Review of Numerical Methods for PDEs 2017

Parabolic equations

- **Initial boundary value problem for the heat equation:**

$$u_t = K\, u_{xx} + f(x,t), \quad 0 < x < L, \quad 0 < t < T,$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for} \quad t \in (0,T),$$

and initial condition

$$u(x,0) = u_0(x).$$

- **Grid points** We choose integers N and M and define $\tau = T/M$ and $h = L/N$. Then we define the grid points (x_k, t_j), where $x_k = hk$ for $k = 0, 1, \ldots, N$ and $t_j = \tau j$ for $j = 0, 1, 2, \ldots, M$.

- **Forward-difference formula for u_t:**

$$u_t(x_k, t_j) \approx \frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\tau} + O(\tau).$$

- **Backward-difference formula for u_t:**

$$u_t(x_k, t_j) \approx \frac{u(x_k, t_j) - u(x_k, t_{j-1})}{\tau} + O(\tau).$$

- **Central-difference formula for u_t:**

$$u_t(x_k, t_j) \approx \frac{u(x_k, t_{j+1}) - u(x_k, t_{r-1})}{\tau} + O(\tau).$$

- **Central-difference formula for u_{xx}:**

$$u_{xx}(x_k, t_j) = \frac{u(x_{k+1}, t_j) - 2u(x_k, t_j) + u(x_{k-1}, t_j)}{h^2} + O(h^2).$$

- **Forward-difference method for the heat equation:**
  Let $w_{k,j} \approx u(x_k, t_j)$. Then for each interior grid point,

$$\frac{w_{k,j+1} - w_{k,j}}{\tau} - K\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(x_k, t_j).$$

- **Boundary conditions:** $w_{0,j} = w_{N,j} = 0$ for $j = 1, \ldots, M$.

- **Initial conditions:** $w_{k,0} = u_0(x_k)$ for $k = 0, 1, \ldots, N$.

- ***Truncation error*** *at a grid point is the amount by which a the solution of the PDE fails to satisfy the difference equation at that point.*

  For forward-difference method the truncation error is

$$\tau_{k,j} = \frac{u_{k,j+1} - u_{k,j}}{\tau} - K\frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{h^2} - f(x_k, t_j) = O(\tau + h^2)$$

  where $u_{k,j} = u(x_k, t_j)$ and $u(x,t)$ is the exact solution.

- **Matrix form of the forward-difference method:**

$$\mathbf{w}^{(j)} = A\mathbf{w}^{(j-1)} + \tau\mathbf{F}^{(j-1)} \quad \text{for} \quad j = 1, 2, \ldots, M,$$

  where

$$A = \begin{pmatrix} 1-2\gamma & \gamma & 0 & \ldots & \ldots & 0 \\ \gamma & 1-2\gamma & \gamma & \ddots & & \vdots \\ 0 & \gamma & 1-2\gamma & \gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \gamma \\ 0 & \ldots & \ldots & 0 & \gamma & 1-2\gamma \end{pmatrix}$$

$$\mathbf{w}^{(j)} = \begin{pmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ \vdots \\ \vdots \\ w_{N-1,j} \end{pmatrix}, \mathbf{F}^{(j)} = \begin{pmatrix} f(x_1, t_j) \\ f(x_2, t_j) \\ \vdots \\ \vdots \\ \vdots \\ f(x_{N-1}, t_j) \end{pmatrix}.$$

- **The (implicit) backward-difference method.**

$$\frac{w_{k,j} - w_{k,j-1}}{\tau} - K\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(x_k, t_j).$$

  for $k = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M$.

- **Matrix form of the backward-difference method:**

$$A\mathbf{w}^{(j)} = \mathbf{w}^{(j-1)} + \tau\mathbf{F}^{(j)} \quad \text{for} \quad j = 1, 2, \ldots, M,$$

  where

$$A = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \ldots & \ldots & 0 \\ -\gamma & 1+2\gamma & -\gamma & \ddots & & \vdots \\ 0 & -\gamma & 1+2\gamma & -\gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\gamma \\ 0 & \ldots & \ldots & 0 & -\gamma & 1+2\gamma \end{pmatrix}.$$

