – *Block diagram of a closed-loop system.*

Solution of example 4

Solution of example 4

We have

$$Y(s) = G(s)E(s)$$

and

$$E(s) = R(s) - B(s)$$

Therefore

$$\begin{aligned} Y(s) &= G(s)\left[R(s) - B(s)\right] \\ Y(s) &= G(s)\left[R(s) - H(s)Y(s)\right] \\ Y(s)\left[1 + G(s)H(s)\right] &= G(s)R(s) \end{aligned}$$

Therefore, the closed-loop transfer function $T(s)$ is given by

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Bode Diagrams

Bode Diagrams

Bode Diagrams

Bode Diagrams or Logarithmic Plots

A Bode diagram consists of two graphs :

- One is a plot of the logarithm of the magnitude of a sinusoidal transfer function
- The other is a plot of the phase angle
- both are plotted against the frequency on a logarithmic scale.

Bode Diagrams

- The standard representation of the logarithmic magnitude of $G(s) = G(j\omega)$ is $20\log|G(j\omega)|$
- The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition
- A simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations

Basic Factors of $G(j\omega)H(j\omega)$

Basic Factors of $G(j\omega)H(j\omega)$

The basic factors that very frequently occur in an arbitrary transfer function $G(j\omega)H(j\omega)$ are :

- Gain K
- Integral and derivative factors $(j\omega)^{\pm 1}$
- First-order factors $(1 + j\omega T)^{\pm 1}$
- Quadratic factors $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{\pm 1}$

The Gain K

The Gain K

$G(s)H(s) = K \Rightarrow G(j\omega)H(j\omega) = K$

- Gain $= |G(j\omega)H(j\omega)| = |K|$
- Phase $= \underline{/G(j\omega)H(j\omega)} = \tan^{-1}(G(j\omega)H(j\omega)) = tan^{-1}(\frac{0}{K})$
  - $= 0 \ if \ \ K > 0$
  - $= -180^\circ \ if \ \ K < 0$

Integral and derivative factors $(j\omega)^{\pm 1}$

Integral and derivative factors $(j\omega)^{\pm 1}$

$L(s) = G(s)H(s) = \frac{1}{s} \Rightarrow L(j\omega) = G(j\omega)H(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega}$

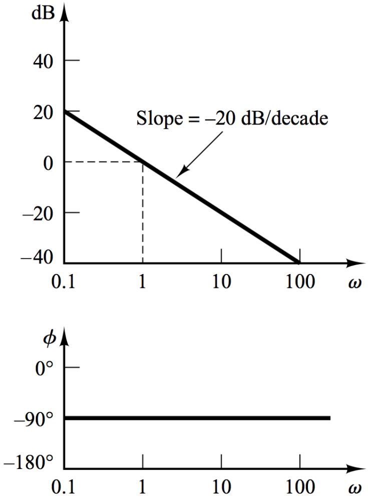
- Gain $= |G(j\omega)H(j\omega)| = \frac{1}{\omega}$
- Phase $(\Phi) = \underline{/G(j\omega)H(j\omega)} = tan^{-1}(\frac{-1/\omega}{0}) = -90°$

Special values of the Gain

- When $\omega = 1rad/s \Rightarrow gain = 1 \rightarrow 20\log(1) = 0$
- When $\omega = 10rad/s \Rightarrow gain = 0.1 \rightarrow 20\log(0.1) = -20\ dB$
- When $\omega = 100rad/s \Rightarrow gain = 0.01 \rightarrow 20\log(0.01) = -40\ dB$
- The slope of the line is $-20\ dB$ per decade.

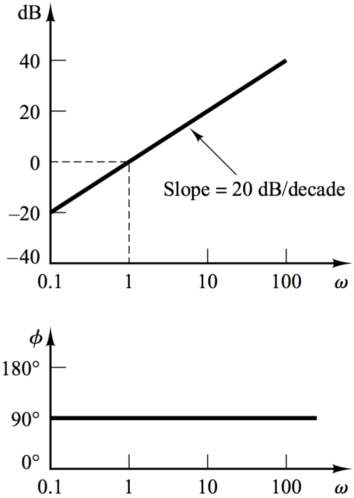
For an integral factor

if $L(s) = G(s)H(s) = s \Rightarrow L(j\omega) = j\omega$
$20log|L(j\omega) = 20log|j\omega| = -log|\frac{1}{j\omega}|$ $\rightarrow$ symmetric with respect to ω axis



Bode diagram of
$G(j\omega) = 1/j\omega$

(a)



Bode diagram of
$G(j\omega) = j\omega$

(b)

First-order factors $(1 + j\omega T)^{\pm 1}$

First-Order Factors $(1 + j\omega T)^{\mp 1}$. The log magnitude of the first-order factor $1/(1 + j\omega T)$ is

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}$$

For low frequencies, such that $\omega \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such that $\omega \gg 1/T$,

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log \omega T \text{ dB}$$

Special values of gain at high ω

- At $\omega = 1/T \Rightarrow -20\log(|gain|) = 0\ dB$
- At $\omega = 10/T \Rightarrow -20\log(|gain|) = -20\log(\omega T)\ dB$
- At $\omega = 100/T \Rightarrow -20\log(|gain|) = -40\log(\omega T)\ dB$
- Thus, the value of $-20\ log(\omega T)\ dB$ decreases by $20\ dB$ for every decade of ω.
- For $\omega >> 1/T$, the log-magnitude curve is thus a straight line with a slope of $-20\ dB/decade$.

