Lecture 2

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<u>For Hw1:</u> *Improper Integrals* (Calc 2)

<u>Ex:</u>

$$\int_0^1 \underbrace{\frac{1}{x}}_{\text{DNE at 0}} dx = \lim_{a \to 0} \int_a^1 \frac{1}{x} \, dx$$
$$= \lim_{a \to 0} \ln(x)\big|_a^1$$

<u>Ex:</u>

$$\int_0^\infty e^{-\lambda t} \, dt = \lim_{b \to \infty} \int_0^b e^{-\lambda t} \, dx$$

Vector Calculus Review

Section 1.2 - Inner Product (Dot Product)

Definition 1.11

If $\vec{x} = (x_1, x_2, \ldots, x_n), \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ then the *inner product,* $\langle \vec{x}, \vec{y} \rangle$ is:

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \langle x_1, x_2, \ldots, x_n \rangle \cdot \langle y_1, y_2, \ldots, y_n \rangle \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R}^n. \end{aligned}$$

<u>Ex.</u>

$\vec{x} = (1, 0, 2), \vec{y} = (2, -6, 3)$. $\langle \vec{x}, \vec{y} \rangle = (1)(2) + (0)(-6) + (2)(3) = 2 + 0 + 6 = 8.$

The *inner product* lets us measure <u>angles</u> and <u>distances</u>.

Lemma 1.12

Let the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and $\lambda, \mu \in \mathbb{R}$.

Properties of the inner product (dot product):

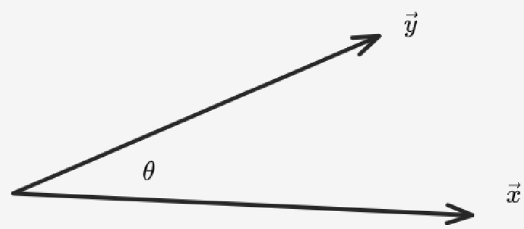
1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetric)
2. $\langle \lambda \mathbf{x} + \mu \mathbf{y}, \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \mu \langle \mathbf{y}, \mathbf{z} \rangle$,
   $\langle \mathbf{x}, \lambda \mathbf{y} + \mu \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle + \mu \langle \mathbf{x}, \mathbf{z} \rangle$ (bilinear)
3. $\langle \mathbf{x}, \mathbf{x} \rangle = |\mathbf{x}|^2$, which equals zero if and only if $\mathbf{x} = 0$.
4. $\langle \mathbf{x}, \mathbf{y} \rangle \leq |\langle \mathbf{x}, \mathbf{y} \rangle| \leq |\mathbf{x}||\mathbf{y}|$ (the Schwarz inequality).

Equation 1.2

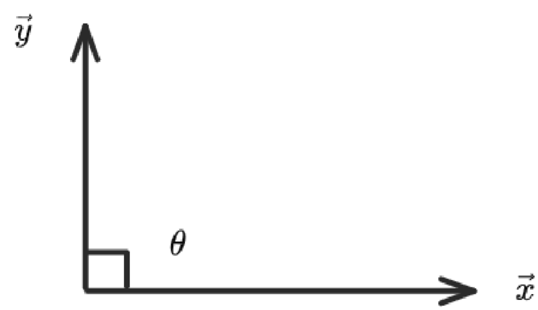
$$\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos(\theta)$$

where $\theta \in [0, 2\pi]$ and we can also say

$$\theta = \cos^{-1}\left(\frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{x}||\vec{y}|}\right)$$



Notice that if $\vec{x} \perp \vec{y}$, then $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\left(\frac{\pi}{2}\right) = 0$. This is because $\cos\left(\frac{\pi}{2}\right) = 0$.

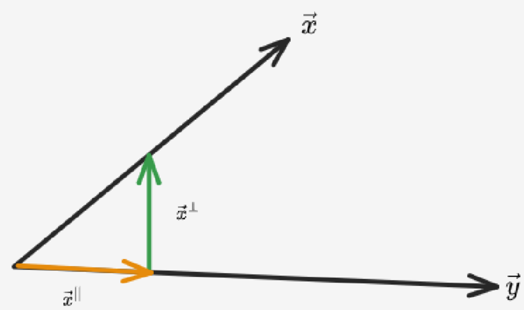


Applications of inner product: Projection and Orthogonal Decomposition

Orthogonal Decomposition

The *Orthogonal Decomposition* of $\vec{x}$ with respect to $\vec{y}$ is:

$$\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$$



- $\vec{x}^{||}$ is parallel to $\vec{y}$
- $\vec{x}^{\perp}$ is orthogonal to $\vec{y}$

To compute orthogonal decomposition:

1. The scalar component, $\lambda = \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{y}|}$
2. The projection $\vec{x}^{||} = \lambda \hat{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{y}|^2} \vec{y}$
3. $\vec{x}^{\perp} = \vec{x} - \vec{x}^{||} = \vec{x} - \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{y}|^2} \vec{y}$