Simulating Quadrotor(X) Dynamics

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0.0 Introduction

The purpose of this document is to establish the mathematical model of quadrotor dynamics. The model provides a foundation for simulating quadrotor motion and offers physical intuition into its behavior.

The state of the quadrotor is defined as:

$$X = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \quad \phi \quad \theta \quad \psi \quad p \quad q \quad r]^T$$

where $[x \quad y \quad z]^T$ are positions (inertial frame), $[\dot{x} \quad \dot{y} \quad \dot{z}]^T$ are linear velocities (inertial frame), $[\phi \quad \theta \quad \psi]^T$ are roll, pitch, and yaw angles (inertial frame), and $[p \quad q \quad r]$ are angular velocities (body frame). The quadrotor is an underactuated system with four independent control inputs, the rotor speeds Ω_i that govern these twelve states.

The modeling process, from motor commands to the full state dynamics, proceeds as follows:

1. **Individual Rotor Speeds:** These form the control inputs to the open-loop dynamic model.
2. **Rotor forces and torques:** Each rotor speed is converted into the corresponding thrust force and torque generated by that motor.
3. **Net forces and moments:** The individual rotor contributions are combined to compute the total forces and moments acting on the quadrotor. These form the control inputs to the closed-loop for the attitude controller.
4. **Translational dynamics:** Using the quadrotor's orientation (roll, pitch, and yaw) and the net thrust, the linear acceleration in the inertial frame is determined. Euler integration is then used to approximate the linear velocities and positions.
5. **Rotational dynamics:** The angular velocities and net moments in the body frame are used to compute angular accelerations. Euler integration provides the updated angular velocities, which, combined with the accelerations, are transformed into the inertial frame to update the orientation.

Following this procedure, the full twelve-state vector of the quadrotor is obtained, capturing it's full behavior. This "web" of equations that make up the quadrotor's dynamics is shown in Figure 0.0.1