- **Double-sweep method for solving tri-diagonal linear system:**

  Consider the tridiagonal system:

$$
\begin{aligned}
&A_i v_{i-1} - C_i v_i + B_i v_{i+1} = F_i \quad \text{for} \quad i = 1, \dots, N-1;\\
&v_0 = v_N = 0;
\end{aligned}
$$

  where the coefficients A_i, B_i and C_i satisfy the conditions

$$
A_i, B_i, C_i > 0, \quad C_i \geq A_i + B_i.
$$

  These equations are solved using the formulae:

$$
\begin{aligned}
&\alpha_0 = 0, \quad \beta_0 = 0,\\
&\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}, \quad \beta_{i+1} = \frac{\beta_i A_i - F_i}{C_i - \alpha_i A_i}
\end{aligned}
$$

  for $i = 1, \dots, N-1$ and

$$
\begin{aligned}
&v_N = 0,\\
&v_{i-1} = \alpha_i v_i + \beta_i \quad \text{for} \quad i = 1, 2, \dots, N.
\end{aligned}
$$

- **Richardson's method:**

$$
\frac{w_{k,j+1} - w_{k,j-1}}{2\tau} - K\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h^2} = f(x_k, t_j)
$$

  for $k = 1, 2, \dots, N-1$ and $j = 1, 2, \dots, M-1$.

- **Notation:**

$$
\delta_x^2\, w_{k,j} = w_{k+1,j} - 2w_{k,j} + w_{k-1,j}
$$

- **Crank-Nicolson method:**

$$
\frac{w_{k,j+1} - w_{k,j}}{\tau} - \frac{K}{2h^2}\left(\delta_x^2\, w_{k,j+1} + \delta_x^2\, w_{k,j}\right) = \frac{1}{2}\left(f(x_k, t_{j+1}) + f(x_k, t_j)\right)
$$

- **Matrix form (Crank-Nicolson method):**

$$
A\,\mathbf{w}^{(j+1)} = B\,\mathbf{w}^{(j)} + \frac{\tau}{2}\left(\mathbf{F}^{(j+1)} + \mathbf{F}^{(j)}\right) \quad \text{for} \quad j = 0, 1, 2, \dots, M-1,
$$

  where

$$
A(\gamma) = \begin{bmatrix}
1+\gamma & -\gamma/2 & 0 & \dots & \dots & 0\\
-\gamma/2 & 1+\gamma & -\gamma/2 & \ddots & & \vdots\\
0 & -\gamma/2 & 1+\gamma & -\gamma/2 & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & & \ddots & \ddots & \ddots & -\gamma/2\\
0 & \dots & \dots & 0 & -\gamma/2 & 1+\gamma
\end{bmatrix}, \quad B = A(-\gamma).
$$

- **Consistency (definition):** a finite-difference approximation to a differential equation is **consistent** if

$$\max_{k,j} |\tau_{k,j}(h,\tau)| \to 0 \quad \text{as} \quad h,\tau \to 0$$

(for each k and j).

Example: the Du Fort - Frankel method for the heat equation, given by

$$\frac{w_{k,j+1} - w_{k,j-1}}{2\tau} - K\frac{w_{k+1,j} - w_{k,j-1} - w_{k,j+1} + w_{k-1,j}}{h^2} = 0,$$

is stable but **not consistent**, because

$$\tau_{k,j} = O\left(\tau^2 + h^2 + \frac{\tau^2}{h^2}\right),$$

- **Convergence (definition):** a finite-difference method is said to be **convergent** if the total error of the method

$$E = \max_{k,j} |u_{k,j} - w_{k,j}|$$

tends to zero as $h \to 0$ and $\tau \to 0$:

$$E \to 0 \quad \text{as} \quad h \to 0 \quad \text{and} \quad \tau \to 0.$$

- **Lax equivalence theorem:** *If we have a well-posed initial boundary value problem and a finite difference approximation to it that satisfies the consistency condition, then stability is the necessary and sufficient condition for convergence.*

- **Non-homogeneous BCs (Dirichlet problem):**

$$\begin{aligned}
&u_t - Ku_{xx} = f(x,t), \quad 0 < x < L, \quad 0 < 0 < T,\\
&u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t),\\
&u(x,0) = u_0(x),
\end{aligned}$$

where $\mu_1(t)$ and $\mu_2(t)$ are given functions.