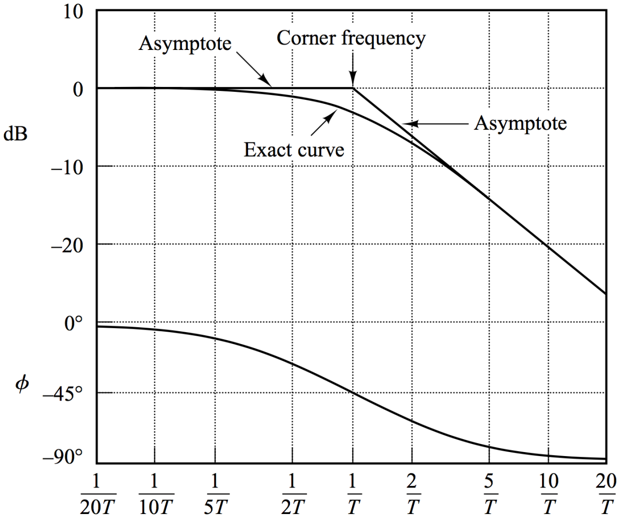
First-order factors $(1 + j\omega T)^{\pm 1}$

Phase Φ

Phase $(\Phi) = \angle(\frac{1}{1+j\omega T}) = -\angle(1 + j\omega T) = -\tan^{-1}(\frac{\omega T}{1}) = -\tan^{-1}(\omega T)$

- When $\omega << 1/T \Rightarrow \Phi = 0$
- When $\omega = 1/T \Rightarrow \Phi = 45^{\circ}$
- When $\omega >> 1/T \Rightarrow \Phi = 90^{\circ}$

Log-magnitude curve, together with the asymptotes, and phase-angle curve of $\frac{1}{(1+j\omega T)}$



First-order $(1 + j\omega T)$ case

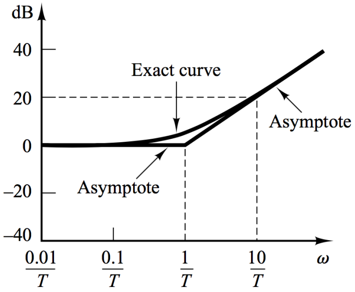
First-order $(1 + j\omega T)$ case

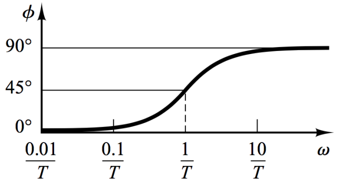
An advantage of the Bode diagram is that for reciprocal factors, for example, the factor $1 + j\omega T$ the log-magnitude and the phase-angle curves need only be changed in sign :

$$20\log|1 + j\omega T| = -20log|\frac{1}{1 + j\omega T}|$$

$$\underline{/1 + j\omega T} = \tan^{-1}(\omega T) = -\underline{/\tfrac{1}{1+j\omega T}}$$

Log-magnitude curve, together with the asymptotes, and phase-angle curve for $1 + j\omega T$





Quadratic factors $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{\pm 1}$

Control systems often possess quadratic factors of the form :

$$G(j\omega)H(j\omega) = \frac{1}{1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2}$$

- If $\zeta > 1$, this quadratic factor can be expressed as a product of two first-order factors with real poles.
- If $0 < \zeta < 1$ this quadratic factor is the product of two complex- conjugate factors.
- Asymptotic approximations to the frequency-response curves are not accurate for a factor with low values of ζ
- This is because the magnitude and phase of the quadratic factor depend on both the corner frequency and the damping ratio ζ.

Quadratic factors $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{\pm 1}$

$$Gain\ (dB) = 20\log\left|\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

- For low frequencies such that $\omega << \omega_n$, the log magnitude becomes

$$-20log(1) = 0\ dB$$

- For high frequencies such that $\omega >> \omega_n$, the log magnitude becomes

$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}$$

The above equation is a straight line having the slope $-40\ dB/decade$

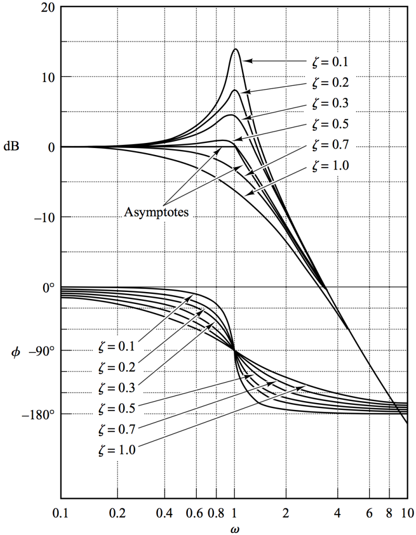
Quadratic factors $\left[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2\right]^{\pm 1}$

