Numerical Methods for PDEs (Spring 2017)

- **Module coordinators**:
  Gustav Delius and Richard Southwell
- **Module aims**:
  - To gain an understanding of basic numerical methods for solving PDEs
  - To be aware of the potential pitfalls and develop experience in avoiding them
  - To implement numerical algorithms in practice
  - To gain competency in using computing tools like R, C++, Git, ...

Teaching and assessment

- **Teaching**:
  - 18 hours of lectures
  - 9 hours of practicals (computer classes)
- **Assessment**:
  - 2 hour exam in week 1 of the Summer term – 75%
  - Coursework – 25%
    - mini-project 1 (10%) Implementing a numerical scheme for solving a given parabolic equation
    - mini-project 2 (15%) Implementing a numerical scheme for a PDE from a published research paper of your choice

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  - 27 hours of discussion of theory and computer work
    - Lecture notes
    - Lecture slides and summary sheets
    - Problem and solution sheets
    - Computer worksheets with example code and exercises
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Course contents

1. Introduction (1 week).
2. Parabolic differential equations. Finite-difference methods. Consistency, stability and convergence of finite-difference schemes. (5 weeks).
3. Finite-difference methods for elliptic differential equations (2 weeks).
4. Finite-difference methods for hyperbolic differential equations (1 week).

Reading list:

1. RL Burden & JD Faires, *Numerical Analysis* (6th ed.), Brooks/Cole Publishing Company, 1997;
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Taylor's theorem for functions of one variable

If $f \in C^{n+1}$ in $(x_0 - \epsilon, x_0 + \epsilon)$, then

$$f(x) = T_n + R_n$$

where T_n in the nth Taylor polynomial

$$T_n = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$

and R_n is the remainder term

$$R_n = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some point ξ between x_0 and x (ξ can be written as $\xi = x_0 + \theta(x - x_0)$ where $0 < \theta < 1$).

Some facts from Calculus

Taylor's theorem for functions of two variables

If $F \in C^{n+1}(D)$ where $D \subset \mathbb{R}^2$, then

$$\begin{aligned}
F(x,y) &= T_n(x,y) + R_n(x,y), \\
T_n(x,y) &= F(x_0,y_0) + \left[\Delta x \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} + \Delta y \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \right] \\
&+ \left[\frac{(\Delta x)^2}{2} \left.\frac{\partial^2 F}{\partial x^2}\right|_{(x_0,y_0)} + \Delta x \Delta y \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(x_0,y_0)} + \frac{(\Delta y)^2}{2} \left.\frac{\partial^2 F}{\partial y^2}\right|_{(x_0,y_0)} \right] \\
&+ \ \cdots\cdots \\
&+ \left[\frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (\Delta x)^{n-j} (\Delta y)^j \left.\frac{\partial^n F}{\partial x^{n-j} \partial y^j}\right|_{(x_0,y_0)} \right]
\end{aligned}$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$ and

$$R_n(x,y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (\Delta x)^{n+1-j} (\Delta y)^j \frac{\partial^{n+1} f}{\partial x^{n+1-j} \partial y^j}(\xi, \mu).$$

Big O notation

Definition. Let $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} f(x) = f_0$. If there exists a constant $K > 0$ such that

$$|f(x) - f_0| \leq K|g(x)|,$$

at least for x sufficiently close to zero, we write

$$f(x) = f_0 + O(g(x)) \quad \text{as} \quad x \to 0.$$

- **Example.** Let us show that $\sin(x)/x = 1 + O(x^2)$ as $x \to 0$.
- **Solution.** The 2nd Taylor polynomial for $sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!}\cos(\xi)$$

where ξ is some number between 0 and x.

We have

$$\left|\frac{\sin x}{x} - 1\right| = \frac{|x|^2}{3!}|\cos(\xi)| \leq \frac{x^2}{3!} = \frac{x^2}{6} \quad \Rightarrow \quad \frac{\sin x}{x} = 1+O(x^2).$$

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Properties of $O(x^n)$:

1. $O(x^n) + O(x^m) = O(x^k)$ for $n, m \geq 0$ and $k = \min\{n, m\}$.
2. $O(x^n)O(x^m) = O(x^{n+m})$ for $n, m \geq 0$.
3. $x^m O(x^n) = O(x^{n+m})$ for $n \geq 0$ and $n + m \geq 0$.

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