- **Reduction to a problem with homogeneous BCs**:

let $g(x,t)$ be such that

$$g(0,t) = \mu_1(t), \quad g(L,t) = \mu_2(t).$$

and $u(x,t) = v(x,t) + g(x,t)$, then $v(x,t)$ satisfies

$$\begin{aligned}
&v_t - K\, v_{xx} = \tilde{f}(x,t), \quad 0 < x < L, \quad 0 < 0 < T,\\
&v(0,t) = 0, \quad v(L,t) = 0,\\
&v(x,0) = v_0(x),
\end{aligned}$$

where

$$\tilde{f}(x,t) = f(x,t) - g_t + K\, g_{xx}, \quad v_0(x) = u_0(x) - g(x,0).$$

- **Non-homogeneous BCs (Neumann problem):**

$$
\begin{aligned}
&u_t - K u_{xx} = f(x,t), \quad 0 < x < L, \quad 0 < 0 < T, \\
&u_x(0,t) = \mu_1(t), \quad u_x(L,t) = \mu_2(t), \\
&u(x,0) = u_0(x).
\end{aligned}
$$

Again, it can be reduced to a problem with

$$\mu_1(t) \equiv 0, \quad \mu_2(t) \equiv 0.$$

- **Forward-difference method for Neumann problem:** we use

$$\frac{w_{k,j+1} - w_{k,j}}{\tau} - K \frac{w_{k+1,j} - 2w_{kj} + w_{k-1,j}}{h^2} = f(k, t_j)$$

for interior grid points and

$$
\begin{aligned}
w_{0,j+1} &= (1-2\gamma) w_{0,j} + 2\gamma w_{1,j} + \tau f(0,t_j), \\
w_{N,j+1} &= (1-2\gamma) w_{N,j} + 2\gamma w_{N-1,j} + \tau f(L,t_j)
\end{aligned}
$$

for boundary grid points.

- **Parabolic equation with variable coefficients:**

$$u_t = a(x,t)\, u_{xx} + b(x,t)\, u_x + c(x,t)\, u + d(x,t)$$

for $0 < x < L$ and $t > 0$, subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \quad u(0,t) = 0, \quad u(L,t) = 0.$$

We assume that

$$a(x,t) > 0 \quad \text{for} \quad 0 \le x \le L, \quad t > 0.$$

- **Explicit forward-difference method:**

$$\frac{w_{k,j+1} - w_{k,j}}{\tau} = a_{kj} \frac{\delta_x^2\, w_{k,j}}{h^2} + b_{kj} \frac{\delta_x\, w_{k,j}}{2h} + c_{kj}\, w_{k,j} + d_{kj},$$

for $k = 1, \ldots, N-1$ and $j = 0, 1, \ldots$. Here

$$
\begin{aligned}
&\delta_x^2 = w_{k+1,j} - 2w_{k,j} + w_{k-1,j}, \quad \delta_x = w_{k+1,j} - w_{k-1,j}; \\
&a_{kj} = a(x_k, t_j), \quad b_{kj} = b(x_k, t_j), \quad \text{etc.}
\end{aligned}
$$

- **Implicit backward-difference scheme:**

$$\frac{w_{k,j} - w_{k,j-1}}{\tau} = a_{kj} \frac{\delta_x^2\, w_{k,j}}{h^2} + b_{kj} \frac{\delta_x\, w_{k,j}}{2h} + c_{kj}\, w_{k,j} + d_{kj},$$

for $k = 1, \ldots, N-1$ and $j = 1, 2, \ldots$.

- **Crank-Nicolson's scheme:**

$$
\begin{aligned}
\frac{w_{k,j+1} - w_{k,j}}{\tau} \quad = \quad & \frac{a_{k,j+1/2}}{2h^2}\, \delta_x^2\, (w_{k,j} + w_{k,j+1}) \\
& + \frac{b_{k,j+1/2}}{4h}\, \delta_x\, (w_{k,j} + w_{k,j+1}) \\
& + \frac{c_{k,j+1/2}}{2}\, (w_{k,j} + w_{k,j+1}) + d_{k,j+1/2},
\end{aligned}
$$

where

$$a_{k,j+1/2} = \frac{a(x_k,t_j) + a(x_k,t_{j+1})}{2}, \quad b_{k,j+1/2} = \frac{b(x_k,t_j) + b(x_k,t_{j+1})}{2}, \quad \text{etc.}$$

