Lecture 3

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Last time:

Consider the following function: $u(x, y)$ (Assume Linear 2nd order eq.) We can write the following:

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_{21}u_{yx} \underbrace{+ b_1 u_x + b_2 u_y + cu + d}_{\text{Other Possibilities}} = 0$$

Lets organize this in matrix form:

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}}_{\nabla^T} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + cu + d = 0.$$

To classify a second order linear PDE, we look at *eigenvalues,* λ of A. We find them by solving the following equation:

$$(A - \lambda I)\vec{v} = 0$$

such that

$$\det(A - \lambda I) = 0$$

The PDE is:

1. *elliptic* if all λ are non-zero and are of the same sign.
2. *hyperbolic* if all λ are non-zero and have the same sign, <u>except one</u>.
3. *parabolic* if any(at least one) $\lambda = 0$.

Why these names?

We can re-write the differential equations as polynomials.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

And geometrically, this describes a sort of *line* in the xy plane. For example, if $A, B, C = 0$, we get a straight line, and so forth. We can say the following:

$$\Delta = B^2 - 4AC$$

From this, we can infer:

$\Delta < 0$: Elliptic
$\Delta > 0$: Hyperbolic
$\Delta = 0$: Parabolic

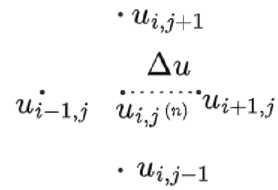
Experiment

Lets derive some equations from physical phenomena, i.e. distribution of candy.

Candy $\rightarrow u(x, y, t)$

- Quotient of candy indicates ∇u
- Magnitude of exchange is proportional to the difference
  - $k(u_i - u_{i-1})$
  - This operation: $\left\lceil \frac{\Delta u}{\underbrace{4}_{\text{Diffusion coefficient}}} \right\rceil$
- Boundary conditions are specified by walls

Mathematics



$$
\begin{aligned}
u_{i,j}^{(n+1)} = u_{i,j}^{(n)} &+ \frac{u_{i+1,j}^{(n)} - u_{i,j}^{(n)}}{4} + \frac{u_{i-1,j}^{(n)} - u_{i,j}^{(n)}}{4} \\
&+ \frac{u_{i,j+1}^{(n)} - u_{i,j}^{(n)}}{4} + \frac{u_{i,j-1}^{(n)} - u_{i,j}^{(n)}}{4}
\end{aligned}
$$

Lets re-write this as a differential equation:

$$
\begin{aligned}
u_{i,j}^{(n+1)} - u_{i,j}^{(n)} = \frac{1}{2}&\left[\frac{u_{i+1,j}^{(n)} - u_{i,j}^{(n)} + u_{i-1,j}^{(n)} - u_{i,j}^{(n)}}{2\Delta x}\right] \\
+ \frac{1}{2}&\left[\frac{u_{i,j+1}^{(n)} - u_{i,j}^{(n)} + u_{i,j-1}^{(n)} - u_{i,j}^{(n)}}{2\Delta y}\right]
\end{aligned}
$$

Recall that

$$
\frac{du}{dt} = \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t}
$$

so we can say

$$
\frac{\partial u}{\partial A} = \frac{1}{2}\left[\frac{\left(\frac{\partial u^+}{\partial x} + \frac{\partial u^-}{\partial x}\right)}{2\Delta x} + \frac{\left(\frac{\partial u^+}{\partial y} + \frac{\partial u^-}{\partial y}\right)}{2\Delta y}\right].
$$