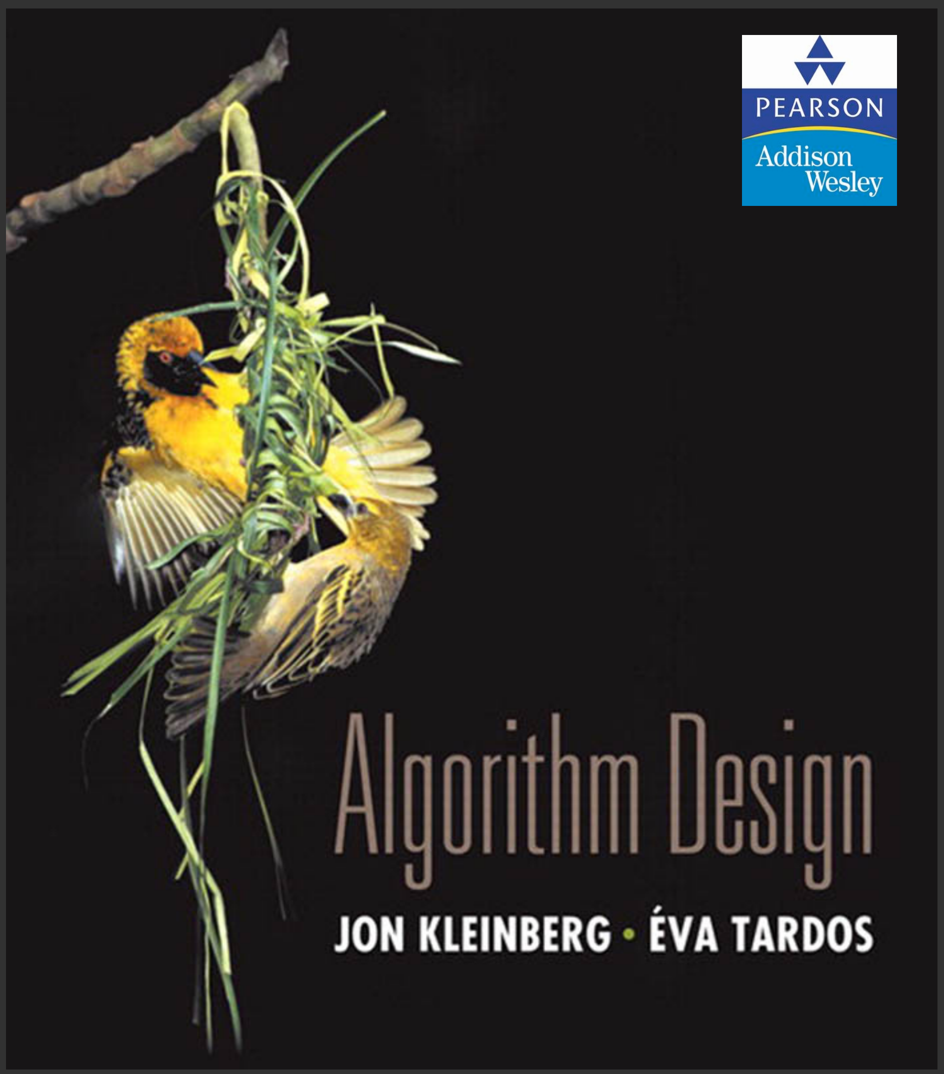


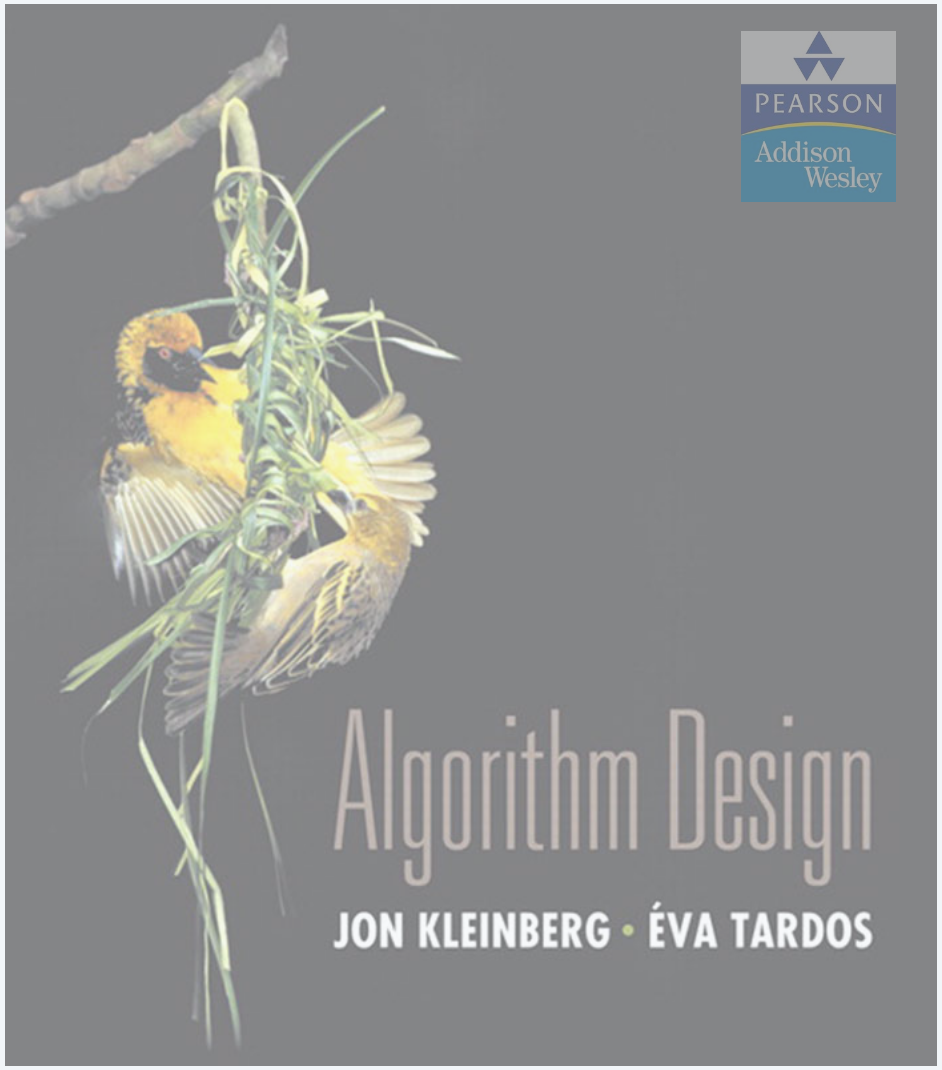
6. Dynamic Programming II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman–Ford–Moore algorithm*
- *distance-vector protocols*
- *negative cycles*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 4/8/18 7:52 PM