Nonlinear heat equation in conservation form

- **Initial boundary value problem for the nonlinear heat equation:**

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x,t,u)\,\frac{\partial u}{\partial x}\right) + f(x,t,u), \quad 0 < x < L, \quad t > 0,\\ &u(x,0) = u_0(x),\\ &u(0,t) = 0, \quad u(L,t) = 0.\end{aligned}$$

- **A useful finite-difference formula:**

$$\begin{aligned}\frac{d}{dx}\left(Q(x)\frac{dg}{\partial x}\right)\bigg|_{x=x_k} \;=\;& \frac{1}{h^2}\Big(Q(x_{k+\frac{1}{2}})\left[g(x_{k+1}) - g(x_k)\right]\\ &-Q(x_{k-\frac{1}{2}})\left[g(x_k) - g(x_{k-1})\right]\Big) + O(h^2)\end{aligned}$$

where

$$Q(x_{k\pm\frac{1}{2}}) = \frac{1}{2}\left[Q(x_k) + Q(x_{k\pm 1})\right].$$

- **An explicit finite-difference scheme:**

$$\frac{w_{k,j+1} - w_{kj}}{\tau} - \frac{1}{h^2}\left(\varkappa_{k+\frac{1}{2},j}\left[w_{k+1,j} - w_{k,j}\right] - \varkappa_{k-\frac{1}{2},j}\left[w_{k,j} - w_{k-1,j}\right]\right) = f_{k,j},$$

where

$$\varkappa_{k\pm\frac{1}{2},j} \equiv \frac{1}{2}\left[K(x_k,t_j,w_{kj}) + K(x_{k\pm1},t_j,w_{k\pm1,j})\right], \quad f_{k,j} \equiv f(x_k,t_j,w_{kj}).$$

- **An implicit finite-difference scheme:**

$$\frac{w_{k,j} - w_{k,j-1}}{\tau} - \frac{1}{h^2}\left(\varkappa_{k+\frac{1}{2},j}\left[w_{k+1,j} - w_{k,j}\right] - \varkappa_{k-\frac{1}{2},j}\left[w_{k,j} - w_{k-1,j}\right]\right) = f_{k,j}.$$

- **Vector form of the implicit scheme:**

$$A\left(\mathbf{w}_j\right)\mathbf{w}_j = \mathbf{w}_{j-1} + \tau\mathbf{F}_j \quad \text{for} \quad j = 1, 2, \ldots,$$

where

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \ldots & \ldots & 0\\ b_1 & a_2 & b_2 & \ddots & & \vdots\\ 0 & b_2 & a_3 & b_3 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & \ddots & b_{N-2}\\ 0 & \ldots & \ldots & 0 & b_{N-2} & a_{N-1}\end{bmatrix}$$

$$a_k = 1 + \frac{\tau}{h^2}(\varkappa_{k+\frac{1}{2},j} + \varkappa_{k-\frac{1}{2},j}), \quad b_k = -\frac{\tau}{h^2}\varkappa_{k+\frac{1}{2},j}.$$

- **A (solvable) modification of the implicit scheme:**

$$A\left(\mathbf{w}_{j-1}\right)\mathbf{w}_j = \mathbf{w}_{j-1} + \tau\mathbf{F}_{j-1} \quad \text{for} \quad j = 1, 2, \ldots,$$

- **Method of successive approximations:** At each time step we compute a sequence $\mathbf{w}_j^{(s)}$ $(s = 0, 1, \dots)$. with

$$\mathbf{w}_j^{(0)} = \mathbf{w}_{j-1}, \quad A\left(\mathbf{w}_j^{(s-1)}\right)\mathbf{w}_j^{(s)} = \mathbf{w}_{j-1} + \tau \mathbf{F}_j^{(s-1)}.$$

