Control engineering

Charbel Bouery, PhD

28 janvier 2020

Contents

Overview on Transient Response Analyses

Overview on Transient Response Analyses

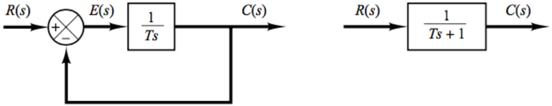
Transient Response of First Order Systems

The closed loop transfer function of a first order system can be written in its standard form as :

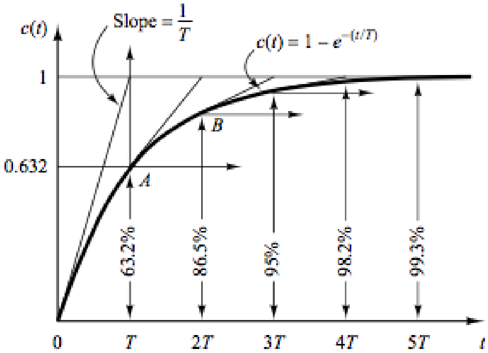
$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \quad (1)$$

The unit step response of the first order system is given by

$$c(t) = 1 - e^{-t/T} \quad (2)$$



Transient Response of First Order Systems

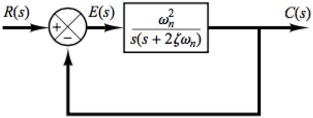


Transient Response of Second Order Systems

The closed loop transfer function of a second order system can be written in its standard form as :

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

The dynamic behaviour of the second-order system can then be described in terms of two parameters ζ and ω_n.



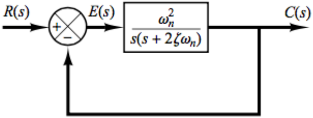
Transient Response of Second Order Systems

- If $0 < \zeta < 1$ the closed-loop poles are complex conjugates and lie in the left-half s plane.
- The system is then called underdamped, and the transient response is oscillatory.
- The closed-loop poles of an underdamped system are located at :

$$p_{1,2} = -\zeta\omega_n \pm j\omega_d \tag{4}$$

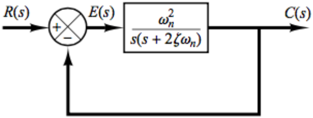
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

- Thus the closed-loop pole position can entirely describe the dynamic behaviour of the system



Transient Response of Second Order Systems

- If $\zeta = 0$ the closed-loop poles lie on the $j\omega$ axis and the transient response does not die out.
- If $\zeta = 1$ the system is called critically damped.
- If $\zeta > 1$ the system is called overdamped.



Transient Response of Second Order Systems

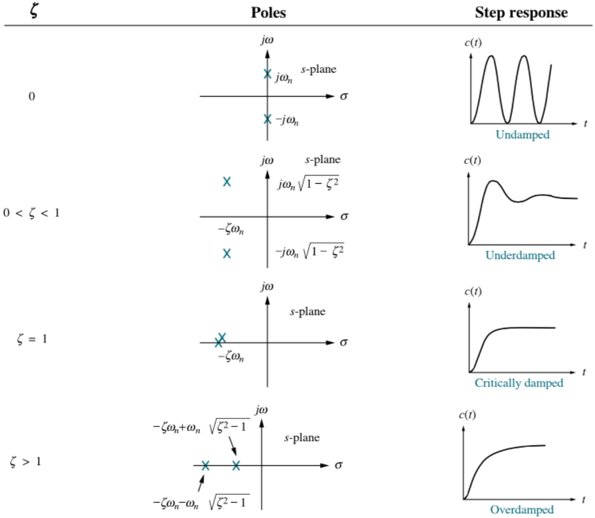
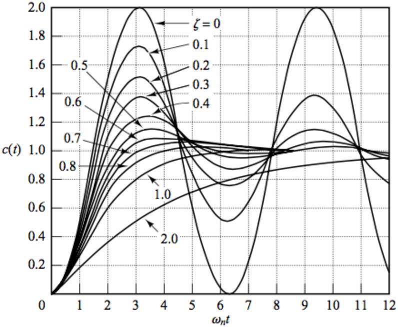


FIGURE Second-order response as a function of damping ratio

Transient Response of Second Order Systems

- Note that an underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than the others.
- An overdamped system is always sluggish in responding to any inputs.



HIGHER-ORDER SYSTEMS

Dominant Poles

- If the ratios of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient- response behavior.
- Those closed-loop poles that have dominant effects on the transient-response behavior are called dominant closed-loop poles.
- Quite often the dominant closed-loop poles occur in the form of a complex-conjugate pair.
- Note that the gain of a higher-order system is often adjusted so that there will exist a pair of dominant complex-conjugate closed-loop poles.

