MATH 189:

Discrete Mathematics

Lecture Notes

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Lecture 1

Lecture 1

What is discrete mathematics?

Brennan Becerra | 2023-01-09

Discrete Mathematics

- Separated
- Integers, $\mathbb{Z}$ $\ldots, -2, -1, 0, 1, 2, \ldots$
- Natural Numbers, $\mathbb{N}$, $0, 1, 2, 3, \ldots$

Continuous

- Calculus, algebra
- Function of <u>real</u> variable
- Real numbers $\mathbb{R}$, $-e, 0, \pi$

Topics

- Combinatorics (counting)
- Sequences
- Graph theory
- Number theory

Language

- Set theory & logic
- First order logic(Quantification logic)
- Functions

Logic

- Propositional logic(Sequential)

Sentences

- Boolean valued statements(True or False)
- Can be denoted by $p, q, r, \ldots$

Examples

$$\begin{array}{ll} 3 = 3 & T \\ 2 < 3 & T \\ F(X) = x^2 \text{ is a continuous function} & T \\ 1 = 0 & F \end{array}$$

Non Examples

- Statements that do not have a truth value

$$\begin{array}{ll} 1 + 1 & \\ 4x = 12 & \text{Depends on what x is} \end{array}$$

Atomic Statements

- A statement that cannot be broken down into smaller parts

Logical Connectives

- Negation $\neg$ "not"
- Conjunction $\wedge$ "and"
- Disjunction $\vee$ "or"
- Implication $\implies$ "if, then"
- Biconditional $\iff$ "If and only if"

Each proposition is represented by a propositional variable $(p, q, r, s, \ldots)$

$p \wedge q$ is true if p,q are true.
$p \vee q$ is true if at least one of p,q is true.
$p \implies q$ is true if p is false or q is true.
$p \iff q$ is true if both p,q are true, or both are false.
$\neg p$ is true if p is false.

Conditional

$$\underbrace{p \implies q}_{\text{if p, then q}}$$

- p is the **hypothesis** or **antecedent**
- q is the **conclusion** or **consequent**

Example

$$\begin{array}{lc} & n < 10 \implies n < 20 \\ n = 5: & \underbrace{5 < 10}_{T} \implies \underbrace{5 < 20}_{T} \\ n = 15: & \underbrace{15 < 10}_{F} \implies \underbrace{15 < 100}_{T} \end{array}$$

Proving a Conditional

How do we know that the statement "if p, then q." is true or false?

- Assume p is true, "do something"
- Conclude q is true

Example

Defenition
A natural number, a is even if there exists a natural nı

Theorem
If a is even, then a^2 is even.

Proof

Assume a is even, so there is a number, n, s.t. $a = 2n$.

$$
\begin{aligned}
a^2 &= a \times a \\
&= (2n)(2n) \\
&= 2(n \cdot 2 \cdot n) \\
&= 2(2n^2)
\end{aligned}
$$

By defenition, there is a number, namely, $2n^2$, so that $a^2$is even.

Converse statements

Given a conditional, $p \implies q$

Converse: $q \implies p$

Contrapositive: $\neg q \implies \neg p$

Example

If f is differentiable at $x = a$ then f is continuous at $x = a$.

Converse:

If f is continuous at $x = a$, then f is differentiable at $x = a$.

FALSE

Contrapositive:

If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

TRUE

Biconditional

$$(p \implies q) \iff (\neg q \implies \neg p)$$
$$(p \implies q) \iff ((p \implies q) \wedge (q \implies p))$$

Lecture 2

Predicate, Quantification

A **Predicate** is a formula that contains variables. $P(x)$, $F(x, y)$

A **Sentence** is a predicate formula with no free variables.

A free variable is something like $4x = 12$, which isn't "bounded."

Sentences have truth values.

Universal Quantifier

For every, for all, $\forall$

$(\forall n)(n < 10 \implies n < 20)$

$(\forall x)(P(x))$

Existential Quantifier

There is, there exists, $\exists$

$(\exists x) 4x = 12$

Consider the collection of even numbers.

0,2,4,6...

$\exists x, 4x = 12$ is false if we are only considering the set of even numbers.

Domain of discourse is the set of values that quantifiers can pull from.

Example

Consider the following formula:

$$(\forall x)(\exists y) y < x.$$

Find a domain where this sentence is true, and one where it is false.

First domain is $\mathbb{N}$.

$(\forall x)$ lets us pick anything.

$(\exists y)$ means we have to find a y but it depends on x.

Choose $x = 5$ s.t. $y = 4$ then $4 < 5$ is true in $\mathbb{N}$.

If we choose $x = 0$, then there is no y so $(\forall x)(\exists y) y < x$ is false in $\mathbb{N}$.

Consider $\mathbb{Z}$.

Set $x = n$, set $y = n - 1$ then $n - 1 < n$ is true.

So $(\forall x)(\exists y) y < x$ is true in $\mathbb{Z}$.

Negating Quantifiers

$\neg(\forall x)(P(x))$ it is not the case that every x has property p. There is an x with property $\neg p$ $(\exists x)\neg P(x)$.

"Proof by counterexample"

$$\neg(\exists x)(P(x)) \iff (\forall x)\neg P(x)$$

e.g

$$\neg(\forall x)(\exists y)y < x \iff (\exists x)\neg(\exists y)y < x$$

$$\iff (\exists x)(\forall y)\neg y < x$$

$$\iff (\exists x)(\forall y, y \geq x)$$

$$(\forall x)(\exists y)y \leq x \text{is false in } \mathbb{N}$$
$$(\exists x)(\forall y)y \geq x \text{ is true in } \mathbb{N}$$

Set Theory

Informally sets as a collection of objects.

e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \{0\}$

$x \in \mathbb{N}, 1 \in \mathbb{N}, 50 \in \mathbb{N}, \pi \notin \mathbb{N}$

$x \in A$ x is a member(element) of the set A.

Example

$A = \{1, 2, 3\}$

$A = \{2, 1, 3\}$

$A = \{2, 1, 3, 3\}$

These are all the same set.

- Sets do not see order or multiplicity.

Sets A, B are equal if each have exactly the same elements.

$$\begin{aligned}&A \subset B\\&\text{A is a subset of B}\\&\text{if every element of A is in B.}\\&(\forall x)(x \in A \implies x \in B)\end{aligned}$$

Example

$$\mathbb{N} \subset \mathbb{Z} \subset \boldsymbol{Q} \subset \mathbb{R} \subset \mathbb{C} \ldots$$

Facts

For all sets A, $A \subset A$. For all sets A, $\emptyset \subset A$.

$\underbrace{x \in \emptyset}_{\text{false}} \implies x \in A$

Set Builder Notation

(Filter sets out by properties)

$\{x \in A \mid P(x)\}$

e.g. even

$\{x \in \mathbb{N} \mid (\exists n) x = 2n\}$

$\{x \in \mathbb{N} \mid x > 1 \wedge (\forall y)(\forall z)(x = y \cdot z) \implies (z = 1 \vee y =$

Union

$$A \cup B \text{ is the set of elements in A or in B.}$$

$$x \in A \cup B \iff x \in A \vee x \in B$$

Example

$$\begin{aligned}&A = \{2,3,4\},\ B = \{4,5,6\}\\&A \cup B = \{2,3,4,5,6\}\end{aligned}$$

Intersection

$A \cap B$ is read "A intersect B"

The set of elements in A and in B.

$$x \in A \cap B \iff x \in A \wedge x \in B.$$

Example

$$\begin{aligned}&A = \{\text{Evens}\},\ B = \{\text{Primes}\}\\&A \cap B = \{2\}\end{aligned}$$

Facts

If $A \subset B$ then $A \cup B = B$

If $A \subset B$ then $A \cap B = A$

Lecture 3

Set operations

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A **set universe** $\mathcal{U}$ is a relative set to do set operations in.

