Домашние задания № 5.

9.

x_1	0	1	0	2	2	2	4	3
x_2	-1	0	0	0	1	0	1	2
y	0	0	0	0	0	1	1	1

$\hat{P}_r\{Y=0\} = 5/8$, $\hat{P}_r\{Y=1\} = 3/8$

$\hat{\mu}_0 = (1,0)^T$, $\hat{\mu}_1 = (3,1)^T$

Выборочные матрицы ковариации для каждого класса:

$$\hat{\Sigma}_0 = \frac{1}{N_0 - 1} \sum_{y^{(i)}=0} (x^{(i)} - \hat{\mu}_0)(x^{(i)} - \hat{\mu}_0)^T =$$

$$= \frac{1}{4}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}(-1,-1) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}(0,0) + \begin{pmatrix} -1 \\ 0 \end{pmatrix}(-1,0) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,0) + \right.$$

$$\left. + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1,1) \right) = \frac{1}{4}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) =$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad \hat{\Sigma}_0^{-1} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\hat{\Sigma}_1 = \frac{1}{N_1 - 1} \sum_{y^{(i)}=1} (x^{(i)} - \hat{\mu}_1)(x^{(i)} - \hat{\mu}_1)^T =$$

$$= \frac{1}{2}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}(-1,-1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0,1) \right) =$$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \qquad \hat{\Sigma}_1^{-1} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} 1/3$$

$$\det \hat{\Sigma}_0 = 1/2 - 1/4 = 1/4$$

$$\det \hat{\Sigma}_1 = 1 - 1/4 = 3/4$$

$$\hat{\Sigma} = \frac{1}{N-K} \sum_k \sum_{y^{(i)}=k} (x^{(i)} - \hat{\mu}_k)(x^{(i)} - \hat{\mu}_k)^T = \frac{1}{6}\left(\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right) =$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2/3 \end{pmatrix} \qquad \hat{\Sigma}^{-1} = \frac{1}{5}\begin{pmatrix} 8 & -6 \\ -6 & 12 \end{pmatrix}$$

① $\delta_0(x) = x^T \hat{\Sigma}^{-1} \hat{\mu}_0 - \frac{1}{2} \mu_0^T \hat{\Sigma}^{-1} \mu_0 + \ln \hat{P}r\{Y=0\} =$

$= (x_1, x_2) \frac{1}{5} \begin{pmatrix} 3 & -6 \\ -6 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \frac{1}{5} \begin{pmatrix} 3 & -6 \\ -6 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \ln 5/8 =$

$= 3/5\, x_1 - 6/5\, x_2 - 3/10 + \ln 5/8.$

$\delta_1(x) = x^T \hat{\Sigma}^{-1} \hat{\mu}_1 - \frac{1}{2} \hat{\mu}_1 \hat{\Sigma}^{-1} \mu_1 + \ln \hat{P}r\{Y=1\} =$

$= (x_1, x_2) \frac{1}{5} \begin{pmatrix} 3 & -6 \\ -6 & 12 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix}^T \frac{1}{5} \begin{pmatrix} 3 & -6 \\ -6 & 12 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \ln 3/8 =$

$= 13/5\, x_1 - 6/5\, x_2 - 43/10 + \ln 3/8$

$\delta_0(x) - \delta_1(x) = 0 \Rightarrow (3/5 - 13/5)x_1 - (\overset{=0}{6/5 - 6/5})x_2 - (3/10 - 43/10)$

$+ \ln 5/8 - \ln 3/8 = \underline{-2x_1 + 4 + \ln 5/3 = 0}$ -> 2eδoc

② $\delta_0(x) = -\frac{1}{2} \ln \det \hat{\Sigma}_0 - \frac{1}{2}(x - \hat{\mu}_0)^T \hat{\Sigma}_0^{-1} (x - \hat{\mu}_0) + \ln \hat{P}r\{Y=0\} =$

$= -\frac{1}{2} \ln \frac{1}{4} + \ln 5/8 - \frac{1}{2} \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} =$

