UNIVERSITY OF YORK

MMath and MSc Examinations 2016
MATHEMATICS
Numerical Methods for PDEs

Time Allowed: 2 hours.

You will receive marks for the best three attempts on the four questions in this paper.
Each question carries 25 marks. The maximal total mark is 75.
Please write your answers in ink; pencil is acceptable for graphs and diagrams. Do not use red ink.
Standard calculators will be provided.
The marking scheme shown on each question is indicative only.

1 (of 4). (a) What does it mean to say that a finite-difference approximation to a differential equation is consistent with this differential equation? **[3]**

(b) What does it mean to say that a finite-difference approximation to a differential equation is convergent? **[3]**

(c) State the Lax equivalence theorem. **[3]**

(d) Consider the parabolic equation

$$u_t = u_{xx} + a(x,t)u \quad \text{for} \quad 0 < x < L,\ \ 0 < t < T \tag{1}$$

where $a(x,t)$ is a given smooth function. Suppose that at an interior grid point (x_k, t_j) this equation is approximated by the finite-difference scheme

$$\frac{w_{k,j+1} - w_{k,j}}{\tau} - \frac{1}{2h^2}\,\delta_x^2\,(w_{k,j} + w_{k,j+1}) - \frac{a(x_k, t_{j+1/2})}{2}\,(w_{k,j} + w_{k,j+1}) = 0$$

where $w_{k,j}$ is an approximation to $u(x_k, t_j)$, τ is the time step, h is the step length in x, $t_{j+1/2} = t_j + \tau/2$ and where

$$\delta_x^2\, w_{k,j} = w_{k+1,j} - 2w_{k,j} + w_{k-1,j}.$$

Find the truncation error of the scheme. **[16]**

2 (of 4). (a) The alternating-direction implicit method for solving the two-dimensional heat equation

$$u_t = K\left(u_{xx} + u_{yy}\right) \quad \text{for} \quad 0 < x < 1,\ 0 < y < 1,\ 0 < t < T$$

is given for interior points by

$$\begin{aligned} w_{k,j}^{n+\frac{1}{2}} &= w_{k,j}^{n} + \frac{\gamma}{2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n}\right), \\ w_{k,j}^{n+1} &= w_{k,j}^{n+\frac{1}{2}} + \frac{\gamma}{2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n+1}\right), \end{aligned}$$

where $w_{k,j}^n$ approximates $u(x_k, y_j, t_n)$; $x_k = hk$ and $y_k = hk$ for $k = 0, 1, \ldots, N$; $h = 1/N$ is the step size in both x and y; $t_n = \tau n$ for $n = 0, 1, \ldots, M$; $\tau = T/M$ is the time step; and where

$$\begin{aligned} &\gamma \equiv \frac{K\tau}{h^2}, \quad \delta_x^2 w_{k,j}^n \equiv w_{k+1,j}^n - 2w_{k,j}^n + w_{k-1,j}^n \\ &\delta_y^2 w_{k,j}^n \equiv w_{k,j+1}^n - 2w_{k,j}^n + w_{k,j-1}^n. \end{aligned}$$

Investigate the stability of this method. **[12]**

(b) Show that the truncation error of this method is $O(\tau^2 + h^2)$. [You may use the fact that the truncation error of the Crank-Nicolson scheme is $O(\tau^2 + h^2)$.] **[13]**

3 (of 4). The Poisson equation

$$u_{xx} + u_{yy} = f(x, y) \quad \text{for} \quad 0 < x < 1, \;\; 0 < y < 1, \tag{2}$$

where f is a given smooth function, subject to the boundary conditions

$$u(0, y) = u(1, y) = 0, \quad u(x, 0) = u(x, 1) = 0, \tag{3}$$

is solved numerically using the finite-difference method:

$$\begin{aligned} &w_{k,0} = 0, \quad w_{k,N} = 0, \quad w_{0,j} = 0, \quad w_{N,j} = 0, \\ &4w_{k,j} - (w_{k+1,j} + w_{k-1,j} + w_{k,j+1} + w_{k,j-1}) = -h^2 f(x_k, y_j) \end{aligned} \tag{4}$$

for $k, j = 1, 2, \ldots, N-1$. Here $w_{k,j}$ approximates $u(x_k, y_j)$; $x_k = kh$ and $y_k = kh$ for $k = 0, 1, \ldots, N$; $h = 1/N$.