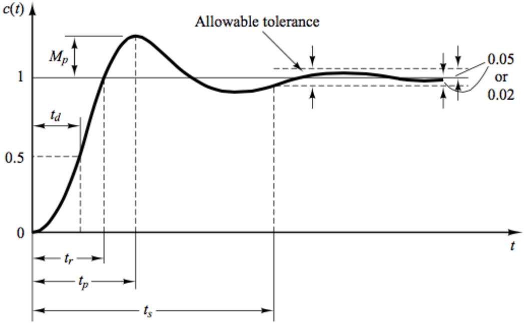
Definitions of Transient-Response Specifications

Definitions of Transient-Response Specifications

Transient-Response Specifications

In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following :

- Delay time, t_d
- Rise time, t_r
- Peak time, t_p
- Maximum overshoot, M_p or $\%OS$
- Settling time, t_s



Transient-Response Specifications

Delay time

Delay time, t_d : The delay time is the time required for the response to reach half the final value the very first time.

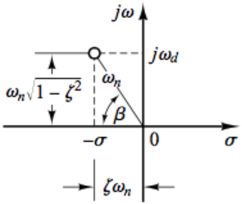
Transient-Response Specifications

Rise time

Rise time, t_r : The rise time is the time required for the response to rise from 10% to 90%.
For second order system the rise time is give by :

$$t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} \tag{5}$$

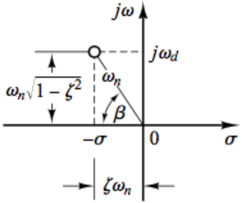
Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$



Transient-Response Specifications

The relation between β and ζ for continuous system is given by :

$$\cos(\beta) = \zeta \tag{6}$$



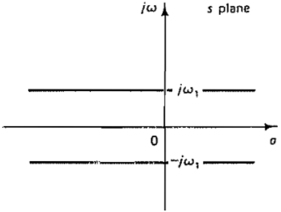
Transient-Response Specifications

Peak Time

Peak time t_p : The peak time is the time required for the response to reach the first peak of the overshoot. For second order system the peak time is given by :

$$t_p = \frac{\pi}{\omega_d} \tag{7}$$

A constant peak time is represented by a horizontal line in the s-plane, where $\omega_1 = \frac{\pi}{t_p}$



Transient-Response Specifications

Maximum (percent) overshoot

Maximum (percent) overshoot, M_p is defined by :

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\% \tag{8}$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

For second order system the Maximum (percent) overshoot is given by :

$$M_p = \exp - \left(\frac{\sigma}{\omega_d} \pi \right) = \exp \left(- \frac{\zeta}{\sqrt{1 - \zeta^2}} \pi \right) \tag{9}$$

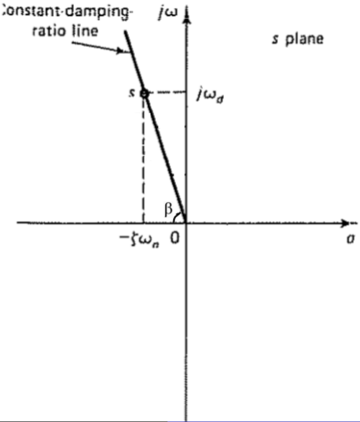
which corresponds to

$$\zeta = - \frac{\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}} \tag{10}$$

Transient-Response Specifications

Constant overshoot

A constant damping ratio (constant overshoot) is represented by a straight line of slope $\beta = \cos^{-1}(\zeta)$, on the left half plane as shown.



Transient-Response Specifications

Settling time

Settling time, t_s : The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).
For second order system the Settling time is given by :

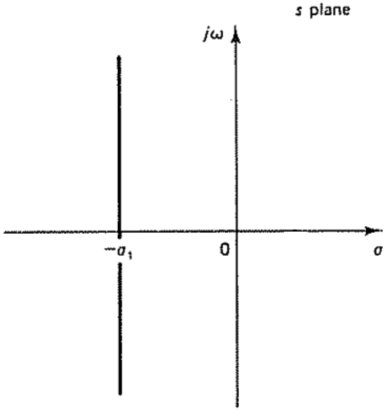
$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad 2\% \text{ criterion} \tag{11}$$

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad 5\% \text{ criterion} \tag{12}$$

