Multilevel Model for Essential Mathematics

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17/10/2021

Contents

Essential Mathematics

Exploring the dataset with basic linear model for each school

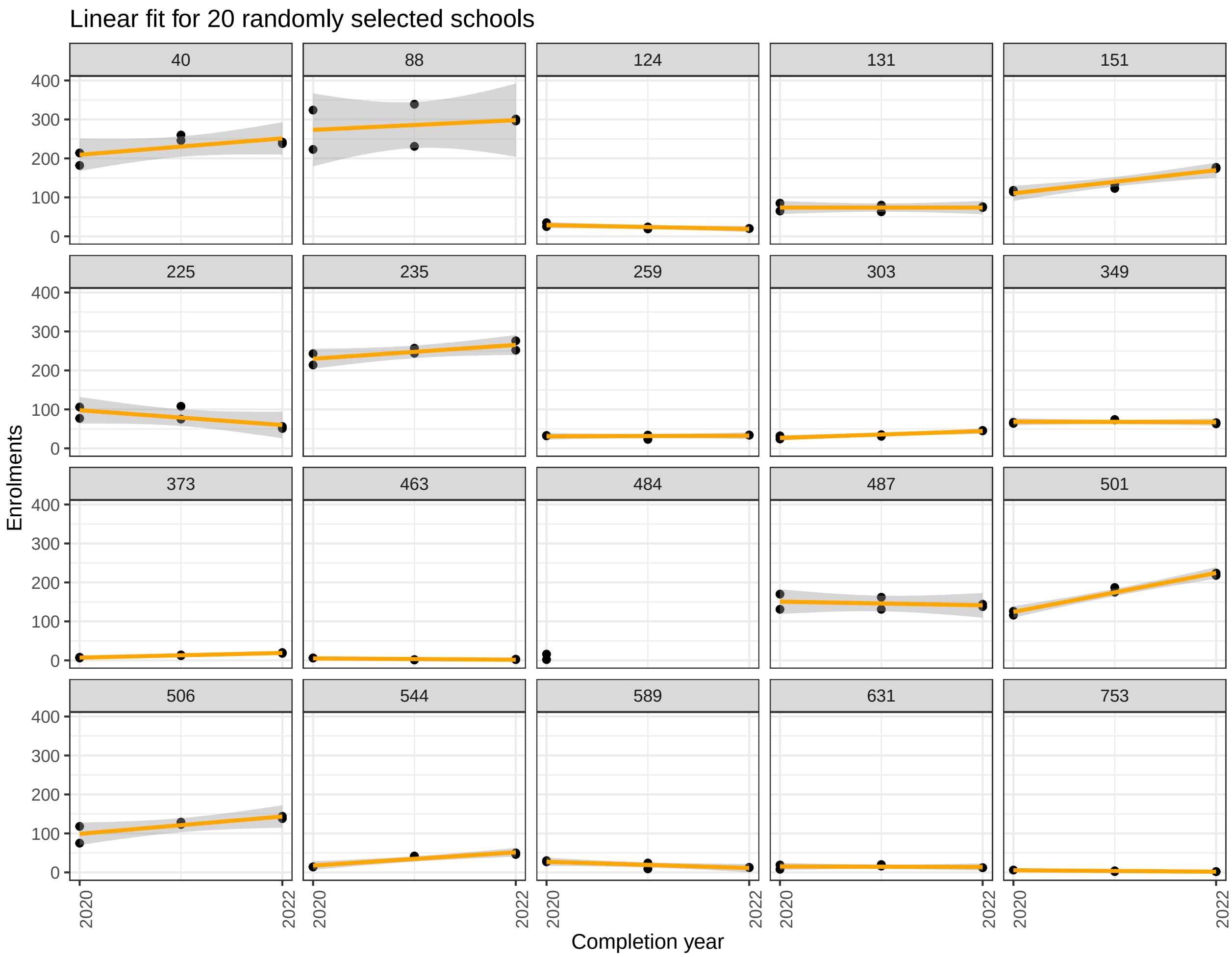
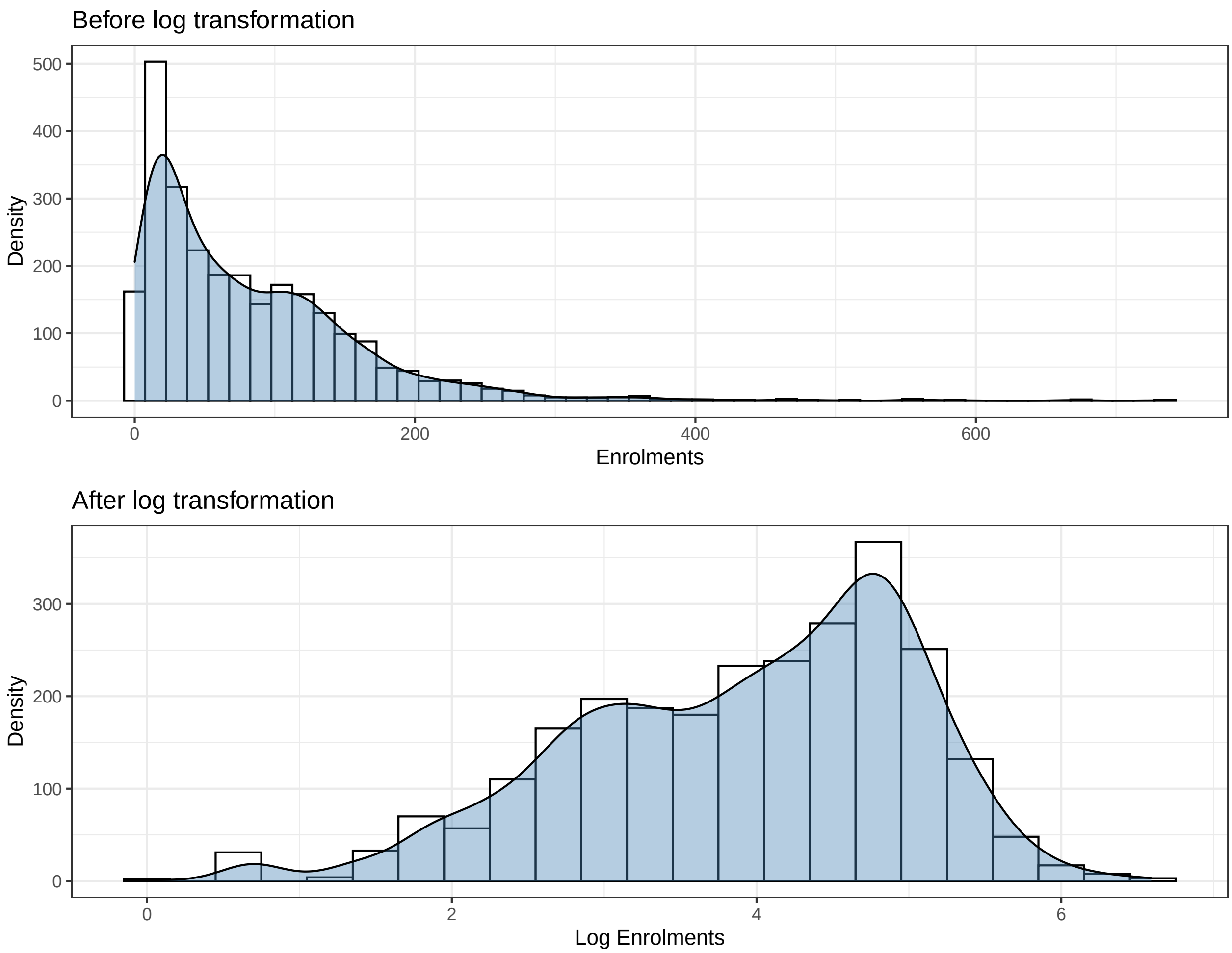