(a) Prove that the system of equations (4) has a unique solution. **[14]**

(b) If the boundary conditions $u(x, 0) = u(x, 1) = 0$ in (3) are replaced by the Neumann conditions

$$u_y(x, 0) = \mu_1(x), \quad u_y(x, 1) = \mu_2(x),$$

where $\mu_1(x)$ and $\mu_2(x)$ are given smooth functions, obtain an appropriate modification of the method with truncation error $O(h^2)$. **[11]**

4 (of 4). At an interior grid point (x_k, t_j), the equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0$$

is approximated by the Lax scheme

$$\frac{1}{\tau}\left(w_{k,j+1} - \frac{1}{2}[w_{k+1,j} + w_{k-1,j}]\right) + \alpha \frac{w_{k+1,j} - w_{k-1,j}}{2h} = 0.$$

Here $w_{k,j}$ approximates $u(x_k, t_j)$, τ is the time step, h is the step length in x.

(a) Find the truncation error of the Lax scheme. **[10]**

(b) Explain why the Lax scheme above has numerical viscosity. **[5]**

(c) Investigate the stability of the Lax scheme. **[10]**

 End of examination.

1. (a) By definition, a finite-difference approximation to a differential equation is **consistent** with this differential equation if local truncation errors tend to zero as the step size goes to zero, i.e.

$$\max_{k,j} |\tau_{ki}(h,\tau)| \to 0 \text{ as } h,\tau \to 0.$$

$\boxed{\text{3 Marks}}$

(b) By definition, a finite-difference method is said to be **convergent** if the total error of the method

$$E = \max_{k,j} |u(x_k,t_j) - w_{k,j}|$$

tends to zero as $h \to 0$ and $\tau \to 0$:

$$E \to 0 \quad \text{as} \quad h \to 0 \quad \text{and} \quad \tau \to 0.$$

$\boxed{\text{3 Marks}}$

(c) **Lax equivalence theorem:** given a well-posed initial boundary value problem and a finite difference approximation to it that satisfies the consistency condition, then stability is the necessary and sufficient condition for convergence. $\boxed{\text{3 Marks}}$

(d) The local truncation error is given by

$$\begin{aligned} \tau_{k,j} &= \frac{u_{k,j+1} - u_{kj}}{\tau} - \frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{2h^2} \\ &\quad - \frac{u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1}}{2h^2} - \frac{a(x_k, t_{j+1/2})}{2} (u_{k,j} + u_{k,j+1}) \,. \end{aligned} \tag{5}$$

where $u_{kj} = u(x_k, t_j)$ and $u(x,t)$ is the exact solution of the problem. Assuming that $u(x,t)$ is smooth enough, we expand $u_{k\pm1,j}$ and $u_{k\pm1,j+1}$ in Taylor's series

$$\begin{aligned} u_{k\pm1,j} &= u(x_k,t_j) \pm h\frac{\partial u}{\partial x}(x_k,t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_j) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k,t_j) + O(h^4), \\ u_{k\pm1,j+1} &= u(x_k,t_{j+1}) \pm h\frac{\partial u}{\partial x}(x_k,t_{j+1}) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_{j+1}) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_k,t_{j+1}) + O(h^4) \end{aligned}$$

It follows that

$$\begin{aligned} u_{k+1,j} - 2u_{kj} + u_{k-1,j} &= 2u(x_k,t_j) + 2\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^4) - 2u_{kj} \\ &= h^2\left(\frac{\partial^2 u}{\partial x^2}(x_k,t_j) + O(h^2)\right). \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned} u_{k+1,j+1} - 2u_{k,j+1} + u_{k-1,j+1} &= h^2 \left(\frac{\partial^2 u}{\partial x^2}(x_k, t_{j+1}) + O(h^2) \right) \\ &= h^2 \left(\frac{\partial^2 u}{\partial x^2}(x_k, t_j) + \tau \frac{\partial^3 u}{\partial x^2 \partial t}(x_k, t_j) + O(\tau^2 + h^2) \right). \quad (7) \end{aligned}$$

