A Distributed Greedy Heuristic for Computing Voronoi Tessellations With Applications Towards Peer-to-Peer Networks

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***Abstract*—Computing Voronoi tessellations in an arbitrary number of dimensions is a computationally difficult task. This problem becomes exacerbated in distributed environments, such as Peer-to-Peer networks and Wireless networks, where Voronoi tessellations have useful applications.**

We present our Distributed Greedy Voronoi Heuristic, which approximates Voronoi tessellations in distributed environments. Our heuristic is fast, scalable, works in any geometric space with a distance and midpoint function, and has interesting applications in embedding metrics such as latency in the links of a distributed network.

I. Introduction

Voronoi diagrams [1] have been used in distributed and peer-to-peer (P2P) applications for some time. They have a wide variety of applications. Voronoi diagrams can be used as part of distributed hash tables[2]. They can be used in coverage detection for wireless networks [3]. Massively Multiplayer Online games (MMOs) can use them to distribute game states and events between players at a large scale [4] [5] [6].

Computing the Voronoi tessellation along with its coprime problem, Delaunay Triangulation, is a well analyzed problem. There are many algorithms to efficiently compute a Voronoi tessellation given all the points on a plane, such as Fortune's sweep line algorithm [7]. However, many network applications are distributed and many of the algorithms to compute Voronoi tessellations are unsuited to a distributed environment.

In addition, trouble occurs when points are located in spaces with more than two dimensions. Computing the Voronoi tessellation of n points in a space with d dimensions takes $O(n^{\frac{2d-1}{d}})$ time [8]. Distributed computations often have to resort to costly Monte-Carlo calculations [9] in order to handle more than two dimensions.

Rather than exactly solving the Voronoi tessellation, we instead a fast and accurate heuristic to approximate each of the regions of a Voronoi tessellation This enables fast and efficient formation of P2P networks. A P2P network built using this heuristic would be able to take advantage of it's available fault-tolerant architecture to route along any inaccuracies that arise. Our paper presents the following contributions:

- We present our Distributed Greedy Voronoi Heuristic (DGVH). The DGVH is a fast, distributed, and highly accurate method whereby nodes calculate their individual regions described by a Voronoi tessellation using the positions of nearby nodes. DGVH can work in an arbitrary number of dimensions and can handle non-euclidean distance metrics. Our heuristic can also handle toroidal spaces. In addition, DGVH can accommodate the calculation of nodes moving their positions and adjust their region accordingly, while maintaining a high degree of accuracy (Section II). Even where small inaccuracies exist, DVGH will create a fully connected graph.
- We discuss what P2P and distributed applications can use DGVH and how (Section $\mathring{)}$. In particular, we show how we can use DGVH to build a distributed hash table with embedded minimal latency.
- We present simulations demonstrating DGVH's efficacy in quickly converging to the correct Voronoi tessellation. We simulated our heuristic in networks ranging between size 500 to 10000. Our simulations show that a distributed network built DGVH accurately determines the region a randomly chosen point falls in 90% of the time within 20 cycles and converges near 100% accuracy by cycle 30.
- We present the previous work we have built upon to create our heuristic and what improvements we made with DGVH.

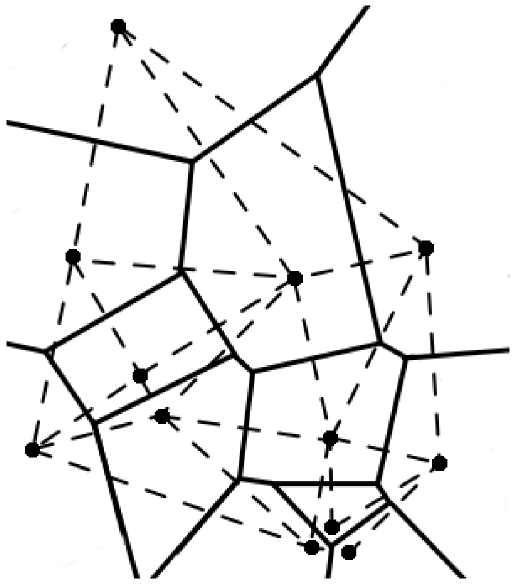

Fig. 1: An example Voronoi diagram for objects on a 2-dimensional space. The black lines correspond to the borders of the Voronoi region, while the dashed lines correspond to the edges of the Delaunay Triangulation.

