Lecture 4

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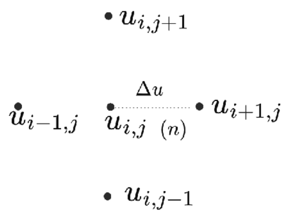
Last time

We observed how distribution of candy was represented by the following model:



With the following equations:

$$
\begin{aligned}
u_{i,j}^{(n+1)} = u_{i,j}^{(n)} &+ \frac{u_{i+1,j}^{(n)} - u_{i,j}^{(n)}}{4} + \frac{u_{i-1,j}^{(n)} - u_{i,j}^{(n)}}{4} \\
&+ \frac{u_{i,j+1}^{(n)} - u_{i,j}^{(n)}}{4} + \frac{u_{i,j-1}^{(n)} - u_{i,j}^{(n)}}{4}
\end{aligned}
$$

Then we wrote:

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{u_{i,j}^{(n+1)} - u_{i,j}^{(n)}}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{4} &\left(\frac{u_{i+1,j}^{(n)} - u_{i,j}^{(n)}}{\Delta x} - \frac{u_{i,j}^{(n)} - u_{i-1,j}^{(n)}}{\Delta x} \right) \\
+ \frac{1}{4} &\left(\frac{u_{i,j+1}^{(n)} - u_{i,j}^{(n)}}{\Delta x} - \frac{u_{i,j}^{(n)} - u_{i,j-1}^{(n)}}{\Delta x} \right).
\end{aligned}
$$

And now we can say $u_{i,j}^{(n)} \to u(x, y, t)$ and in fact,

$$
\lim_{\Delta t \to 0} \frac{u_{i,j}^{(n+1)} - u_{i,j}^{(n)}}{\Delta t} = \cdots = \frac{\partial u}{\partial t}.
$$

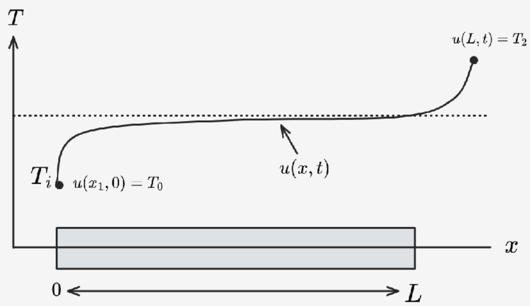
Continuing on, we can say

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \lim_{\Delta x, \Delta y \to 0} \frac{1}{4} \left(\frac{\frac{\partial u}{\partial x}\big|^{+} - \frac{\partial u}{\partial x}\big|^{-}}{\Delta x} + \frac{\frac{\partial u}{\partial y}\big|^{+} - \frac{\partial u}{\partial y}\big|^{-}}{\Delta y} \right) \\
&= \frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
&= \frac{1}{4} \nabla^2 u.
\end{aligned}
$$

Alas, we have arrived at the *Laplacian*! We should note that $\frac{1}{4}$ is just a sort of *diffusion coefficient*.

A brief tour through temperature modeling

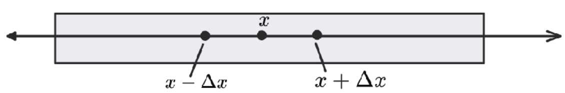
Metal Strip Temperature Model



$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}.$$

Note that we must state some *initial*, and *boundary* conditions conditions, i.e. $x \in [0, L], t \in [0, \infty)$. More on this later...

We can think about modeling this with limits...



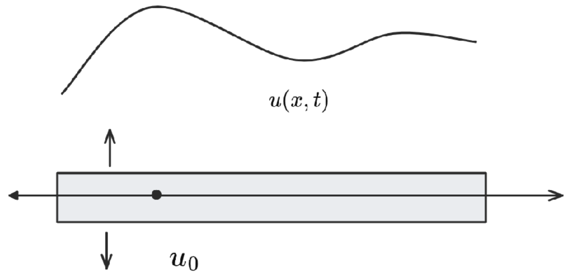
further described by

$$\frac{\partial y}{\partial t} = k^2 \lim_{\Delta t \to 0} \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x}.$$

This becomes

$$k^2 \lim_{\Delta x \to 0} \frac{2}{\Delta x} \left(\frac{u(x + \Delta x, t) + u(x - \Delta x)}{2} - u(x, t) \right).$$

Suppose we're observing a scenario where the rod can emit heat into the surrounding environment.



We can write:

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} + \beta(\underbrace{u_0}_{\text{Environment}} - u).$$

- u_0: Temperature of the environment