SECTION 6.8

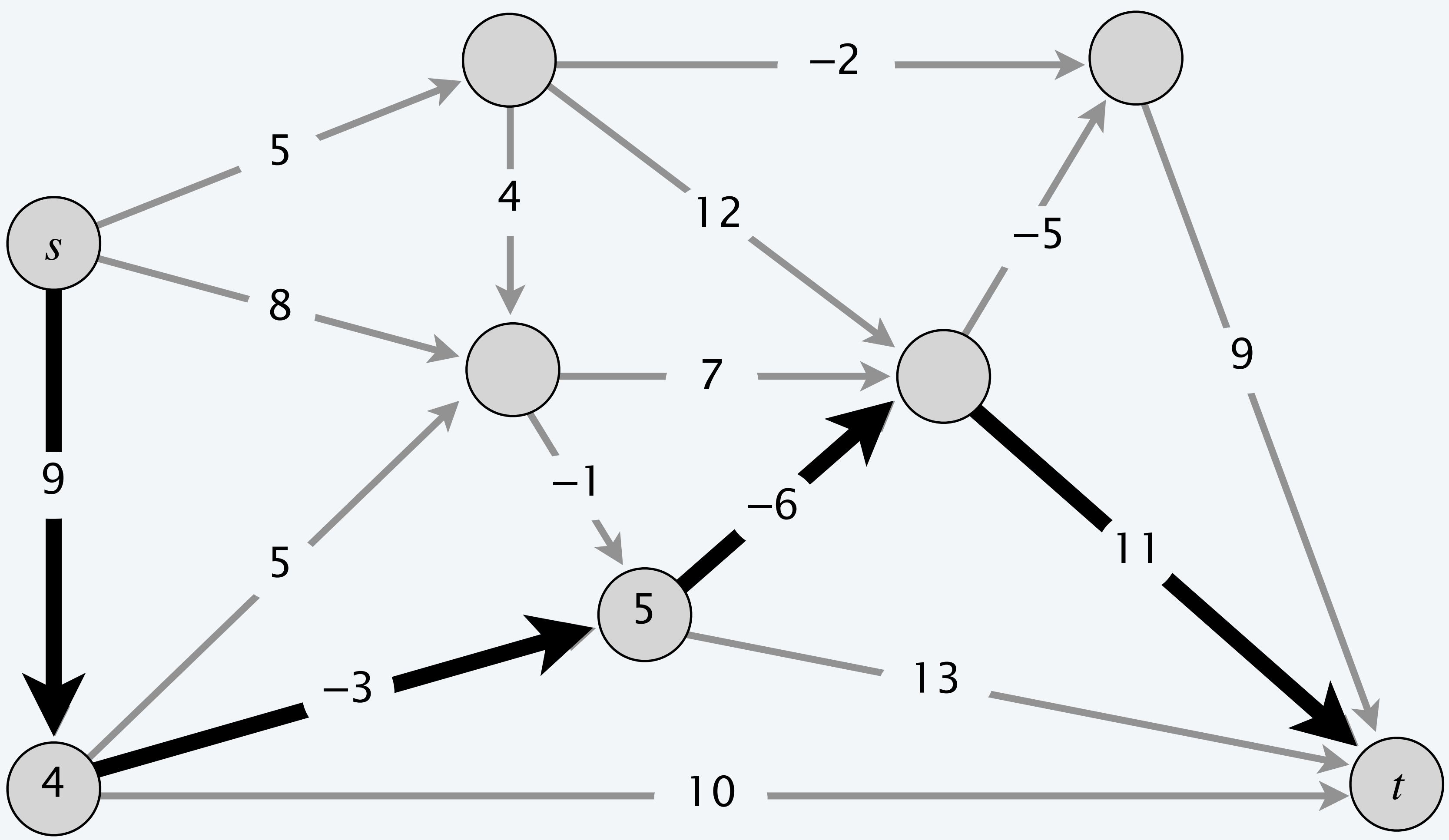
6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman–Ford–Moore algorithm*
- *distance-vector protocols*
- *negative cycles*

Shortest paths with negative weights

Shortest-path problem. Given a digraph $G = (V, E)$, with arbitrary edge lengths ℓ_{vw}, find shortest path from source node s to destination node t.

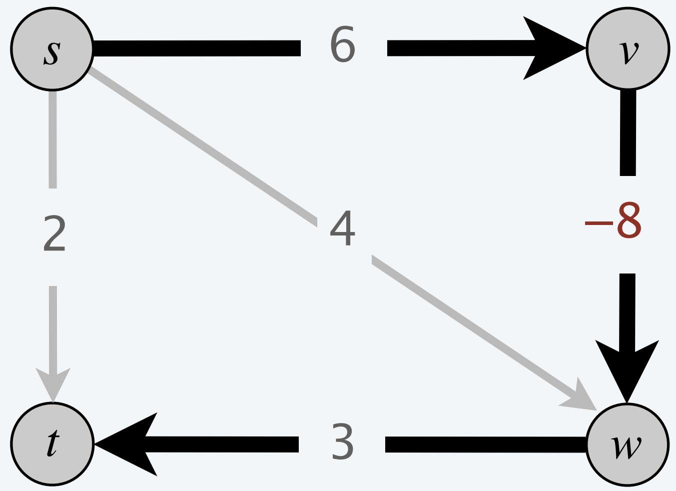
assume there exists a path from every node to t



length of shortest path from s to t = 9 − 3 − 6 + 11 = 11

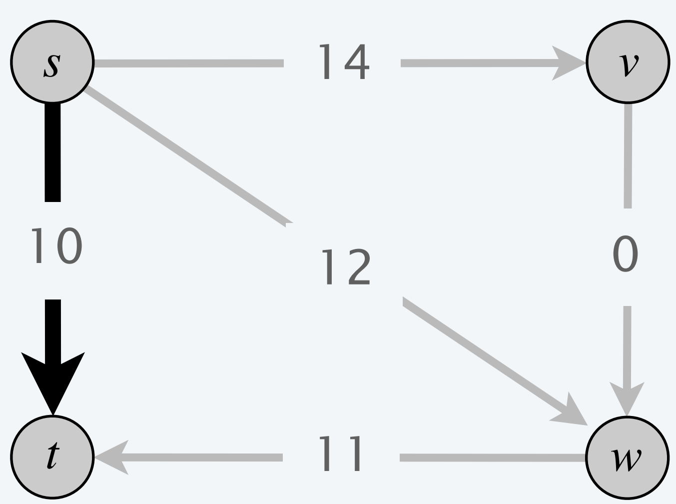
Shortest paths with negative weights: failed attempts

Dijkstra. May not produce shortest paths when edge lengths are negative.



Dijkstra selects the vertices in the order s, t, w, v
But shortest path from s to t is $s \to v \to w \to t$.