II. Distributed Greedy Voronoi Heuristic

A Voronoi diagram is the partition of a space into cells or regions along a set of objects O such that all the points in a particular region are closer to one object than any other object. We refer to the region owned by an object as that object's Voronoi region. The Voronoi diagram and Delaunay Triangulation are dual problems, as an edge between two objects in a Delaunay Triangulation exists if and only if those object's Voronoi regions border each other. This means that solving either problem will yield the solution to both. An example Voronoi diagram is shown in Figure 1. For additional information, Aurenhammer [10] provides a formal and extremely thorough description of Voronoi tessellation, as well as their applications.

A. The Heuristic

The Distributed Greedy Voronoi Heuristic (DGVH) is a fast method for nodes to define their individual Voronoi region (Algorithm 1). This is done by selecting the nearby nodes that would correspond to the points connected to it by a Delaunay triangulation. The rationale for this heuristic is that, in the majority of cases, the midpoint between to nodes falls on the common boundary of their Voronoi regions.

Each cycle, nodes exchange their peer lists with a current neighbor and then recalculate their neighbors. A node combines their neighbor's peer list with its own to create a list of candidate neighbors. This combined list is sorted from closest to furthest. A new peer list is then created starting with the closest candidate. The node then examines each of the remaining candidates in the sorted list and calculates the midpoint between the node and the candidate. If any of the nodes in the new peer list are closer to the midpoint than the candidate, the candidate is set aside. Otherwise the candidate is added to the new peer list.

Algorithm 1 Distributed Greedy Voronoi Heuristic

```
1: Given node $n$ and its list of $candidates$.
2: Given the minimum $table\_size$
3: $short\_peers \leftarrow$ empty set that will contain $n$'s one-hop peers
4: $long\_peers \leftarrow$ empty set that will contain $n$'s two-hop peers
5: Sort $candidates$ in ascending order by each node's distance to $n$
6: Remove the first member of $candidates$ and add it to $short\_peers$
7: for all $c$ in $candidates$ do
8:     $m$ is the midpoint between $n$ and $c$
9:     if Any node in $short\_peers$ is closer to $m$ than $n$ then
10:        Reject $c$ as a peer
11:    else
12:        Remove $c$ from $candidates$
13:        Add $c$ to $short\_peers$
14:    end if
15: end for
16: while $|short\_peers| < table\_size$ and $|candidates| > 0$ do
17:     Remove the first entry $c$ from $candidates$
18:     Add $c$ to $short\_peers$
19: end while
20: Add $candidates$ to the set of $long\_peers$
21: if $|long\_peers| > table\_size^2$ then
22:     $long\_peers \leftarrow$ random subset of $long\_peers$ of size $table\_size^2$
23: end if
```

The following paragraphs may need reordering: DGVH never actually solves for the actual polytopes that describe a node's Voronoi region. This is unnecessary and prohibitively expensive [9]. Rather, once the heuristic has been run, nodes can determine whether a given point would fall in it's region.

Nodes do this by calculating the distance of the given point to itself and othe nodes it knows about. The point