- There is no set of all sets

Let $U = \{x : x = x\}$ be the set of all things. This cannot exist.

Russell's Paradox.

"If we have a universe of sets, can this set contain itself?"

$R = \{x|x \notin x\}$ is the set of sets that don't contain themselves. Is $R \in R$?

$R \in R \implies R \notin R$

$R \notin R \implies R \in R$

- $\neg p \wedge p$ is not possible

The compliment of a set

The **compliment** of a set A, denoted $\overline{A}$ relative to a universe, $\mathcal{U}$ is the set of all $x \in \mathcal{U}$ not in A.

$$\overline{A} = \{x \in \mathcal{U}|x \notin A\}$$

e.g.

$$\mathcal{U} = \{x \in \mathbb{N} | 0 \le x \le 20\}$$
$$A = \{\text{Evens between 0 and 20}\}$$
$$\overline{A} = \{1, 3, 5, \ldots, 19\}$$

Set Difference

The set $A \backslash\ B$ is the set of all elements of A not in B.

$\{x \in A | x \notin B\} = \overline{A} \cap \overline{B}$

$$A = \{1, 2, \ldots, 10\}$$
$$B = \{2, 4, 6, 8, 10\}$$
$$A/B = \{1, 3, 5, 7, 9\}$$

Cardinality

The cardinality of a set A, denote $\mid A|$ is the numbers of elements of A.

e.g. $\mid \{1, 2, 3\} \mid= 3$

$\mid \{\text{evens between 0 and 20}\} \mid= 11$

A set is **finite** if $\mid A \mid= n$, $n \in \mathbb{N}$.

A set is **countable** if it is finite or is the size of $\mid \mathbb{N} \mid$.

Power Set

If A is a set, then $\mathcal{P}(A)$, the **power set,** is the set of all subsets of A.

$$\mathcal{P}(A) = \{B | B \subseteq A\}$$

If $\mid A \mid = n$ then $\mid \mathcal{P}(A) \mid = 2^n$

e.g.

$A = \{1, 2\}$

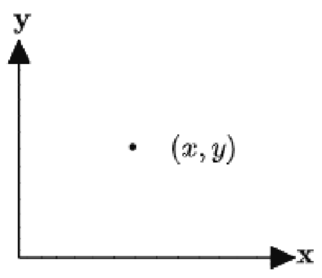
$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$

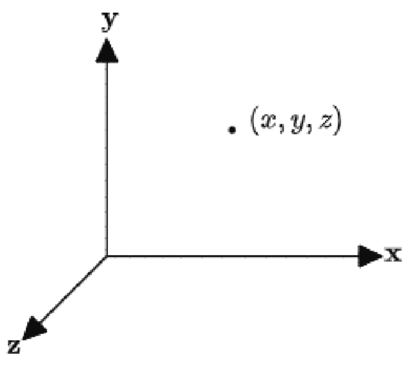
If A is set $\emptyset \in \mathcal{P}(A)$.

Cartesian Product

$A \times B$ is $\{(x, y) | x \in A \wedge y \in B\}$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \qquad \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$





$A = \{1, 2\}$ $B = \{3,$

$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 4), (2, 5)\}$

If $\mid A \mid= n, \mid B \mid= m$ then $\mid A \times B \mid= n \cdot m$.

It is rarely the case that $A \times B = B \times A$.

Lecture 4

0.4

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Functions

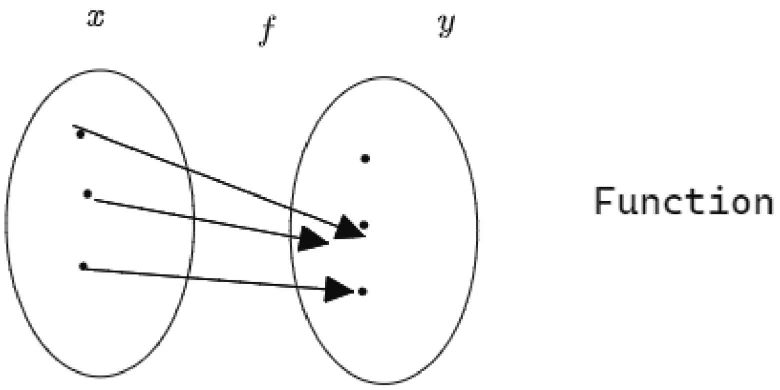
A function (map) is a relationship between sets that associates one input with exactly one output.

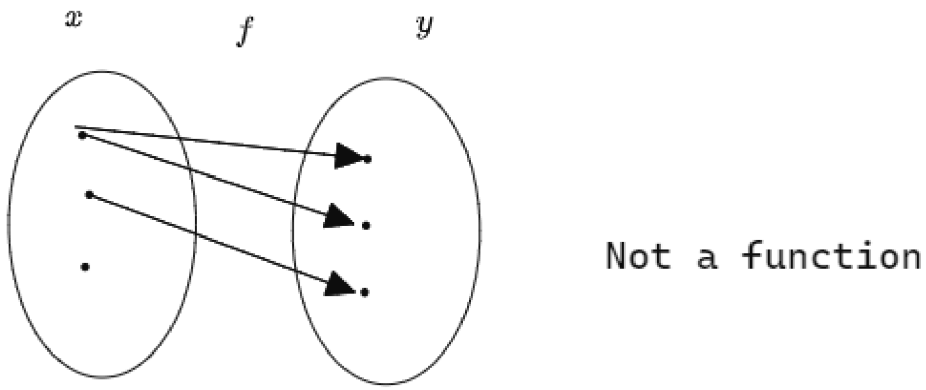
Notation

$f : x \implies y$ x is the domain and x is the codomain.

If $x = y$, then $f(x) = f(y)$.

Example





A function must be defined on its entire domain.

Image

Images can be defined by:

- Points
- Domain
- Subsets of the domain

$$\text{if} f : x \implies y \text{ is a function}$$
$$x \in X \implies f(x) = y \text{ in the image point of } x.$$
$$\text{The image of the domain, X, f(x), isthe range.}$$

$$\text{If } A \subseteq X, \text{then } f(A) = \{y \in Y | \exists x \in A f(x) = y\}$$

e.g.

$f : \mathbb{R} \implies \mathbb{R}\ f(x) = x^2$

$f(2) = 4$

$f(\mathbb{R}) = 0 \leq x < \infty$

Suppose $A = \{x \in \mathbb{R} | \mid x \mid \geq 1\} \subseteq \mathbb{R}$

$f(A) = 0 \leq x < \infty$

$f : \mathbb{N} \implies \mathbb{N}\ f(n) = 2n$

dom = cod = $\mathbb{N}$

$f(\mathbb{N}) = \{\text{Even}\}$

$f(\{1, 2, 3\}) = \{2, 4, 6\}$

---

define $g : \{1, 2, 3, 4\} \implies \mathbb{N}$

x	$g(x)$
1	1
2	4
3	9
4	16

non example

$f : \mathbb{N} \implies \mathbb{N}\ f(n) = \frac{n}{2}$

$f(1) = \frac{1}{2} \notin \mathbb{N}$

$$f : \mathbb{N} \implies \mathbb{Z}$$

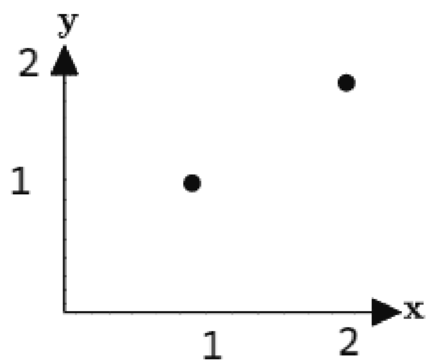
$$f(n) = \begin{cases} -\frac{n}{2} \text{ if n is even} \\ \frac{n+1}{2} \text{ if n is odd} \end{cases}$$

n	$f(n)$
0	0
1	1
2	-1
3	3
4	-3

Definition by cases.