Figure 1: Basic linear model for 20 randomly selected schools to provide an at-a-glance visualisation of enrolment trends within schools for biology subject

As stated above, Essential Mathematics is one of the new subject introduced in the QCE system. Thus, there are only observations for three years, spanning from 2020 to 2022, as shown in Figure 1. The various sizes of schools can be seen in the plot, where schools 589, 631, 753 (bottom-right) have relatively low enrolments as compared to school 40 or school 88 (top-right). School 484 introduced the school in 2020, but discontinued offering the subject since.

Getting the data ready for modelling

Linearise response variable using log transformation



The enrolments were right skewed, which is likely to be attributed to the various school sizes (as seen in Figure 1). A log transformation was implemented to the response variable (*i.e.* `enrolments`) to allow the the multilevel model to better capture the enrolment patterns.

Unconditional means model

Table 1: AIC values for all candidate models for Essential Mathematics

	df	AIC
Model0.2: Schools nested within districts	4	3614.649
Model0.0: Within schools	3	3619.822
Model0.1: Schools nested within postcodes	4	3621.822

Referring back to step 3, three candidate models are fitted, with the AIC shown in Table 1. `Model0.2`, corresponding to having schools nested within districts is the best model, with optimised (lowest) AIC and will be used in the subsequent analysis.

Intraclass correlation (ICC)

```
## Random effects:

##  Groups                        Name        Variance Std.Dev.
##  qcaa_school_id:qcaa_district (Intercept) 1.148377 1.07162
##  qcaa_district                (Intercept) 0.047705 0.21841
##  Residual                                 0.113995 0.33763

##
##  Fixed effects:

##             Estimate Std. Error  t value
## (Intercept) 3.841442 0.07919555 48.50577

##
##  Number of schools (level-two group) = 458
##  Number of district (level-three group) = 13
```

This model takes into account 458 schools nested in 13 districts. In a three-level multilevel model, two intraclass correlations can be obtained using the model summary output above:

The **level-two ICC** relates to the correlation between school i from a certain district k in time t and in time t^*:

$$\text{Level-two ICC} = \frac{\tau_{00}^2}{\tau_{00}^2 + \phi_{00}^2 + \sigma^2} = \frac{1.1484}{(1.1484 + 0.0477 + 0.1140)} = 0.8766$$

This can be conceptualised as the correlation between enrolments of two random draws from the same school at two different years. In other words, 87.66% of the total variability is attributable to the differences between schools from the same district rather than changes over time within schools.

