# On the Asympthotic Behavior of the Coefficient Field of Newforms modulo $p$

Breki Pálsson

15. September 2023

Let $f \in \mathcal{S}_k(N)$ be a cusp form of level $N$ and weight $k$. There are $\mathbb{C}$ linear maps for $n \in \mathbb{N}, T_n : \mathcal{S}_k(N) \to \mathcal{S}_k(N)$, which commute. These maps are called Hecke operators

We have two choices:

1. Consider only $n$ such that $\gcd(n, N) = 1$
2. Or we work with the "new space"

Then $T_n$ is diagonilisable. As they commute the $T_n$ are simutaneously diagonilisable.

There are common eiganvectors called "Hecke eigenforms" then we have a nice formula for the coefficients.

**Remark 0.1**

Vary $f$ and see how $\mathbb{Q}_f$ varies.

**Theorem 0.1**

(Shimura, Eichler) $\mathbb{Q}_f := \mathbb{Q}(a_n(f) | n \in \mathbb{N})$ is a number field.

## Conjectures

**Remark 0.2**

Maeda's conjecture:
For any normalized Hecke eiganform $f \in \mathcal{S}_k(1)$, we have

- $[\mathbb{Q}_f : \mathbb{Q}] = d_k$, where $d_k = \dim(\mathcal{S}_k(1))$ in other words "as big as possible"
- For any $p$, $\mathbb{Q}_f = \mathbb{Q}(a_p(f))$ for some $f \in \mathcal{S}_k(1)$.
- $\mathbb{Q}_f/\mathbb{Q}$ is not *Galois*, tha Galois group of the Galois closure of $\mathbb{Q}_f/\mathbb{Q}$ is $S_{d_k}$, the symmetric group on $d_k$ letters. Or in other words "as big as possible"

We will first look at the case where $N = 1$, and vary the weight $k$.

**Remark 0.3**

Van der Waeiden's Conjecture:
100% of all the irredicoble integral polynomials have a symmetric group as a Galois group.

## Results

**Remark 0.4**

Paper by kimball Martin: Generalsasipn of maeda's Conjecture. In particular he treats the case of $\mathcal{S}_2(\Gamma_0(N))$ for varying squarefree $N$ (or simply $N$ prime).

Let $E/\mathbb{Q}$ be an elliptic cure. Then by Wiles we can associate with it a Hecke eiganform of weight 2 and level $\Gamma_0(N)$ where $N$ is the conclsion? of $E$ which is squarefree for semi-stable elliptic curves.

$$f_E(z) = \sum_{n=1}^{\infty} a_n(E) q^n$$

Where $a_p(E) = p + 1 - \#E(\mathbb{F}_p) \in \mathbb{Z}$

This means that $\mathbb{Q}_f = \mathbb{Q}$

In $S_k(\Gamma_0(N))$ with $N$ squarefree, for every $p|N$ we have a Artin-Lehner involuiton $\iota_p$ that commutes with $T_p$

$$\mathcal{S}_k(N) = \mathcal{S}_k(N)^+ \oplus \mathcal{S}_k(N)^-$$

Where the Hecke oppetators give a $+$ and a $-$ eiganspace

If we restrict ourselfs to $N = 1$ i.e. "Normal Maeda":

$\mathbb{Q}_f = \mathbb{Q}(a_p(f)) \implies a_p(f)$ is a zero of tne

# 1 Preliminaries

# 2 Newforms

**Definition 2.1**

Let $\mathcal{M}_k$ be the space of entire modular forms of weight $k$ and $\mathcal{S}_k$ be the space of cusp forms. The mapping

$$\langle \cdot, \cdot \rangle : \mathcal{M}_k \times \mathcal{S}_k \to \mathbb{C}, \quad \langle f, g \rangle := \int_{\mathcal{F}} f(\tau)\overline{g(\tau)}(\operatorname{im}\tau)^k d\nu(\tau)$$

is called the Petersson inner product, where $\mathcal{F}$ is the fundamental region of the modular group and if we denote $\tau = x + iy$ we let $d\nu(\tau) = y^{-2}dxdy$ be the hyperbolic volume form.

Let

$$\alpha_t : \mathcal{S}_k(\Gamma_1(M)) \to \mathcal{S}_k(\Gamma_1(N))$$

be the *degeneracy map*, which is given by $f(q) \to f(q^t)$.

**Definition 2.2**

We will define the *old subspace* of $\mathcal{S}_k(\Gamma_1(N))$ denoted by $\mathcal{S}_k(\Gamma_1(N))_{old}$, is the sum of the images of all maps $\alpha_t$ with $M$ a proper divisor of $N$ and $t$ any divisor of $N/M$, note that $\alpha_t$ depends on $t, N$ and $M$ so there is a slight abuse of notation. the new subspace of $\mathcal{S}_k(\Gamma_1)$, which we denote by $\mathcal{S}_k(\Gamma_1)_{new}$, is the intersection if the kernel of all the maps $\beta_t$ with $M$ a proper divisor of $N$.

One can use the Petersson inner product to show that

$$\mathcal{S}_k(\Gamma_1) = \mathcal{S}_k(\Gamma_1)_{new} \oplus \mathcal{S}_k(\Gamma_1)_{old}$$

Moreover the new and old subspaces are preserved by all Hecke operators.

**Theorem 2.1**

(Atkin, Lehner, Li). We have a decomposition

$$\mathcal{S}_k(\Gamma_1 N) = \bigoplus_{M|N} \bigoplus_{d|N/M} \alpha_d(\mathcal{S}_k(\Gamma_1(M)))_{new}$$

Each space $\mathcal{S}_k(\Gamma_1(M)_{new})$ is a direct sum of distinct (nonisomorphic) simple $\mathbb{T}_{\mathbb{C}}^{(N)}-$modules.

**Definition 2.3**

A *newform* is a $\mathbb{T}$-eigenform $f \in \mathcal{S}_k(\Gamma_1(N))_{new}$ that is normalized so that the coefficient of $q$ is 1

## Computing the Coefficient field of a newform

## 3 Conjectures

Idea we want to look at is the mod $p$ version of maedas conjecture.

As $\mathbb{Q}_f$ is a number field, we can factorize $p$ for any prime into prime ideals

$$p\mathcal{O}_f = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

We want to study the sequence

$$\{[\mathcal{O}_f/\mathfrak{p}_1 : \mathbb{F}_p], \cdots, [\mathcal{O}_f/\mathfrak{p}_r : \mathbb{F}_p]\}$$