Finite sets

Graphically



$$f : \{0, 1, 2\} \implies \{0, 1, 2\}$$

Matrix notation

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

x	y
0	2
1	1
2	2

Recursion

- Used for defining functions on $\mathbb{N}$
- Defines values of a procedure that depend on previous values of that procedure

Example

$f(n) = n!$

$$\begin{aligned} f(0) &= 1 \text{ Initial condition or base case} \\ f(n+1) &= (n+1) \cdot f(n) \text{ Recursive stop} \end{aligned}$$

$$\begin{aligned} & f(3) \\ & = f(2+1) \\ & = (2+1)(f(2)) \\ & = 3 \cdot f(1+1) \\ & = 3(1+1) \cdot f(1) \\ & = 3 \cdot 2 \cdot f(0+1) \\ & = 3 \cdot 2 \cdot (0+1) \cdot f(0) \\ & = 3 \cdot 2 \cdot 1 \cdot 1 \\ & = 6 \end{aligned}$$

The $\mathbb{N}$ is a recursively defined set.

$0 \in \mathbb{N}, \text{if } n \in \mathbb{N}, \text{then } n+1 \in \mathbb{N} \text{ no other natural numb}$

Sentences in propositional logic are recursively defined.

Base $p, q, r, \ldots$

recursive step if ϕ, ψ are propositional sentences then so are

$\phi \wedge \psi, \phi \vee \psi, \phi \implies \psi, \phi \iff \psi, \neg\phi$

Lecture 5

Properties of functions

Subtitle

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A **function** $f : x \implies y$ is **surjective** if every element of y is the image point of an element of x.

$(\forall y \in y)(\exists x \in x)\, f(x) = y$

$$f : \mathbb{N} \implies \mathbb{Z} \qquad f(x) = \begin{cases} -\frac{x}{2} \text{ x is even} \\ \frac{x+1}{2} \text{ x is odd} \end{cases}$$

Let $n \in \mathbb{Z}$ suppose n is negative. Let $x = -2n.$ We will get a positive value for x such that $x \in \mathbb{N}$.

Suppose n is positive. Set $x = 2n - 1$

$f(2n-1) = \frac{2n-1+1}{2} = n$

If $n = 0$ then $f(0) = 0$.

Injections

$(\forall x)(\exists y)(f(x) = f(y) \implies x = y)$

$f : \mathbb{N} \implies \mathbb{N}\, f(n) = 2n$

Let $n, m \in \mathbb{N}$

$f(n) = 2n\ f(m) = 2m$

If $2n = 2m$ then $n = m$.

$$f : \mathbb{N} \implies \mathbb{Z} f(x) = \begin{cases} -\frac{x}{2} \text{ x is even} \\ \frac{x+1}{2} \text{ x is odd} \end{cases}$$

Suppose $f(n) = f(m)$

$$-\frac{n}{2} = -\frac{m}{2} \implies n = m$$
$$\frac{n+1}{2} = \frac{m+1}{2} \implies n = m$$

This function is injective.

Bijections

A function is a bijection if it is both injective and surjective.

e.g.

$$f : \mathbb{N} \implies \mathbb{Z} \qquad f(x) = \begin{cases} -\frac{x}{2} & \text{x is even} \\ \frac{x+1}{2} & \text{x is odd} \end{cases}$$

e.g $f : \mathbb{R} \implies \mathbb{R}\, f(x) = x^3$ is a bijection

e.g. $f : \mathbb{N} \implies \mathbb{N}\, f(n) = n^3$ **is not a bijection**

If $f : x \implies y$ is a bijection, then $|x| = |y|$.

$f : \mathbb{N} \implies \mathbb{Z}$ proves that $|\mathbb{N}| = |\mathbb{Z}|$.

If A, B are finite in $|A| = |B| = n$ means there are bijective functions

Lecture 6

Counting(combinatorics)

0.5

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Counting

An **event** is a set of outcomes.

The Additive principle

If event A has m outcomes and event B has n outcomes, and A and B are disjoint, then the total outcomes for A or B is $n + m$.

A **string** is a finite list of characters from some alphabet.

e,g. How many strings of length 2 in the English alphabet are there that start with C or D?

C *or D*

26 outcomes

CA CB ... CZ

26 outcomes

DA DB ... DZ

26+26 =52 outcomes

e.g. 6 different flavors of ice cream and 4 different toppings. How many different ice cream topic combinations can you get?

4+4+4+4+4+4=24 $= 6 \cdot 4$

Multiplication principle

Event A has m outcomes and each outcome of A has n many outcomes of B, then the event $A \wedge B$ has $m \cdot n$ many outcomes.

License Plates

Idaho L-plates

$44 \_ \_ \_ \_ \_$

44 36 36 36 36 36

$44 \cdot 36^5$

ATM PINs

$\underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 10^4 = 10,000$

How many functions there are from one finite set to another

$$f : \{1, 2, 3\} \to \{a, b, c, d\}$$

- $f(1)$ has 4 options
- $f(2)$ has 4 options
- $f(3)$ has 4 options

$4^3 = 64$

$f = \underbrace{\{1, 2, 3, \dots, n\}}_{\text{Total repetitions}} \to \underbrace{\{1, 2, 3, \dots, k\}}_{\text{Total options}}$

k^n

e.g.

Suppose a friend is over

- Two movies sequentially, horror first, then action
- One movie
- 7 Horror, 6 Action

Let $H = \{\text{Horror}\},\ \text{and } A = \{\text{Action}\}$

$|H| = 7,\ |A| = 6$

$|H \cup A| = 13 = |H| + |A|$ because $H \cap A = \emptyset$

$H = \{h_1, h_2, \ldots, h_7\}$

$A = \{a_1, a_2, \ldots, a_6\}$

$$\begin{array}{ll} (h_1, a_1), (h_1, a_2), \ldots, (h_1, a_6) & \\ (h_2, a_1), (h_2, a_2), \ldots, (h_2, a_6) & \\ \vdots & \\ (h_7, a_1) & (h_7, a, 6) \end{array}$$

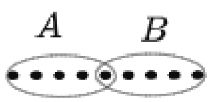
The set of these ordered pairs is $H \times A$