The **level-three ICC** refers to the correlation between different schools i and i^* from a specific school j.

$$\text{Level-three ICC} = \frac{\phi_{00}^2}{\tau_{00}^2 + \phi_{00}^2 + \sigma^2} = \frac{0.0477}{(1.1484 + 0.0477 + 0.1140)} = 0.0364$$

Similarly, it can be inferred that the correlation between enrolments of two randomly selected schools from different districts are 3.64%, where the total variability can be attributed to the difference between districts.

Unconditional Growth model

```
##  Groups                        Name        Variance   Std.Dev. Corr
##  qcaa_district:qcaa_school_id (Intercept) 1.14444636 1.069788
##                                year20      0.04903558 0.221440 -0.088
##  qcaa_district                 (Intercept) 0.04361994 0.208854
##                                year20      0.00082473 0.028718 0.230
##  Residual                                  0.07323625 0.270622
```

```
##               Estimate Std. Error   t value
## (Intercept) 3.78369587 0.07738087 48.897045
## year20      0.05820918 0.01485465  3.918583
```

```
##  Number of Level Two groups =  458
##  Number of Level Three groups =  13
```

The unconditional growth model adds the systematic changes over time, the model specification can be found in step 4. This allows for assessing within-school variability which can be attributed to the linear changes over time. Based on the model output:

- $\pi_{0ij} = 3.7837$: Initial status for school i in district j (*i.e.* expected log enrolments when time = 0)
- $\pi_{1ij} = 0.0582$: Growth rate for school i in district j
- $\epsilon_{tij} = 0.0732$: Variance in within-school residuals after accounting for linear growth overtime

Essential Mathematics was first introduced in 2020, and schools are expected to have 43.9785 ($e^{3.7837}$), on average. Furthermore, enrolments were expected to increase by 5.9937% ($(e^{0.0582092} - 1) \times 100$) every year. The estimated within-school variance decrease by 35.7577% (0.1140 to 0.0732362), implying that 35.7577% of the within-school variability can be explained by the linear growth over time.

Testing fixed effects

Table 2: AIC for all possible models with different combinations of fixed effects

model	npar	AIC	BIC	logLik
model4.0	19	2946.979	3058.468	-1454.489
model4.6	15	2951.546	3039.564	-1460.773
model4.2	14	2951.814	3033.964	-1461.907
model4.8	14	2951.814	3033.964	-1461.907
model4.9	14	2951.814	3033.964	-1461.907
model4.10	14	2951.814	3033.964	-1461.907
model4.1	17	2955.390	3055.143	-1460.695
model4.7	16	2955.666	3049.552	-1461.833
model4.3	13	2981.319	3057.601	-1477.659
model4.4	14	2985.158	3067.308	-1478.579
model4.5	15	2985.186	3073.204	-1477.593

As detailed in step 6, level-two predictors (`sector` and `unit`) are added to the model. The largest possible model will be fitted, before removing each fixed effect one by one whilst recording the AIC for each model. `model4.0` corresponds to the largest model while `model4.10` is the smallest possible model. The model with the optimal (lowest) AIC is the largest possible model `model4.0`, and will be used in subsequent sections.

Parametric bootstrap to test random effects

Table 3: Parametric Bootstrap to compare larger and smaller, nested model

npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr_boot(>Chisq)
17	2944.454	3044.208	-1455.227	2910.454	NA	NA	NA
19	2946.979	3058.468	-1454.489	2908.979	1.47565	2	0.256

