$$G_1(\vec{q},\hat{\beta}) = \frac{\underset{3D\,\delta}{\delta}(\vec{q})}{[\Delta S]^3}\, e^{-\alpha}\, \boxed{e^{\alpha\hat{\beta}\cdot\hat{n}_0}}$$

$$G_2(\vec{q},\hat{\beta}) = \frac{e^{-2\alpha}}{[\Delta S]^3}\int d\Omega' \,\boxed{\delta(\vec{q}-\hat{\beta}')}\, e^{\alpha\hat{\beta}'\cdot\hat{n}_0} \times e^{\alpha\hat{\beta}'\cdot\hat{\beta}}$$

$$= e^{-\alpha}\int d\Omega'\, G_1(\vec{q}-\hat{\beta}',\hat{\beta}')\, e^{\alpha\hat{\beta}'\cdot\hat{\beta}}$$

$$^{3D}\delta(\vec{X}-\vec{X}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$= \underbrace{\delta(\hat{X}-\hat{X}_0)}_{\text{2D polar delta}} \frac{\delta(|\vec{X}|-|\vec{X}_0|)}{|\vec{X}|^2} \quad \text{Polar coord}$$

$$\delta(\vec{q}-\hat{\beta}') = \delta(\hat{q}-\hat{\beta}')\, \frac{\boxed{\delta(q-1)}}{q^2}$$

$$\delta(1-q)$$

Identities

$$\delta(f(x)) = \delta(-f(x))$$

$$f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$$

$$\delta(f(x)) = \frac{\delta(x-x_0)}{\left|\frac{df}{dx}\right|} \qquad x_0 = \text{root of } f(x)$$

$$\delta(a(x-x_0)) = \frac{\delta(x-x_0)}{|a|}$$

$$G_2(\vec{q},\hat{\beta}) = \frac{e^{-2\alpha}\,\delta(1-q)}{(\Delta S)^3} \int d\Omega'\, \delta(\hat{q}-\hat{\beta}') \times \underbrace{e^{\alpha \hat{\beta}'\cdot\hat{n}_0}\; e^{\alpha\hat{\beta}\cdot\hat{\beta}'}}$$

$$G_2 = \frac{e^{-2\alpha}\,\delta(1-q)}{(\Delta S)^3}\, e^{\alpha\vec{q}\cdot(\hat{n}_0+\hat{\beta})}$$

$$\int dx\, \delta(x-x_0)\, f(x)$$

$$f(x_0)$$

$$= f(x_0) \int dx\, \delta(x-x_0) = f(x_0)$$

$$\boxed{f(x_0) = \int dx\, \underline{f(x)}\, \delta(x-\underline{x_0})}$$

$$G_3(\vec{q},\hat{\beta}) = e^{-\alpha} \int d\Omega'\, e^{\alpha \hat{\beta}\cdot\hat{\beta}'}$$

$$\times \frac{e^{-2\alpha}\, \delta(1-|\vec{q}-\hat{\beta}'|)}{(\Delta S)^3}\, e^{\alpha (\vec{q}-\hat{\beta}')\cdot(\hat{n}_0+\hat{\beta}')}$$

$$= \frac{e^{-3\alpha}}{(\Delta S)^3} \int d\Omega'\, \delta(1-|\vec{q}-\hat{\beta}'|)\, e^{\alpha\left[\hat{\beta}\cdot\hat{\beta}' + \alpha\vec{q}\cdot(\hat{n}_0+\hat{\beta}') - \hat{n}_0\cdot\hat{\beta}' - 1\right]}$$

$$= \frac{e^{-4\alpha}}{|\Delta s|^3}\, e^{\alpha \vec{q}\cdot\hat{n}_0} \int d\Omega\, \delta\left(|\vec{q}-\hat{\beta}'| - 1\right) \times e^{\alpha \hat{\beta}'\cdot \underbrace{[\hat{\beta}+\vec{q}-\hat{n}_0]}_{\vec{D}}}$$

$$\boxed{\delta\left(|\vec{q}-\hat{\beta}'|-1\right)}$$

$$|\vec{q}-\hat{\beta}'| = \sqrt{q^2+1-2q\underbrace{(\hat{q}\cdot\hat{\beta}')}_{s}}$$

$$\int d\Omega' = \int_{-1}^{1} ds \int_{0}^{2\pi} d\theta$$

$$\hat{\beta}' = \hat{q}\, s + \sqrt{1-s^2}\,\hat{\beta}_\perp$$

$$\hat{q}\cdot\hat{\beta}_\perp = 0$$

$$\boxed{\vec{D} = \hat{\beta}+\vec{q}-\hat{n}_0}$$

$$\vec{q} = \frac{\vec{x}_i-\vec{x}_j}{\Delta s} - (\hat{n}_i+\hat{n}_j)$$

$$\int d\Omega' \, \delta\left(\sqrt{q^2+1-2qs} - 1\right) e^{\alpha \hat{\beta}' \cdot \vec{D}}$$