- **Newton method:** Consider a system of nonlinear equations

$$F_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, n.$$

Compute a sequence of approximations $\mathbf{x}^{(s)}$ $(s = 0, 1, 2, \dots)$ using the formula

$$\mathbf{x}^{(s)} = \mathbf{x}^{(s-1)} + \mathbf{r}^{(s)},$$

where $\mathbf{r}^{(s)}$ is the solution of the linear system

$$J(\mathbf{x}^{(s-1)})\mathbf{r}^{(s)} = -\mathbf{F}(\mathbf{x}^{(s-1)})$$

with

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}.$$

- **Newton's method applied to the case when K = constant:**

$$-\gamma r_{k-1,j}^{(s)} - \gamma r_{k+1,j}^{(s)} + \left(1 + 2\gamma - \tau \frac{\partial f(x_k, t_j, w_{kj}^{(s-1)})}{\partial w_{kj}^{(s-1)}}\right) r_{k,j}^{(s)} =$$
$$= -(1 + 2\gamma) w_{kj}^{(s-1)} + \gamma\left(w_{k+1,j}^{(s-1)} + w_{k-1,j}^{(s-1)}\right) + \tau f\left(x_k, t_j, w_{kj}^{(s-1)}\right) + w_{k,j-1}$$

where $\gamma = K\tau/h^2$.

Two-dimensional heat equation

- **Initial boundary value problem for 2d heat equation:** Let $\mathcal{D}$ be any connected domain in the x, y plane and S its boundary. The heat equation:

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t) \quad \text{for} \quad (x, y) \in \mathcal{D},$$

with boundary condition

$$u(x, y, t) = g(x, y, t) \quad \text{for} \quad (x, y) \in S$$

and initial condition

$$u(x, y, 0) = u_0(x, y) \quad \text{for} \quad (x, y) \in \mathcal{D},$$

where $f(x, y, t), g(x, y, t)$ and $u_0(x, y)$ are given functions.

- **Grid points:** On a rectangular domain

$$\mathcal{D} = \{\, (x, y) \,|\, a < x < b,\ c < y < d \,\}.$$

introduce

$$x_k = a + kh_1 \ \text{ for } \ k = 0, 1, \ldots, N_1, \quad y_j = c + jh_2 \ \text{ for } \ j = 0, 1, \ldots, N_2,$$

where $h_1 = (b - a)/N_1$ and $h_2 = (d - c)/N_2$.

Let w_{kj}^n be the discrete approximation to $u_{kj}^n \equiv u(x_k, y_j, t_n)$.

- **Forward difference method:**

$$w_{kj}^{n+1} = w_{kj}^n + \tau K \left(\frac{\delta_x^2}{h_1^2} + \frac{\delta_y^2}{h_2^2} \right) w_{kj}^n + \tau f_{kj}^n,$$

This is stable if

$$K\tau \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right) \le \frac{1}{2}.$$

The truncation error is of $O(\tau + h_1^2 + h_2^2)$.

- **Backward difference method:**

$$\frac{w_{kj}^n - w_{kj}^{n-1}}{\tau} - K \left(\frac{\delta_x^2}{h_1^2} + \frac{\delta_y^2}{h_2^2} \right) w_{kj}^n = f_{kj}^n$$

Unconditionally stable and $O(\tau + h_1^2 + h_2^2)$.

- **Crank-Nicolson method:**

$$\frac{w_{kj}^{n+1} - w_{kj}^n}{\tau} - \frac{K}{2} \left(\frac{1}{h_1^2}\delta_x^2 + \frac{1}{h_2^2}\delta_y^2 \right) \left(w_{kj}^n + w_{kj}^{n+1} \right) = f_{kj}^{n+\frac{1}{2}}$$

where

$$f_{kj}^{n+\frac{1}{2}} = \frac{f(x_k, y_j, t_n) + f(x_k, y_j, t_{n+1})}{2} + O(\tau^2).$$

Unconditionally stable and $O(\tau^2 + h_1^2 + h_2^2)$.

- **ADI method:**

$$\begin{aligned}
\frac{w_{k,j}^{n+\frac{1}{2}} - w_{k,j}^n}{\tau} &= \frac{K}{2h^2} \left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^n \right) + \frac{1}{2} f_{k,j}^{n+\frac{1}{2}}, \\
\frac{w_{k,j}^{n+1} - w_{k,j}^{n+\frac{1}{2}}}{\tau} &= \frac{K}{2h^2} \left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n+1} \right) + \frac{1}{2} f_{k,j}^{n+\frac{1}{2}}.
\end{aligned}$$

Unconditionally stable and $O(\tau^2 + h_1^2 + h_2^2)$. Can be solved using double-sweep method.