The phase angle Φ is :

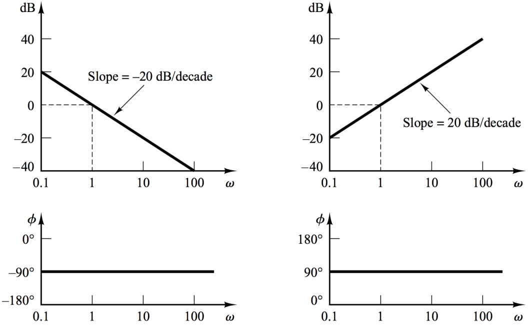
$$\Phi = \angle \frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2} = -tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

- The phase angle is a function of both ζ and ω
- At $\omega = 0 \Rightarrow \Phi = 0$
- At the corner frequency $\omega = \omega_n \Rightarrow \Phi = -90^\circ$ regardless of ζ
- At $\omega = \infty \Rightarrow \Phi = -180^\circ$

Log-magnitude curves, together with the asymptotes, and phase-angle curves of $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]$



Summary Bode Diagram - Derivative and Integral



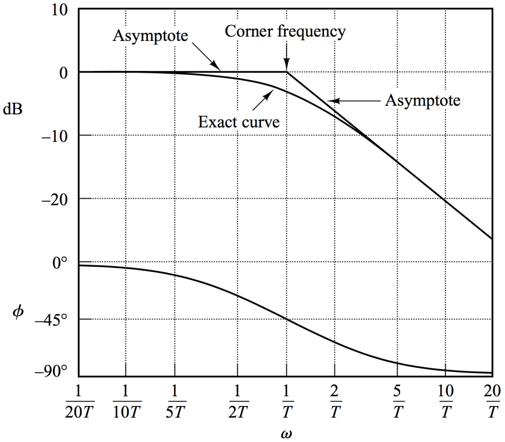
Bode diagram of $G(j\omega) = 1/j\omega$

(a)

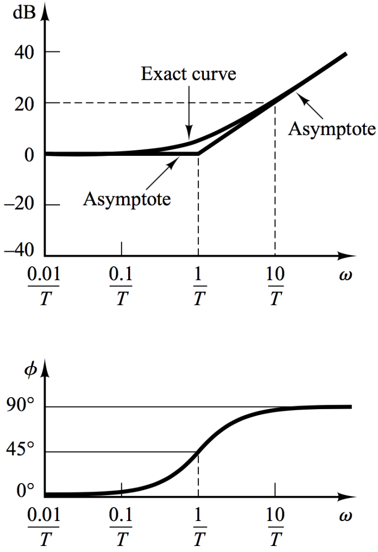
Bode diagram of $G(j\omega) = j\omega$

(b)

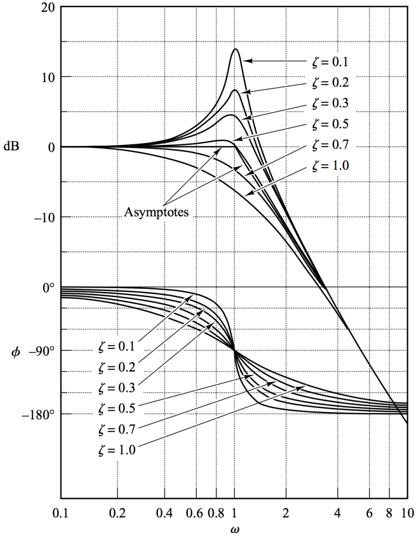
Summary Bode Diagram - First Order $\frac{1}{1+j\omega T}$



Summary Bode Diagram - First order $1 + j\omega T$



Summary Bode Diagram - Quadratic Factor $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]$



Example 5

Draw the Bode Diagram for the transfer function :

$$G(s) = \frac{100}{s + 30}$$

Solution of Example 5

Step 1 : Rewrite the transfer function in proper form

Make both the lowest order term in the numerator and denominator unity :

$$
\begin{aligned}
G(s) &= \frac{100}{s+30} \\
&= \frac{100}{30}\,\frac{1}{\frac{s}{30}+1} \\
&= 3.3\,\frac{1}{\frac{s}{30}+14}
\end{aligned}
$$

Solution of Example 5

Step2 : Separate the transfer function into its constituent parts.

The transfer function has 2 components :

- A constant of 3.3
- A pole at s=-30

Solution of Example 5

Step 3 : Draw the Bode diagram for each part.

The constant $K = 3.3$

- Gain (dB) $=-20\log 3.3 = 10.4\ dB$
- Phase of a positive constant = 0 degrees.

First Order : the pole at $10\ rad/sec$ (break frequency)

- Gain is 0 dB up to the break frequency then drops off with a slope of -20 dB/dec
- The phase is 0 degrees up to 1/10 the break frequency (3 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (300 rad/sec).

Solution of Example 5