Truncation error of the ADI method

The ADI method:

$$\frac{w_{k,j}^{n+\frac{1}{2}} - w_{k,j}^n}{\tau} = \frac{K}{2h^2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^n\right) + \frac{1}{2} f_{k,j}^{n+\frac{1}{2}}, \tag{1}$$

$$\frac{w_{k,j}^{n+1} - w_{k,j}^{n+\frac{1}{2}}}{\tau} = \frac{K}{2h^2}\left(\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2 w_{k,j}^{n+1}\right) + \frac{1}{2} f_{k,j}^{n+\frac{1}{2}}. \tag{2}$$

To find the local truncation error of the ADI method, we first eliminate the intermediate values from Eqs. (1), (2). Adding the two equations, we obtain

$$\frac{w_{k,j}^{n+1} - w_{k,j}^n}{\tau} = \frac{K}{2h^2}\left(2\delta_x^2 w_{k,j}^{n+\frac{1}{2}} + \delta_y^2\left[w_{k,j}^n + w_{k,j}^{n+1}\right]\right) + f_{k,j}^{n+\frac{1}{2}}.$$

Subtracting (2) from (1), we find that

$$\frac{2}{\tau} w_{k,j}^{n+\frac{1}{2}} = \frac{w_{k,j}^{n+1} + w_{k,j}^n}{\tau} + \frac{K}{2h^2}\delta_y^2\left[w_{k,j}^n - w_{k,j}^{n+1}\right].$$

It follows that

$$\frac{w_{k,j}^{n+1} - w_{k,j}^n}{\tau} = \frac{K}{2h^2}\left(\delta_x^2 + \delta_y^2\right)\left(w_{k,j}^n + w_{k,j}^{n+1}\right) + f_{k,j}^{n+\frac{1}{2}} + \frac{K^2\tau}{4h^4}\delta_x^2\delta_y^2\left[w_{k,j}^n - w_{k,j}^{n+1}\right]. \tag{3}$$

If the last term on the right side of this equation were absent, the equation would coincide with the Crank-Nicolson method whose local truncation error is $O(\tau^2 + h^2)$.

We will show that the last term in (3), evaluated on the exact solution $u(x, y, t)$, is $O(\tau^2)$. To do this, we first observe that

$$\frac{1}{h^2}\delta_x^2\, u_{kj}^n = u_{xx}(x_k, y_j, t_n) + \frac{h^2}{12}\, u_{xxxx}(x_k, y_j, t_n) + O(h^4),$$

$$\frac{1}{h^2}\delta_y^2\, u_{kj}^n = u_{yy}(x_k, y_j, t_n) + \frac{h^2}{12}\, u_{yyyy}(x_k, y_j, t_n) + O(h^4). \tag{4}$$

It follows from (4) that

$$\frac{1}{h^4}\delta_x^2\delta_y^2\, u_{k,j}^n = u_{xxyy}(x_k, y_j, t_n) + O(h^2), \quad \frac{1}{h^4}\delta_x^2\delta_y^2\, u_{k,j}^{n+1} = u_{xxyy}(x_k, y_j, t_{n+1}) + O(h^2).$$

Further, we have

$$\begin{aligned} \frac{1}{h^4}\delta_x^2\delta_y^2\left(u_{k,j}^n - u_{k,j}^{n+1}\right) &= u_{xxyy}(x_k, y_j, t_n) - u_{xxyy}(x_k, y_j, t_{n+1}) + O(h^2), \\ &= -\tau u_{xxyy}(x_k, y_j, t_n) + O(\tau^2) + O(h^2). \end{aligned} \tag{5}$$

It follows that

$$\frac{K^2\tau}{4h^4}\,\delta_x^2\delta_y^2\left(u_{k,j}^n - u_{k,j}^{n+1}\right) = \frac{K^2\tau^2}{4}\left[-u_{xxyyt}(x_k, y_j, t_n) + O(\tau)\right] + O(\tau h^2) = O(\tau^2).$$

Therefore, the local truncation error of the ADI method is $O(\tau^2 + h^2)$.