Hyperbolic equations

Wave equation

- **Initial boundary value problem for wave equation:**

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = F(x,t), \quad a < x < b, \quad 0 < t < T,$$

subject to the boundary conditions

$$u(a,t) = u(b,t) = 0 \quad \text{for} \quad t \in [0,T],$$

and initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad x \in [a,b],$$

- **Initial conditions:**

$$\begin{aligned} w_{k,0} &= f(x_k) \\ w_{k,1} &= f(x_k) + \tau g(x_k) + \frac{\tau^2}{2}\left[\alpha^2 f''(x_k) + F(x_k,0)\right] \end{aligned}$$

for each $k = 1, 2, \ldots, N-1$. Truncation error is $O(\tau^2)$.

- **Explicit forward-difference scheme:**

$$w_{k,j+1} = 2\left(1-\gamma^2\right) w_{kj} + \gamma^2\left(w_{k+1,j} + w_{k-1,j}\right) - w_{k,j-1} + \tau^2 F_{kj}$$

with boundary conditions $w_{0,j} = w_{N,j} = 0$ for each $j = 1, 2, \ldots, M$. Truncation error is $O(\tau^2 + h^2)$.

- **Matrix form of forward-difference scheme:**

$$\mathbf{w}^{(j+1)} = A\mathbf{w}^{(j)} - \mathbf{w}^{(j-1)} + \tau^2 \mathbf{F}^{(j)},$$

where

$$A = \begin{bmatrix} 2(1-\gamma^2) & \gamma^2 & 0 & \cdots & \cdots & 0 \\ \gamma^2 & 2(1-\gamma^2) & \gamma^2 & \ddots & & \vdots \\ 0 & \gamma^2 & 2(1-\gamma^2) & \gamma^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \gamma^2 \\ 0 & \cdots & \cdots & 0 & \gamma^2 & 2(1-\gamma^2) \end{bmatrix}.$$

- **Courant stability condition**

$$\tau < \frac{h}{\alpha}.$$

- **Unconditionally stable implicit scheme:**

$$w_{k,j+1} - 2w_{kj} + w_{k,j-1} - \gamma^2\left[\sigma\delta_x^2 w_{k,j+1} + (1-2\sigma)\delta_x^2 w_{k,j} + \sigma\delta_x^2 w_{k,j-1}\right] = \tau^2 F_{kj}$$

with $\sigma \geq 1/4$. Truncation error is $O(\tau^2 + h^2)$.

System of 1st-order conservation laws:

$$\mathbf{U}_t + [\mathbf{F}(\mathbf{U})]_x = 0$$

- **Example: wave equation**

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ s \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} -\alpha s \\ -\alpha r \end{pmatrix} = 0$$

- **Lax scheme:**

$$\mathbf{U}_{k,j+1} = \frac{1}{2}\left(\mathbf{U}_{k+1,j} + \mathbf{U}_{k-1,j}\right) - \frac{\tau}{2h}\left[\mathbf{F}(\mathbf{U}_{k+1,j}) - \mathbf{F}(\mathbf{U}_{k-1,j})\right].$$

  This is stable if $\tau < h/\alpha$ and has trunctation error of $O(\tau + h^2)$.

- **Leapfrog method:**

$$\mathbf{U}_{k,j+1} = \mathbf{U}_{k,j-1} - \frac{\tau}{h}\left[\mathbf{F}(\mathbf{U}_{k+1,j}) - \mathbf{F}(\mathbf{U}_{k-1,j})\right].$$

  This is stable if $\tau < h/\alpha$ and has trunctation error of $O(\tau^2 + h^2)$.

- **Two-step Lax-Wendroff scheme:**

$$\begin{aligned}
\mathbf{U}_{k+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{2}\left[\mathbf{U}_{k+1,j} + \mathbf{U}_{k,j}\right] - \frac{\tau}{2h}\left[\mathbf{F}(\mathbf{U}_{k+1,j}) - \mathbf{F}(\mathbf{U}_{k,j})\right], \\
\mathbf{U}_{k,j+1} &= \mathbf{U}_{k,j} - \frac{\tau}{h}\left[\mathbf{F}(\mathbf{U}_{k+\frac{1}{2},j+\frac{1}{2}}) - \mathbf{F}(\mathbf{U}_{k-\frac{1}{2},j+\frac{1}{2}})\right].
\end{aligned}$$

  This is stable if $\tau < h/\alpha$ and has trunctation error of $O(\tau^2 + h^2)$.