Also, we have

$$\begin{aligned} u_{k,j+1} - u_{kj} &= u(x_k, t_j) + \tau \frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(\tau^3) - u_{kj} \\ &= \tau \left(\frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau}{2} \frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(\tau^2) \right), \\ u_{k,j+1} + u_{kj} &= 2u(x_k, t_j) + \tau \frac{\partial u}{\partial t}(x_k, t_j) + O(\tau^2) \quad (8) \end{aligned}$$

and

$$a(x_k, t_{j+1/2}) = a(x_k, t_j) + \frac{\tau}{2} \frac{\partial a}{\partial t}(x_k, t_j) + O(\tau^2) \quad (9)$$

Substituting (6)–(9) into the formula for τ_{kj}, we obtain

$$\begin{aligned} \tau_{k,j} &= \frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau}{2} \frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(\tau^2) \\ &\quad - \frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2}(x_k, t_j) + O(h^2) \right) \\ &\quad - \frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2}(x_k, t_j) + \tau \frac{\partial^3 u}{\partial x^2 \partial t}(x_k, t_j) + O(\tau^2 + h^2) \right) \\ &\quad - \frac{1}{2} \left(a(x_k, t_j) + \frac{\tau}{2} \frac{\partial a}{\partial t}(x_k, t_j) + O(\tau^2) \right) \left(2u(x_k, t_j) + \tau \frac{\partial u}{\partial t}(x_k, t_j) + O(\tau^2) \right) \\ &= \frac{\partial u}{\partial t}(x_k, t_j) - \frac{\partial^2 u}{\partial x^2}(x_k, t_j) - a(x_k, t_j) u(x_k, t_j) \\ &\quad + \frac{\tau}{2} \left(\frac{\partial^2 u}{\partial t^2}(x_k, t_j) - \frac{\partial^3 u}{\partial x^2 \partial t}(x_k, t_j) - \frac{\partial a}{\partial t}(x_k, t_j)\, u(x_k, t_j) - a(x_k, t_j) \frac{\partial u}{\partial t}(x_k, t_j) \right) \\ &\quad + O(\tau^2 + h^2). \end{aligned}$$

The first three terms disappear because u satisfies Eq. (1). Also, differentiating Eq. (1) with respect to t, we find that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial a}{\partial t}\, u - a\, \frac{\partial u}{\partial t} = 0.$$

Hence, $\tau_{k,j} = O(\tau^2 + h^2)$.

$\boxed{\text{16 Marks}}$

Remarks. *Parts (a)-(c) were discussed in lectures, part (d) is similar to a homework problem.*

Total: 25 Marks

2. (a) Let $z_{k,j}^n = w_{k,j} - \tilde{w}_{k,j}$ be the perturbation at the grid point (x_k, y_j, t_n) for each $k, j = 0, 1, 2, \ldots, N$ and $n = 0, 1, \ldots$.

z_{kj} satisfies the difference equation

$$
\begin{aligned}
z_{k,j}^{n+\frac{1}{2}} &= z_{k,j}^n + \frac{\gamma}{2}\left(\delta_x^2 z_{k,j}^{n+\frac{1}{2}} + \delta_y^2 z_{k,j}^n\right), \\
z_{k,j}^{n+1} &= z_{k,j}^{n+\frac{1}{2}} + \frac{\gamma}{2}\left(\delta_x^2 z_{k,j}^{n+\frac{1}{2}} + \delta_y^2 z_{k,j}^{n+1}\right),
\end{aligned}
\tag{10}
$$

for $k = 1, 2, \ldots, N$ and $j = 1, 2, \ldots$. We seek a particular solution of (10) in the form

$$z_{k,j}^n = \rho_{qp}^n e^{iqx_k + ipy_j}$$

for $q, p \in \mathbb{R}$ and $n = 0, 1, \ldots$ The finite-difference method is stable with respect to perturbations of initial condition, if

$$|\rho_{qp}| \leq 1 \quad \text{for all} \quad q, p \in \mathbb{R}$$

Let

$$z_{k,j}^{n+\frac{1}{2}} = z_{k,j}^n \rho_{pq}^{(1)} \quad \text{and} \quad z_{k,j}^{n+1} = z_{k,j}^{n+\frac{1}{2}} \rho_{pq}^{(2)}$$

