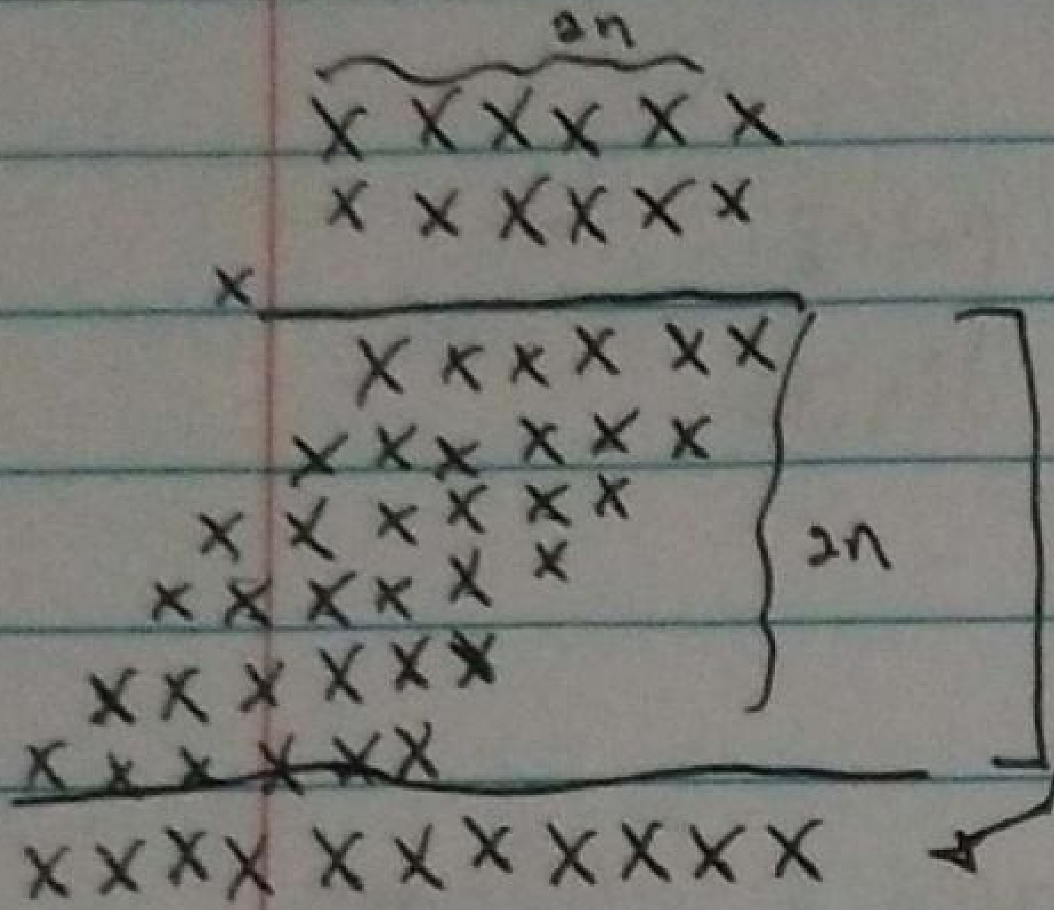
CPTS 350 HW 7 bonus

Method 1 - grammar school. Method 2 — Speed-up version

Goal: verify Method 2 is at least 2x faster than Method 1.

Assumption: Input size is N. $N = 2n + 2n = 4n$

total time (# of steps) for method 1:



$T_1(N)$ = total time for these steps

result, $T_2(N)$ = total time for last step

$T_1(N) = O(n^2)$

$T_2(N) = O(n)$

① $T_{m1}(N) = T_1(N) + \cancel{T_2(N)}\, O(n)$

② Guess. $T_{m1}(N) = O(n^2)$. That is, there's a constant C, such that $T_{m1}(N) \leq C \cdot n^2$ for all n.

③ Check.

I.H: $\forall i < n, T_{m1}(i) \leq C \cdot i^2$

$$
\begin{aligned}
T_{m1}(N) &= T_1(N) + a \cdot n \\
&\leq C(n)^2 + a \cdot n \\
&= C \cdot n^2 + a \cdot n \\
&\leq C \cdot n^2 \text{ when } C >> a.
\end{aligned}
$$

Conclusion: Method 1 has a worst-case time complexity of $O(n^2)$.

total time (# of steps) for Method 2:

$$\begin{array}{r} (xx)\ (xx)\ (xx) \\ \times\ (xx)\ (xx)\ (xx) \\ \hline \end{array}$$
$\Big\}$ $T_0(N)$ = time spent grouping 2 digits to 1 big digit
$T_0(N) = O(n)$

$\Big]$ $T_1(N)$ = time spent here doing addition
$T_1(N) = O\left(\frac{n}{2}\right)^2$

$\}$ $T_2(N)$ = final result in pairs

xx xx xx xx xx $\}$ $T_3(N)$ = final result in digits

① $T_{m2}(N) = \underbrace{T_0(N)}_{O(n)} + T_1(N) + \underbrace{T_2(N)}_{O(n)} + \underbrace{T_3(N)}_{O(n)}$

② Guess: $T_{m2}(N) = O\left(\frac{n}{2}\right)^2$. That is, theres a constant C such that $T_{m2}(N) \leq C \cdot \left(\frac{n}{2}\right)^2$ for all n.

③ Check:

I.H: $\forall i < n,\ T_{m2}(i) \leq C \cdot \left(\frac{i}{2}\right)^2$

$$T_{m2}(N) = \underbrace{O(n)}_{a \cdot n} + T_1(N) + \underbrace{O(n)}_{a \cdot n} + \underbrace{O(n)}_{a \cdot n}$$
$$\leq C\left(\frac{n}{2}\right)^2 + 3an$$
$$= \frac{Cn^2}{4} + 3an$$
$$\leq \frac{Cn^2}{4} \text{ when } C >> a.$$

$T_{m1} = O(n^2)$

$T_{m2} = O\left(\frac{n^2}{4}\right)$

$$\lim_{n\to\infty} \frac{n^2}{\frac{n^2}{4}} = \lim_{n\to\infty} \frac{4n^2}{n^2}$$

$$\lim_{n\to\infty} 4 = 4 < \infty. \checkmark$$

Conclusion: M_2 is at least 2x faster than M_1 clearly. If we group together the numbers into groups of 3 or more, we can make the algorithm run even faster. Clearly grouping numbers together before performing multiplication (Method 2) is much superior to grammar school rules (Method 1). the result follows.