UNIVERSITY OF YORK

MMath and MSc Examinations 2015
MATHEMATICS
Numerical Methods for PDEs

Time Allowed: 2 hours.

Answer Question 1 and any **two** *of the remaining three questions.*
Question 1 carries 40 marks. Questions 2, 3 and 4 carry 30 marks each.
The marking scheme shown on each question is indicative only.
Standard calculators will be provided but are not necessary.
Please write your answers in ink; pencil is acceptable for graphs and diagrams. Do not use red ink.

The following notation is used throughout the paper: K, L and T are positive real constants; N and M are natural numbers; $\tau = T/M$ is the time step; $h = L/N$ is the step length in x; (x_k, t_j), where $x_k = kh$ for $k = 0, 1, \dots, N$ and $t_j = j\tau$ for $j = 0, 1, \dots, M$, are the grid points; $w_{k,j}$ is an approximation to $u(x_k, t_j)$.

1 (of 4). The non-homogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad \text{for} \quad 0 < x < 1, \;\; 0 < t < T,$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u_0(x)$$

is solved numerically using the finite-difference method:

$$\begin{gathered} w_{k,0} = u_0(x_k), \quad w_{0,j} = 0, \quad w_{N,j} = 0, \\ \frac{w_{k,j} - w_{k,j-1}}{\tau} - \frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(x_k, t_j), \end{gathered}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$. Here $w_{k,j}$ is an approximation to $u(x_k, t_j)$ and $x_k = kh$ $(k = 0, 1, \ldots, N)$, $t_j = j\tau$ $(j = 0, 1, \ldots, M)$, $h = 1/N$, $\tau = T/M$.

(a) Investigate the stability of the method.

[20]

(b) Describe the double-sweep method for solving the difference equations

$$A_i v_{i-1} - C_i v_i + B_i v_{i+1} = F_i \quad \text{for} \quad i = 1, \ldots, N-1; \quad v_0 = v_N = 0;$$

where the coefficients A_i, B_i and C_i satisfy the conditions

$$A_i, B_i, C_i > 0, \quad C_i \geq A_i + B_i.$$

[15]

(c) Explain how to modify the double-sweep method in the case of non-zero boundary conditions

$$v_0 = \mu_1, \quad v_N = \mu_2.$$

[5]

2 (of 4). Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a(x,t)\frac{\partial u}{\partial x} + g(x,t)u + f(x,t), \quad 0 < x < 1, \;\; 0 < t < T,$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = u_0(x).$$

Here $a(x,t)$, $g(x,t)$ and $f(x,t)$ are given continuous functions of x and t for $x \in [0,1]$, $t \in [0,T]$.

(a) Describe the forward-difference scheme for this boundary-value problem. Compute the local truncation error, and show that the scheme is consistent with the differential equation. **[18]**

(b) If the boundary condition $u(0,t) = 0$ in part (a) is replaced by the condition

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

obtain a finite difference approximation to this condition such that it is consistent with your approximation of the differential equation.

[12]

3 (of 4). Consider the following boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{for} \quad 0 < x < 1, \;\; 0 < y < 1; \\
&u(0,y) = -y^2, \quad u(1,y) = 1 - 2y - y^2 \quad \text{for} \quad 0 < y < 1; \\
&u(x,0) = x^2, \quad u(x,1) = x^2 - 2x - 1 \quad \text{for} \quad 0 < x < 1.
\end{aligned}$$

(a) Obtain a finite-difference approximation to this problem and find the truncation error at interior grid points. **[12]**

(b) For $f(x,y) = 0$, find approximations $w_{k,j}$ to $u(x_k, y_j)$ at the grid points $(x_k, y_j) = (kh, jh)$ for $k, j = 0, 1, 2, 3$, with $h = 1/3$.

