Lecture 1

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What is an equation? Equations generally take the form of equating variables, like the following:

$$2x - 3 = 0$$

This has a trivial solution. We are looking for the value of this variable that satisfies the expression, i.e $x = \frac{3}{2}$. We know this is true because we can test it. If I say $x = -1$, we know this is not true, because we can test it, and get something else.

This class aims to study differential equations, like

$$\frac{du}{dt} = \lambda u.$$

Here the solution to this problem takes the form of $u(t)$, which is a function of the independent variable, t. Sometimes we'll be given other information like $u(0) = u_0 =$ initial value. This might take the form,

$$u(t) = Ce^{\lambda t}$$

We need to check that this works.

$$\frac{du}{dt} = C\lambda e^{\lambda t}$$

Upon substitution, we find

$$C\lambda e^{\lambda t} = \lambda \cdot Ce^{\lambda t} \checkmark$$

This works!

Partial Differential Equation

We are looking for solutions of multiple variables,

$$u(\overline{x}, t).$$

$u(\overline{x}, t)$ does not necessarily have to correspond to linear space, but abstract space. Think of things like machine learning, and so forth.

Shorthands:

- $\frac{\partial u}{\partial t} = U_t$
- $\frac{\partial U}{\partial x} = U_x$
- $\frac{\partial^2}{\partial x^2} = U_{xx}$
- $\frac{\partial^2 u}{\partial y^2} = U_{yy}$

Canonical PDE's

1. Wave equation: $U_{tt} = C^2 U_{xx}$. $U_{tt} = C^2 \nabla^2 U$
2. Heat equation: $U_t = k^2 U_{xx}$ $U_t = k^2 \nabla^2 U$
3. Laplace's equation: $U_{xx} + U_{yy} = 0$ $\nabla^2 u = 0$

$$\nabla = \left[\frac{\partial .}{\partial x}, \frac{\partial .}{\partial y}, \frac{\partial .}{\partial z}\right]^T$$

$$\nabla U = \left[\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right]^T = \operatorname{grad}(U)$$

$$\nabla \overline{U} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}(\overline{U})$$

Laplacian

$$\nabla^2 U = \nabla \cdot \nabla U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Properties of Canonical PDE's

1. Linearity: No products of U, U_x, U_{xx}.

$\implies$ If U_1 is a solution to a linear PDE, and U_2 is a solution to the same PDE, then $\underbrace{\alpha U_1 + \beta U_2}_{\text{Superposition}}$ is also a solution to that same PDE.

2. Second Order.
3. Homogenous.

Linear Operator

- $L_1(u) = 5u$
- $L_2(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$\underbrace{L(\alpha u_1 + \beta u_2) = \alpha L(u_1) + \beta L(u_2)}_{\text{Linearity}}$$

Lets think of some nonlinear equations. Consider *Burgers Equation*:

$$u_t + u u_{xx} = \nu u_{xx}.$$