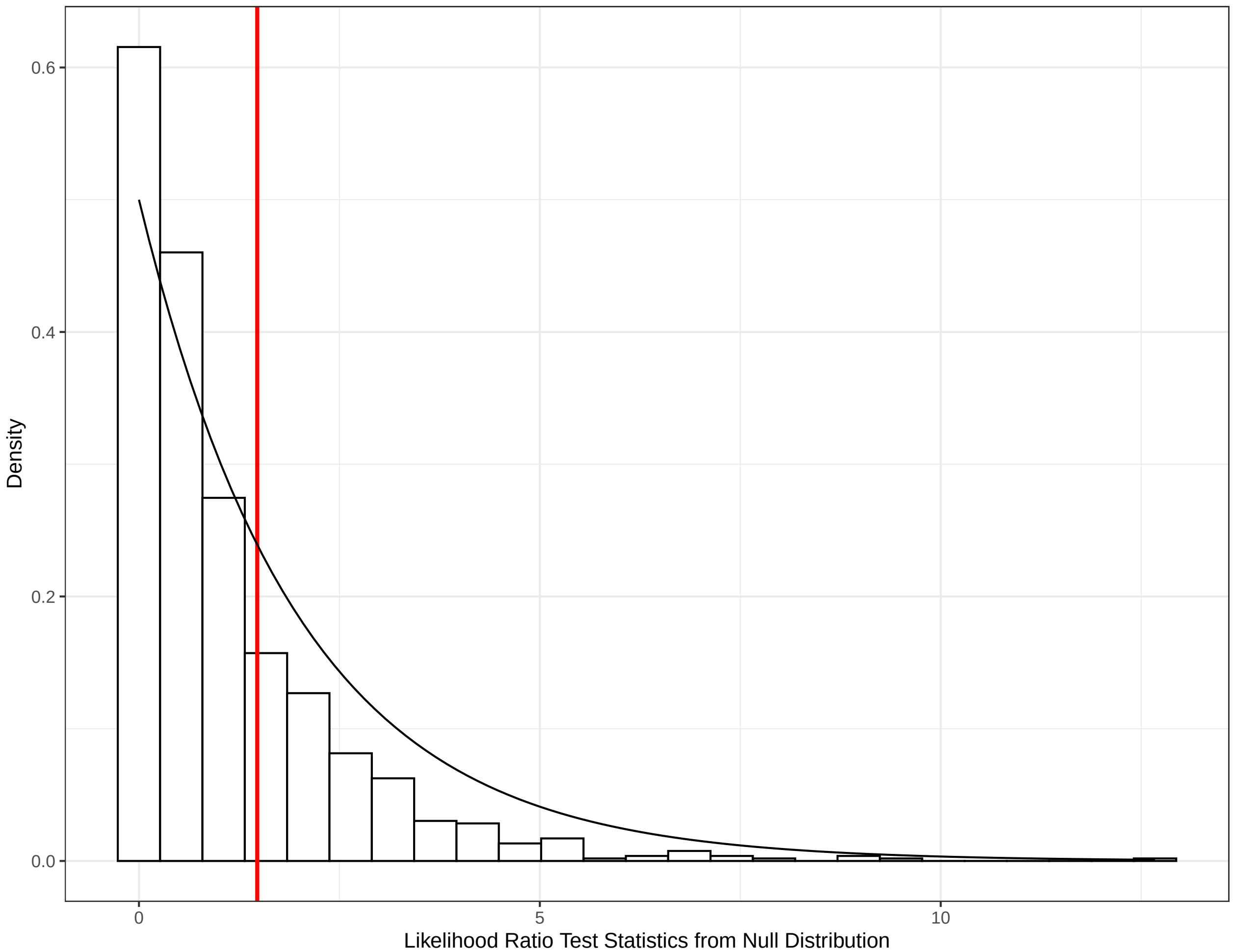


Figure 2: Histogram of likelihood ratio test statistic, with a red vertical line indicating the likelihood ratio test statistic for the actual model

The parametric bootstrap is used to approximate the likelihood ratio test statistic to produce a more accurate p-value by simulating data under the null hypothesis (detailed explanation can be found in step 7. Figure 2 displays the likelihood ratio test statistic from the null distribution, with the red line representing the likelihood ratio test statistic using the actual data.

The p-value of 25.6% (Table 3) indicates the proportion of times in which the bootstrap test statistic is greater than the observed test statistic. The large estimated p-value is $0.256 < 0.05$ fails to reject the null hypothesis at the 5% level, indicating that the smaller model (without random slope at level three) is preferred.

Confidence interval

Table 4: 95% confidence intervals for fixed and random effects in the final model

var	2.5 %	97.5 %
sd_(Intercept)|qcaa_district:qcaa_school_id	0.7489372	0.8615391
cor_year20.(Intercept)|qcaa_district:qcaa_school_id	-0.2253224	-0.0015329
sd_year20|qcaa_district:qcaa_school_id	0.2044296	0.2476368
sd_(Intercept)|qcaa_district	0.1525776	0.4495489
sigma	0.2577367	0.2748534
(Intercept)	3.3929124	3.8944034
year20	0.0209839	0.1432844
sectorGovernment	0.5992458	1.0092179
sectorIndependent	-1.1713273	-0.7385895
unityear_12_enrolments	0.0544900	0.1923526
year20:sectorGovernment	-0.1098924	0.0381604
year20:sectorIndependent	-0.0548805	0.1066265
year20:unityear_12_enrolments	-0.1250023	-0.0091553
sectorGovernment:unityear_12_enrolments	-0.2852633	-0.1176833
sectorIndependent:unityear_12_enrolments	-0.0772452	0.1178977
year20:sectorGovernment:unityear_12_enrolments	0.0179056	0.1611691
year20:sectorIndependent:unityear_12_enrolments	-0.0740500	0.0683533

The parametric bootstrap is utilised to construct confidence intervals (as detailed in step 8). If the confidence intervals for the random effects does not include 0, it provides statistical evidence that the p-value is less than 0.5. In other words, it suggests that the random effects and the correlation between the random effects are significant at the 5% level. The 95% confidence interval is shown above (Table 4), and the random effects all exclude 0, further reiterating that they are statistically significant at the 5% level.

