DS GROUP SET 1

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First, I apologize for some minor notation differences. Through undergrad I always used P for probability and I for indicators, and thus while typing this document I defaulted back to using them. I'll make sure to be more careful for the next document.

Problem 1 (Lemma 1.11)**.**

a Here we need to prove that $P(\Omega|A) = 1$. $P(\Omega|A) = \frac{P(\Omega)P(A|\Omega)}{P(A)}$ by Bayes' rule. Although this can also be proven with the definition, I think this method is simpler. From the measure P, we know that $P(\Omega) = 1$. We can also reason that $P(A|\Omega) = P(A)$ since knowing that something happened doesn't give any information on which outcome happened. (independence) Thus $P(\Omega|A) = \frac{P(A)}{P(A)} = 1$ as desired.

b First, define $B, C \in \mathcal{F}$ such that $B \cap C = \emptyset$. We then need to prove that $P(B \cup C|A) = P(B|A) + P(C|A)$. By the definition of conditional probability, and then using simple set math, we can take

$$P(B \cup C|A) = \frac{P(A \cap (B \cup C))}{P(A)} = \frac{P((A \cap B) \cup P(A \cap C))}{P(A)}.$$

Next, we know that $A \cap B$ and $A \cap C$ are disjoint, since $A \cap B \subseteq B$ and $A \cap C \subseteq C$ and by assumption B and C are disjoint. Thus we know that the above

$$= \frac{P(A \cap B) + P(A \cap C)}{P(A)} = \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap C)}{P(A)} = P(B|A) + P(C|A).$$

By definition of conditional probability. Thus we are done.

Problem 2 (Lemma 2.7)**.**

a First, we need to show that for some $A \in \Omega$, $I_A = 1 - I_{A^c}$. To achieve this, we start by looking at the definitions of I_A and I_{A^c}:

$$I_A = \begin{cases} 1 & w \in A \\ 0 & w \notin A \end{cases}, \quad I_{A^c} = \begin{cases} 1 & w \in A^c \\ 0 & w \notin A^c \end{cases}$$

Since $A^c = \Omega \setminus A$, if $w \in A$, $w \notin A^c$ and if $w \notin A$, $w \in A^c$. Thus we can rewrite I_A as:

$$I_A = \begin{cases} 1 & w \notin A^c \\ 0 & w \in A^c \end{cases}$$

However, this is simply the inverse of A_{A^c}. When $I_A = 1$, $I_{A^c} = 0$, and when $I_A = 0$, $I_{A^c} = 1$ Thus we can rewrite I_A as:

$$I_A = \begin{cases} 1 - I_{A^c} & w \notin A^c \\ 1 - I_{A^c} & w \in A^c \end{cases}$$

Since these cases are the same, we can compact this down into simply $I_A = 1 - I_{A^c}$ as desired.

b Here, we again start by looking at the definition of $I_{A \cap B}$:

$$I_{A\cap B} = \begin{cases} 1 & w \in (A \cap B) \\ 0 & w \notin (A \cap B) \end{cases}$$

Looking at $w \in (A \cap B)$, this can be rewritten $(w \in A) \cap (w \in B)$. Aka to be in the union of two sets is equivalent to being in both sets. Further, $w \notin (A \cap B)$ can be rewritten as $(w \notin (A)) \cup (w \notin (B))$. Here not being in the union of two sets is equivalent to minimally not being in one of the sets. Thus $I_{A\cap B}$ can be written as:

$$I_{A\cap B} = \begin{cases} 1 & (w \in A) \cap (w \in B) \\ 0 & (w \notin (A)) \cup (w \notin (B)) \end{cases}$$

Next, consider the two indicators I_A and I_B. For $I_A \cdot I_B = 1$, $I_A = I_B = 1$, thus $w \in A$ and $w \in B$,. For $I_A \cdot I_B = 0$, $I_A = 0$ or $I_B = 0$, thus $w \notin A$ or $w \notin B$. However this simply gives us:

$$I_A \cdot I_B = \begin{cases} 1 & (w \in A) \cap (w \in B) \\ 0 & (w \notin (A)) \cup (w \notin (B)) \end{cases}$$