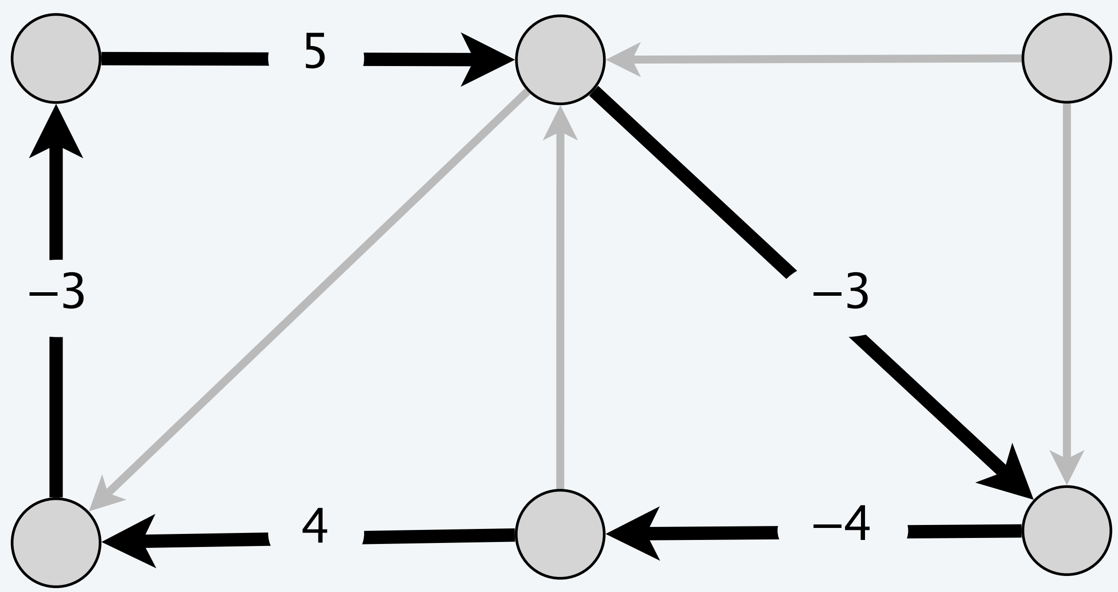
Reweighting. Adding a constant to every edge length does not necessarily make Dijkstra's algorithm produce shortest paths.



Adding 8 to each edge weight changes the shortest path from $s \to v \to w \to t$ to $s \to t$.

Negative cycles

Def. A negative cycle is a directed cycle for which the sum of its edge lengths is negative.

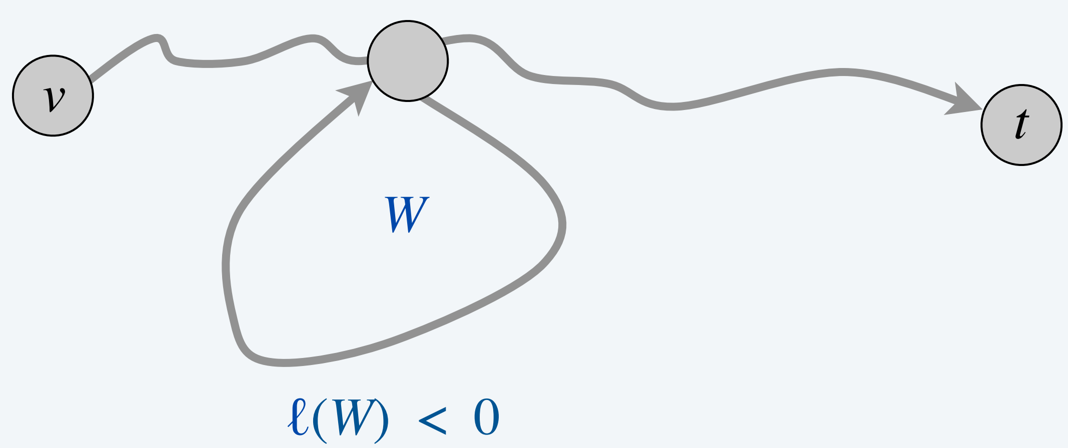


a negative cycle W : $\ell(W) \;=\; \sum_{e \in W} \ell_e \;<\; 0$

Shortest paths and negative cycles

Lemma 1. If some $v \rightsquigarrow t$ path contains a negative cycle, then there does not exist a shortest $v \rightsquigarrow t$ path.

Pf. If there exists such a cycle W, then can build a $v \rightsquigarrow t$ path of arbitrarily negative length by detouring around W as many times as desired. ▪

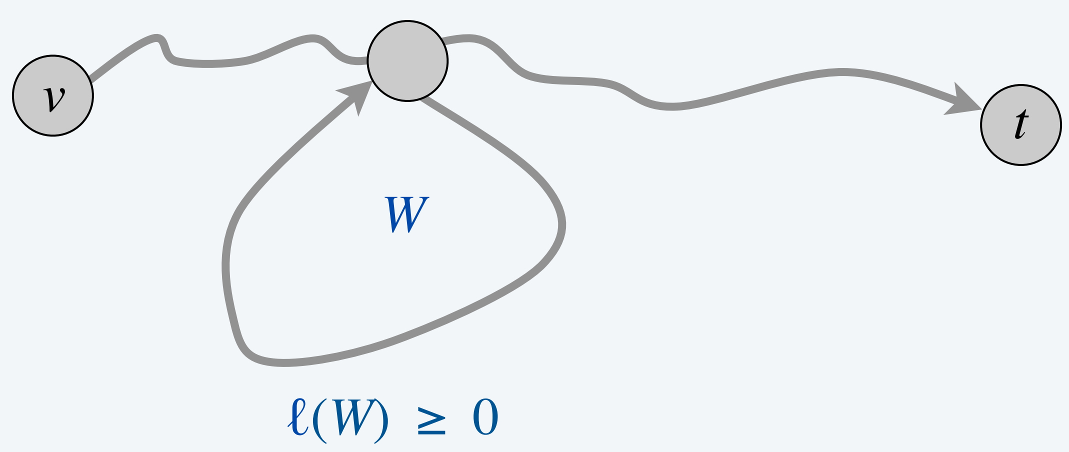


Shortest paths and negative cycles

Lemma 2. If G has no negative cycles, then there exists a shortest $v \rightsquigarrow t$ path that is simple (and has $\leq n-1$ edges).

Pf.

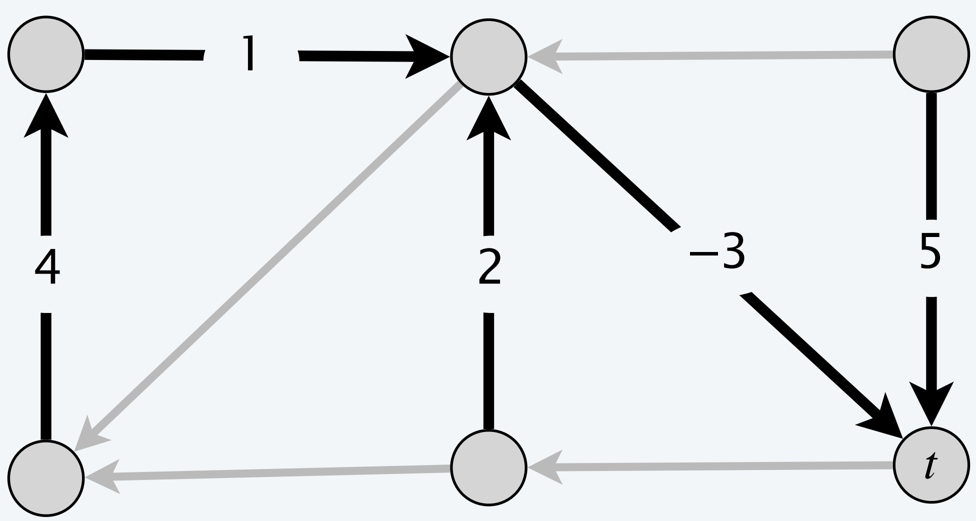
- Among all shortest $v \rightsquigarrow t$ paths, consider one that uses the fewest edges.
- If that path P contains a directed cycle W, can remove the portion of P corresponding to W without increasing its length. ▪