$= \cancel{-\frac{1}{2} \ln 5/32} - (x_1^2 - 2x_1 + 2x_2^2 - 2x_1 x_2 + 2x_2 + 1) - \frac{1}{2} \ln \frac{1}{4} + \ln 5/8$

$\delta_1(x) = -\frac{1}{2} \ln \det \hat{\Sigma}_1 - \frac{1}{2}(x - \hat{\mu}_1)^T \hat{\Sigma}_1^{-1} (x - \hat{\mu}_1) + \ln \hat{P}r\{Y=1\} =$

$= -\frac{1}{2} \ln 3/4 + \ln 3/8 - \frac{1}{6} \begin{pmatrix} x_1 - 3 \\ x_2 - 1 \end{pmatrix}^T \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 1 \end{pmatrix} =$ $Y=1$

$= \cancel{-\frac{1}{2} \ln 9/32} - \frac{1}{3}(2x_1^2 - 10x_1 + 2x_2^2 - 2x_1 x_2 + 2x_2 + 14) - \frac{1}{2} \ln 3/4 + \ln 3/8$

$\delta_0(x) - \delta_1(x) = 0 \Rightarrow \underline{x_1^2 + 4x_1 + 4x_2 - 4x_1 x_2 + 4x_2^2 - 3/2 \ln 3 - 3 \ln 5/3 - 11 = 0}$

2roc.

15.

x_1	0	0	1	1	0	0	1	1	1	0
x_2	0	1	0	1	1	1	1	1	1	1
y	0	0	0	0	0	1	1	1	1	1

$\hat{P_r}\{Y=0\} = 1/2$
$\hat{P_r}\{Y=1\} = 1/2$

$\hat{P_r}\{X_1=0 \mid Y=0\} = 3/5$
$\hat{P_r}\{X_1=1 \mid Y=0\} = 2/5$

$\hat{P_r}\{X_2=0 \mid Y=0\} = 2/5$
$\hat{P_r}\{X_2=1 \mid Y=0\} = 3/5$

$\hat{P_r}\{X_1=0 \mid Y=1\} = 2/5$
$\hat{P_r}\{X_1=1 \mid Y=1\} = 3/5$

$\hat{P_r}\{X_2=0 \mid Y=1\} = 0$
$\hat{P_r}\{X_2=1 \mid Y=1\} = 1$

$$P_r\{Y=0 \mid X_1=1, X_2=1\} = \frac{\hat{P_r}\{X_1=1 \mid Y=0\} \cdot \hat{P_r}\{X_2=1 \mid Y=0\} \cdot \hat{P_r}\{Y=0\}}{P_r\{X_1=1, X_2=1\}}$$

$$= \frac{2/5 \cdot 3/5 \cdot 1/2}{3/25 + 3/10} = \frac{2}{7}$$

$$P_r\{Y=1 \mid X_1=1, X_2=1\} = \frac{\hat{P_r}\{X_1=1 \mid Y=1\} \cdot \hat{P_r}\{X_2=1 \mid Y=1\} \hat{P_r}\{Y=1\}}{P_r\{X_1=1, X_2=1\}} =$$

$$= \frac{3/5 \cdot 1 \cdot 1/2}{3/25 + 3/10} = \frac{5}{7}$$

16.

x_1	4	0	-2	2
x_2	3	1	-3	-1

$\bar{x}_1 = 1$

$\bar{x}_2 = 0$

$$X_c = \begin{pmatrix} 3 & -1 & -3 & 1 \\ 3 & 1 & -3 & -1 \end{pmatrix}^T$$

$$C = X_c^T X_c = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$$

$$\begin{pmatrix} 20-\lambda & 16 \\ 16 & 20-\lambda \end{pmatrix} = (20-\lambda)^2 - 16^2 =$$

