Lecture 1

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Differential Geometry of curves & surfaces

$f(x, y) = 0$

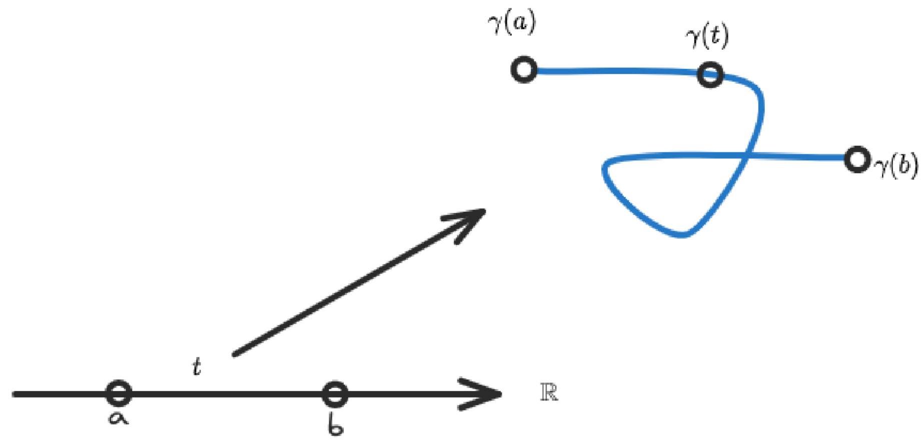
$x^2 + y^2 - a^2 = 0$

$y = \pm\sqrt{a^2 - x^2}$

when its working, its beautiful...

Sec 1.1 - Parameterized Curves

$\gamma : I \to \mathbb{R}^n$



Component Form

$$\gamma(t) = (\underbrace{x_1(t), x_2(t), \ldots, x_n(t)}_{\text{Component (coordinate) functions}})$$

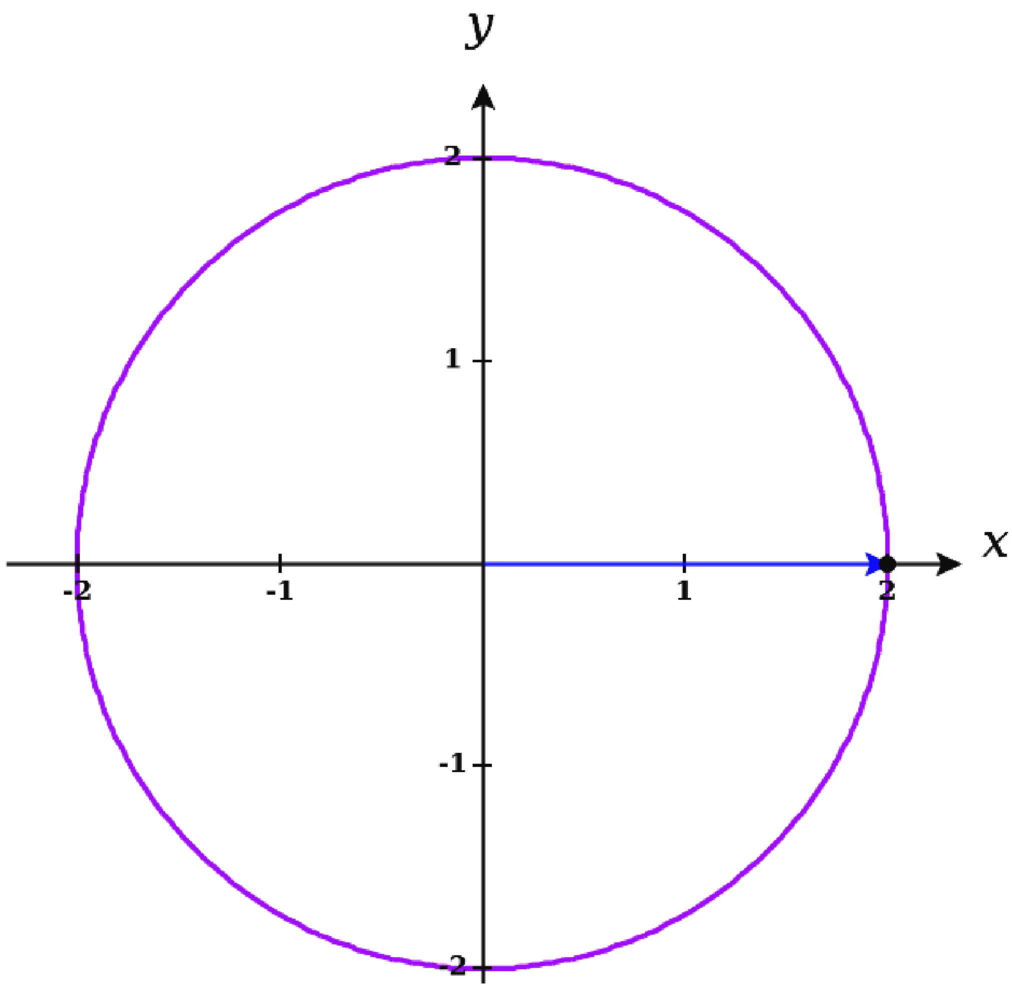
Another name for this is the *position vector*.

