Numerical Methods for PDEs (Spring 2017)

Problems 2

Hand in written solutions to problems 6 and 7 at the start of the lecture on 21 February.
Consider the heat equation

$$u_t - K u_{xx} = 0 \quad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{3}$$

Problem 4. Show that the Du Fort - Frankel method for Eq. (1), given by

$$\frac{w_{k,j+1} - w_{k,j-1}}{2\tau} - K\frac{w_{k+1,j} - w_{k,j-1} - w_{k,j+1} + w_{k-1,j}}{h^2} = 0,$$

has the local truncation error $O\left(\tau^2 + h^2 + \tau^2/h^2\right)$.

Problem 5. The initial boundary value problem (1)–(3) is solved numerically using the finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &\frac{w_{k,j+1} - w_{k,j}}{\tau} - K(1-\sigma)\frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} - K\sigma\frac{w_{k+1,j+1} - 2w_{k,j+1} + w_{k-1,j+1}}{h^2} = 0, \end{aligned} \tag{4}$$

for $k = 1, 2, \ldots, N-1$ and $j = 0, 1, \ldots$. Here w_{kj} is an approximation to $u(x_k, y_j)$ and $x_k = kh$ $(k = 0, 1, \ldots, N)$, $t_j = j\tau$ $(j = 0, 1, \ldots)$, $h = \frac{1}{N}$. In Eqs. (4), σ is a real parameter such that $0 \le \sigma \le 1$. Show that the method is stable if

$$\sigma \ge \frac{1}{2}\left(1 - \frac{1}{2\gamma}\right)$$

where $\gamma = K\tau/h^2$.

Problem 6. Show that if

$$\gamma \equiv \frac{K\tau}{h^2} = \frac{1}{6}$$

in the explicit forward-difference method for Eq. (1):

$$\frac{w_{k,j+1} - w_{kj}}{\tau} - K\frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} = 0,$$

then the local truncation error is $O(\tau^2)$ or, equivalently $O(h^4)$.

Problem 7. Devise a backward difference scheme of $O(\tau + h^2)$ for the non-homogeneous heat equation eq.(2.73) in the notes with boundary conditions as given in eq.(2.80). This means you need to derive an expression for $w_{0,j+1}$ similar to eq.(2.78) and also a similar expression for $w_{N,j+1}$.

Problem 8. Consider the equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} - 2a(x,t)\frac{\partial u}{\partial x} + b(x,t) \quad \text{for} \quad 0 < x < 1, \quad t > 0,$$

subject to the initial and boundary conditions

$$u(0,t) = \mu_1(t), \quad u(1,t) = \mu_2(t), \quad u(x,0) = u_0(x).$$

Obtain a finite-difference approximation to this boundary-value problem and show that your finite-difference method is consistent with the equation, i.e. that the local truncation errors tend to zero as step sizes in x and in t go to zero.