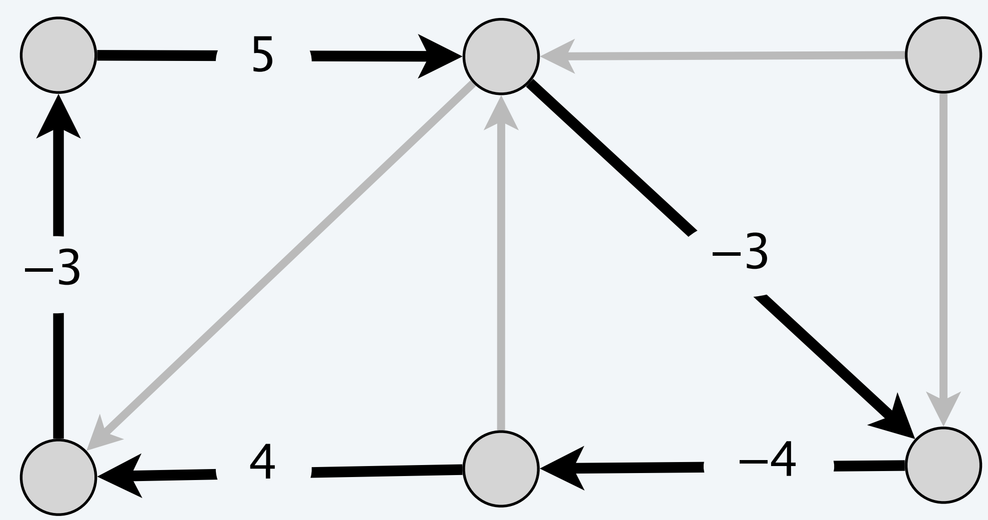
Shortest-paths and negative-cycle problems

Single-destination shortest-paths problem. Given a digraph $G = (V, E)$ with edge lengths ℓ_{vw} (but no negative cycles) and a distinguished node t, find a shortest $v \rightarrow t$ path for every node v.

Negative-cycle problem. Given a digraph $G = (V, E)$ with edge lengths ℓ_{vw}, find a negative cycle (if one exists).



shortest-paths tree



negative cycle

Shortest paths with negative weights: dynamic programming

Def. $OPT(i, v)$ = length of shortest $v \rightsquigarrow t$ path that uses $\le i$ edges.

Goal. $OPT(n-1, v)$ for each v.

by Lemma 2, if no negative cycles, there exists a shortest $v \rightsquigarrow t$ path that is simple

Case 1. Shortest $v \rightsquigarrow t$ path uses $\le i-1$ edges.

- $OPT(i, v) = OPT(i-1, v)$.

optimal substructure property (proof via exchange argument)

Case 2. Shortest $v \rightsquigarrow t$ path uses exactly i edges.

- if (v, w) is first edge in shortest such $v \rightsquigarrow t$ path, incur a cost of ℓ_{vw}.
- Then, select best $w \rightsquigarrow t$ path using $\le i-1$ edges.

Bellman equation.

$$OPT(i,v) \;=\; \begin{cases} 0 & \text{if } i = 0 \text{ and } v = t \\ \infty & \text{if } i = 0 \text{ and } v \neq t \\ \min\left\{ OPT(i-1,v),\; \min\limits_{(v,w)\in E} \{OPT(i-1,w) + \ell_{vw}\} \right\} & \text{if } i > 0 \end{cases}$$

Shortest paths with negative weights: implementation

SHORTEST-PATHS(V, E, ℓ, t)

FOREACH node $v \in V$:

  $M[0, v] \leftarrow \infty$.

$M[0, t] \leftarrow 0$.

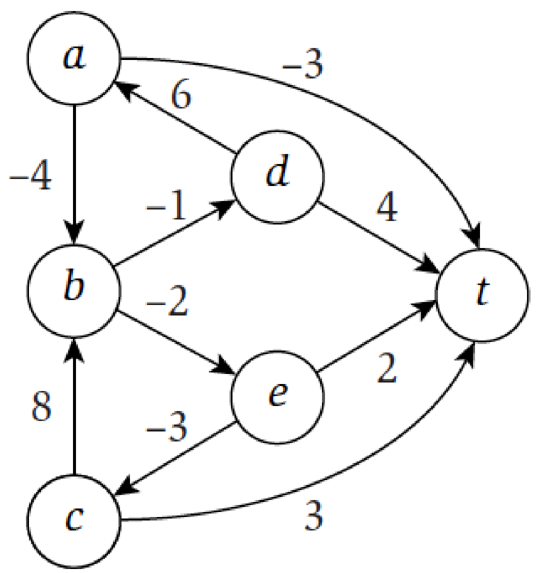
FOR $\mathrm{i} = 1$ TO $n - 1$

  FOREACH node $v \in V$:

    $M[i, v] \leftarrow M[i-1, v]$.

    FOREACH edge $(v, w) \in E$:

      $M[i, v] \leftarrow \min \{ M[i, v],\ M[i-1, w] + \ell_{vw} \}$.



(a)

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	−3	−3	−4	−6	−6
b	∞	∞	0	−2	−2	−2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

(b)

Figure 6.23 For the directed graph in (a), the Shortest-Path Algorithm constructs the dynamic programming table in (b).

Shortest paths with negative weights: implementation

Theorem 1. Given a digraph $G = (V, E)$ with no negative cycles, the DP algorithm computes the length of a shortest $v \leadsto t$ path for every node v in $\Theta(mn)$ time and $\Theta(n^2)$ space.

Pf.

- Table requires $\Theta(n^2)$ space.
- Each iteration i takes $\Theta(m)$ time since we examine each edge once. ▪

Finding the shortest paths.

- Approach 1: Maintain $successor[i, v]$ that points to next node on a shortest $v \leadsto t$ path using $\leq i$ edges.
- Approach 2: Compute optimal lengths $M[i, v]$ and consider only edges with $M[i, v] = M[i-1, w] + \ell_{vw}$. Any directed path in this subgraph is a shortest path.

Dynamic programming: quiz 6

It is easy to modify the DP algorithm for shortest paths to…

- **A.** Compute lengths of shortest paths in $O(mn)$ time and $O(m + n)$ space.
- **B.** Compute shortest paths in $O(mn)$ time and $O(m + n)$ space.
- **C.** Both A and B.
- **D.** Neither A nor B.

Shortest paths with negative weights: practical improvements

Space optimization. Maintain two 1D arrays (instead of 2D array).

- $d[v]$ = length of a shortest $v \rightsquigarrow t$ path that we have found so far.
- $successor[v]$ = next node on a $v \rightsquigarrow t$ path.

Performance optimization. If $d[w]$ was not updated in iteration $i - 1$, then no reason to consider edges entering w in iteration i.

Bellman–Ford–Moore: efficient implementation

BELLMAN–FORD–MOORE(V, E, c, t)

FOREACH node $v \in V$:

  $d[v] \leftarrow \infty$.

  $successor[v] \leftarrow null$.

$d[t] \leftarrow 0$.