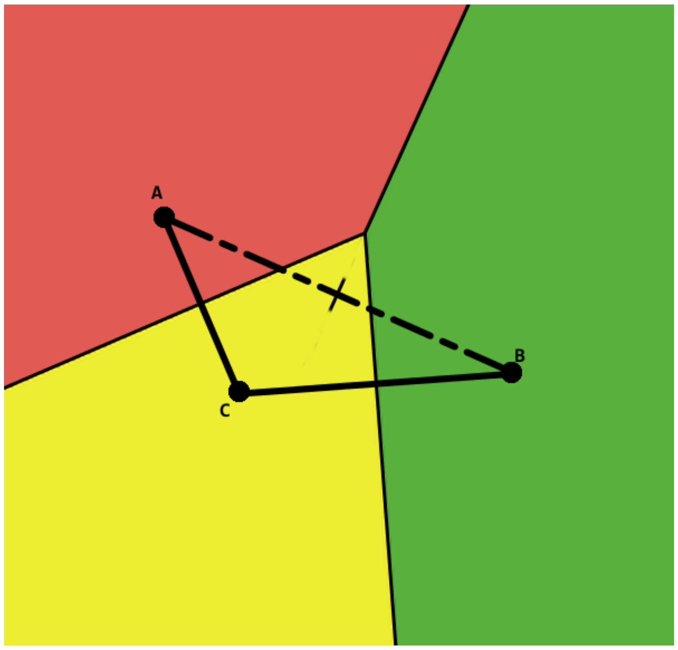

Fig. 2: The edge between A and B is not detected by DGVH, as node C is closer to the midpoint than B is. This is mitigated by peer management polices.

falls into a particular node's Voronoi region if it is the node to which it has the shortest distance. This process continues recursively until a node determines that itself to be the closest node to the point. Thus, a node defines its Voronoi region by keeping a list of the peers that bound it.

This heuristic has the benefit of being fast and scalable into any geometric space where a distance function and midpoint can be defined. The distance metric used for this paper is the minimum distance in a multidimensional unit toroidal space. Where $\vec{a}$ and $\vec{b}$ are locations in a d-dimensional unit toroidal space:

$$distance = \sqrt{\sum_{i \in d} (\min(|\vec{a}_i - \vec{b}_i|, 1 - |\vec{a}_i - \vec{b}_i|))^2}$$

In two-dimensional spaces, there is quick hack to achieve 100% accuracy distributed.

Our heuristic can be overaggressive in removing candidate nodes. For example, if a node is located between two other nodes, such that their midpoint does not fall upon the shared face of their Voronoi regions, then this heuristic will not link the blocked peers. This is demonstrated in Figure 2. Our algorithm handles these cases via our method of peer management (Section II-B).

B. Peer Management

Nodes running the heuristic maintain two peer lists: *Short Peers* and *Long Peers*. This is done to mitigate the error induced by DGVH and providing robustness against churn[1] in a distributed system.

Short Peers are the set of peers DGVH judged to have Voronoi regions adjacent to the node's own. Using a lower bound on the length of *Short Peers* corrects for errors in the approximation as it force nodes to include peers in that would otherwise be omitted. Previous work by Beaumont *et al.* [9] has found a useful lower bound on short peers to be $3d + 1$. Should the number of short peers generated by DGVH be less than the lower bound, the nearest peers not already included in *Short Peers* are added to it, until *Short Peers* is of sufficient size.

There is no upper bound to the number of short peers a node can have. This means in contrived cases, such as a single node surrounded by other nodes forming a hypersphere, this number can grow quite high. Bern *et al.* [11] found that the expected maximum degree of a vertex in a Delaunay Triangulation is

$$\Theta(\frac{\log n}{\log \log n})$$

where n is the number of nodes in the Delaunay Triangulation. This bound applies to a Delaunay Triangulation in any number of dimensions. Thus, the maximum expected size of *Short Peers* is bounded by $\Theta(\frac{\log n}{\log \log n})$, which is a highly desirable number in many distributed systems [12] [13].