$$= (20-\lambda-16)(20-\lambda+16) = (4-\lambda)(36-\lambda)$$

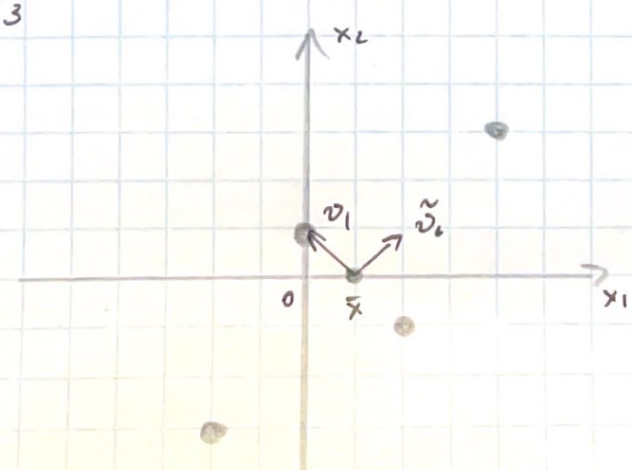
$$\left[\begin{array}{l} \lambda_1 = 4 \\ \lambda_2 = 36 \end{array}\right.$$

λ_1: $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \tilde{v}_1 = 1/\sqrt{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\to$ главные

λ_2: $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \tilde{v}_2 = 1/\sqrt{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\to$ компоненты направления

$\frac{1}{N-1}\lambda_1 = \frac{4}{3}$ $\to$ дисперсии

$\frac{1}{N-1}\lambda_2 = \frac{36}{3}$ $\to$ по гл. компонентам.



19.

① $\dfrac{\partial g_k}{\partial s_l} = g_k\left(I(k=l) - g_l\right)$

$$\frac{\partial g_k}{\partial s_l} = \frac{\partial}{\partial s_l}\left(\frac{e^{s_k}}{\sum_{l^*=1}^{K} e^{s_{l^*}}}\right) = \overset{= g_k}{\frac{e^{s_k}}{\sum_{l^*=1}^{K} e^{s_{l^*}}}} \cdot \overset{-g_l}{\frac{-e^{s_l}}{\sum_{l^*=1}^{K} e^{s_{l^*}}}} \; \cancel{= g_k g_l}$$

] $k \neq l \Rightarrow g_k(-g_l)$

] $k = l \Rightarrow \dfrac{\partial g_k}{\partial s_l} = \ldots = \dfrac{e^{s_k}}{\sum_{l^*=1}^{K} e^{s_{l^*}}} \cdot \left(1 - \dfrac{e^{s_k}}{\sum_{l^*=1}^{K} e^{s_{l^*}}}\right) = g_k(1-g_k)$

$\Rightarrow \dfrac{\partial g_k}{\partial s_l} = g_k\left(I(k=l) - g_l\right)$

② $\dfrac{\partial R^{(i)}}{\partial g_k} = -\dfrac{I(y^{(i)}=k)}{g_k}$

$$\frac{\partial R^{(i)}}{\partial g_k} = \frac{\partial}{\partial g_k}\left(-\sum_{m=1}^{K} I(y^{(i)}=m)\ln g_m\right) = -\frac{I(y^{(i)}=k)}{g_k}$$

③ $\dfrac{\partial R^{(i)}}{\partial g_l} = g_l - I(l = y^{(i)})$

$$\frac{\partial R^{(i)}}{\partial s_l} = \frac{\partial}{\partial s_l}\left(-\sum_{m=1}^{K} I(y^{(i)}=m)\ln g_m(\ldots)\right) =$$

$$= -\sum_{m=1}^{K} \frac{I(y^{(i)}=m)}{g_m}\cdot\frac{\partial g_m}{\partial s_l} = -\sum_{m=1}^{K}\frac{I(y^{(i)}=m)}{g_m} g_m\left(I(m=l) - g_l\right)$$

$$= g_l \underbrace{\sum_{m=1}^{K} I(y^{(i)}=m)}_{=1} - \sum_{m=1}^{K} I(m=l)\, I(y^{(i)}=m) =$$

$$= g_l - \sum_{m=1}^{K} I(m=l)\, I(y^{(i)}=m) = g_l - I(y^{(i)}=l)$$