FOR $i = 1$ TO $n - 1$

  FOREACH node $w \in V$:

    IF ($d[w]$ was updated in previous pass)

      FOREACH edge $(v, w) \in E$:

        IF ($d[v] > d[w] + \ell_{vw}$)

          $d[v] \leftarrow d[w] + \ell_{vw}$.

          $successor[v] \leftarrow w$.

  IF (no $d[\cdot]$ value changed in pass i) STOP.

---

pass i
$O(m)$ time

Dynamic programming: quiz 7

Which properties must hold after pass i of Bellman–Ford–Moore?

A. $d[v]$ = length of a shortest $v \rightsquigarrow t$ path using $\leq i$ edges.

B. $d[v]$ = length of a shortest $v \rightsquigarrow t$ path using exactly i edges.

C. Both A and B.

D. Neither A nor B.

Bellman–Ford–Moore: analysis

Lemma 3. For each node v : $d[v]$ is the length of some $v \rightsquigarrow t$ path.

Lemma 4. For each node v : $d[v]$ is monotone non-increasing.

Lemma 5. After pass i, $d[v] \le$ length of a shortest $v \rightsquigarrow t$ path using $\le i$ edges.

Pf. [by induction on i]

- Base case: $i = 0$.
- Assume true after pass i.
- Let P be any $v \rightsquigarrow t$ path with $\le i + 1$ edges.
- Let (v, w) be first edge in P and let P' be subpath from w to t.
- By inductive hypothesis, at the end of pass i, $d[w] \le c(P')$ because P' is a $w \rightsquigarrow t$ path with $\le i$ edges.
- After considering edge (v, w) in pass $i + 1$:

and by Lemma 4, $d[w]$ does not increase

$$
\begin{aligned}
d[v] &\le \ell_{vw} + d[w] \\
&\le \ell_{vw} + c(P') \\
&= \ell(P) \quad \blacksquare
\end{aligned}
$$

and by Lemma 4, $d[v]$ does not increase

Bellman–Ford–Moore: analysis

Theorem 2. Assuming no negative cycles, Bellman–Ford–Moore computes the lengths of the shortest $v \rightsquigarrow t$ paths in $O(mn)$ time and $\Theta(n)$ extra space.

Pf. Lemma 2 + Lemma 5. ▪

shortest path exists and has at most $n-1$ edges

after i passes, $d[v] \le$ length of shortest path that uses $\le i$ edges

Remark. Bellman–Ford–Moore is typically faster in practice.

- Edge (v, w) considered in pass $i + 1$ only if $d[w]$ updated in pass i.
- If shortest path has k edges, then algorithm finds it after $\le k$ passes.

Bellman–Ford–Moore: finding the shortest paths

Lemma 6. Any directed cycle W in the successor graph is a negative cycle.

Pf.

- If $successor[v] = w$, we must have $d[v] \geq d[w] + \ell_{vw}$.
  (LHS and RHS are equal when $successor[v]$ is set; $d[w]$ can only decrease; $d[v]$ decreases only when $successor[v]$ is reset)
- Let $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$ be the sequence of nodes in a directed cycle W.
- Assume that (v_k, v_1) is the last edge in W added to the successor graph.
- Just prior to that:

$$
\begin{array}{lcll}
d[v_1] & \geq & d[v_2] & + \ \ell(v_1, v_2) \\
d[v_2] & \geq & d[v_3] & + \ \ell(v_2, v_3) \\
\vdots & \vdots & & \vdots \\
d[v_{k-1}] & \geq & d[v_k] & + \ \ell(v_{k-1}, v_k) \\
d[v_k] & > & d[v_1] & + \ \ell(v_k, v_1)
\end{array}
$$

holds with strict inequality since we are updating $d[v_k]$

- Adding inequalities yields $\underbrace{\ell(v_1, v_2) + \ell(v_2, v_3) + \ldots + \ell(v_{k-1}, v_k) + \ell(v_k, v_1) < 0}_{W \text{ is a negative cycle}}$. ▪

Bellman–Ford–Moore: finding the shortest paths

Theorem 3. Assuming no negative cycles, Bellman–Ford–Moore finds shortest $v \rightsquigarrow t$ paths for every node v in $O(mn)$ time and $\Theta(n)$ extra space.

Pf.

- The successor graph cannot have a directed cycle. [Lemma 6]
- Thus, following the successor pointers from v yields a directed path to t.
- Let $v = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = t$ be the nodes along this path P.
- Upon termination, if $successor[v] = w$, we must have $d[v] = d[w] + \ell_{vw}$.
  (LHS and RHS are equal when $successor[v]$ is set; $d[\cdot]$ did not change)

  since algorithm terminated
- Thus,

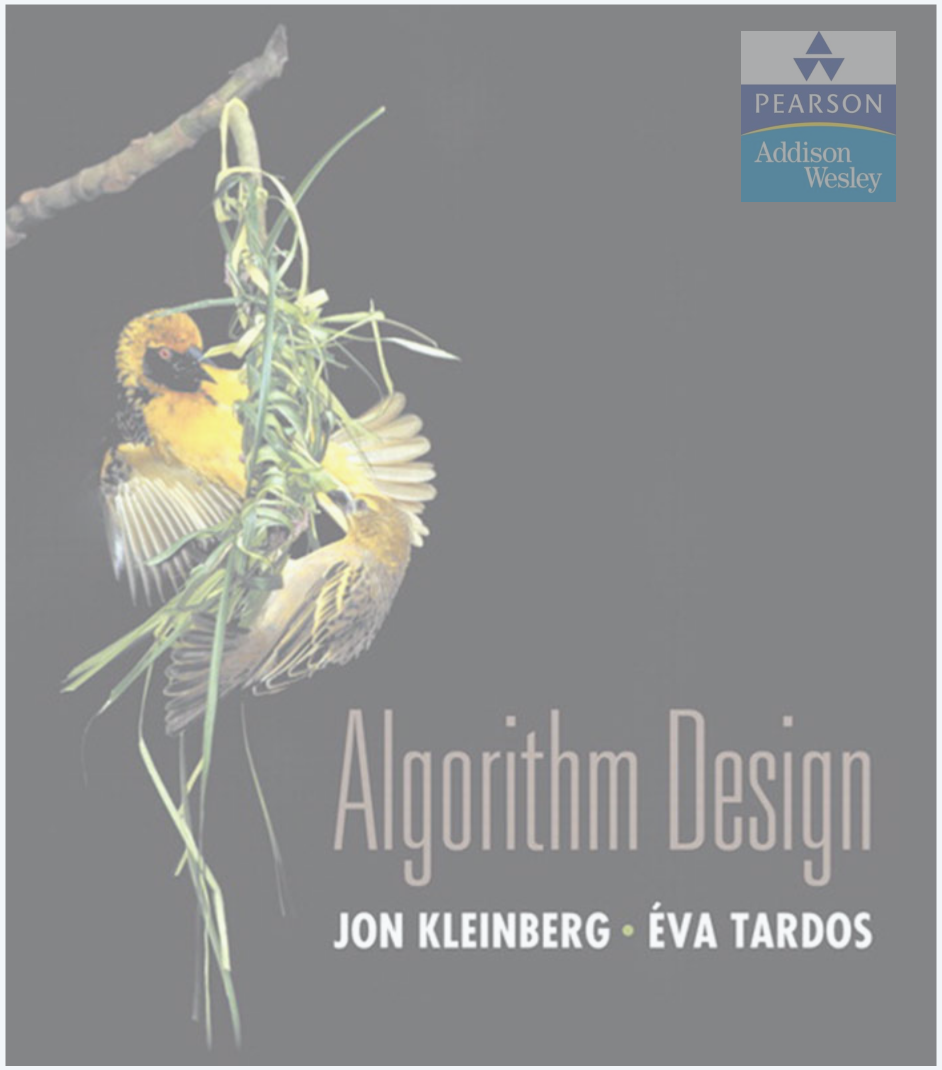
$$
\begin{aligned}
d[v_1] &= d[v_2] + \ell(v_1, v_2) \\
d[v_2] &= d[v_3] + \ell(v_2, v_3) \\
&\vdots \\
d[v_{k-1}] &= d[v_k] + \ell(v_{k-1}, v_k)
\end{aligned}
$$