Long Peers is the list of two-hop neighbors of the node. When a node learns about potential neighbors, but are not included in the short peer list, they may be included in the long peer list. *Long Peers*has a maximum size of $(3d+1)^2$, although this size can be tweaked to the user's needs. For example, if *Short Peers* has a minimum size of 8, then *Long Peers* has a maximum of 64 entries. We recommend that members of *Long Peers* are not actively probed during maintenance to minimize the cost of maintenance. A maximum size is necessary, as leaving it unbounded would result in a node eventually keeping track of all the nodes in the network, which would be counter to the design of a distributed and scalable system

How nodes learn about peers is up to the application. We experimented using a gossip protocol, whereby a node selects peer from *Short Peers* at random to "gossip" with. When two nodes gossip with each other, they

[1]The disruption caused to an overlay network by the continuous joining, leaving, and failing of nodes.

exchange their *Short Peers* with each other. The node combines the lists of short peers[2] and uses DGVH to determine which of these candidates correspond to its neighbors along the Delaunay Triangulation. The candidates determined not to be short peers become long peers. If resulting number of long peers exceeds the maximum size of *Long Peers*, a random subset of the maximum size is kept.

The formal algorithm for this process is described in Algorithm 2. This maintenance through gossip process is very similar to the gossip protocol used in Beaumont et al.'s RayNet [9].

Algorithm 2 Gossiping

```
1: Node $n$ initiates the gossip.
2: $neighbor \leftarrow$ random node from $n.short\_peers$
3: $n\_candidates \leftarrow n.short\_peers \cup n.long\_peers \cup neighbor.short\_peers$
4: $neighbor\_candidates \leftarrow neighbor.short\_peers \cup neighbor.long\_peers \cup n.short\_peers.$
5: $n$ and $neighbor$ each run Distributed Greedy Voronoi Heuristic using their respective $candidates$
```

C. Algorithm Analysis

DVGH is very efficient in both terms of space and time. Suppose a node n is creating its short peer list from k candidates in an overlay network of N nodes. The candidates must be sorted, which takes $O(k \cdot \lg(k))$ operations. Node n must then compute the midpoint between itself and each of the k candidates. Node n then compares distances to the midpoints between itself and all the candidates. This results in a cost of

$$k \cdot \lg(k) + k \text{ midpoint computations} + k^2 \text{ distance computations}$$

Since k is bounded by $\Theta(\frac{\log N}{\log \log N})$ [11] (the expected maximum degree of a node?), we can translate the above to

$$O(\frac{\log^2 N}{\log^2 \log N})$$

In the vast majority of cases, the number of peers is equal to the constant minimum table size. This yields $k = (3d+1)^2 + 3d + 1$ in the expected case, where the lower bound and expected complexities are $\Omega(1)$.

[2]Nodes remove themselves and repetitions from the candidates they receive.

Previous work [9] claims constant time approximation. The reality is that Raynet's leading constant is in the order of thousands. Our algorithm has a greater asymptotic worst case cost, but for all current realistic network sizes it will be more time efficient then RayNet's approximation.

III. Applications

A. Distributed Hash Tables

Algorithm 3 Lookup in a Voronoi-based DHT

```
1: Given node $n$
2: Given $m$ is a message addressed for $loc$
3: $potential\_dests \leftarrow n \cup n.short\_peers$
4: $c \leftarrow$ node in $potential\_dests$ with shortest distance to $loc$
5: if $c == n$ then
6:     return $n$
7: else
8:     return $c.lookup(loc)$
9: end if
```

This routing algorithm is extremely similar to the lookup algorithm used in CAN [14], which is bounded by a runtime of $O(n^{\frac{1}{d}})$.

B. Wireless Coverage

IV. Experiments

We implemented two sets of experiments for DGVH. The first compares the Voronoi tessellations created by DGVH to actual Voronoi tessellation. Our second set of experiments demonstrate that DGVH's errors are of little consequence when building a distributed and fault-tolerant systems.

A. Experiment 1: Voronoi Accuracy

B. Experiment 2: P2P Convergence and Routing

Our second set of experiments examines how DGVH could be used to create a DHT and how well it would perform in this task. Our simulation demonstrates how DGVH can be used to create a stable overlay from a chaotic starting topology after a sufficient number of gossip cycles. We do this by showing that the rate of successful lookups approaches 1.0. We compare these results to RayNet [9], which proposed that a random k-connected graph would be a good, challenging starting configuration for demonstrating convergence of a DHT to a stable network topology.