$\gamma(t)$ is *smooth* (C^∞) meaning its infinitely differentiable.

In $\mathbb{R}^2 : \gamma(t) = (x_1(t), x_2(t))$
In $\mathbb{R}^3 : \gamma = (x_1(t), x_2(t), x_3(t))$

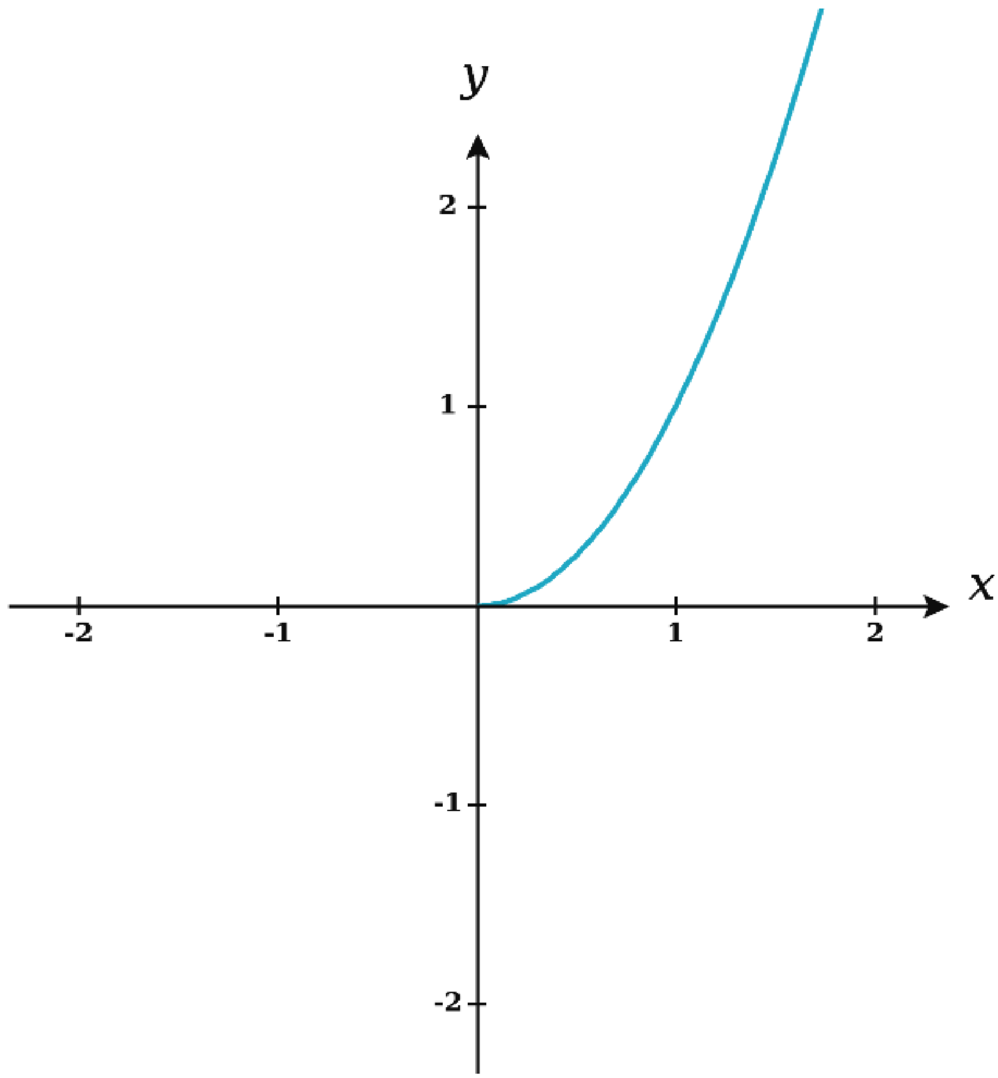
Examples:

Circle: $\gamma(t) = (2\cos(t), 2\sin(t))$



Spiral: $\gamma(t) = (2\cos(t), 2\sin(t), t)$

Parabola: $\gamma(t) = (t, t^2)$


y
x
2
1
-1
-2
-2
-1
1
2

Definition 1.6

The *derivative* of a curve, $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is:

$$\gamma'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)).$$

Ex:

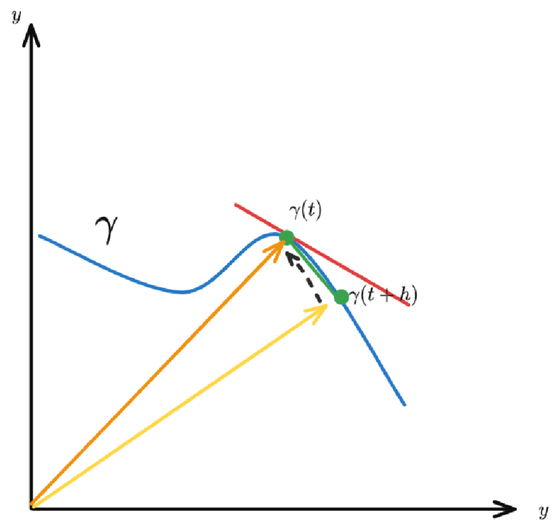
$\gamma(t) = (2\cos(t), 2\sin(t))$, $\gamma'(t) = (-2\sin(t), 2\cos(t))$

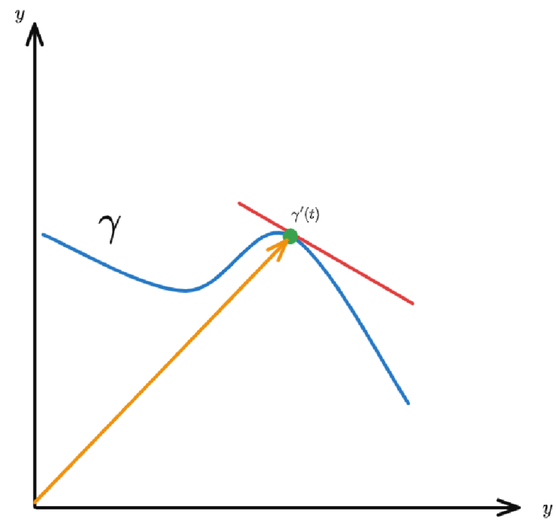
$\gamma(t) = (t, t^2)$, $\gamma'(t) = (1, 2t)$

$\gamma(t) = (t, |t|) = (t, \sqrt{t^2})$ (Try showing that γ is not differentiable at the origin)

Proposition 1.7

$$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$





$\gamma'(t)$ is *tangent* to $\gamma(t)$. $\gamma'(t) = \frac{d\,\text{position}}{d\,\text{time}}$, otherwise known as *velocity*.

Definition 1.8 ***Speed*** **and** ***Arclength***

For a curve $\gamma : I \to \mathbb{R}^n$,

- *Speed* is $|\gamma'(t)|$
- Arclength of $\gamma(t)$ between $t_1, t_2 : S = \int_{t_1}^{t_2} |\gamma'(t)|\, dt$

Ex.

$\gamma(t) = (\cos(t), \sin(t)), t \in [0, 2\pi)$
$\gamma'(t) = (-\sin(t), \cos(t))$
$|\gamma'(t)| = \sqrt{(-\sin)^2 + (\cos(t))^2} = 1.$

$$S = \int_0^{2\pi} 1\, dx = 2\pi.$$

$$\gamma(t) = (x(t), y(t), z(t))$$
$$|\gamma(t)| = [x(t)^2 + y(t)^2 + z(t)^2]^{1/2}$$

Definition 1.9

A curve, $\gamma : I \to \mathbb{R}^n$ is:

- *Regular* if $|\gamma'(t)| \neq 0$ for any t.
- *Unit Speed* (aka *parameterized by arclength*) if $|\gamma'(t)| = 1$ for all t.