Lecture 2

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In this course we will work on:

- Diffusion-type problems (parabolic equations)
- Wave-like problems (hyperbolic equations)
- Boundary value problems (elliptic equations)

What software do you need?

- Matlab
- Python
- Wolfram Mathematica
- etc.

Assessments

- Homework: roughly every two weeks, 40%
- Two midterm exams: 40%
- Final exam: 20%

Last time:

1. Wave equation: $u_{tt} = C^2 u_{xx}$. $u_{tt} = C^2 \nabla^2 u$
2. Heat equation: $u_t = k^2 u_{xx}$. $u_t = k^2 \nabla^2 u$
3. Laplace's equation: $u_{xx} + u_{yy} = 0$. $\nabla^2 u = 0$

Consider the following function: $u(x, y)$ (Assume Linear 2nd order eq.) We can write the following:

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_{21}u_{yx}b_1u_x + b_2u_y + cu + d = 0$$

Lets organize this in matrix form:

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}}_{\nabla^T} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + cu + d = 0.$$

To classify a second order linear PDE, we look at *eigenvalues,* λ of A. We find them by solving the following equation:

$$(A - \lambda I)\vec{v} = 0$$

such that

$$\det(A - \lambda I) = 0$$

The PDE is:

1. *elliptic* if all λ are non-zero and are of the same sign.
2. *hyperbolic* if all λ are non-zero and have the same sign, <u>except one</u>.
3. *parabolic* if any $\lambda = 0$.

Example 1

Laplace's equation: $u_{xx} + u_{yy} = 0$. We can re-write this as: $1u_{xx} + 0u_{xy} + 1u_{yy} + 0u_{yx} = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lets find our $\lambda's$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)^2 = 0 \\ \lambda &= 1 \end{aligned}$$

So we can see by definition, the Laplace equation is *elliptic*.

Example 2

Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$ We can write this as

$$u_{tt} - c^2 + u_{xx} - c^2 u_{yy} = 0.$$

Now lets find our $\lambda's$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}$$

We can see that our $\lambda's$ will be non-zero except one of them, so this is *hyperbolic*.

Example 3

$u_t = k^2\nabla^2 u = k^2(u_{xx} + u_{yy} + u_{zz})$ in canonical form, we get

$$k^2 u_{xx} + k^2 u_{yy} + k^2 u_{zz} - u_t = 0$$

And we can write

$$A = \begin{bmatrix} k^2 & 0 & 0 & 0 \\ 0 & k^2 & 0 & 0 \\ 0 & 0 & k^2 & 0 \\ 0 & 0 & 0 & \underline{0} \end{bmatrix}$$

We can see that this has a $\lambda = 0$, so this PDE is *parabolic*.