Interpreting final model

Composite Model

- Level one (measurement variable)

$$Y_{tij} = \pi_{0ij} + \pi_{1ij} year20_{tij} + \epsilon_{tij}$$

- Level two (schools within districts)

$$\pi_{0ij} = \beta_{00j} + \beta_{01j} sector_{ij} + \beta_{02j} unit_{ij} + \beta_{03j} sector_{ij} unit_{ij} + u_{0ij}$$
$$\pi_{1ij} = \beta_{10j} + \beta_{11j} sector_{ij} + \beta_{12j} unit_{ij} + \beta_{03j} sector_{ij} unit_{ij} + u_{1ij}$$

- Level three (districts)

$$\beta_{00j} = \gamma_{000} + r_{00j}$$
$$\beta_{01j} = \gamma_{010} + r_{01j}$$
$$\beta_{02j} = \gamma_{020} + r_{02j}$$
$$\beta_{03j} = \gamma_{030} + r_{03j}$$
$$\beta_{10j} = \gamma_{100}$$
$$\beta_{11j} = \gamma_{110}$$
$$\beta_{12j} = \gamma_{120}$$
$$\beta_{13j} = \gamma_{130}$$

Therefore, the composite model can be written as

$$\begin{aligned}
Y_{tij} &= \pi_{0ij} + \pi_{1ij} year92_{tij} + \epsilon_{tij} \\
&= (\beta_{00j} + \beta_{01j} sector_{ij} + \beta_{02j} unit_{ij} + \beta_{03j} sector_{ij} unit_{ij} + u_{0ij}) + \\
&\quad (\beta_{10j} + \beta_{11j} sector_{ij} + \beta_{12j} unit_{ij} + \beta_{13j} sector_{ij} unit_{ij} + u_{1ij}) year20_{tij} + \epsilon_{tij} \\
&= \left[\gamma_{000} + r_{00j} + (\gamma_{010} + r_{01j}) sector_{ij} + (\gamma_{020} + r_{02j}) unit_{ij} + (\gamma_{030} + r_{03j}) sector_{ij} unit_{ij} + u_{0ij}\right] + \\
&\quad \left[\gamma_{100} + \gamma_{110} sector_{ij} + \gamma_{120} unit_{ij} + \gamma_{130} sector_{ij} unit_{ij} + u_{1ij}\right] year20_{tij} + \epsilon_{tij} \\
&= \left[\gamma_{000} + \gamma_{010} sector_{ij} + \gamma_{020} unit_{ij} + \gamma_{030} sector_{ij} unit_{ij} + \gamma_{100} year20_{tij} + \gamma_{110} sector_{ij} year_{tij} + \gamma_{120} unit_{ij} year_{tij} + \gamma_{130} s\right. \\
&\quad \left[r_{00j} + r_{01j} sector_{ij} + r_{02j} unit_{ij} + r_{03j} sector_{ij} unit_{ij} + u_{0ij} + u_{1ij} year20_{tij} + \epsilon_{tij}\right]
\end{aligned}$$