Since this is identical to $I_{A\cap B}$ we can conclude that $I_{A\cap B} = I_A \cdot I_B$.

c Here we will again start with the definition of $I_{A\cup B}$:

$$I_A \cdot I_B = \begin{cases} 1 & w \in (A \cup B) \\ 0 & w \notin (A \cup B) \end{cases}$$

As in part b, we can rewrite $w \in (A \cup B)$ as $(w \in A) \cup (w \in B)$. Being in the union is equivalent to being minimally in one of the sets. Also, $w \notin (A \cup B)$ can be rewritten as $(w \notin A) \cap (w \notin B)$. To not be in the union you must be in neither of the sets. This gives us a rewritten $I_{A\cup B}$:

$$I_{A\cup B} = \begin{cases} 1 & (w \in A) \cup (w \in B) \\ 0 & (w \notin A) \cap (w \notin B) \end{cases}$$

Since we are using set union, simply $I_A + I_B$ seems logical, however if we explicitly write this out we get:

$$I_A + I_B = \begin{cases} 2 & (w \in A) \cap (w \in B) \\ 1 & ((w \in A) \cup (w \in B)) \setminus ((w \in A) \cap (w \in B)) \\ 0 & (w \notin A) \cap (w \notin B) \end{cases}$$

This is due to lack of independence. We can thus apply a similar solution as in probability by subtracting $I_A \cdot I_B$. This will give us $I_A + I_B - I_A \cdot I_B$:

$$I_A + I_B - I_A \cdot I_B = \begin{cases} 1 & (w \in A) \cup (w \in B) \\ 0 & (w \notin A) \cap (w \notin B) \end{cases}$$

This adjustment subtracts 1 from the previous 2 case, preserves the previous 1 case since here explicitly $I_A \cdot I_B$ will be 0. Finally, the previous 0 case also remains unchanged. This also finally gives us an equivalent function to $I_{A\cap B}$, thus they are equivalent. (I apologize if this was way longer than needed. I wanted to be rigorous but might have overdone it.)

Problem 3 (Theorem 2.15.4)**.**

$$\int_{-\infty}^{\infty} f(x)dx = F(\infty) - F(-\infty)$$

by part 3 of the theorem. However, by definition of the CDF $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$. Thus (this is marginally sloppy, treating limits as equal, but I think it's "okay") $F(\infty) - F(-\infty) = 1$.

Problem 4 (Problem 8)**.**
To show that $g(x)$ is a $\mathbb{R}^1$ random vector, thus just a random variable, we need to show that for any $x \in \mathbb{R}$,

$$g(X)^{-1}((-\infty, x])) := \{\omega : g(X)(\omega) \leq x\} \in \mathcal{F}.$$

Since g is Borel, we know that $\mathcal{F}$ is a Borel sigma algbra, thus we can write this as:

$$g(X)^{-1}((-\infty, x])) := \{\omega : g(X)(\omega) \leq x\} \in \Sigma.$$

We also know by definition of a Borel sigma algebra that the set consists of all the half spaces of Ω. Thus when we have 1 dimension, the set consists of elements of the form $(-\infty, x]$ for $x \in \mathbb{R}$. Further, from Definition 2.16 we are given that for any $A \in \Sigma$, $g(A)^{-1} \in \Sigma$. Since the elements of Σ are of the form $(-\infty, x]$, this means that for any x,

$$g(X)^{-1}((-\infty, x])) \in \Sigma.$$

Thus $g(X)$ is a random variable.

Problem 5 (2.50 Tower)**.**
Following the proof for the continuous case, we start with

$$E[E[X|Y]] = E[g(Y)] = \sum_y g(y) f_y(y) = \sum_y E[X|Y] f_y(y)$$
$$= \sum_y (\sum_x x \cdot f_{x|y}(x|y)) f_y(y).$$

Distributing, applying Lemma 2.49, and then using the definition of marginal density we then get

$$= \sum_y \sum_x x \cdot f_{X|Y}(x|y) f_y(y) = \sum_y \sum_x x \cdot f_{X,Y}(x,y) = \sum_x x \cdot f_X(x)$$

. This is then by definition $E[X]$, so we are done.