[18]

4 (of 4). Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t < T,$$

subject to the boundary and initial conditions

$$\begin{aligned} &u(0,t) = u(1,t) = 0 \text{ for } 0 < t < T, \\ &u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \le x \le 1. \end{aligned}$$

Let (x_k, t_j) be the grid points where $x_k = kh$ ($k = 0, 1, \ldots, N$), $h = 1/N$, $t_j = \tau j$ ($j = 0, \ldots, M$), $\tau = T/M$ and let the equation be approximated by the finite-difference scheme

$$w_{k,j+1} - 2w_{k,j} + w_{k,j-1} - \frac{\gamma^2}{2}\left(\delta_x^2 w_{k,j+1} + \delta_x^2 w_{k,j-1}\right) = 0$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$, where

$$\gamma = \frac{\alpha\tau}{h}, \quad \delta_x^2 w_{k,j} = w_{k+1,j} - 2w_{k,j} + w_{k-1,j}.$$

(a) Investigate the stability of the scheme. **[18]**

(b) Obtain a finite-difference approximation to the initial condition for $\partial u/\partial t$ with error $O(\tau^2)$. **[12]**

1. (a) If we introduce the error $z_{k,0} = w_{k,0} - \tilde{w}_{k,0}$ into the initial condition, it will propagate with each step in time. Let $z_{k,j} = w_{k,j} - \tilde{w}_{k,j}$ be the error at the mesh point (x_k, t_j) for each $k = 0, 1, 2, \ldots, N$ and $j = 0, 1, \ldots$. It follows from the equations for $w_{k,j}$ that $z_{k,j}$ satisfies the difference equation

$$\frac{z_{k,j} - z_{k,j-1}}{\tau} - \frac{z_{k+1,j} - 2z_{k,j} + z_{k-1,j}}{h^2} = 0 \tag{1}$$

for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$. We seek a particular solution of (1) in the form

$$z_{k,j} = \rho_q^j e^{iqx_k}, \quad q \in \mathbb{R}. \tag{2}$$

The finite-difference method is stable, if

$$|\rho_q| \le 1$$

for all q.

Substitution of (2) into (1) yields

$$e^{iqx_k} \left(\rho_q^j - \rho_q^{j-1}\right) - \gamma \rho_q^j \left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) = 0, \qquad \gamma = \frac{\tau}{h^2},$$

or

$$1 - \frac{1}{\rho_q} - \gamma \left(e^{iqh} - 2 + e^{-iqh}\right) = 0.$$

Since

$$e^{iqh} - 2 + e^{-iqh} = \left(e^{iqh/2} - e^{-iqh/2}\right)^2 = -4\sin^2 \frac{qh}{2},$$

we obtain

$$\rho_q = \frac{1}{1 + 4\gamma \sin^2 \frac{qh}{2}}.$$

Evidently, $|\rho_q| \le 1$ for all q, and therefore, the method is unconditionally stable.

Remarks. *Discussed in lectures.*

$\boxed{\text{20 Marks}}$

(b) To solve the system, we seek α_i and β_i such that

$$v_{i-1} = \alpha_i v_i + \beta_i \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{3}$$

Substitution of (3) into our system yields

$$(\alpha_i A_i - C_i) v_i + B_i v_{i+1} + \beta_i A_i - F_i = 0 \quad \text{for} \quad i = 1, \ldots, N-1. \tag{4}$$

From (3), we also have

$$v_i = \alpha_{i+1} v_{i+1} + \beta_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

Substituting this into (4), we find that

$$[(\alpha_i A_i - C_i)\alpha_{i+1} + B_i]v_{i+1} + [(\alpha_i A_i - C_i)\beta_{i+1} + \beta_i A_i - F_i] = 0 \quad \text{for} \quad i = 1, \dots, N-1.$$

The last equation is satisfied if the two expressions in the square brackets are both zero. This leads to the following recursive formulas:

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}, \quad \beta_{i+1} = \frac{\beta_i A_i - F_i}{C_i - \alpha_i A_i}, \quad \text{for} \quad i = 1, \dots, N-1. \quad (5)$$