$|H \times A| = 42 = |H| \cdot |A|$

Additive property

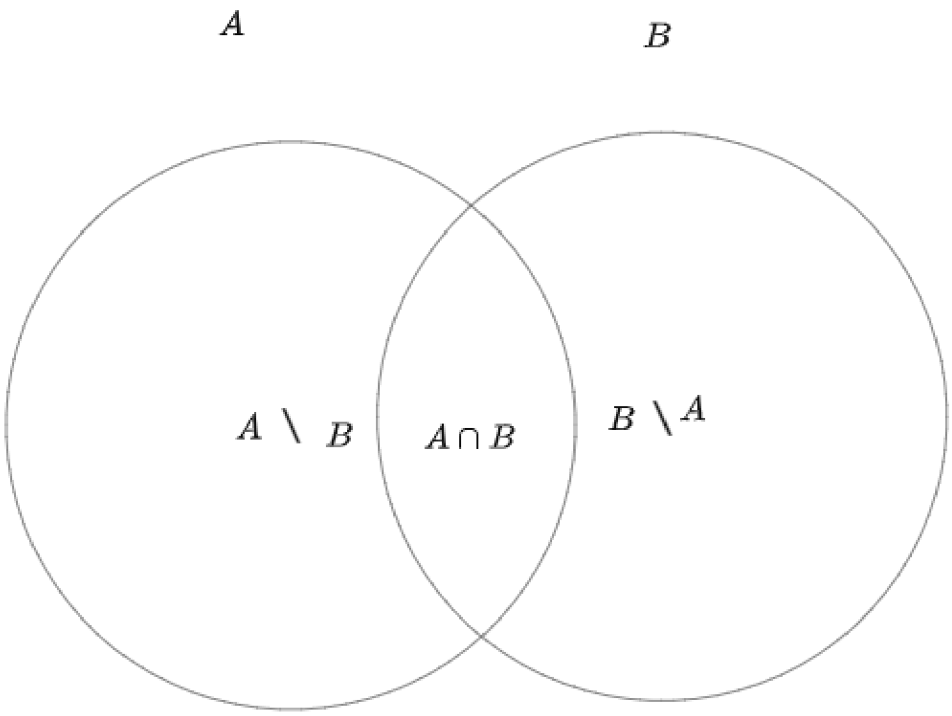
If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$

Suppose $A \cap B \neq 0$



$|A| = 4 = |B|$

Disjointigius

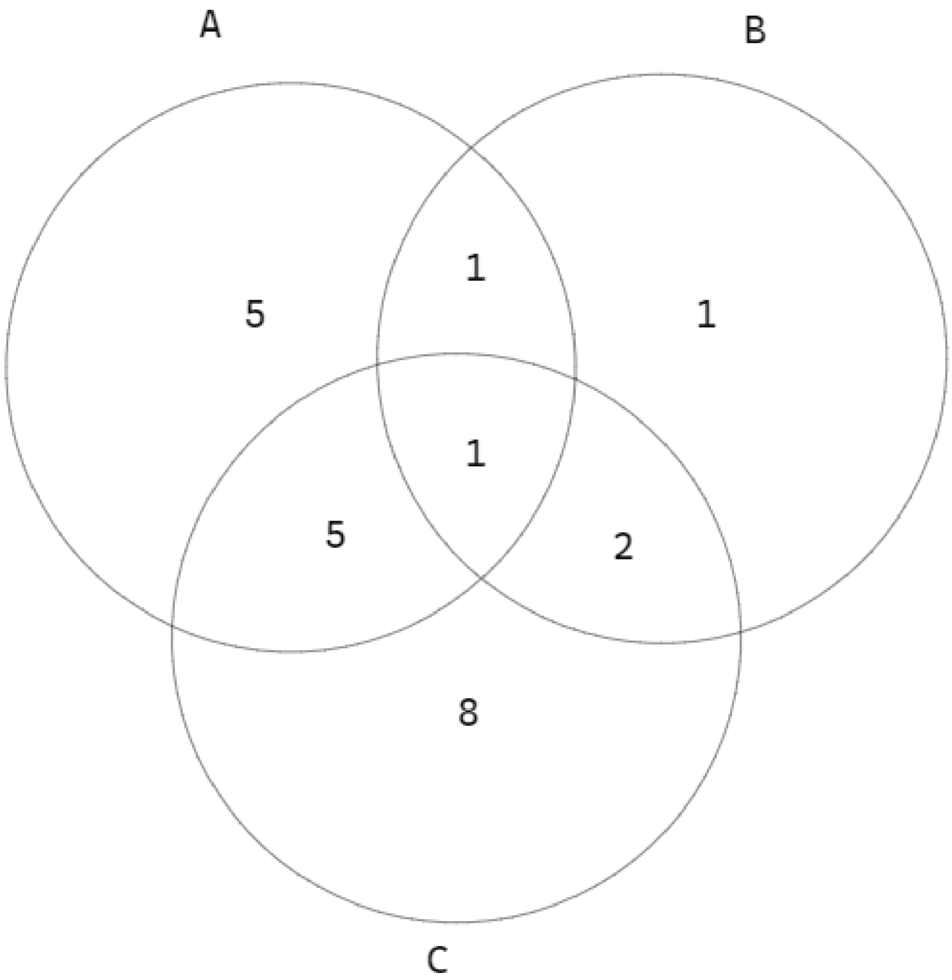


Principle of Inclusion/Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

e.g. Exam for 3 subjects Algebra, Biology and Chemistry taken by 41 students. How many failed at least 1 exam?

Subject	A	B	C	AB	AC	BC	ABC
Failed	12	5	8	2	6	3	1



15 People failed at least one exam.

PIE

$$
\begin{aligned}
&|A \cup B \cup C| = |A| + |B| + |C| \\
&- (|A \cup B| + |A \cup C| + |B \cap C|) \\
&+ |A \cap B \cap C|
\end{aligned}
$$

Lecture 7

Binomial Coefficients

1.2

Brennan Becerra | 2023-01-25

Binary Strings

- Ordered lists of 0s and 1s

The set of all binary sequences are functions from $\leq n$ to the set $\{0, 1\}$.

e.g

$f : \{1, 2, 3, \ldots 9\} \to \{0, 1\}$ sequence: $(1, 0, 1, 0, 0, 1, 1, 1, 1)$

$\sigma = \{0, 1\}$

σ^* words on 0,1

def: empty word exists if w is a word then wo, w1 are words.

Characteristic Functions

$$f : A \to \{0, 1\} s.t.$$
$$f(n) = \begin{cases} 1 & \text{if n} \in A \\ 0 & \text{if n } \notin A \end{cases}$$

If A is a set, then $|\mathcal{P}(A)| = |\text{characteristic functions on subsets of A}|$

If $|A| = n$ then $\mathcal{P}(A) = 2^n$

Motivating Question

If A has $|A| = n$ and $k \leq n$ how many subsets of A have size k?

If $k = n$, how many subsets of size n does A have?

A: 1 subset of size n, which is A itself.

If $k = 0$ how many subsets?

A: 1, $\emptyset$.

If $|A| = n$ then A has n many subsets of size 1.

Suppose that $|A| = 3$. This can have 8 subsets.

1 subset of size 0

1 subset of size 3

3 subsets of size 1

3 subsets of size 2.

$\{a, b, c\}$

There is a one-to-one correspondence between subsets of size 1 and size 2.

Suppose that $|A| = 4$. This can have 16 subsets.

1 subset of size 0

1 subset of size 4

4 subsets of size 1

4 subsets of size 3

6 subsets of size 2 (by process of elimination)

Suppose that $|A| = 5$ this has 32 subsets

1 subset of size 0

1 subset of size 5

5 subsets of size 1

4 subsets of size 4

(20 subsets of size 2 or 3)

10 subsets of size 2

10 subsets of size 3

Let B^n be the set of all binary sequences of length n. Let B^n_k be the set of all binary sequences of length n with exactly $\underbrace{k \text{ many 1s.}}_{\text{weight}}$

e.g. $B_1^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Corresponds to the subsets of size 1 with a set of size 1.

What is $|B_k^n$?

e.g.

Count B_3^5

B_3^5

Each sequence starts with a 0 or a 1.