(so that $\rho_{qp} = \rho_{pq}^{(1)} \rho_{pq}^{(2)}$). Substituting these in (10), we find that

$$
\begin{aligned}
\rho_{pq}^{(1)} &= 1 + \frac{\gamma}{2}\rho_{pq}^{(1)}\left(e^{iqh} - 2 + e^{-iqh}\right) + \frac{\gamma}{2}\left(e^{iph} - 2 + e^{-iph}\right), \\
\rho_{pq}^{(2)} &= 1 + \frac{\gamma}{2}\left(e^{iqh} - 2 + e^{-iqh}\right) + \rho_{pq}^{(2)}\frac{\gamma}{2}\left(e^{iph} - 2 + e^{-iph}\right).
\end{aligned}
$$

Since

$$e^{iqh} - 2 + e^{-iqh} = \left(e^{iqh/2} - e^{-iqh/2}\right)^2 = -4\sin^2\frac{qh}{2},$$

we obtain

$$
\begin{aligned}
\rho_{pq}^{(1)} &= \frac{1 - 2\gamma\sin^2\frac{ph}{2}}{1 + 2\gamma\sin^2\frac{qh}{2}}, \\
\rho_{pq}^{(2)} &= \frac{1 - 2\gamma\sin^2\frac{qh}{2}}{1 + 2\gamma\sin^2\frac{ph}{2}}.
\end{aligned}
$$

It follows that

$$\rho_{qp} = \rho_{pq}^{(1)}\rho_{pq}^{(2)} = \frac{\left(1 - 2\gamma\sin^2\frac{ph}{2}\right)\left(1 - 2\gamma\sin^2\frac{qh}{2}\right)}{\left(1 + 2\gamma\sin^2\frac{ph}{2}\right)\left(1 + 2\gamma\sin^2\frac{qh}{2}\right)}.$$

Evidently, $|\rho_{qp}| \le 1$ and, therefore, the method is unconditionally stable. $\boxed{\text{12 Marks}}$

(b) We have

$$w_{k,j}^{n+\frac{1}{2}} = w_{k,j}^{n} + \frac{\gamma}{2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n}\right), \tag{11}$$

$$w_{k,j}^{n+1} = w_{k,j}^{n+\frac{1}{2}} + \frac{\gamma}{2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n+1}\right), \tag{12}$$

To find the local truncation error of the ADI method, we first eliminate the intermediate values from Eqs. (11), (12). Adding the two equations (divided by τ), we obtain

$$\frac{w_{k,j}^{n+1} - w_{k,j}^{n}}{\tau} = \frac{K}{2h^2}\left(2\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2\left[w_{k,j}^{n} + w_{k,j}^{n+1}\right]\right).$$

Subtracting (12) from (11) and dividing the result by τ, we find that

$$\frac{2}{\tau} w_{k,j}^{n+\frac{1}{2}} = \frac{w_{k,j}^{n+1} + w_{k,j}^{n}}{\tau} + \frac{K}{2h^2}\delta_y^2\left[w_{k,j}^{n} - w_{k,j}^{n+1}\right].$$

It follows that

$$\frac{w_{k,j}^{n+1} - w_{k,j}^{n}}{\tau} = \frac{K}{2h^2}\left(\delta_x^2 + \delta_y^2\right)\left(w_{k,j}^{n} + w_{k,j}^{n+1}\right) + \frac{K^2\tau}{4h^4}\delta_x^2\delta_y^2\left[w_{k,j}^{n} - w_{k,j}^{n+1}\right]. \tag{13}$$

If the last term on the right side of this equation were absent, the equation would coincide with the Crank-Nicolson method whose local truncation error is $O(\tau^2 + h^2)$.

Let us show that the last term in (13) is $O(\tau^2)$ [if we replace $w_{k,j}^n$ with $u_{k,j}^n = u(x_k, y_j, t_n)$]. Since

$$\frac{1}{h^2}\delta_x^2\, u_{kj}^n = u_{xx}(x_k, y_j, t_n) + \frac{h^2}{12}\, u_{xxxx}(x_k, y_j, t_n) + O(h^4),$$

$$\frac{1}{h^2}\delta_y^2\, u_{kj}^n = u_{yy}(x_k, y_j, t_n) + \frac{h^2}{12}\, u_{yyyy}(x_k, y_j, t_n) + O(h^4),$$

we have

$$\frac{1}{h^4}\delta_x^2\delta_y^2\, u_{k,j}^n = u_{xxyy}(x_k, y_j, t_n) + O(h^2), \qquad \frac{1}{h^4}\delta_x^2\delta_y^2\, u_{k,j}^{n+1} = u_{xxyy}(x_k, y_j, t_{n+1}) + O(h^2).$$

Hence,

$$\frac{K^2\tau}{4h^4}\,\delta_x^2\delta_y^2\,\left(u_{k,j}^n - u_{k,j}^{n+1}\right) = \frac{K^2\tau^2}{4}\left[-u_{xxyyt}(x_k, y_j, t_n) + O(\tau)\right] + O(\tau h^2) = O(\tau^2).$$

Therefore, the local truncation error of the ADI method is $O(\tau^2 + h^2)$.