Now if α_1 and β_1 are known, then α_i and β_i for $i = 2, 3, \dots, N$ can be computed from Eqs. (5). α_1 and β_1 can be determined from Eq. (3) and the fact that $v_0 = 0$. Indeed

$$v_0 = \alpha_1 v_1 + \beta_1 \quad \text{and} \quad v_0 = 0 \quad \Rightarrow \quad \alpha_1 v_1 + \beta_1 = 0.$$

To satisfy the last equation, we choose $\alpha_1 = 0$ and $\beta_1 = 0$. Once we know all α_i and β_i, we compute $v_{N-1}, v_{N-2}, \dots, v_1$ using formula (3). 15 Marks

(c) In the double-sweep method, we use boundary conditions in order to (i) determine α_1 and β_1 and (ii) compute v_{N-1} when we apply formula (3) for $i = N$. Evidently, the latter will also work for $v_N \neq 0$. As for α_1 and β_1, these should be changed in order to satisfy the condition $v_0 = \mu_1$. So,

$$v_0 = \alpha_1 v_1 + \beta_1 \quad \text{and} \quad v_0 = \mu_1 \quad \Rightarrow \quad \alpha_1 v_1 + \beta_1 = \mu_1.$$

To satisfy this equation, we choose $\alpha_1 = 0$ and $\beta_1 = \mu_1$. This is the only modification of the double-sweep method needed to deal with non-zero boundary conditions.

Remarks. *Discussed in lectures.*

5 Marks

Total: 40 Marks

2. (a) First we choose positive integers N and M and define the grid points (x_k, t_j):

$$x_k = kh, \quad i = 0, 1, \dots, N; \quad t_j = j\tau, \quad i = 0, 1, \dots, M; \quad h = 1/N, \quad \tau = T/M.$$

Let $w_{k,j}$ be an approximation to the solution at the grid point (x_k, t_j), i.e. $w_{k,j} \approx u(x_k, t_j)$. Employing the centre difference formulae for the first and second derivatives with respect to x and the forward-difference formula for u_t, we obtain

$$\begin{aligned} \frac{w_{k,j+1} - w_{k,j}}{\tau} &- \frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} \\ &- a_{k,j}\frac{w_{k+1,j} - w_{k-1,j}}{2h} - g_{k,j}w_{k,j} - f_{k,j} = 0 \qquad (6) \end{aligned}$$

for $k = 1, \ldots, N-1$, $j = 0, \ldots, M-1$. Here $a_{k,j} = a(x_k, t_j)$, $g_{k,j} = g(x_k, t_j)$ and $f_{k,j} = f(x_k, t_j)$. It follows from the initial condition and the boundary conditions that

$$w_{k,0} = u_0(x_k), \quad k = 0, \ldots, N; \quad w_{0,j} = w_{N,j} = 0, \quad j = 1, \ldots, M.$$

The local truncation error of the difference equation (6) is

$$\begin{aligned}\tau_{i,j} &= \frac{u_{k,j+1} - u_{k,j}}{\tau} - \frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{h^2} \\ &- a_{k,j}\frac{u_{k+1,j} - u_{k-1,j}}{2h} - g_{k,j}u_{k,j} - f_{k,j}. \end{aligned} \tag{7}$$

Here $u_{k,j} = u(x_k, t_j)$ and $u(x,t)$ is the exact solution of the problem. Expanding $u_{k\pm1,j}$ and $u_{k,j\pm1}$ in Taylor's series at point (x_k, t_j), we find that

$$u_{k\pm1,j} = u_{k,j} \pm h\frac{\partial u}{\partial x}(x_k, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k, t_j) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k, t_j) + O(h^4),$$
$$u_{k,j\pm1} = u_{k,j} \pm \tau\frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k, t_j) \pm \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_k, t_j) + O(\tau^4).$$