Transient-Response Specifications

Settling time

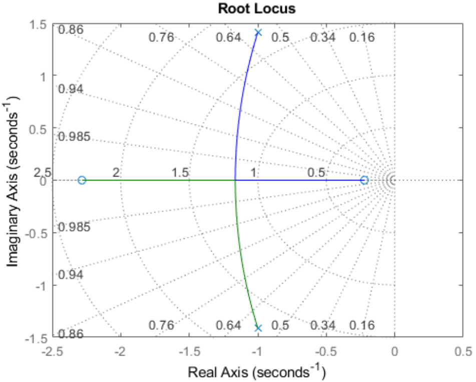
A Constant settling time is represented by a vertical line at $-\sigma_1 = -\frac{4}{ts}$ (for the 2% criterion)



Transient-Response Specifications

Constant Natural Frequency ω_n

Matlab uses damping ratio (ζ) and natural frequency (ω_n) to draw the desired pole location. It can be shown that a constant ω_n $(in\ rad/sec)$ represents an ellipse (*half an ellipse in the left-half plane*), centred at the origin and intersects the real axis at ω_n



Transient-Response Specifications

The maximum overshoot and the rise time conflict with each other. In other words, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other necessarily becomes larger.

Example (classwork)

Example 1 (classwork)

Find the desired pole location to satisfy the following characteristics :

- Maximum overshoot : $M_p \leq 10\%$
- settling time : $t_s \leq 4\ sec$ (use 2% criterion)

Stability of Control Systems

Stability of Control Systems

Stability of Control Systems

Review

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable ? If it is unstable, how should we stabilize the system ?

Stability of Control Systems

Review

Recall that stability can be determined from the location of the poles :

- If the poles are only in the left half-plane, the system is stable.
- If any poles are in the right half-plane, the system is unstable.
- If the poles are on the $j\omega$-axis and in the left half-plane, the system is marginally stable as long as the poles on the $j\omega$-axis are of unit multiplicity ; it is unstable if there are any multiple $j\omega$ poles.

ROUTH'S STABILITY CRITERION

Routh-Hurwitz criterion can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the jv-axis. (Notice that we say how many, not where.) We can find the number of poles in each section of the s-plane, but we cannot find their coordinates.

ROUTH'S STABILITY CRITERION

Consider for example that the characteristic equation is given by $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$.
The next step is to draw the table shown beneath. Each sign change in the first column correspond to a pole on the right half-plane.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example (classwork)

Example 2 (classwork)

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0 \tag{13}$$

ROUTH'S STABILITY CRITERION

Special Case

If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated.

Example 3 (classwork)

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = s^3 + 2s^2 + s + 2 = 0 \tag{14}$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (14) has two roots at $s = \pm j$.

ROUTH'S STABILITY CRITERION

If, however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$P(s) = s^3 - 3s + 2 = (s-1)^2(s+2) = 0 \tag{15}$$

the array of coefficients is

$$\begin{array}{lccc}
\text{One sign change:} & s^3 & 1 & -3 \\
 & s^2 & 0 \approx \epsilon & 2 \\
 & s^1 & -3 - \dfrac{2}{\epsilon} & \\
\text{One sign change:} & s^0 & 2 &
\end{array}$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

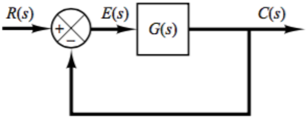
Steady-State Errors Overview

Steady-State Errors Overview

Steady-State Errors

The steady-state error of the system shown is defined as :

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} \quad (16)$$



Steady-State Errors

Static Position Error Constant K_p

The steady-state error in terms of the static position error constant K_p is :

$$e_{ss} = \frac{1}{1+K_p} \tag{17}$$

Where

$$K_p = \lim_{s\to 0} G(s) = G(0) \tag{18}$$

Static Velocity Error Constant K_v

The steady-state error in terms of the static velocity error constant K_v is :

$$e_{ss} = \frac{1}{K_v} \tag{19}$$

Where

$$K_v = \lim_{s\to 0} sG(s) \tag{20}$$

Steady-State Errors

Static Acceleration Error Constant K_a

The steady-state error in terms of the static acceleration error constant K_a is :

$$e_{ss} = \frac{1}{K_a} \tag{21}$$

Where

$$K_a = \lim_{s \to 0} s^2 G(s) \tag{22}$$

The type of the system (in the table beneath) represent the number of poles at the origin, i.e. a type 1 system represent one pole at the origin (pole of $G(s)$) and a type 2 system represent two poles at the origin.

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Figure – *Steady-State Error in Terms of Gain* K

Example 4 (classwork)

Find the steady-state errors for inputs of $u(t)$, $tu(t)$, and $\frac{1}{2}t^2u(t)$ to the system shown beneath. The function $u(t)$ is the unit step.