# Numerical Methods for PDEs (Spring 2017)

## Problems 1

Attempt each problem as soon as the material comes up in your assigned reading. The deadline for handing in the written solution for a problem will be at the start of the session on Tuesday following the session in which we discuss the material.

**Problem 1.** Derive a finite-difference formula for the mixed derivative

$$\frac{\partial^2 u}{\partial x \partial t}$$

at $(x_k, t_j)$ based on the grid points $(x_k, t_j)$, $(x_{k+1}, t_j)$, $(x_k, t_{j+1})$ and $(x_{k+1}, t_{j+1})$, where $t_{j+1} = t_j + \tau$ and $x_{k+1} = x_k + h$.

**Problem 2.** The heat equation

$$\frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{for} \quad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u_0(x), \tag{2}$$

is solved numerically using the Crank-Nicolson finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &w_{k,j+1} - w_{k,j} - \frac{\gamma}{2}\left(w_{k+1,j} - 2w_{kj} + w_{k-1,j} + w_{k+1,j+1} - 2w_{k,j+1} + w_{k-1,j+1}\right) = \tau f(x_k, t_j + \tau/2), \end{aligned} \tag{3}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$. Here $w_{kj}$ is an approximation to $u(x_k, y_j)$ and

$$\gamma = K\tau/h^2, \quad x_k = kh \;\; (k = 0, 1, \ldots, N), \quad t_j = j\tau \;\; (j = 0, 1, \ldots, M), \quad h = \frac{1}{N}, \quad \tau = \frac{T}{M}.$$

Show that the local truncation error, given by

$$\tau_{k,j}(h) = \frac{1}{\tau}\left[u_{k,j+1} - u_{kj} - \frac{\gamma}{2}\left(u_{k+1,j} - 2u_{kj} + u_{k-1,j} + u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1}\right)\right] - f(x_k, t_j + \tau/2),$$

is $O(\tau^2 + h^2)$. (Here $u_{k,j} = u(x_k, t_j)$.)

**Problem 3.** The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} \quad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{4}$$

where $\alpha$ is a real constant, subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{5}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{6}$$

is solved numerically using the finite-difference method:

$$\begin{aligned} &w_{k0} = u_0(x_k), \quad w_{0j} = 0, \quad w_{Nj} = 0, \\ &\frac{w_{k,j} - w_{k,j-1}}{\tau} - \frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} - \alpha\,\frac{w_{k+1,j} - w_{k-1,j}}{2h} = 0, \end{aligned} \tag{7}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots$. Here $w_{kj}$ is an approximation to $u(x_k, y_j)$ and $x_k = kh$ $(k = 0, 1, \ldots, N)$, $t_j = j\tau$ $(j = 0, 1, \ldots)$, $h = \frac{1}{N}$.

(a) Find the local truncation error of this finite-difference scheme.

(b) Investigate the stability of the scheme (using the Fourier method).