It follows that

$$\frac{u_{k-1,j} - 2u_{k,j} + u_{k+1,j}}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_k, t_j) + O(h^2), \tag{8}$$
$$\frac{u_{k+1,j} - u_{k-1,j}}{2h} = \frac{\partial u}{\partial x}(x_k, t_j) + O(h^2). \tag{9}$$

Substituting (8), (9) in the formula for $\tau_{i,j}$, we obtain

$$\begin{aligned}\tau_{k,j} &= \frac{\partial u}{\partial t}(x_k, t_j) + O(\tau) - \frac{\partial^2 u}{\partial x^2}(x_k, t_j) \\ &\quad -a_{k,j}\frac{\partial u}{\partial x}(x_k, t_j) - g(x_i, t_j)u(x_k, t_j) + f(x_k, t_j) + O(h^2).\end{aligned}$$

Using the fact that $u(x,t)$ satisfies the differential equation, we conclude that

$$\tau_{i,j} = O(\tau + h^2).$$

Thus, the truncation errors tend to zero as $\tau, h \to 0$, which means that the proposed finite-difference method is consistent.

Remarks. *Similar to a homework exercise.*

$\boxed{\text{18 Marks}}$

(b) To approximate the boundary condition $u_x(0,t) = 0$, we add a 'false' boundary at $x = x_{-1} = x_0 - h$ and assume that the difference formula (6) approximates the equation at points (x_0, t_j) ($j = 0, \ldots, M-1$). Then, we have

$$\begin{aligned}\frac{w_{0,j+1} - w_{0,j}}{\tau} &- \frac{w_{1,j} - 2w_{0,j} + w_{-1,j}}{h^2} \\ &- a_{0,j}\frac{w_{1,j} - w_{-1,j}}{2h} - g_{0,j}w_{0,j} - f_{0,j} = 0.\end{aligned} \tag{10}$$

Approximating, $u_x(x_0, t_j)$ by the centre difference formula (whose truncation error is $O(h^2)$), we obtain

$$\frac{w_{1,j} - w_{-1,j}}{2h} = 0. \tag{11}$$

Eliminating $w_{-1,j}$ from Eqs. (10) and (11), we find that

$$w_{0,j+1} = \left(1 - 2\frac{\tau}{h^2} + \tau g_{0,j}\right) w_{0,j} + 2\frac{\tau}{h^2} w_{1,j} + \tau f_{0,j}. \tag{12}$$

This is an explicit formula that relates the boundary values at the time levels t_j and t_{j+1}.

In approximating the boundary condition at $x = 0$ by Eq. (12), we used formulae (10) and (11). As we already know, the truncation error for Eq. (10) is $O(\tau + h^2)$ and, in view of (9), the error for Eq. (11) is $O(h^2)$. Therefore, the truncation error of Eq. (12) is $O(\tau + h^2)$, which is consistent with the truncation error of the difference equation (6).

Remarks. *Discussed in lectures.*

$\boxed{\text{12 Marks}}$

$\boxed{\text{Total: 30 Marks}}$

3. (a) First we define the grid points (x_k, y_j) for $k, j = 0, 1, \ldots, N$ where $x_k = kh$, $y_j = jh$, $h = 1/N$. Approximating u_{xx} and u_{yy} by the central difference formula for the second derivative at all interior grid points, we obtain

$$\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} + \frac{w_{k,j+1} - 2w_{k,j} + w_{k,j-1}}{h^2} = f(x_k, y_j)$$

or, equivalently,

$$4w_{k,j} - w_{k+1,j} - w_{k-1,j} - w_{k,j+1} - w_{k,j-1} = -h^2 f(x_k, y_j) \tag{13}$$

for $k, j = 1, 2, \ldots, N-1$. Boundary conditions imply that

$$w_{0,j} = -y_j^2, \quad w_{N,j} = 1 - 2y_j - y_j^2 \quad \text{for} \quad j = 1, \ldots, N-1; \tag{14}$$

$$w_{k,0} = x_k^2, \quad w_{k,N} = x_k^2 - 2x_k - 1 \quad \text{for} \quad k = 1, \ldots, N-1. \tag{15}$$