Elliptic equations

- **Boundary value problem for the Poisson equation:**

  Let $\mathcal{D}$ be any connected domain in the x, y plane and S its boundary, and $f : \mathcal{D} \to \mathbb{R}$ a given function.

  The Poisson equation:

$$u_{xx} + u_{yy} = f \quad \text{for} \quad (x, y) \in \mathcal{D},$$

  Boundary condition:

$$u(x, y) = g(x, y) \quad \text{for} \quad (x, y) \in S.$$

- **Grid points:** On a rectangular domain

$$\mathcal{D} = \{\,(x, y) \,|\, a < x < b,\ c < y < d\,\}.$$

  introduce

$$x_k = a + kh_1 \ \text{ for } \ k = 0, 1, \ldots, N_1, \quad y_j = c + jh_2 \ \text{ for } \ j = 0, 1, \ldots, N_2,$$

  where $h_1 = (b - a)/N_1$ and $h_2 = (d - c)/N_2$.

- **Finite-difference scheme:**

$$\frac{w_{k+1,j} - 2w_{k,j} + w_{k-1,j}}{h_1^2} + \frac{w_{k,j+1} - 2w_{k,j} + w_{k,j-1}}{h_2^2} = f_{kj}$$

  for each interior grid point. Here $f_{kj} = f(x_k, y_j)$.

  The boundary condition:

$$\begin{aligned} w_{0,j} &= g(x_0, y_j), \quad w_{N_1,j} = g(x_{N_1}, y_j) \quad \text{for each} \quad j = 1, \ldots, N_2 - 1 \\ w_{k,0} &= g(x_k, y_0), \quad w_{k,N_2} = g(x_k, y_{N_2}) \quad \text{for each} \quad k = 1, 2, \ldots, N_1 - 1. \end{aligned}$$

  Generalises to irregularly shaped domains straighforwardly.

- **Square domain** ($c = a$, $d = b$): Let

$$N_1 = N_2 \equiv N \quad \text{and} \quad h_1 = h_2 \equiv h$$

  Then

$$4w_{k,j} - (w_{k+1,j} + w_{k-1,j} + w_{k,j+1} + w_{k,j-1}) = -h^2 f_{k,j},$$

  for each $k, j = 1, 2, \ldots, N - 1$.

- **Existence of unique solution.** Maximum principle can be used to show that the only solution to homogeneous equation

$$4w_{k,j} - (w_{k+1,j} + w_{k-1,j} + w_{k,j+1} + w_{k,j-1}) = 0$$

  with homogeneous boundary conditions is the trivial solution $w_{jk} = 0$ for all j, k.

- **Relaxation methods:** Find solution of Poisson equation by evolving heat equation until it is close to steady state solution with $u_t = 0$.

- **Neuman boundary conditions:** Consider the Laplace equation

$$u_{xx} + u_{yy} = 0$$

  in the unit square ($0 < x < 1,\ 0 < y < 1$) with boundary conditions for normal derivative:

$$u_x(0, y) = g_0(y), \quad u_x(1, y) = g_1(y), \quad u_y(x, 0) = h_0(x), \quad u_y(x, 1) = h_1(x).$$

  Then, in addition to the usual difference equations at interior points, we get the equations

$$\begin{aligned} &4w_{0,j} - (2w_{1,j} + w_{0,j+1} + w_{0,j-1}) = -2hg_0(y_j), \\ &4w_{N,j} - (2w_{N-1,j} + w_{N,j+1} + w_{N,j-1}) = 2hg_1(y_j), \\ &4w_{k,0} - (w_{k+1,0} + w_{k-1,0} + 2w_{k,1}) = -2hh_0(x_k), \\ &4w_{k,N} - (w_{k+1,N} + w_{k-1,N} + 2w_{k,N-1}) = 2hh_1(x_k), \end{aligned}$$

  for $j = 1, \ldots, N - 1$ and $k = 1, \ldots, N - 1$, giving us a system of $(N + 1)^2 - 4$ coupled equations.