- Adding equations yields $\underset{\substack{\text{min length of any } v \rightsquigarrow t \text{ path} \\ \text{(Theorem 2)}}}{d[v]} = \underset{0}{d[t]} + \underbrace{\ell(v_1, v_2) + \ell(v_2, v_3) + \ldots + \ell(v_{k-1}, v_k)}_{\text{length of path } P}$. ▪

Single-source shortest paths with negative weights

year	worst case	discovered by
1955	$O(n^4)$	Shimbel
1956	$O(m\, n^2\, W)$	Ford
1958	$O(m\, n)$	Bellman, Moore
1983	$O(n^{3/4}\, m \log W)$	Gabow
1989	$O(m\, n^{1/2} \log(nW))$	Gabow–Tarjan
1993	$O(m\, n^{1/2} \log W)$	Goldberg
2005	$O(n^{2.38}\, W)$	Sankowsi, Yuster–Zwick
2016	$\tilde{O}(n^{10/7} \log W)$	Cohen–Mądry–Sankowski–Vladu
20xx	???	

single-source shortest paths with weights between –W and W



SECTION 6.9

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman–Ford–Moore algorithm*
- *distance-vector protocols*
- *negative cycles*

Distance-vector routing protocols

Communication network.

- Node ≈ router.
- Edge ≈ direct communication link.
- Length of edge ≈ latency of link. ← non-negative, but Bellman–Ford–Moore used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman–Ford–Moore. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each edges are processed in Bellman–Ford–Moore is not important. Moreover, algorithm converges even if updates are asynchronous.

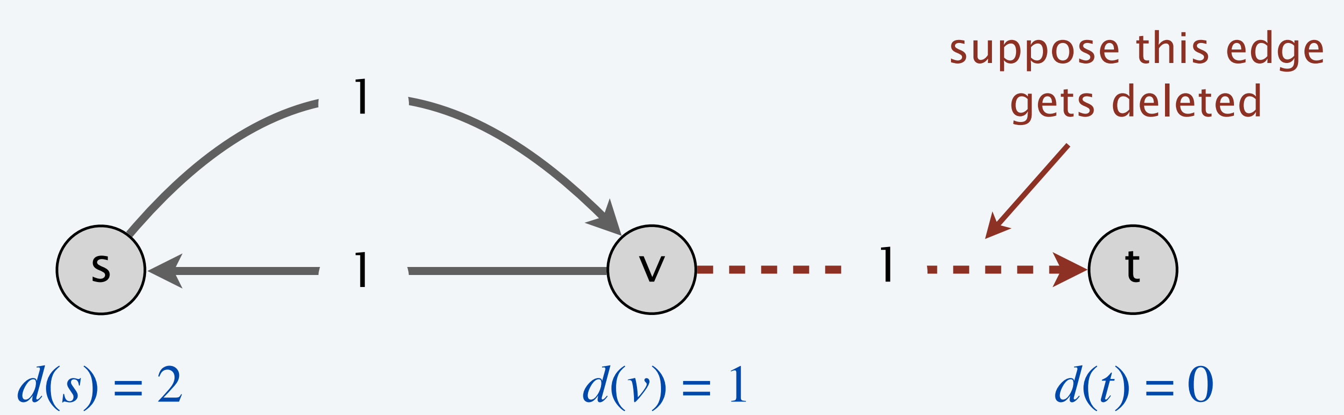
Distance-vector routing protocols

Distance-vector routing protocols. ["routing by rumor"]

- Each router maintains a vector of shortest-path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs n separate computations, one for each potential destination node.

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge lengths may change during algorithm (or fail completely).



"counting to infinity"

Path-vector routing protocols

Link-state routing protocols.

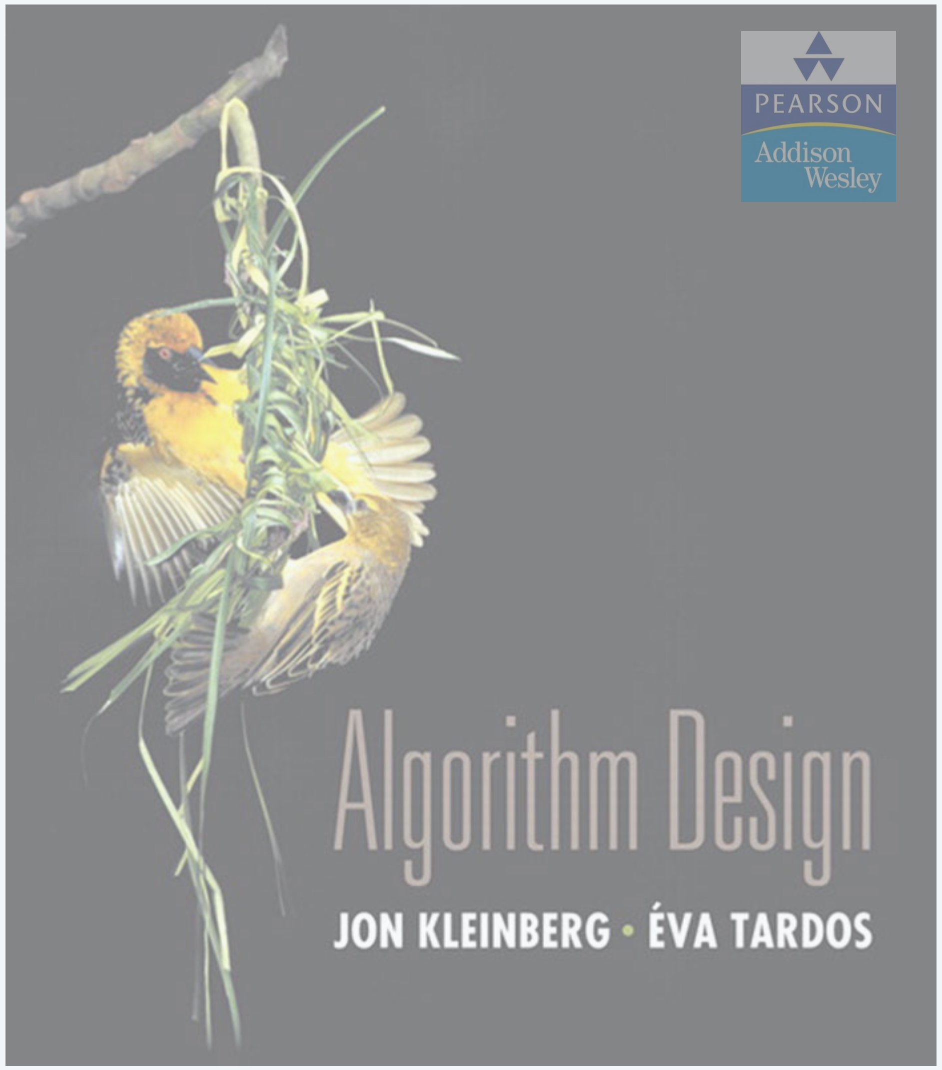
not just the distance and first hop

- Each router stores the whole network topology.
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).