The truncation error of the scheme at (x_k, y_j) is

$$\tau_{k,j} = \frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{h^2} + \frac{u_{k,j+1} - 2u_{k,j} + u_{k,j-1}}{h^2} - f(x_k, y_j) \tag{16}$$

where $u_{k,j} = u(x_k, y_j)$ and $u(x, y)$ is the exact solution of the problem. Since

$$u(x_{k\pm1}, y_j) = u(x_k, y_j) \pm h u_x(x_k, y_j) + \frac{h^2}{2} u_{xx}(x_k, y_j) \pm \frac{h^3}{6} u_{xxx}(x_k, y_j) + O(h^4),$$

$$u(x_k, y_{j\pm1}) = u(x_k, y_j) \pm h u_y(x_k, y_j) + \frac{h^2}{2} u_{yy}(x_k, y_j) \pm \frac{h^3}{6} u_{yyy}(x_k, y_j) + O(h^4),$$

we obtain

$$\frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{h^2} = u_{xx}(x_k, y_j) + O(h^2),$$
$$\frac{u_{k,j+1} - 2u_{k,j} + u_{k,j-1}}{h^2} = u_{yy}(x_k, y_j) + O(h^2).$$

Hence,

$$\tau_{k,j} = u_{xx}(x_k, y_j) + u_{yy}(x_k, y_j) - f(x_k, y_j) + O(h^2) = O(h^2).$$

Remarks. *Discussed in lectures.*

12 Marks

(b) At the interior grid points, we solve equations (13) (with $f = 0$) for each $k, j = 1, 2$ or, equivalently,

$$\begin{aligned} 4w_{1,1} - (w_{2,1} + w_{0,1} + w_{1,2} + w_{1,0}) &= 0, \\ 4w_{2,1} - (w_{3,1} + w_{1,1} + w_{2,2} + w_{2,0}) &= 0, \\ 4w_{1,2} - (w_{2,2} + w_{0,2} + w_{1,3} + w_{1,1}) &= 0, \\ 4w_{2,2} - (w_{3,2} + w_{1,2} + w_{2,3} + w_{2,1}) &= 0. \end{aligned} \tag{17}$$

For the grid points on the boundary, we have

$$\begin{aligned} &w_{0,1} = -h^2, \quad w_{0,2} = -4h^2, \\ &w_{3,1} = 1 - 2h - h^2, \quad w_{3,2} = 1 - 4h - 4h^2, \\ &w_{1,0} = h^2, \quad w_{2,0} = 4h^2, \\ &w_{1,3} = h^2 - 2h - 1, \quad w_{2,3} = 4h^2 - 4h - 1. \end{aligned} \tag{18}$$

Substitution of (18) in (17) yields

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} w_{1,1} \\ w_{2,1} \\ w_{1,2} \\ w_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 3h^2 - 2h + 1 \\ -3h^2 - 2h - 1 \\ -8h \end{pmatrix} = \begin{pmatrix} 0 \\ 2/3 \\ -2 \\ -8/3 \end{pmatrix}. \tag{19}$$

Our problem is reduced to solving system (19) of 4 linear equations for 4 unknowns. On subtracting the 4th equation from the 1st one, we find that

$$4w_{1,1} - 4w_{2,2} = 8h \quad \Rightarrow \quad w_{2,2} = w_{1,1} - 2h = w_{1,1} - \frac{2}{3}.$$

Using this in the 2nd and 3rd equations, we obtain

$$w_{2,1} = \frac{w_{1,1}}{2}, \quad w_{1,2} = \frac{w_{1,1}}{2} - \frac{2}{3}.$$

Substitution of these into the 1st equation yields

$$w_{1,1} = -\frac{2}{9}.$$

Hence,

$$w_{2,2} = -\frac{8}{9}, \quad w_{2,1} = -\frac{1}{9}, \quad w_{1,2} = -\frac{7}{9}.$$