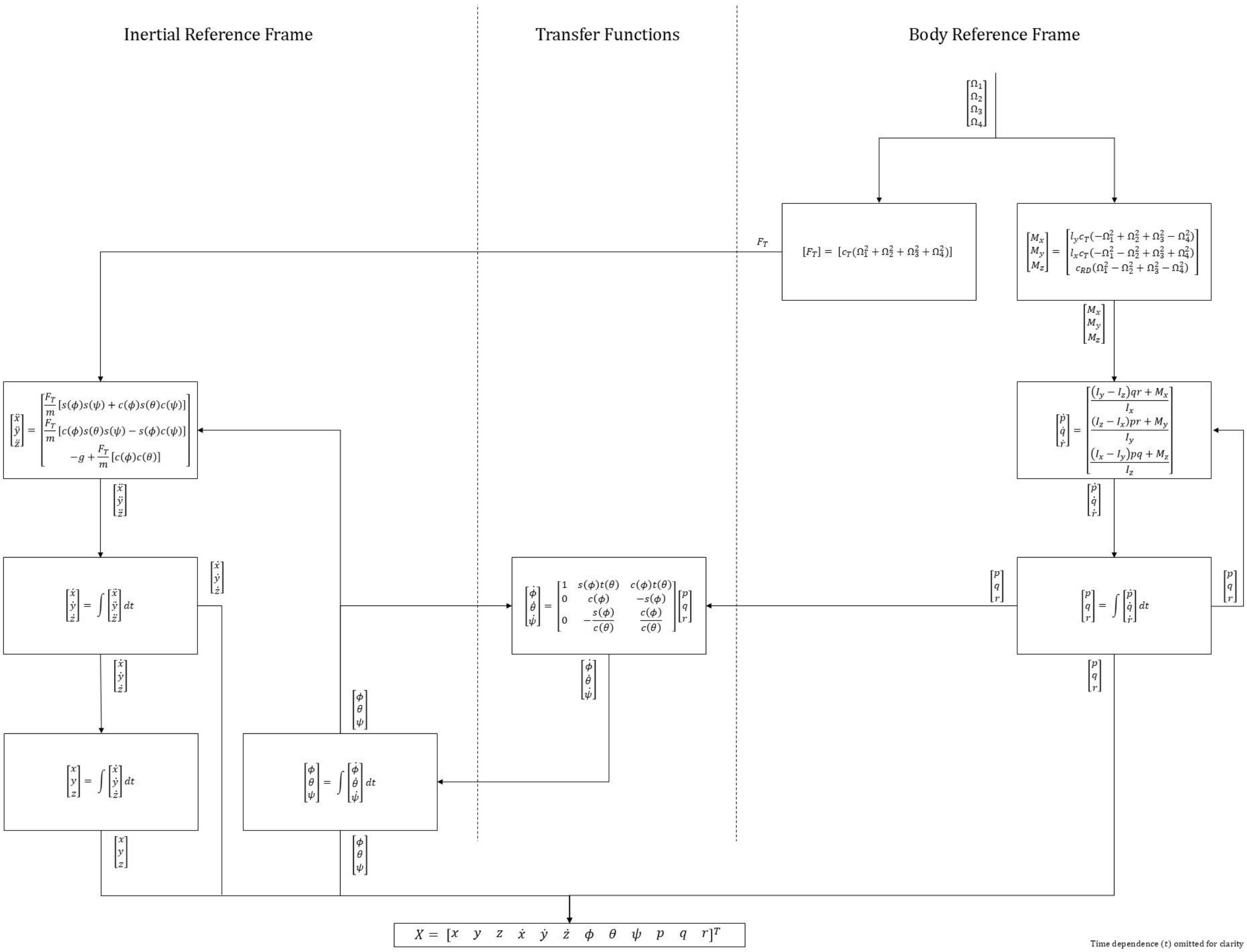

Figure 0.0.1. the “web” of equations that make up the quadrotor’s dynamics.

1.0 Nomenclature

Variables			
Symbol	**Simulation**	**Units**	**Description**
x, y, z	x, y, z	m	Position in the inertial frame
$\dot{x}, \dot{y}, \dot{z}$	dx, dy, dz	$\frac{m}{s}$	Linear velocity in the inertial frame
$\ddot{x}, \ddot{y}, \ddot{z}$	ddx, ddy, ddz	$\frac{m}{s^2}$	Linear acceleration in the inertial frame
u, v, w	u, v, w	$\frac{m}{s}$	Linear velocity in the body frame
$\dot{u}, \dot{v}, \dot{w}$	du, dv, dw	$\frac{m}{s^2}$	Linear acceleration in the body frame

ϕ, θ, ψ	phi, theta, psi	rad	Roll, pitch, yaw angles in the inertial frame
$\dot{\phi}, \dot{\theta}, \dot{\psi}$	dphi, dtheta, dpsi	$\frac{rad}{s}$	Roll, pitch, yaw rates in the inertial frame
$\ddot{\phi}, \ddot{\theta}, \ddot{\psi}$	ddphi, ddtheta, ddpsi	$\frac{rad}{s^2}$	Roll, pitch, yaw accelerations in the inertial frame
p, q, r	p, q, r	$\frac{rad}{s}$	Angular velocity in body frame
$\dot{p}, \dot{q}, \dot{r}$	dp, dq, dr	$\frac{rad}{s^2}$	Angular acceleration in body frame
$\Omega_1, \Omega_2, \Omega_3, \Omega_4$	omega_1, omega_2, omega_3, omega_4	$\frac{rad}{s}$	Rotor angular speeds
Ω_h	omega_h	$\frac{rad}{s}$	Rotor speed at hover
u_1, u_2, u_3, u_4	u1, u2, u3, u4		Control inputs
F_T	F_T	N	Total thrust force
M_x, M_y, M_z	M_x, M_y, M_z	$N * m$	Roll, pitch, yaw moments

Parameters			
Symbol	**Simulation**	**Units**	**Description**
m	m	kg	Quadrotor mass
I_x, I_y, I_z	I_x, I_y, I_z	$Kg * m^2$	Principal moments of inertia
l	l	m	Arm length
l_x, l_y	L_x, l_y	m	Effective roll and pitch lever arms
$\theta_{motor1-2}$	angle_motor1_2	deg	Angle between front motors
$\theta_{motor2-3}$	angle_motor2_3	deg	Angle between side motors
c_T	c_T	$\frac{N * s^2}{rad^2}$	Thrust coefficient
c_{RD}	c_RD	$\frac{N * m * s^2}{rad^2}$	Rotor drag coefficient