If it starts with a 0, then the last 4 bits must have thre

If it starts with a 1, then the last 4 bits must have 2 ones.
B_2^4

$|B_3^5| = |B_2^4| + |B_3^4| = |B_1^3| + |B_2^3| + 4 = 3 + 3 + 4 = 10$

B_4^2
If it starts with a 0, then the last four bits must have t
B_1^3
If it starts with a 1 the last 3 bits must have one 1.

Consider the set $\{a,b,c,d,e\}$.

Binomial coefficient

If $n \geq 0$ $k \leq n$ then $\binom{n}{k}$ "n choose k" is defined as $|B_n^k$.

- The number of k element subsets of an n element set
- The number of ways to choose objects from n objects

Def by recursion

$$\binom{n}{0} = 1 = \binom{n}{n}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Lecture 8

Permutations and Combinations

Brennan Becerra | 2023-01-27

Permutations

- A re-arrangement of objects.

e.g. $A = \{1, 2, 3, 4, 5\}$ We want to select three of them

$\underline{5}\ \underline{4}\ \underline{3} = 60$ over counts

$\{a, b, d\}, \{a, d, b\}, \{b, a, d\}, \{b, d, a\}, \{d, a, b\}, \{d, b, a\}$

These are all <u>permutations</u> or rearrangements of a, b, d

There are $n!$ different permutations of n distinct objects.

$$\begin{aligned}
&f(0) = 0! = 1 \\
&f(n+1) = (n+1)n! = (n+2) \cdot f(n)
\end{aligned}$$

Counting functions

$$f : \{1, \ldots, n\} \to \{1, \ldots, k\}$$

$$\frac{k}{f(1)} \frac{k}{f(2)} \cdots \frac{k}{f(n)} = k^n$$

If $|x| = n$ and $|y| = k$ then the set of $f : x \to y$

Lecture 9

1.4 Combinatorial Proofs

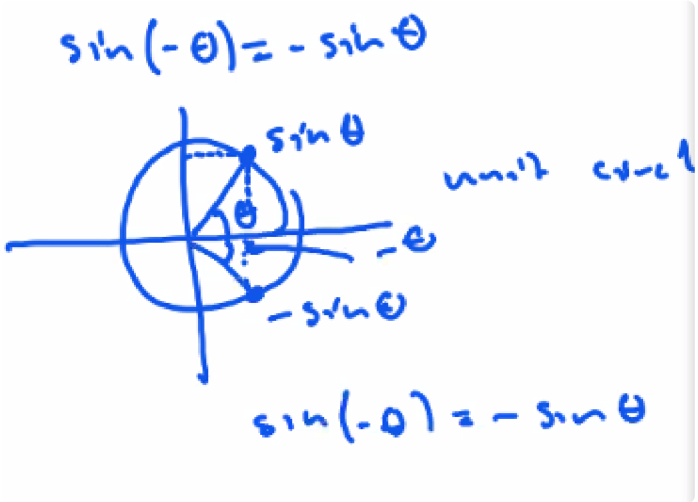
Trig "verify identities"

Algebraic

$$\begin{aligned}
&\sin^2\theta + \cos^2\theta = 1 \\
&\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \\
&\tan^2\theta + 1 = sec^2\theta
\end{aligned}$$

Geometric

$\sin(-\theta) = -\sin\theta$



Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots +$$

Interpret some Facts

Coefficients of x^n and y^n that are both 1, $\binom{n}{o}$, $\binom{n}{n}$

Coefficients are symmetric $\binom{n}{k} = \binom{n}{n-k}$

$1+4+6+4+7 = 16 = 2^4$

$$\sum \binom{n}{k} = 2^n$$

Summary of Binomial coefficients

- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{k} = \binom{n}{n-k}$
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ recursion
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Algebraically verify the recursion using the factorial definition

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} +$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k_1(n-k-1)!}$$

$$= \frac{(n-1)! \cdot k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}$$

$$= \frac{(n-1)!k + (n-1)!(n-k)}{k!(n-k)!}$$

$$= \frac{(n-1)!(k+n-k)}{k!(n-k)}$$

$$= \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}$$

Analogy

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$
$$= \sum_{n=1}^{\infty}\frac{1}{n!}$$
$$= \text{ Unique function that satisfies f'=f up to constants}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{0} = |\mathbf{B}_0^n| = |\{\underbrace{(0,0,\ldots,0)}_{\text{n-slots}}\}| = 1$$

$$\binom{n}{n} = |\{(1,1,\ldots,1)\}| = 1$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Define a function f on the n-bit strings $f : B^{n\to} B^n$

eg $f((1,0,1,1)) = (0,1,0,0)$, this many is a bijection

Reconstruct this function to B_k^n if a weight of the string is k

f of that string has weight $n-k$, range is B_{n-k}^n

$f : B_k^n \to B_{n-k}^n$ is a bijection

$$\binom{n}{k} = |B_k^n| = |B_{n-k}^n| = \binom{n}{n-k}$$

Explain $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ If $|A| = k$, then $\mathcal{P}(A) = 2^{n.}$

- $\binom{n}{0}$ select all 0 element subsets.
- $\binom{n}{1}$ select all 1 element subsets.
- $\binom{n}{2}$ select all 2 element subsets. ...
- $\frac{\binom{n}{1}}{|\mathcal{P}(A)|=2^n}$ select all 1 element subsets.

Slick way

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{n=0}^{n} \binom{n}{k}$$

Schema of Combinatorial Proofs

Binomial identity that you want to verify $A = B$

Come up with a combinatorial problem with solutions A and B.

Anagrams

Suppose we have 4 A's, 3 B's 2 C's 1 D. Form 10 letter anagrams with the letters.

$$-----------$$

$$\binom{10}{4} \text{ options for A}$$
$$\binom{6}{3} \text{ options for B}$$
$$\binom{3}{2} \text{ options for C}$$
$$\binom{1}{1} \text{ options for D}$$

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = 12600$$

Less than $\frac{n!}{n_1!n_1!...n_k!}$ in the situation $\frac{10!}{4!3!2!1!} = 12600$

Verify the following identity

$$1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 =$$

Come up with a combinatorial problem that has both sides as solutions.

$\binom{n+2}{3}$ is the number of 3 element subsets of a size $n+2$ set. $\{1, 2, \ldots, n+2\}$.

The three element subsets have the form $\{a, b, c\}$ assume $a < b < c$.

Count these sets with different fixed b values.

$b \neq 1$ because there is always an $a < b$ in the 3 element subset.

$b=2 \;\; a<2<c, \;\; 1<2<c$ there are n options for c(1 option for a)

$b=3$ $a<3<c$, 2 options, $n-1$ options for c. $2(n-1)$

$b=4$ $a<4<c$, 3 options for n, $n-2$ options for c. $3(n-2)$

...

$b=n+1$ $a<n+1<c$ n options for a, 1 option for c $n \cdot 1$

by the addition property $1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n \cdot 1) = \binom{n+2}{3}$

Lecture 10

Distribute 7 cookies to 4 kids

- don't break the cookies
- repetitions are allowed
- it is possible for a kid to get no cookies

4^7 assumes order matters, however, order does <u>not</u> matter

---

Encode the number f cookies given t o each kid as a 4-digit number.

3112 corresponds to :

- A: 3 cookies
- B: 1
- C: 1
- D: 2

The digits are not free.