$\boxed{\text{13 Marks}}$

Remarks. *Discussed in lectures.*

Total: 25 Marks

3. (a) The existence of a unique solution of this system is equivalent to non-existence of nontrivial (i.e. nonzero) solutions of the homogeneous equations

$$4w_{ij} - (w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) = 0, \tag{14}$$

for each $i, j = 1, 2, \ldots, N-1$. Equations (14) are supplemented with homogeneous boundary conditions

$$w_{0j} = w_{N,j} = 0 \quad \text{and} \quad w_{i0} = w_{i,N} = 0 \quad \text{for} \quad i, j = 1, 2, \ldots, N-1. \tag{15}$$

Let (m, n) be a point at which the maximum of w_{kj} is attained. There may be several points at which the maximum is attained. If the maximum is attained at a boundary point, then $\max_{0\le k,j\le N} w_{kj} = 0$. Assume that the maximum is attained at an interior point (m, n). Since, again, there may be several such points, we choose a point (m, n) such than it corresponds to the maximum value of index m, i.e.

$$w_{m,n} = \max_{0\le k,j\le N} w_{kj} \quad \text{and} \quad w_{m,n} > w_{m+1,n}.$$

Then, Eq. (14) yields

$$\begin{aligned} w_{mn} - w_{m+1,n} \quad &+ \quad (w_{mn} - w_{m-1,n}) + (w_{mn} - w_{m,n+1}) \\ &+ \quad (w_{mn} - w_{m,n-1}) \ge w_{mn} - w_{m+1,n} > 0. \end{aligned} \tag{16}$$

Evidently, (16) is in contradiction with (14). Thus, our assumption that the maximum is attained at an interior mesh point is wrong. Thus, we have

$$\max_{0\le k,j\le N} w_{kj} = 0. \tag{17}$$

Similarly, it can be shown that

$$\min_{0\le k,j\le N} w_{kj} = 0. \tag{18}$$

It follows from Eqs. (17) and (18) that $w_{kj} = 0$ for each $k, j = 0, 1, \ldots, N$. Thus, the homogeneous system has only zero solution, which proves that the corresponding non-homogeneous system has a unique solution.

14 Marks

(b) First, we add 'false' boundaries at $y = y_{-1} = y_0 - h$ and $y = y_{N+1} = y_N + h$ and assume that the finite-difference scheme (4) approximates the Poisson equation at points (x_k, y_0) and (x_k, y_N) ($k = 1, 2, \ldots, N-1$). Then, we have

$$4w_{k,0} - (w_{k+1,0} + w_{k-1,0} + w_{k,1} + w_{k,-1}) = -h^2 f(x_k, y_0) \tag{19}$$

and

$$4w_{k,N} - (w_{k+1,N} + w_{k-1,N} + w_{k,N+1} + w_{k,N-1}) = -h^2 f(x_k, y_N) \tag{20}$$

for $k = 1, 2, \ldots, N-1$. Approximating $u_y(x_k, 0)$ and $u_y(x_k, 1)$ by the central difference formula (whose truncation error is $O(h^2)$), we obtain

$$\frac{w_{k,1} - w_{k,-1}}{2h} = \mu_1(x_k), \quad \frac{w_{k,N+1} - w_{k,N-1}}{2h} = \mu_2(x_k) \tag{21}$$

for $k = 1, 2, \ldots, N-1$. Eliminating $w_{k,-1}$ and $w_{k,N+1}$ from Eqs. (19)–(21), we find that

$$\begin{aligned} 4w_{k,0} - (w_{k+1,0} + w_{k-1,0} + 2w_{k,1}) &= -h^2 f(x_k, y_0) - 2h\mu_1(x_k), \\ 4w_{k,N} - (w_{k+1,N} + w_{k-1,N} + 2w_{k,N-1}) &= -h^2 f(x_k, y_N) + 2h\mu_2(x_k) \end{aligned}$$

for $j = 1, 2, \ldots, N-1$. These and the given difference equations at the interior grid points represent the required modification of the method. $\boxed{\text{11 Marks}}$