During the first two cycles of the simulation, each node bootstraps its short peer list by appending 10 nodes, selected uniformly at random from the entire network. In each cycle, the nodes gossip (Algorithm 2) and run DGVH using the new information. We then calculate the hit rate of successful lookups by simulating 2000 lookups from random nodes to random locations, as described in Algorithm 4. A lookup is successful when the network correctly determines which Voronoi region contains a randomly selected point.

Our experimental variables for this simulation were the number of nodes in VHash overlay and the number of dimensions. We tested network sizes of 500, 1000, 2000, 5000, and 10000 nodes each in 2, 3, 4, and 5 dimensions. The hit rate at each cycle is $\frac{hits}{2000}$, where $hits$ are the number of successful lookups.

Algorithm 4 Routing Simulation Sample

```
1: $start \leftarrow$ random node
2: $dest \leftarrow$ random set of coordinates
3: $ans \leftarrow$ node closest to $dest$
4: if $ans == start.lookup(dest)$ then
5:     increment $hits$
6: end if
```

Our results are shown in Figures 3a, 3b, 3c, and 3d. Our graphs show that the created overlay quickly constructs itself from a random configuration and that our hit rate reached 90% by cycle 20, regardless of dimension. Lookups consistently approached a hit rate of 100% by cycle 30. In comparison, RayNet's routing converged to a perfect hit rate at around cycle 30 to 35 [9] As the network size and number of dimensions each increase, convergence slows, but not to a significant degree.

Routing Speed: CAN

V. Related Work

While there has been previous work on applying Voronoi regions to DHTs and peer-to-peer (P2P) applications, we have found no prior work on how to perform embedding of an inter-node latency graph.

Backhaus et al.'s VAST [6] is a Voronoi-based P2P protocol designed for handling event messages in a massively multiplayer online video game. Each node finds its neighbors by constructing a Voronoi diagram using Fortune's sweepline algorithm [7]. VAST demonstrated that Voronoi diagrams could be used as the backbone to large-scale applications, although their work focused specifically on using 2-dimensional Voronoi diagrams. VAST could use VHash approximates the Voronoi region rather than solving it, as higher dimension Voronoi regions are computationally expensive to solve.

The two DHT protocols developed by Beumont et al., VoroNet [15] and RayNet [9] are the closest comparisons to VHash. VoroNet is based off Kleinberg's small world model [16] and achieves polylogarithmic lookup time. Each node in Voronet solves its Voronoi region to determine its neighbors and also maintains a link to a randomly chosen distant node. Voronet focused specifically on the two-dimensional Voronoi computations and the techniques used would be too expensive in higher dimensions and were not resilient to churn [9].

RayNet [9] was based on the work done on Voronet and used a heuristic to calculate Voronoi tessilations. Like our DGVH, RayNet's heuristic does not solve for Voronoi regions, as that is prohibitively expensive. RayNet uses a Monte-Carlo method to approximate the volume of a node's Voronoi region in constant time. While effective at estimating the Voronoi region, the volume-based Monte-Carlo approximation is expensive and requires multiple samples. This gives the runtime of RayNet's heuristic an enormous leading constant. RayNet does mention the idea of mapping attributes to each axis, but how this can be exploited is left as future work.

Unlike ACE, we handle unbounded simultaneous join and leave operations.

VI. Conclusion

We ran two Our first experiments demonstrate that our heuristic is "mostly correct" and our second set demonsrates that "mostly correct" is sufficient to build a P2P network which can route accurately.

Another venue for exploration is application of caching and replication strategies to a functional distributed file system running on top of VHash. Such extensions seek to improve upon existing work done on file replication and caching schemes [17].