7000 is valid code but 7123 is not valid

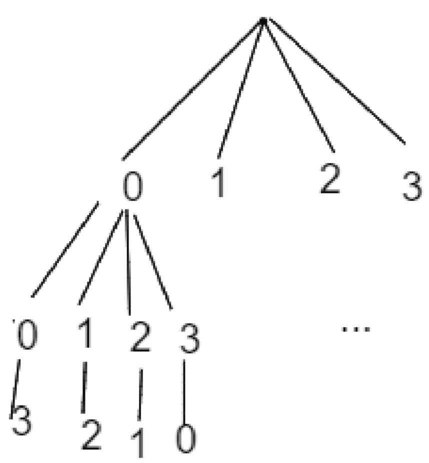
We're really asking the sum f the digits to be 7.

If $x_1, x_2, x_3, x_4 \geq 0$

The number of solutions is

$x_1 + x_2 + x_3 + x_4 = 7$

$$x_1, x_2, x_3 \geq 0$$
$$x_1 + x_2 + x_3 = 3$$



10 solutions

Think of the kids as bins!

Example:

- In the case of kids, A, B,C, D and 7 cookies, we could have:

ABAADC

Might as well list in order

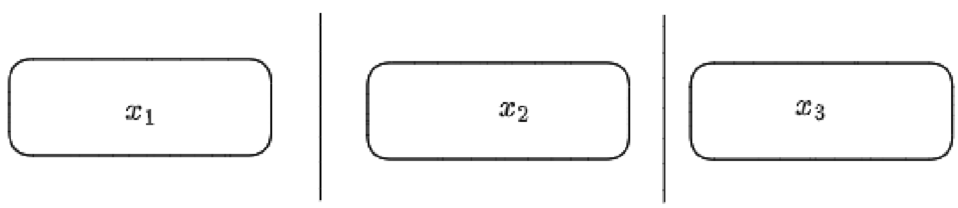
AAABCDD

In terms of starts and bars,

How man ways can we arrange 7 stars and 3 bars?

$15\binom{10}{3}$

3 stars and 2 bars



**|*|

$\binom{5}{2}$

Distribute 7 cookies to 4 kids, with the stipulation that every kid must have at least one cookie

We are now thinking about distributing 3 cookies to 4 kids

$\binom{6}{3}$

**|*||

If we want to distribute n many indistinguishable objects into k many bins, then the formula is

$\binom{n+k-1}{k-1}$

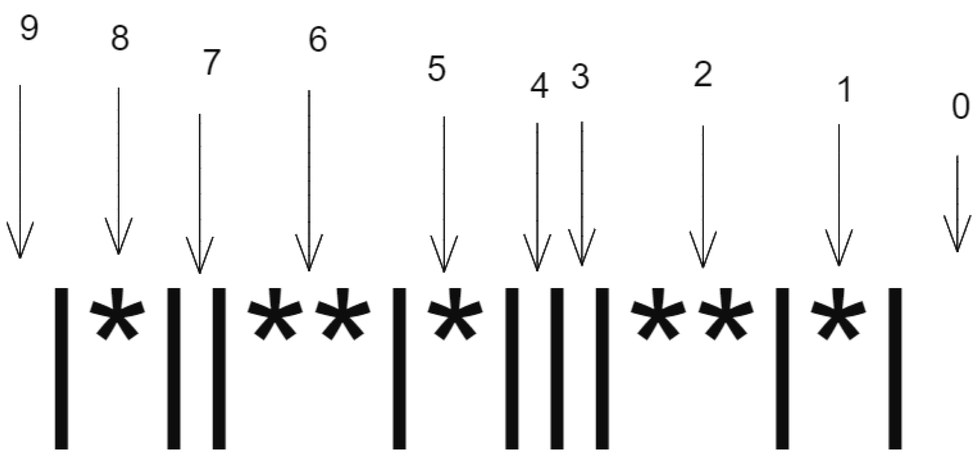
$k-1$ is the number of bars and n is the number of stars.

Example:

How many 7-digit phone numbers are there in which the numbers are non-increasing?

Note: Non-increasing means that if $x < y$ then $f(x) \geq f(y)$. Decreasing means that $x < y$ then $f(x) \geq f(y)$

Bins: are the possible $\{0, \ldots, 9\}$ digits.



Lecture 11

3.1

Logic

Brennan Becerra | 2023-02-13

Truth and deductions(proof)

Truth: Semantics

- Start with *true* statements and have a way to generate *new* statements in a "truth preserving way"

Deductions: Syntax

- "symbol pushing"

Argument

- List of statements (premises) where the last statement is the conclusion
- An argument is **valid** if whenever the *premises* are true, the conclusion must be true

"If It's raining outside, then I carry an umbrella"

"It's raining outside" therefore "I carry an umbrella"

Modus Ponens

$p \implies q$

p therefore q

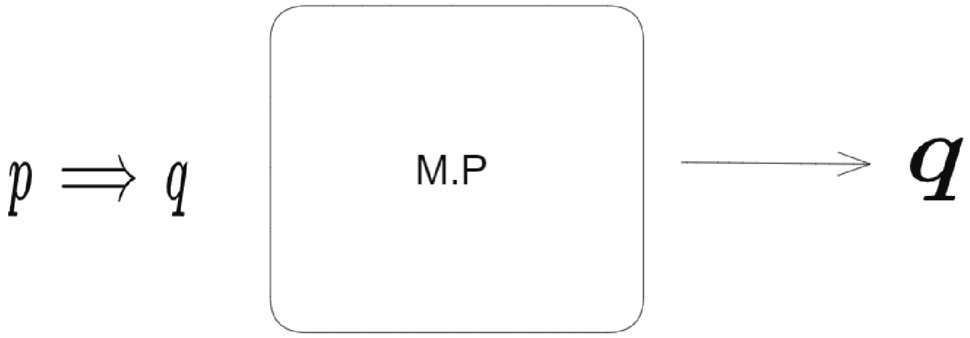
Invalid reasoning "I carry an umbrella" therefore "It's raining outside"

- Affirming the consequent
- Carrying an umbrella does not determine the weather

$p \implies q$

q therefore p

Modus Ponens is an example of the deduction rule (rule of inference).



1. $p \implies (q \implies r)$
2. p
3. q

therefore

4. $q \implies r$ 1,2 Mp
5. r 3,4 Mp

Predicate logic

All humans are mortal. Socrates is human. Therefore Socrates is mortal

- $H(x)$ "x is human"
- $M(x)$ "x is mortal"
- s "Socrates"

Syntax of "verbal" proof

$$1.(\forall x)(H(x) \implies M(x))$$
$$2.H(s)$$
$$3.H(s) \implies M(s) \text{ Universal instantiation 1.}$$
$$4.M(s)3, 2MP$$

There are a zoo of different deduction rules.

- $\neg\neg p \implies p$ Double negation elimination
- $(p \implies (q \implies r)) \implies ((p \wedge q) \implies r)$ Importation
- $\frac{q}{p \wedge q}$ conjunction introduction

Truth

Recall that the truth of a propositional statement depends only on the atomic statements and the logical connectives that make up the statement.