Remarks. *Part (a) was discussed in lectures, part (b) is unseen but similar to a homework problem.*

$\boxed{\text{Total: 25 Marks}}$

4. (a) The truncation error of the Lax scheme at the grid point (x_k, t_j) is

$$\tau_{kj} = \frac{1}{\tau}\left(u_{k,j+1} - \frac{1}{2}[u_{k+1,j} + u_{k-1,j}]\right) + \alpha\frac{u_{k+1,j} - u_{k-1,j}}{2h} \tag{22}$$

where $u_{kj} = u(x_k, t_j)$. Expanding $u_{k\pm1,j}$ and $u_{k,j+1}$ in Taylor's series at point (x_k, t_j), we find that

$$\begin{aligned} u_{k\pm1,j} &= u_{k,j} \pm h\frac{\partial u}{\partial x}(x_k, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_k, t_j) + O(h^3), \\ u_{k,j+1} &= u_{k,j} + \tau\frac{\partial u}{\partial t}(x_k, t_j) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_k, t_j) + O(\tau^3). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{u_{k+1,j} - u_{k-1,j}}{2h} &= \frac{\partial u}{\partial x}(x_k, t_j) + O(h^2), \\ \frac{u_{k+1,j} + u_{k-1,j}}{2} &= u(x_k, t_j) + O(h^2), \end{aligned}$$

Substituting these in Eq. (22) we find that

$$\begin{aligned} \tau_{kj} &= \frac{1}{\tau}\left(\tau u_t(x_k, t_j) + O(\tau^2) + O(h^2)\right) + \alpha u_x(x_k, t_j) + O(h^2) \\ &= O(\tau + h^2 + h^2/\tau). \end{aligned}$$

$\boxed{\text{10 Marks}}$

(b) The Lax scheme, given by

$$\frac{1}{\tau}\left(w_{k,j+1}-\frac{1}{2}[w_{k+1,j}+w_{k-1,j}]\right)+\alpha\frac{w_{k+1,j}-w_{k-1,j}}{2h}=0,$$

can also be written as

$$\frac{w_{k,j+1}-w_{k,j}}{\tau}+\alpha\frac{w_{k+1,j}-w_{k-1,j}}{2h}=\frac{h^2}{2\tau}\frac{w_{k+1,j}-2w_{k,j}+w_{k-1,j}}{h^2}. \tag{23}$$

For small τ and h the difference equation (23) approximates the equation

$$u_t+\alpha u_x=\frac{h^2}{2\tau}u_{xx}. \tag{24}$$

The term on the right side of (24) describes diffusion of u. This is why the Lax scheme is said to have *numerical dissipation* or *numerical viscosity*. $\boxed{\text{5 Marks}}$

(c) Since the difference equation of the Lax method is linear and homogeneous, the perturbation z_{kj} satisfies the same equation

$$\frac{1}{\tau}\left(z_{k,j+1}-\frac{1}{2}[z_{k+1,j}+z_{k-1,j}]\right)+\alpha\frac{z_{k+1,j}-z_{k-1,j}}{2h}=0.$$

Substituting $z_{k,j}=\rho_q^j e^{iqx_k}$ into this equation, we obtain

$$\rho_q=\frac{1}{2}\left(e^{iqh}+e^{-iqh}\right)-\frac{\gamma}{2}\left(e^{iqh}-e^{-iqh}\right)\quad\Rightarrow\quad\rho_q=\cos(qh)-i\gamma\sin(qh)$$

where $\gamma=\tau\alpha/h$. Hence,

$$|\rho_q|^2=\cos^2(qh)+\gamma^2\sin^2(qh).$$

The stability condition $|\rho_q|\le 1$ leads to the requirement

$$\gamma\le 1\quad\text{or}\quad\tau\le\frac{h}{\alpha}. \tag{25}$$

$\boxed{\text{10 Marks}}$

Remarks. *Part (a) - unseen but standard, parts (b) and (c) - discussed in lectures.*

$\boxed{\text{Total: 25 Marks}}$