Fixed effects

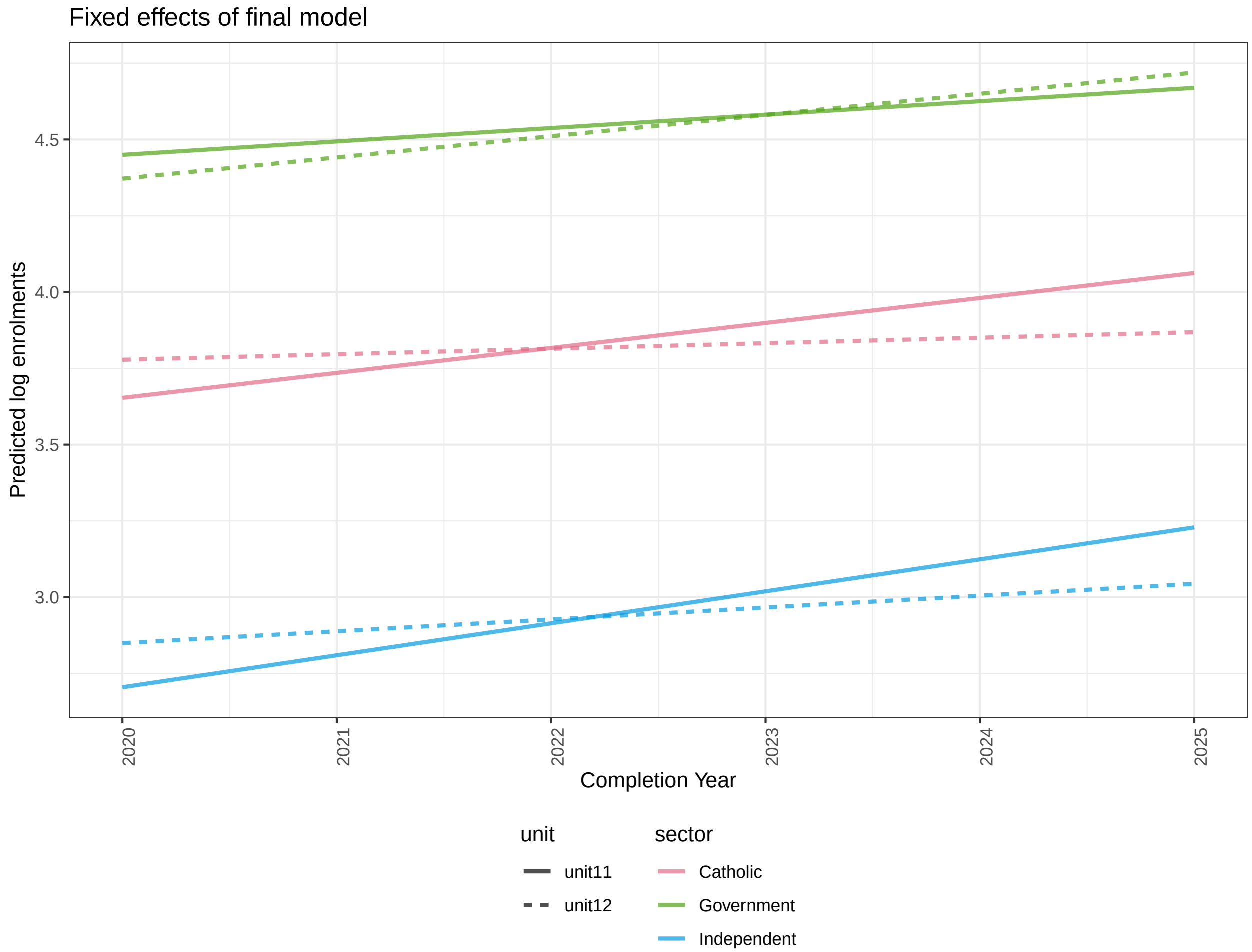


Figure 3: Fixed effects of the final model for Essential Mathematics

With a three-way interaction, it is easier to visualise the fixed effects (Figure 3). All sectors demonstrated the same trend with units, where unit 11 showed a larger increase in enrolments, on average. This implies that over the years, more students are enrolling in unit 11 than unit 12.