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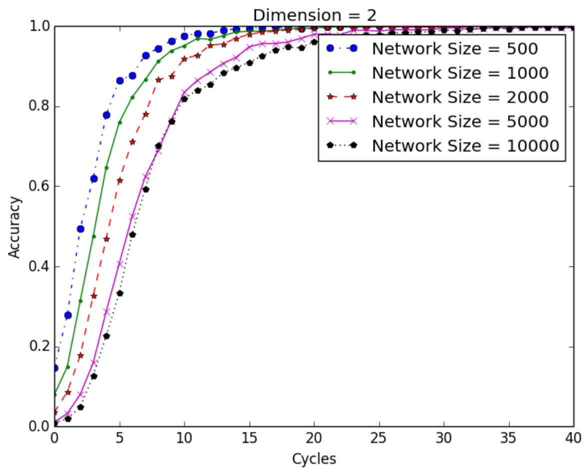
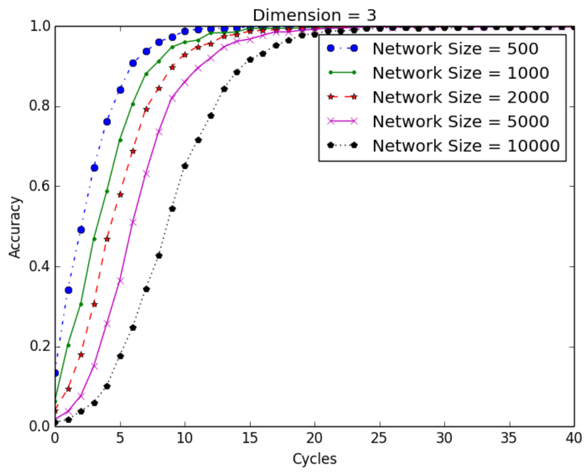
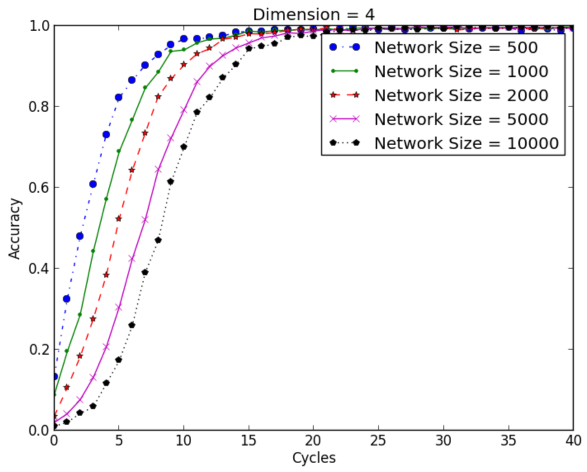


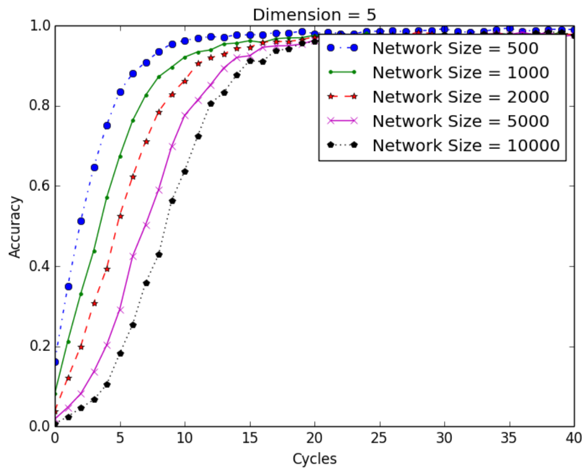
(a) This plot shows the accuracy rate of lookups on a 2-dimensional VHash network as it self-organizes.



(b) This plot shows the accuracy rate of lookups on a 3-dimensional VHash network as it self-organizes.



(c) This plot shows the accuracy rate of lookups on a 4-dimensional VHash network as it self-organizes.



(d) This plot shows the accuracy rate of lookups on a 5-dimensional VHash network as it self-organizes.

Fig. 3: These figures show that, starting from a randomized network, VHash forms a stable and consistent network topology. The Y axis shows the success rate of lookups and the X axis show the number of gossips that have occurred. Each point shows the fraction of 2000 lookups that successfully found the correct destination.

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