5 Truth Tables

Negation

$$\begin{array}{c|c} p & \neg p \\ \hline t & f \\ f & t \end{array}$$

And

p	p	$p \land q$
t	t	t
t	f	f
f	t	f
f	f	f

Or

p	p	$p \lor q$
t	t	t
t	f	t
f	t	t
f	f	f

Conditional

p	q	$p \implies q$
t	t	t
t	f	f
f	t	t
f	f	t

Biconditional

p	q	$p \iff q$
t	t	t
t	f	f
f	t	f
f	f	t

Example

Truth table of $\neg p \lor q$

p	q	$\neg p$	$\neg p \vee q$
t	t	f	t
t	f	f	f
f	t	t	t
f	f	t	t

$\neg p \vee q$ is logically equivalent to $p \implies q$

Two statements with the same column in the TT are called **logically equivalent.**

Example

$p \implies p$

p	$p \implies p$
t	t
f	t

All $t's$ in a column is called a **tautology**

$p \wedge \neg p$

p	$\neg p$	p
t	f	f
f	t	f

Check whether $p \implies (q \implies r)$ is logically equivalent to $(p \wedge q) \implies r$

p	q	r	$q \implies r$	$p \implies (q \implies r)$	$(p \wedge q) \implies$
t	t	t	t	t	t
t	t	f	f	t	f
t	f	t	t	f	

Demorgaus Laws

$\neg(p \wedge q) \iff \neg p \vee \neg q$

$\neg(p \vee q) \iff \neg p \wedge \neg q$

In terms of sets

$X \notin A \cap B <> x \in A$ or $x \in B$

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
t	t	t	f	f	f	f
t	f	f	f	t	t	t
f	t	f	t	f	t	t
f	f	f	t	t	t	t

Using Modus Ponens

p	q	$p \implies q$
t	t	t
t	f	f
f	t	t
f	f	t

$p \implies q$ is equivalent to saying p therefore q

Decide whether $p \implies r$ $q \implies r$ r therefore $p \vee q$

p	q	r	$p \implies r$	$q \implies r$	$p \vee q$
t	t	t	t	t	t
t	t	f	f	f	t
t	f	t	t	t	t
t	f	f	f	t	t
f	t	t	t	t	t
f	t	f	t	f	t
f	f	t	t	t	f
f	f	f	t	t	t

Predicate Logic

- We don't have truth tables
- We have domains of discourse

$$(\forall x)(\exists y)x < y \qquad \text{and} \qquad (\exists x)(\forall y)x \leq y$$

- Both true in $\mathbb{N}$
  - Consider $x = 0$
- Have different truth values in $\mathbb{R}$

Two predicate logic statements are **not** logically equivalent if they have different truth values in the same domain.

Rule of Inference

$P(a) \implies (\forall x)(P(x))$

> a is some element
>
> *universal generalization*
>
> $a = a \implies (\forall x) x = x$

> Lecture 12
> ## Proofs
>
> ---
>
> Brennan Becerra | 2023-02-15
>
> ---
>
> ### Informal Proofs
>
> - In English
>
> ### Proofs
>
> - Foundation of mathematical practice
> - Every statement in math must be proven except the axioms
>
> ### Axioms
>
> - Laws of arithmetic
> - e.g.
>   - $a \cdot 0 = 0$
>   - $a + b = b + a$
>   - $a + (b + c) = (a + b) + c$

Universal and existential statements

- Proofs of universal cannot be done with examples
- e.g.
  - Consider the polynomial $P(n) = n^2 + n + 41$
  - We could claim that every value of $P(n)$ is prime
  - $P(0) = 41$
  - $P(1) = 43$
  - $P(2) = 47$
  - $P(3) = 53$
    - Fails at $n = 40$
    - $P(40) = 40^2 + 40 + 41$
    - $= 41^2$ which is not prime

Goldbach Conjecture

- Every natural number is the sum of two primes
  - True for every value up to 10^{20}

Existential Statements

$(\forall x)(\exists y) x < y$ is true $\mathbb{N}$

- Things need to be "constructed"

Weierstrass Function

- claim: there is a function from $\mathbb{R} \rightarrow \mathbb{R}$ that is everywhere continuous but nowhere differentiable

Proof techniques

Direct Proof

$p \to q$

- Assume p
- Do some stuff
- Conclude q

Recall

If a is even, then a^2 is even.

Theorem: If a^2 is even, then a is even

Proof:

- Assume a^2 is even.
- So there is a $k \in \mathbb{N}$ such that $a^2 = 2k$.
- $a \cdot a = 2k$.
- So 2 divides a.
- Then a is even

Contradiction

- Shows that a statement is true, by showing the negation of the statement and proving a contradiction
- Something like $0 = 1$

Logic law

- Principle of explosion

> - Contradictions imply everything

$p \land \neg p \implies q$

1. p
2. $\neg p$
3. $p \lor q$
4. $\neg p \implies q$
5. q

<u>Theorem:</u> $\sqrt{2}$ is irrational.

A number, n is rational if there are integers, a and b such that $n = \frac{a}{b}$ $b \neq 0$.

<u>Proof:</u>

Assume that $\sqrt{2}$ is rational. We should be able to find two integers, a and b such that $\sqrt{2} = \frac{a}{b}$. Let a, b be in the lowest terms $\left(\underbrace{\frac{3}{7}}_{lowest}, \frac{6}{14}, \frac{9}{20}\right)$.

- $2 = \frac{a^2}{b^2}$
- $2b^2 = a^2$
- a^2 is even
- a must also be even
- There is a k such that $a = 2k$
- $2b^2 = (2k)^2 = 4k^2$
- This is a contradiction because a and b are the lowest terms, thus, $\sqrt{2}$ is irrational

Theorem: There are infinitely many prime numbers.

Proof: Assume there are finitely many prime numbers. List all of them, $p_1, p_2, \ldots, p_k$. Consider the number, $N = p_1 \cdot p_2 \cdot p_3 \cdots \cdot p_k + 1$. What are the prime divisors of N? What if N is Prime? If N is prime, we're done, as that would be a contradiction. Suppose N is composite(Not prime). Is N divisible by p_1? No. p_2? No. p_3? No. ... p_k? No. So, since N has prime divisors, not in the list, $p_1, \ldots, p_k$ there are other primes not in the list.

Fundamental Theorem of Arithmetic

- Every natural number is factorable in a unique product of powers of primes.
- $900 = 9 \cdot 100$
  - $= 3^2 \cdot 5^2 \cdot 2^2$

Proof by Contrapositive

- If a^2 is even, then a is even

Contrapositive

- If a is odd, then a^2 is odd

Proof:

- Assume a is odd, so $a = 2k + 1$
- $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

- So a^2 is odd

Proof by Cases

Theorem: For every natural number, n, $n^3 - n$ is even.

Proof:

- Suppose n is even, then $n = 2k$.
- $n^3 - n = 8k^3 - 2k = 2(2k^3 - 2)$
- So $n^3 - n$ is even
- Suppose n is odd, then $n = 2k + 1$
- $n^3 - n = (2k+1)^3 - (2k+1)$
  - $= 8k + 12k^2 + 6k + 1 - 2k - 1$
  - $= 2(4k^3 + 6k^2 + 2k)$
- So $n^3 - n$ is even

Alternatively

- $n^3 - n = n(n^2 - 1) = n(n+1)(n-1)$

Lecture 16

Title

Brennan Becerra | 2023-03-03

2.2 13,15

2.3 12

Lecture 17

Lecture 17

2.5

Brennan Becerra | 2023-03-06

Induction

A proof technique that allows us to convince ourselves that a mathematical statement is always true.

- Universal statements about natural numbers
- Helpful for things defined by recursion

Given a predicate about natural numbers, $P(n)$, it allows us to prove $(\forall n)(P(n))$.

Method

1. Base case: Prove that $P(0)$ is true. You do this directly. This is the easy step.
2. Inductive case: Prove that $P(k) \to P(k+1)$ for all $k \geq 0$. That is, prove that for any $k \geq 0$ if $p(k)$ is true, then $p(k+1)$ is true as well. This is the proof of an if...then... statement, so you can assume $P(k)$ is true then you must explain why $P(k+1)$ is also true, given the assumption.