Thus, the solution of (19) is

$$\begin{pmatrix} w_{1,1} \\ w_{2,1} \\ w_{1,2} \\ w_{2,2} \end{pmatrix} = \begin{pmatrix} -2/9 \\ -1/9 \\ -7/9 \\ -8/9 \end{pmatrix}. \tag{20}$$

Remarks. *Similar to a homework exercise.*

$\boxed{\text{18 Marks}}$
$\boxed{\text{Total: 30 Marks}}$

4. (a) We will investigate the stability of the scheme by the Fourier method. The perturbation $z_{k,j}$ satisfies the equation

$$z_{k,j+1} - 2z_{k,j} + z_{k,j-1} - \frac{\gamma^2}{2}\left(\delta_x^2 z_{k,j+1} + \delta_x^2 z_{k,j-1}\right) = 0$$

for $k = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots$ Substituting $z_{k,j} = \rho_q^j e^{iqx_k}$ into this equation, we obtain

$$e^{iqx_k}\left(\rho_q^{j+1} - 2\rho_q^j + \rho_q^{j-1}\right) - \frac{\gamma^2}{2}\left(\rho_q^{j+1} + \rho_q^{j-1}\right)\left(e^{iqx_{k+1}} - 2e^{iqx_k} + e^{iqx_{k-1}}\right) = 0$$

or

$$\left(\rho_q^2 - 2\rho_q + 1\right) + 2\gamma^2 \sin^2\frac{qh}{2}\left(\rho_q^2 + 1\right) = 0.$$

Hence, ρ_q satisfies the quadratic equation

$$\rho_q^2 - 2a\rho_q + 1 = 0, \tag{21}$$

where

$$a \equiv \frac{1}{1 + 2\gamma^2 \sin^2\frac{qh}{2}}.$$

Its roots are $\rho_q^{\pm} = a \pm \sqrt{a^2 - 1}$, so that the product of the roots is equal to 1 ($\rho_q^+ \rho_q^- = 1$). It follows that the stability condition $|\rho_q| \le 1$ can be satisfied only if $|\rho_q^+| = |\rho_q^-| = 1$. This means that the roots must be either complex conjugate or both equal to 1. In terms of the discriminant, the last condition is equivalent to $a^2 - 1 \le 0$.

Since
$$a = \frac{1}{1 + 2\gamma^2 \sin^2 \frac{qh}{2}} \leq 1$$
for all q, so holds $a^2 - 1 \leq 0$. Therefore the implicit method in question is unconditionally stable. $\boxed{\text{18 Marks}}$

Remarks. *Unseen, but standard.*

(b) Expanding $u(x_k, t_1)$ in Taylor's series in t at $(x_k, 0)$, we obtain
$$\frac{u(x_k, t_1) - u(x_k, 0)}{\tau} = u_t(x_k, 0) + \frac{\tau}{2} u_{tt}(x_k, 0) + O(\tau^2).$$
Suppose that the wave equation also holds on the initial line, i.e.
$$u_{tt}(x_k, 0) - \alpha^2 u_{xx}(x_k, 0) = 0 \quad \text{for} \quad k = 0, 1, \ldots, N.$$
Then
$$u_{tt}(x_k, 0) = \alpha^2 u_{xx}(x_k, 0) = \alpha^2 f''(x_k).$$
Therefore,
$$u(x_k, t_1) = u(x_k, 0) + \tau g(x_k) + \frac{\alpha^2 \tau^2}{2} f''(x_k) + O(\tau^3).$$
Hence,
$$w_{k,1} = w_{k,0} + \tau g(x_k) + \frac{\alpha^2 \tau^2}{2} f''(x_k). \tag{22}$$
This approximates the initial condition for u_t with truncation error $O(\tau^2)$ for each $k = 1, 2, \ldots, N - 1$. $\boxed{\text{12 Marks}}$

Remarks. *Discussed in lectures.*

$\boxed{\text{Total: 30 Marks}}$