2.0 Assumptions

Several simplifying assumptions are adopted to reduce model complexity while retaining the dominant physical effects relevant to control and simulation, resulting in a sufficiently accurate representation of quadrotor dynamics. The key assumptions are:

- All state variables are directly measurable (perfect observability)
- Sensor dynamics, noise, and filtering are neglected (ideal sensors)
- Rotor thrust and drag are modeled using average coefficients , c_T and c_{RD}, with air density and blade geometry lumped into these terms
- The quadrotor is modeled as a rigid, unreformable body
- Aerodynamic body drag, ground effects, rotor inertia (gyroscopic effects), and rotor flapping are neglected

3.0 Reference Frames

Reference frames are essential in quadrotor modeling because translational motion is naturally expressed in the inertial frame, while forces and torques are generated in the body frame. To capture both, two primary frames are defined:

- Inertial (Earth) frame — a fixed frame used to express the vehicle's position and orientation relative to the environment
- Body frame — a moving frame attached to the quadrotor, used to express forces and torques generated by the rotors

While additional frames are sometimes introduced in detailed modeling, this two-frame formulation is sufficient. Figure 3.0.1 illustrates the inertial and body frames as applied to a quadrotor in the X-configuration.

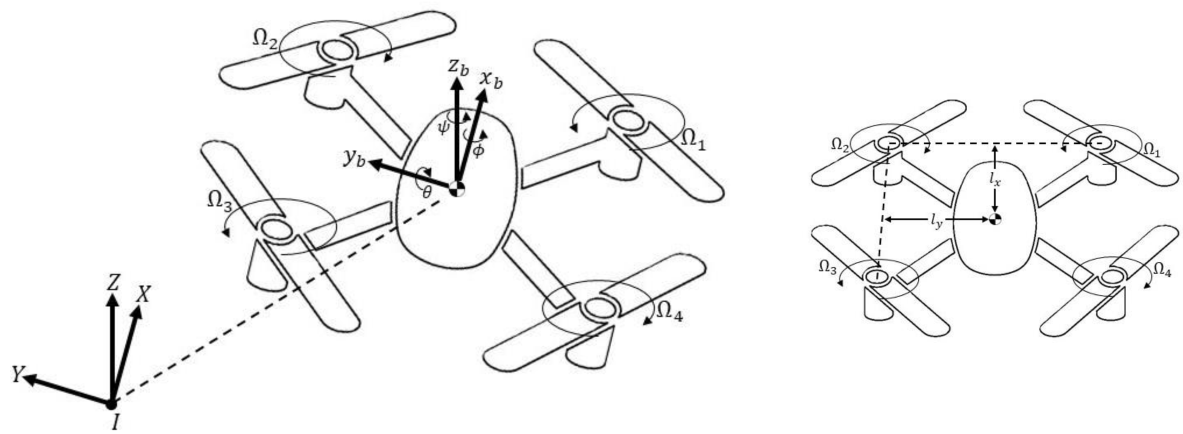

Figure 3.0.1. Reference frames and the Quadrotor in the X Configuration.

Euler rotation matrices define the transformation between the body and inertial frames by applying successive rotations about the roll, pitch, and yaw axes. They provide a compact representation of the quadrotor's orientation in three-dimensional space.