Random effects

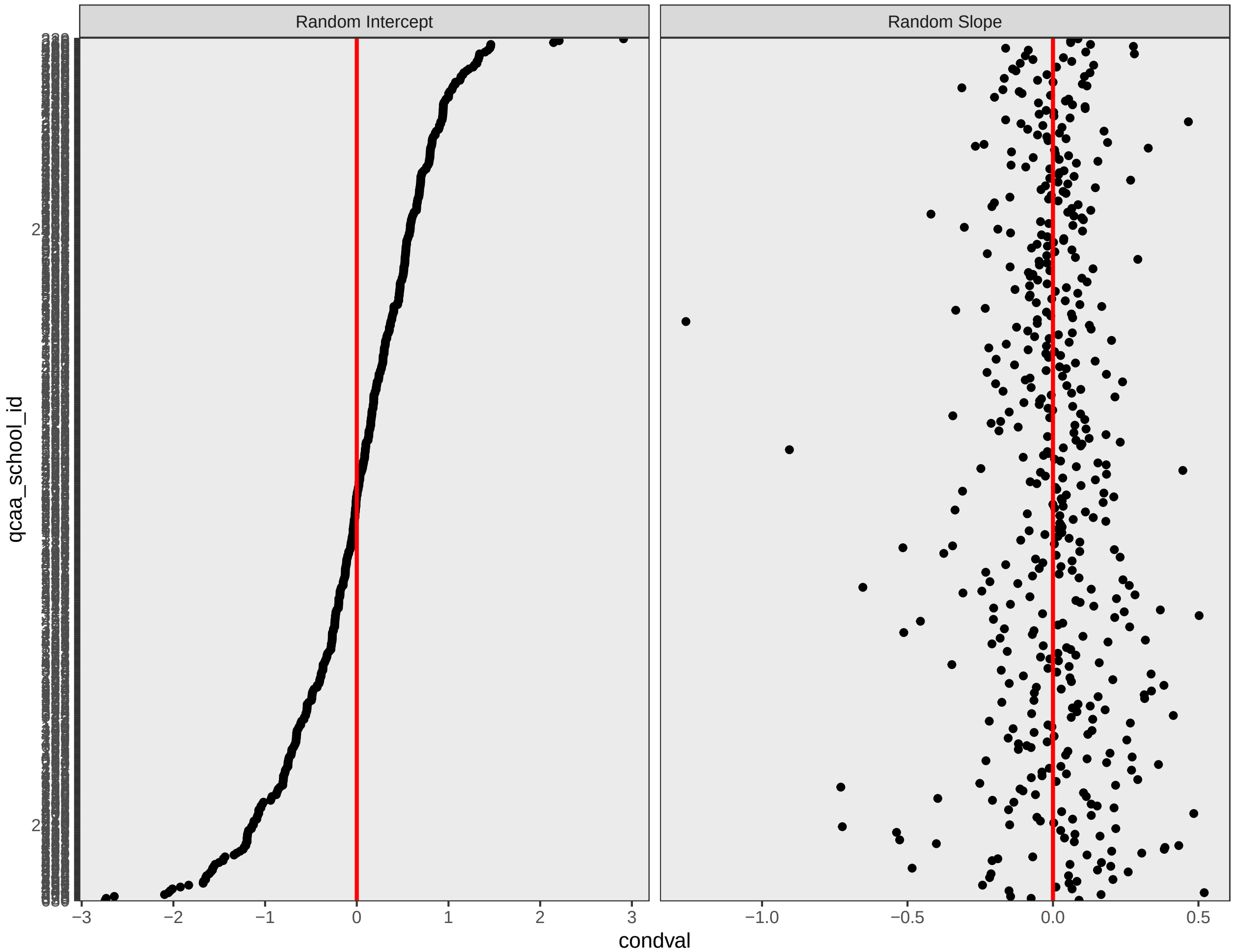


Figure 4: Random effects for schools

As shown in the model output and in Figure 4, there was little correlation (-0.12) between the random intercept and slope. Schools with appear to decrease or increase in enrolments at different rates; This may be attributed to the fact that the data only consists of observations for three years and there is not enough data to evaluate the enrolment patterns in schools.