Dominos

Base case: I can knock over the first domino. Inductive step: For any given domino, it will knock over the next one, then all dominoes fall over.

$$(\forall n)P(n)$$

- Most powerful tool for proving things about natural numbers

Can't prove without induction

$$(\forall n)(\forall m)(n+m=m+n)$$

Theorem: $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

Proof:

Base case $n=0$. $0=0$.

Induction: Assume that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Goal: $\frac{(n+1)(n+2)}{2}$

$\underbrace{1+2+3+\cdots+n}_{=\frac{n(n+1)}{2}}+(n+1)=\frac{n(n+1)}{2}+(n+1)$
(Induction hypothesis)

$1+2+3+\cdots+n+(n+1)=\frac{n(n+1)+2(n+1)}{2}=\frac{(n+1)(n}{2}$

Theorem: $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

Proof

Base case: $n=0$. $0=0$.

Induction: Assume $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ holds.

$$\underbrace{1^2+2^2+3^2+\cdots+n^2}_{\frac{n(n+1)(2n+1)}{6}}+(n+1)^2=\frac{n(n+1)(2n+1)}{6}$$

Goal:

Show that $(n+1)(n+2)(n+3)$ will happen in the numerator.

Theorem: $(\forall n\geq 4)(2^n<n!)$.

Proof:

Base case: $n=4$

$2^4=16<24=4!$

Inductive step:

Assume $(\forall n\geq 4)(2^n<n!)$.

Goal: Show that $2^{n+1}<(n+1)!$

- $2^{n+1} = \underbrace{2^n \cdot 2}_{<n! \cdot 2}$
- $(n+1)! = n!(n+1)$
- $2^{n+1} = \underbrace{2^n \cdot 2}_{<n! \cdot (n+1)}$

Theorem: $(\forall n \in \mathbb{N})(6^n - 1 \text{ is a factor of } 5)$
$(\exists k) : 6^n - 1 = 5k$

Proof:

Base case: $n = 1$ $6^0 - 1 = 0 = 5(0)$

Induction:

Assume $(\forall n \in \mathbb{N})((\exists k) : 6^n - 1 = 5k)$

Goal: Find j such that $6^{n+1} - 1 = \sqrt{5}j$.

$6^{n+1} - 1 = 6^n \cdot 6 - 1$

$6^n = 5k + 1 = (5k + 1) \cdot 6 - 1$ (Induction hypothesis)

$= 30k + 6 - 1$

$= 30k + 5$

$= 5(6k + 1)$

Strong Induction

1. Base case: $P(0)$
2. Inductive case: Assume $P(k)$ holds for all $k < n$, show that $P(n)$ is true.

Theorem: Every natural number has a prime factorization. Every natural number, $n \geq 2$ is either prime or the product of primes.

Proof:

Base case $n = 2$. 2 is prime.

Inductive step: For a given natural number, n, assume that for all $2 \leq k < n$ that k is either prime or a product of primes.

If n is prime, we're done.

If n is not prime, there are m_1, m_2 that are 1 or n itself such that $n = m_1 \cdot m_2$. So $2 = m_1, m_2 \leq k$

By induction hypothesis m_1, m_2 are either prime or a product of primes.

$n = m_1 \cdot m_2$

Well Ordering Property

Every subset of natural numbers has a least element.

Lecture 18

Lecture 18

Subtitle

Brennan Becerra | 2023-03-08

Fibonacci Numbers

Recall Fibonacci numbers

$F_0 = 0, F_1 = 1$

$F_{n+2} = F_{n+1} + F_n$

<u>Prove:</u> $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$

Base: $n = 0$

$F_0 = 0$

$F_{2(0)+1} - 1 = F_1 - 1 = 1 - 1 = 0$

Inductive step: Assume $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ (Induction hypothesis)

$F_0 + F_2 + F_4 + \cdots + F_{2n} + F_{2n+2}$

Goal: $F_{2(n+1)+1} - 1 = F_{2n+3} - 1$

$$\underbrace{F_0 + F_2 + F_4 + \cdots + F_{2n}} + F_{2n+2} = F_{2n+1} - 1 = F_{2n+}$$

$$= F_{2n+3} - 1$$

Prove: $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$

Base case: $n = 0$ $F_0^2 = 0$ $F_0 \cdot F_1 = 0 \cdot 1 = 0$

Inductive step: Assume $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$ holds

Goal $F_{n+1} \cdot F_{n+2}$

Consider $F_0^2 + F_1^2 + \cdots + F_n^2 + F_{n+1}^2 = F_n + F_{n+1} + F_{n+1}^2$

$= F_{n+1}(F_n + F_{n+1}) = F_{n+1} \cdot F_{n+2}$

Golden Ratio

<u>Def:</u> $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$

$\frac{x+1}{x} = \frac{x}{1}$

$x + 1 = x^2$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2}$

$x = \frac{\sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}$ Golden ratio

$x = \frac{1-\sqrt{5}}{2} = \Psi$

$\Psi = -\frac{1}{\phi}$

$-\frac{1}{\phi} = -\frac{1}{\frac{1+\sqrt{5}}{2}}$

$= \Psi$

Binet's Formula

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

Show that $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$

Base case 1: $F_0 = \frac{\phi^0-\psi^0}{\phi-\psi} = \frac{1-1}{\phi-\Psi} = \frac{0}{\phi-\psi}$

$F_0 = \frac{\phi^1-\psi^1}{\phi-\psi} = 1$

Inductive Hypothesis:

$F_{n+1} = \frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi}$

$F_n = \frac{\phi^n-\psi^n}{\phi-\psi}$

Goal: $F_{n+1} = \frac{\phi^{n+2}-\psi^{n+2}}{\phi-\psi}$

$$F_{n+2} = F_{n+1} + F_n$$

$$= F_{n+1} = \frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} + F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

$$= \frac{\phi^{n+1} + \phi^n - (\psi^{n+1} + \psi^n)}{\phi - \psi}$$

$$= \frac{\phi^n(\phi + 1) - \psi^n(\psi + 1)}{\phi - \psi}$$

$$= \frac{\phi^n \cdot \phi^2 - \psi^n \cdot \psi^2}{\phi - \psi}$$

$$\frac{\phi^n(\phi + 1) - \psi^n(\psi + 1)}{\phi - \psi}$$

$$= \frac{\phi^{n+2} - \psi^{n+2}}{\phi - \psi}$$

$$\lim_{n \to \infty} \frac{(\phi^{n+1} - \psi^{n+1})}{\frac{\frac{\phi - \psi}{\phi^n - \psi^n}}{\phi - \psi}}$$

$$\lim_{n \to \infty} \frac{\phi^{n+1} - \psi^{n+1}}{\phi^n - \psi^n}$$

$$\lim_{n \to \infty} \frac{\phi^{n+1}}{\phi^n} = \phi$$

Object x

Def A

Def B

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Lecture 19
Lecture 19

Subtitle

Brennan Becerra | 2023-03-10

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1. Show $n! < n^n$ for all $n \geq 2$
2. #19 section 2.5
3. A polygon is <u>convex</u> if every two points in the polygon can be connected by a straight line in the polygon. Show the sum of interior angles of an $n-$ sided convex polygon is $(n-2) \cdot 180^\circ$ for $n \geq 3$.
4. Show that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\left(\frac{n(n+1)}{2}\right)\right)^2$
5. Use induction to show $n^3 + 3n^2 + 2n$ is divisible by 3 for all $n \geq 1$.