$$\hat{\beta}' = s\hat{q} + \sqrt{1-s^2}\,\hat{\beta}_\perp \qquad \hat{\beta}_\perp \cdot \hat{q} = 0$$

$$\hat{\beta}' \cdot \vec{D} = s\,\hat{q}\cdot\vec{D} + \sqrt{1-s^2}\,\underbrace{\hat{\beta}_\perp' \cdot \vec{D}}$$

$$\vec{D}_\perp = \vec{D} - \hat{q}(\hat{q}\cdot\vec{D}) \qquad \hat{\beta}_\perp \cdot \vec{D} = \hat{\beta}_\perp \cdot \vec{D}_\perp = |\vec{D}_\perp| \cos\theta$$

$$\hat{\beta}' \cdot \vec{D} = s\,\hat{q}\cdot\vec{D} + \sqrt{1-s^2}\,|\vec{D}_\perp| \cos\theta$$

$$\int_{-1}^{+1} ds \int_0^{2\pi} d\theta \, \delta\left(\sqrt{q^2+1-2qs} - 1\right) e^{\alpha \hat{q}\cdot\vec{D} s} \, \underbrace{e^{\alpha\sqrt{1-s^2}\,|\vec{D}_\perp|\cos\theta}}$$

$$= \int_{-1}^{+1} ds \, \delta\left(\sqrt{q^2+1-2qs} - 1\right) e^{\alpha \hat{q}\cdot\vec{D} s}$$

$$Q = \times \int_0^{2\pi} d\theta \, e^{\alpha\sqrt{1-s^2}\,|\vec{D}_\perp|\cos\theta}$$

Russians 3.195.4

$$\int_0^{\pi} dx\, e^{\beta \cos x} = \sqrt{\pi}\, \underbrace{\Gamma(1/2)}_{\sqrt{\pi}} I_0(\beta)$$

$$Q = 2\pi\, I_0\left(\alpha \sqrt{1-s^2}\, |\vec{D}_\perp|\right)$$

$$\int_{-1}^{+1} ds\, \delta\left(\sqrt{q^2+1-2qs} - 1\right) f(s)$$

$$\delta(f(x)) = \frac{\delta(x-x_0)}{\left|\frac{df}{dx}\right|} \qquad x_0 = \text{root}$$

$$\sqrt{q^2+1-2qs} = 1$$

$$q^2 \cancel{+1} - 2qs = \cancel{1} \qquad s = q/2$$

$$-2 \le q \le 2 \qquad 0 \le q \le 2$$

$$H(2-q)$$

$$f(s) = \sqrt{q^2+1-2qs} - 1$$

$$\frac{df}{ds} = \frac{1}{2}\frac{-2q}{\sqrt{q^2+1-2qs}} = \frac{-q}{\sqrt{q^2+1-2qs}} = -q$$

$$\left|\frac{df}{ds}\right| = q$$

$$\int_{-1}^{+1} ds \int_0^{2\pi} d\theta\, \delta\left(\sqrt{q^2+1-2qs} - 1\right) e^{\alpha \hat{q}\cdot\vec{D} s} \; \underbrace{e^{\alpha\sqrt{1-s^2}\,|\vec{D}_\perp|\cos\theta}}$$

$$= 2\pi \frac{H(2-q)}{q} e^{\frac{\alpha}{2}\vec{q}\cdot\vec{D}} I_0\left(\alpha\sqrt{1-q^2/4}\,|\vec{D}_\perp|\right)$$

$$|\vec{D}_\perp| = |D| - (\hat{q}\cdot\vec{D})^2$$

$$\vec{D}_\perp = \hat{\beta} + \vec{q} - \vec{n}_0$$

$$- \left(\hat{q}\cdot\left(\hat{\beta} + \vec{q} - \hat{n}_0\right)\right)\hat{q}$$

$$= \left(\hat{\beta} - \hat{n}_0\right) - \hat{q}\left(\left(\hat{\beta} - \hat{n}_0\right)\cdot\hat{q}\right)$$

$$|\vec{D}_\perp| \to \left|\left(\hat{\beta} - \hat{n}_0\right)_\perp\right|$$

$$G_3^{(\vec{q},\hat{\beta})} = \frac{e^{-4\alpha}}{|\Delta S|^3}\, e^{\alpha \vec{q}\cdot\hat{n}_0}\, e^{\frac{\alpha}{2}\vec{q}\cdot\vec{D}}\, \frac{\theta(2-q)}{q}\, I_0\left(\alpha\sqrt{1-\frac{q^2}{4}}\, |\vec{D}_\perp|\right)$$