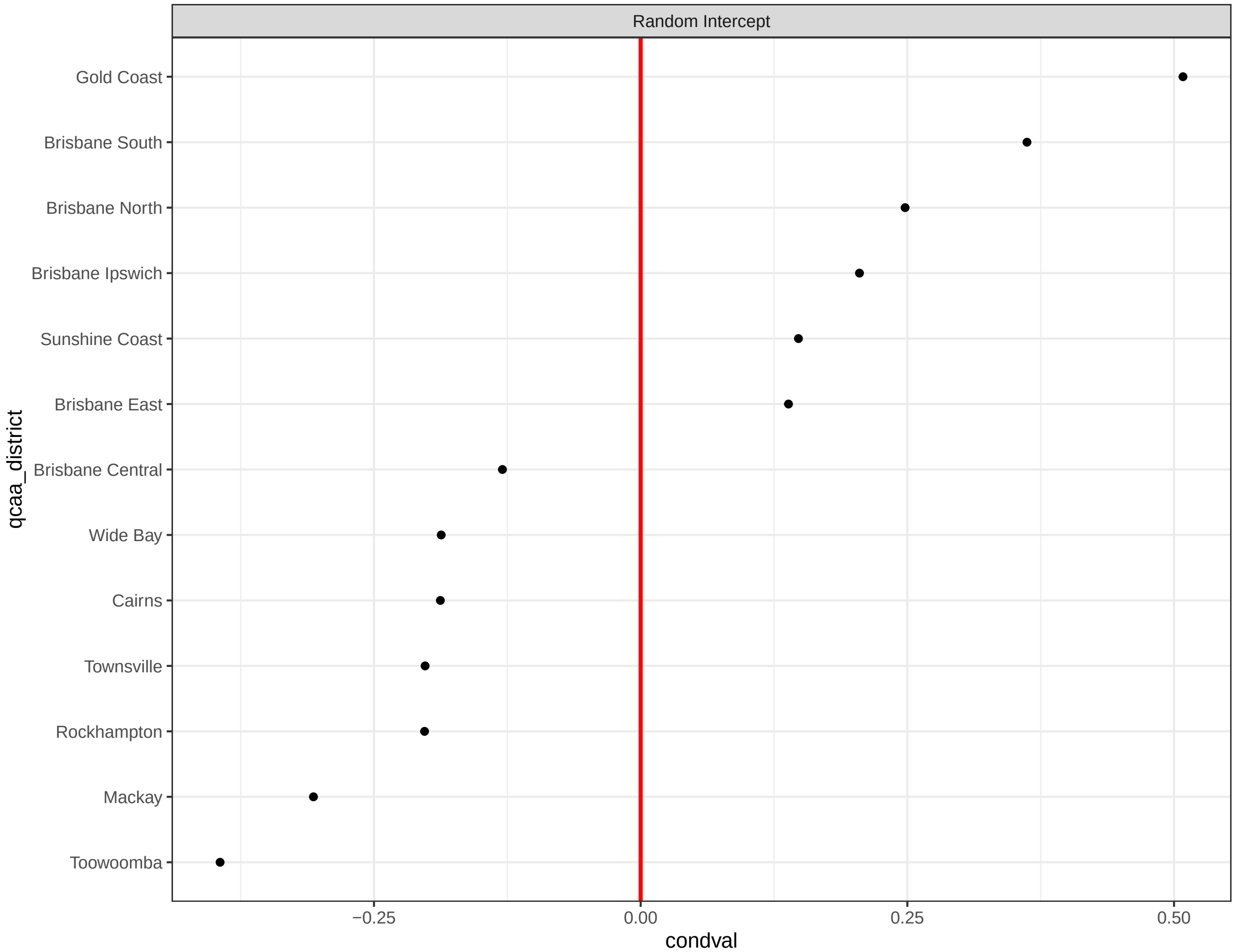


Figure 5: Random intercept for districts

As the random slopes are removed, all districts are predicted to have the same increase in enrolments over the years; And as was discussed previously, this was a reasonable assumption or an otherwise perfect correlation with random slope and intercept will be fitted. Figure 5 demonstrates that schools in Gold Coast has the largest enrolments while Mackay have the lowest enrolments in Essential Mathematics subject, on average.

Predictions

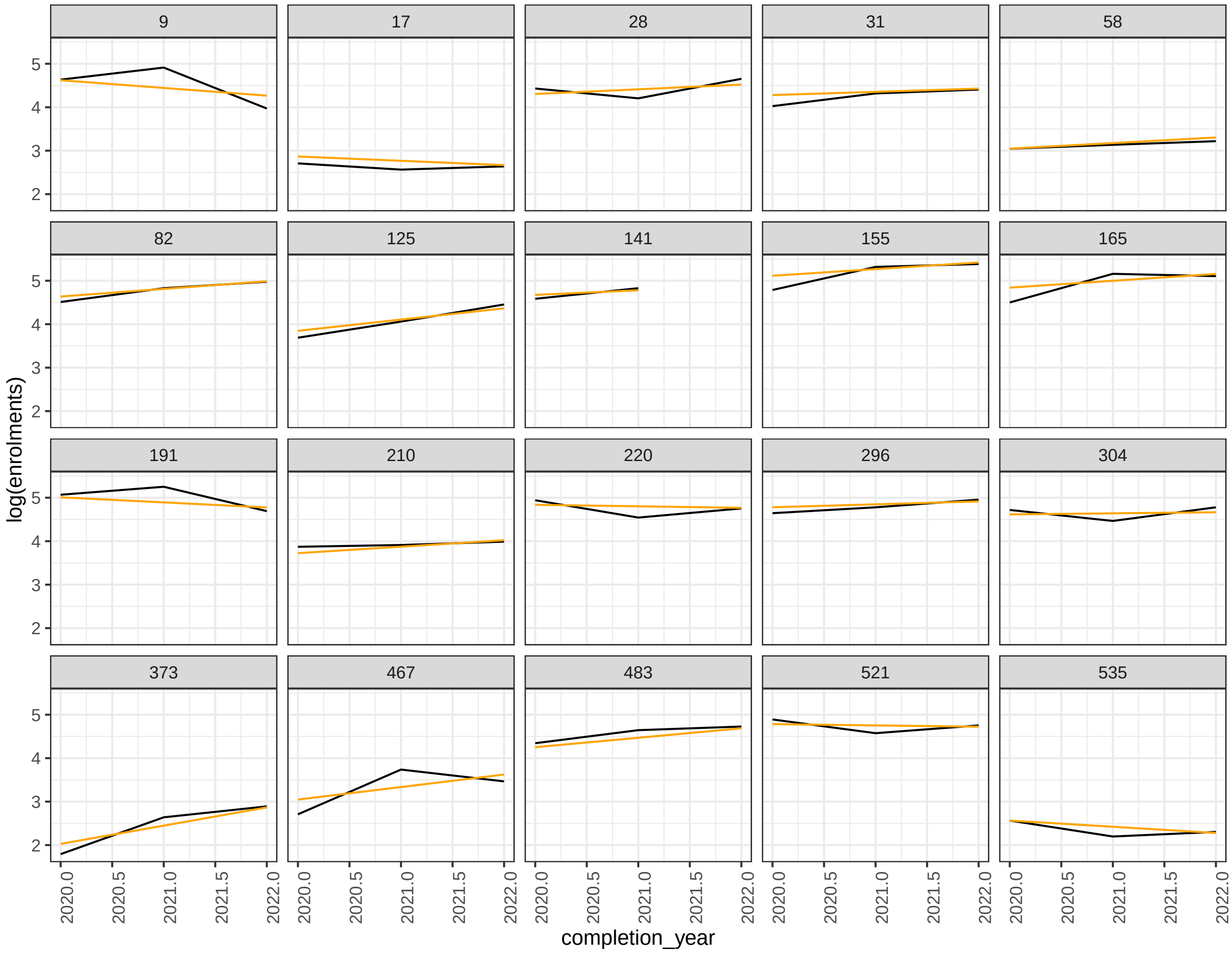


Figure 6: Model predictions for 20 randomly selected schools

Figure 6 above shows the predictions for 20 randomly selected schools.