6. Dynamic Programming II

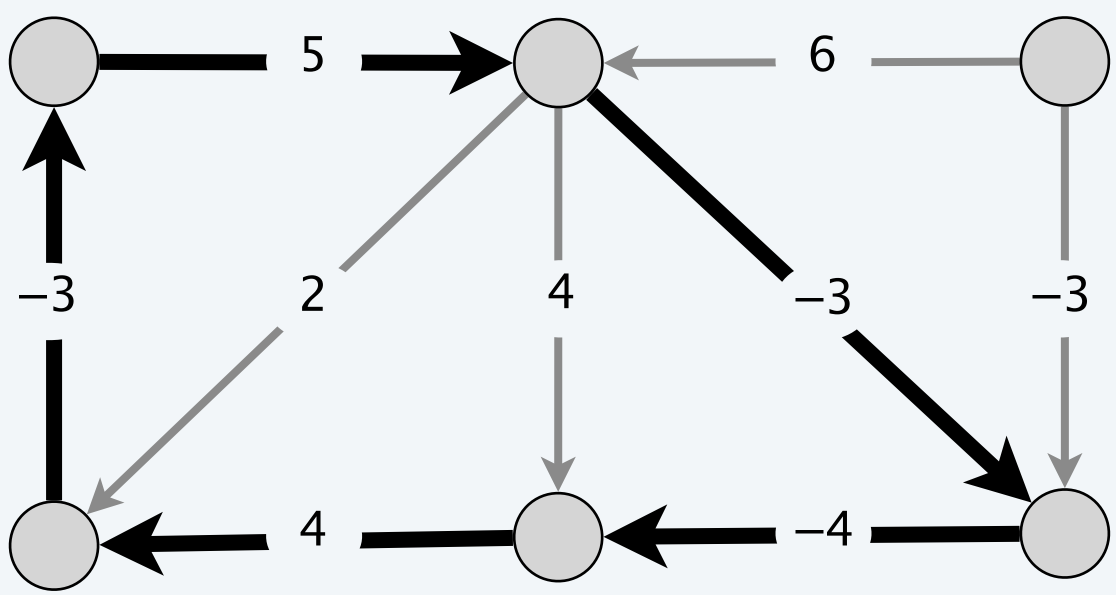
- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman–Ford–Moore algorithm*
- *distance vector protocol*
- *negative cycles*



Section 6.10

Detecting negative cycles

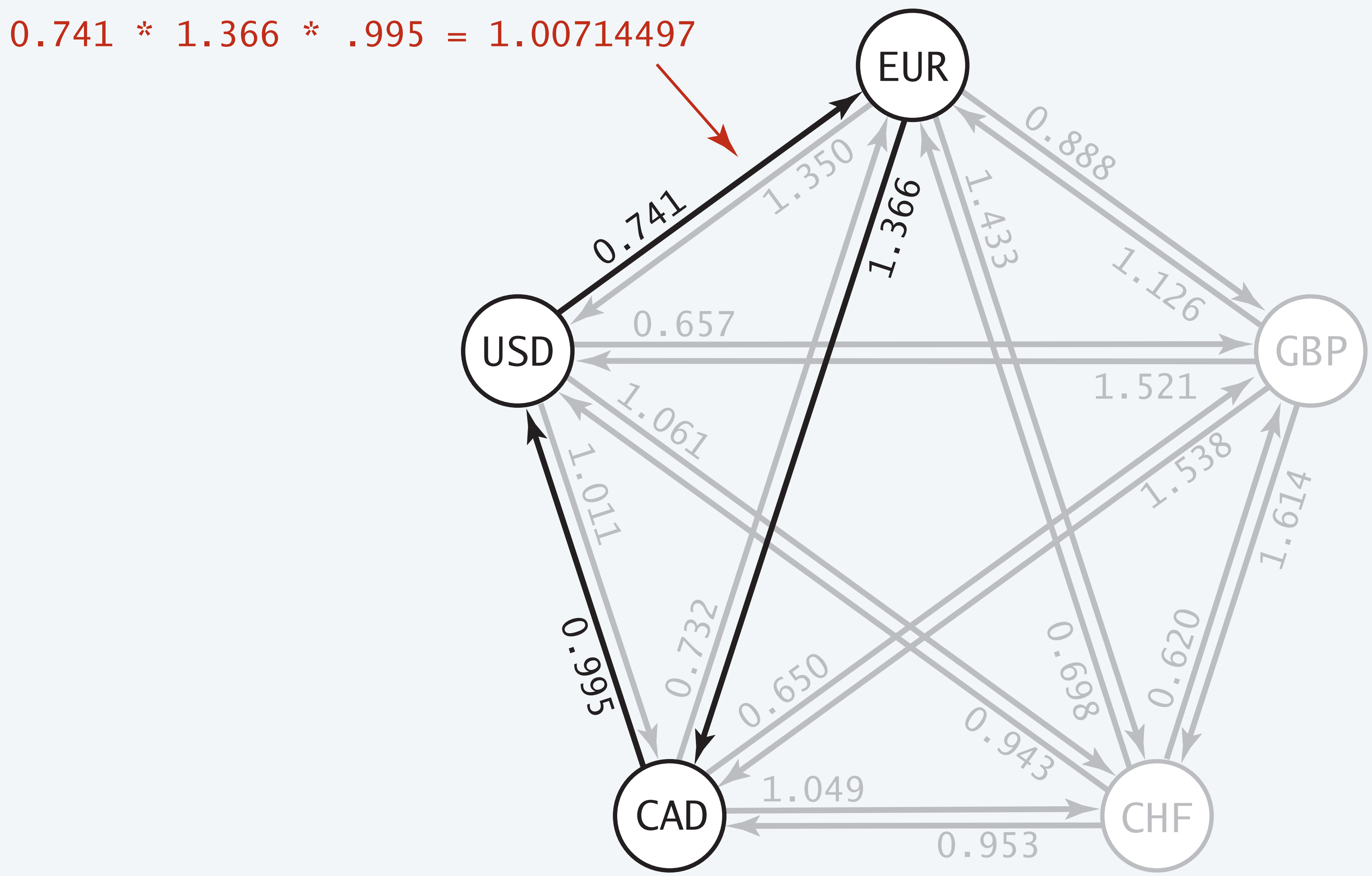
Negative cycle detection problem. Given a digraph $G = (V, E)$, with edge lengths ℓ_{vw}, find a negative cycle (if one exists).