$$\vec{q}\cdot\vec{n}_0 + \frac{1}{2}\vec{q}\cdot\vec{D} = \vec{q}\cdot\hat{n}_0 + \frac{1}{2}\vec{q}\cdot\left(\vec{q} + \hat{\beta} - \hat{n}_0\right)$$

$$= \frac{1}{2}\vec{q}\cdot\hat{n}_0 + \frac{1}{2}q^2 + \frac{1}{2}\vec{q}\cdot\hat{\beta} - \cancel{\frac{1}{2}\vec{q}\cdot\hat{n}_0}$$

$$= \frac{1}{2}\vec{q}\cdot\left(\vec{q} + \hat{\beta} + \hat{n}_0\right)$$

$$\frac{A(N-q)}{q} \longrightarrow \frac{A(N+1-q)}{q} \checkmark$$

$$s = \hat{q}\cdot\hat{\beta}$$

$$(s, \theta)$$

$$G_{N+1}(\vec{q}, \hat{\beta}) = e^{-\alpha} \int d\Omega' \; G_N(\vec{q}-\hat{\beta}', \hat{\beta}') e^{\alpha \hat{\beta}\cdot\hat{\beta}'}$$

Change of variables

$$\bar{q} = |\vec{q} - \hat{\beta}'|$$

$$= \sqrt{q^2+1-2qs}$$

$$\int d\Omega' \rightarrow \underbrace{\int_{-1}^{+1} ds \int_0^{2\pi} d\theta}_{\hat{q},\ \hat{\beta}_\perp}$$

$$ds \to d\bar{q}\, J \qquad -1 \le s \le 1$$

$$J = \frac{\bar{q}}{q} \qquad \downarrow$$

$$|1-q| \le \bar{q} \le 1+q$$

$$G_3 = \frac{\Theta(2-q)}{q}$$

$$G_4 \to \int_{|1-q|}^{1+q} d\bar{q}\, \frac{\bar{q}}{q} \frac{\Theta(2-\bar{q})}{\bar{q}} \int_0^{2\pi} d\theta \sim\sim\sim$$

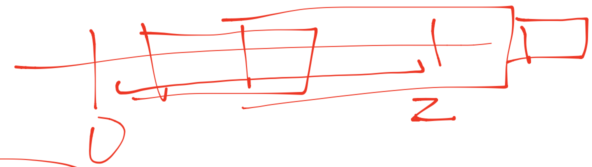
$$\bar{q} \le 2$$

$$G_4 \sim \frac{1}{q}$$

$$1+q \ge 2$$

$$\left(\int_{|1-q|}^{1+q} d\bar{q}\, \boxed{\Theta(2-\bar{q})} \right. - \sim\sim\sim$$

$0 \le \bar{q} < 2$



$$\boxed{|1-q| < 2}$$

$1-q < 2$ for $q<1$

$\boxed{q-1<2}$ $\boxed{\text{for } q>1}$

$$\underbrace{1-2}_{-1} < q$$

$\boxed{q<3}$ $\boxed{\Theta(3-q)}$

$$1+q > |1-q|$$

1

$$G_4 \sim \frac{\Theta(3-q)}{q} \left(\int_{|1-q|}^{1+q} dq_1 \int_0^{2\pi} d\phi \, (\vec{q}, \hat{p}_1) \right)$$

$G(\vec{q}, \hat{\beta})$

$$G_N \sim \frac{\#(N-1-q)}{q}$$

G_{N+1}

$$G_{N+1} \sim \frac{\#(N-q)}{q}$$

$q < N$

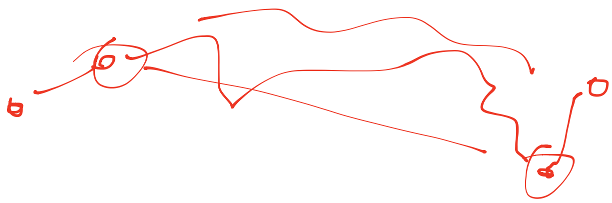
$$\left| \frac{\vec{x}-\vec{x}_0'}{\Delta s} - (\hat{n}+\hat{n}_0) \right| < N$$

$$\left| \vec{x} - \vec{x}' - \Delta s(\hat{n}+\hat{n}_0) \right|$$

$$|\vec{x} - \hat{n}\Delta s - (\vec{x}' - \hat{n}_0 \Delta s)| < N\Delta s$$

arclength of interior nodes

N



Paper

1. PI formalism for optically dense model

2. $G_1, G_1, G_3, G_4, \ldots, G_N \sim O(N-9)$

3. Monte Carlo normalization

4. Compare MCN w/ code
   - 6 nodes including endpoints $N=4$
   - Weights $= e^{-\alpha} e^{\alpha \hat{\beta}_i \cdot \hat{\beta}_{i+1}}$

$\alpha = 1$

$$\alpha = \frac{1}{\Delta S \mu}$$

$\Delta S = S/N$