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix} \qquad EQ\ 3.0.1$$

$$R_y(\theta) = \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix} \qquad EQ\ 3.0.2$$

$$R_z(\psi) = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad EQ\ 3.0.3$$

$$R_{zyx}(\phi,\theta,\psi) = R_z(\psi)\cdot R_y(\theta)\cdot R_x(\phi)$$
$$= \begin{bmatrix} c(\theta)c(\psi) & s(\phi)s(\theta)c(\psi) - c(\phi)s(\phi) & c(\phi)s(\theta)c(\psi) + s(\phi)s(\psi) \\ c(\theta)s(\psi) & s(\phi)s(\theta)s(\psi) + c(\phi)c(\psi) & c(\phi)s(\theta)s(\psi) - s(\phi)c(\psi) \\ -s(\theta) & s(\phi)c(\theta) & c(\phi)c(\theta) \end{bmatrix} \qquad EQ\ 3.0.4$$

The relationship between body-frame angular rates, $[p \quad q \quad r]^T$, and the time derivatives of the Euler angles, $[\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}]^T$, is given by a nonlinear transformation matrix. This mapping accounts for the fact that roll, pitch, and yaw rates are not independent but instead coupled through the following sequence:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & s(\phi)t(\theta) & c(\phi)t(\theta) \\ 0 & c(\phi) & -s(\phi) \\ 0 & -\frac{s(\phi)}{c(\theta)} & \frac{c(\phi)}{c(\theta)} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad EQ\ 3.0.5$$

The inverse transformation expresses the body angular rates in terms of Euler angle derivatives:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s(\theta) \\ 0 & c(\phi) & s(\phi)c(\theta) \\ 0 & -s(\phi) & c(\phi)c(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \qquad EQ\ 3.0.6$$

These relations are necessary for the rotational dynamics, as they connect measurable body rates to the Euler angle representation of orientation.

4. Forces and Moments

To define the forces and moments, the motor configuration of the quadrotor must first be specified. The numbering of each rotor in the X-configuration is shown in Figure 3.0.1. The quadrotor is an underactuated system, possessing four control inputs (the motor speeds) to regulate twelve states, including linear position, linear velocity, angular orientation, and angular velocity. Control is achieved indirectly by mapping motor speeds to the net forces and moments generated on the body.

4.1 Thrust and Drag

Thrust is generated by each rotor as it accelerates air downward, producing an equal and opposite upward force in the body frame. For rotor i, the thrust is modeled as:

$$T_i = c_{Ti}\rho A_{ri} r_i^2 \Omega_i^2 \qquad EQ\ 4.1.1$$

where $c_{Ti}$is the thrust coefficient, ρ is the air density, A_{ri} is the rotor disk area, r_i is the rotor radius, and Ω_i is the angular velocity. In practice, the geometric and aerodynamic terms (ρ, A_{ri}, r_i) are typically lumped into an effective thrust coefficient, giving the simplified form

$$T_i = c_{Ti}\Omega_i^2 \qquad EQ\ 4.1.2$$

Using the average coefficient c_T, the total thrust force acting along the body z-axis is:

$$F_T = c_T(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \qquad EQ\ 4.1.3$$

The hover condition occurs when this thrust balances the quadrotor's weight, corresponding to the mean rotor speed

$$\Omega_h = \sqrt{\frac{mg}{4c_T}} \qquad EQ\ 4.1.4$$

Where m is the mass of the quadrotor and g is the gravitational constant. By increasing or decreasing the speed of all four motors relative to the angular speed required to hover, Ω_h, we can translate the quadrotor in the z-direction.

Like equation 4.1.1 the torque induced by rotor drag can be expressed in full form as:

$$\tau_i = c_{RDi}\rho A_{ri} r_i^3 \Omega_i^2 \qquad EQ\ 4.1.5$$

where c_{RDi} is the rotor drag coefficient. As with the thrust formulation of EQ 4.1.1–4.1.2, the aerodynamic and geometric terms are typically lumped into an effective drag coefficient, giving the simplified relation:

$$\tau_i = c_{RD}\Omega_i^2 \qquad EQ\ 4.1.6$$

where c_{RD} captures the combined influence of blade geometry, aerodynamic properties, and air density.