Detecting negative cycles: application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



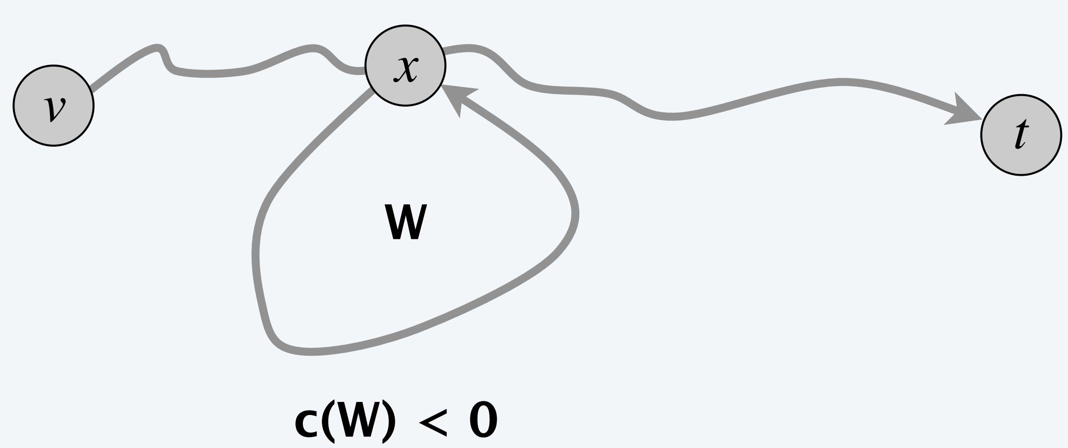
Detecting negative cycles

Lemma 7. If $OPT(n, v) = \mathrm{OPT}(n-1, v)$ for every node v, then no negative cycles.
Pf. The $OPT(n, v)$ values have converged $\Rightarrow$ shortest $v \rightsquigarrow t$ path exists. ▪

Lemma 8. If $OPT(n, v) < \mathrm{OPT}(n-1, v)$ for some node v, then (any) shortest $v \rightsquigarrow t$ path of length $\leq n$ contains a cycle W. Moreover W is a negative cycle.

Pf. [by contradiction]

- Since $OPT(n, v) < \mathrm{OPT}(n-1, v)$, we know that shortest $v \rightsquigarrow t$ path P has exactly n edges.
- By pigeonhole principle, the path P must contain a repeated node x.
- Let W be any cycle in P.
- Deleting W yields a $v \rightsquigarrow t$ path with $< n$ edges $\Rightarrow$ W is a negative cycle. ▪

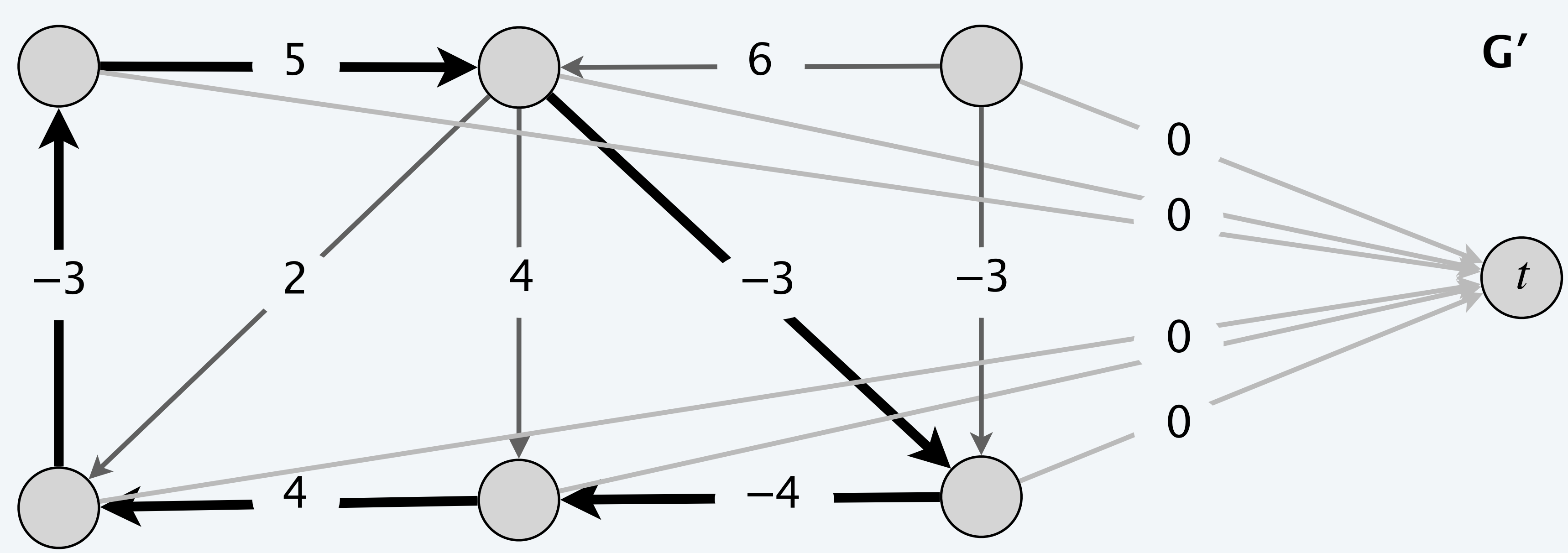


Detecting negative cycles

Theorem 4. Can find a negative cycle in $\Theta(mn)$ time and $\Theta(n^2)$ space.

Pf.

- Add new sink node t and connect all nodes to t with 0-length edge.
- G has a negative cycle iff G' has a negative cycle.
- Case 1. [$OPT(n, v) = OPT(n-1, v)$ for every node v]
  By Lemma 7, no negative cycles.
- Case 2. [$OPT(n, v) < OPT(n-1, v)$ for some node v]
  Using proof of Lemma 8, can extract negative cycle from $v \rightsquigarrow t$ path.
  (cycle cannot contain t since no edge leaves t) ▪



Detecting negative cycles

Theorem 5. Can find a negative cycle in $O(mn)$ time and $O(n)$ extra space.

Pf.

- Run Bellman–Ford–Moore on G' for $n' = n + 1$ passes (instead of $n' - 1$).
- If no $d[v]$ values updated in pass n', then no negative cycles.
- Otherwise, suppose $d[s]$ updated in pass n'.
- Define $pass(v)$ = last pass in which $d[v]$ was updated.
- Observe $pass(s) = n'$ and $pass(successor[v]) \geq pass(v) - 1$ for each v.
- Following successor pointers, we must eventually repeat a node.
- Lemma 6 $\Rightarrow$ the corresponding cycle is a negative cycle. ▪

Remark. See p. 304 for improved version and early termination rule.
(Tarjan's subtree disassembly trick)

Dynamic programming: quiz 9

How difficult to find a negative cycle in an undirected graph?

- **A.** $O(m \log n)$
- **B.** $O(mn)$
- **C.** $O(mn + n^2 \log n)$
- **D.** $O(n^{2.38})$
- **E.** No poly-time algorithm is known.