4.2 Moments

The roll and pitch moments are generated by differential rotor speeds, which create torque about the body-frame x- and y-axes, respectively. The roll-inducing moment, M_x, results from the speed difference between the left pair of motors (2 and 3) and the right pair (1 and 4). Similarly, the pitch-inducing moment, M_y, arises from the difference between the front pair (1 and 2) and the rear pair (3 and 4). These moments are leveraged through the effective arm lengths l_x and l_y, which represent the perpendicular distances from the quadrotor center of mass to the corresponding rotor force lines of action. While many models simplify by using the physical arm length, this formulation computes l_x and l_y more precisely using the measured arm length and the inter-arm angles:

$$l_x = l\sin(\theta_{motor1-2}) \qquad EQ\ 4.2.1$$

$$l_y = l\sin(\theta_{motor2-3}) \qquad EQ\ 4.2.2$$

The resulting body moments are then expressed as:

$$M_x = l_y c_T(-\Omega_1^2 + \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \qquad EQ\ 4.2.3$$

$$M_y = l_x c_T(-\Omega_1^2 - \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \qquad EQ\ 4.2.4$$

This approach captures the geometric asymmetry of the X-configuration, ensuring more accurate modeling of roll and pitch dynamics.

In addition to roll and pitch, the quadrotor experiences a yaw-inducing moment, M_z , which arises from the reactive drag torque generated by each rotor. Since rotors spin in opposite directions to balance angular momentum, their drag torques partially cancel. By commanding unequal speeds between clockwise and counterclockwise rotors, a net yaw torque is produced.

For the X-configuration, rotors 1 and 3 rotate counterclockwise, while rotors 2 and 4 rotate clockwise. The total yaw moment is therefore:

$$M_z = c_{RD}(\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \qquad EQ\ 4.2.5$$

This formulation ensures that increasing the relative speed of one rotor pair over the other generates the desired yaw motion. Unlike roll and pitch, which leverage rotor thrust and arm length, yaw control is directly tied to rotor drag effects.

Together, equations EQ 4.1.3 and EQ 4.2.3–4.2.7 fully define the net force and moments acting on the quadrotor body. In compact form,

$$\begin{bmatrix} F_T \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} c_T(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ l_y c_T(-\Omega_1^2 + \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \\ l_x c_T(-\Omega_1^2 - \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ c_{RD}(\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \end{bmatrix} \qquad EQ\ 4.2.6$$

5.0 Newton-Euler Dynamics

The complete motion of the quadrotor is governed by the Newton–Euler equations, which describe the translational and rotational dynamics of a rigid body under applied forces and moments. These equations couple the vehicle's translational motion in the inertial frame with its rotational motion in the body frame.

5.1 Translational Dynamics

The translational dynamics follow directly from Newton's Second Law. The net force consists of gravity, expressed in the inertial frame, and thrust, expressed in the body frame and mapped to the inertial frame via the rotation matrix $R_{zyx}(\phi, \theta, \psi)$:

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = R_{zyx}(\phi, \theta, \psi) \begin{bmatrix} 0 \\ 0 \\ F_T \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \qquad EQ\ 5.1.1$$

Showing that while thrust always acts along the body z-axis, its effect in the inertial frame depends on the vehicle's attitude. Expanding and rearranging (5.1.1) gives:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \frac{F_T}{m}[s(\phi)s(\psi) + c(\phi)s(\theta)c(\psi)] \\ \frac{F_T}{m}[c(\phi)s(\theta)s(\psi) - s(\phi)c(\psi)] \\ -g + \frac{F_T}{m}[c(\phi)c(\theta)] \end{bmatrix} \qquad EQ\ 5.1.2$$

5.2 Rotational Dynamics

The rotational dynamics are expressed by the Euler rigid-body equation:

$$I\omega + \omega \times (I\omega) = M \qquad EQ\ 5.2.1$$

where I is the inertia matrix in the body frame, $\omega = [p \quad q \quad r]^T$ is the angular velocity vector, and $M = [M_x \quad M_y \quad M_z]^T$ is the applied moment vector defined in Section 4.2. Expanding and rearranging EQ 5.2.1 gives:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dfrac{\left(I_y - I_z\right)qr + M_x}{I_x} \\ \dfrac{(I_z - I_x)pr + M_y}{I_y} \\ \dfrac{\left(I_x - I_y\right)pq + M_z}{I_z} \end{bmatrix} \qquad EQ\ 5.2.2$$