The Risch Differential Equation on an Algebraic Curve

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Abstract

We present a new rational algorithm for solving Risch differential equations over algebraic curves. This algorithm can also be used to solve n^{th}-order linear ordinary differential equations with coefficients in an algebraic extension of the rational functions. In the general ("mixed function") case, this algorithm finds the denominator of any solution of the equation.

1 Introduction

Since Risch ([10]) reduced the problem of integrating exponentials to solving a first order linear differential equation in a differential field, there has been considerable interest in solving

$$z' + fz = g \tag{R}$$

for z in a given differential field K, where $f, g \in K$. While there are effective algorithms for solving equation (R) when f and g are purely transcendental elementary functions ([4]), the published algorithms for the case where f and g are algebraic functions ([5, 7, 9]) require computing with the local series expansions of f and g at their poles, which is the source of extensive computations in the algebraic closure of K. In fact, no implementation of an algorithm that solves equation (R) over an algebraic curve has been reported.

We present here a new algorithm for solving equation (R) when K is an algebraic extension of the rational function field. Our algorithm uses only rational operations and never needs to compute in an extension of K. This algorithm has been implemented in the Maple and Scratchpad computer algebra systems.

Our algorithm proceeds by (1) reducing equation (R) to a system of first order linear ordinary differential equations over the rational functions, (2) reducing the system to n^{th}-order linear ordinary differential equations over the rational functions, (3) solving these equations separately. Steps (1) and (2) can be carried out in the "mixed function case" (where K is an algebraic extension of an elementary function field), and step (3) will return the denominator of any rational solution in that case. Also, while we present here an implementation for the Risch differential equation, this algorithm can be used to solve a linear differential equation of any order over an algebraic curve.

2 Reducing to a linear system

Let E be a field of characteristic 0, and V a finite dimensional algebra over E. Let n be the dimension of V over E and $\mathbf{b} = (b_1, ..., b_n)$ be any basis for V over E. Given any $u \in V$, we write $\mathbf{u_b}$ for the column vector of the coordinates of u in the basis $\mathbf{b}$ ($\mathbf{u_b} \in E^n$). We then have $u = \mathbf{b} \cdot \mathbf{u_b}$ for any $u \in V$. Let $u \in V$ and define the *regular representation* of u (w.r.t $\mathbf{b}$) to be

$$M_{\mathbf{b}}(u) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u_{1b}} & \mathbf{u_{2b}} & \cdots & \mathbf{u_{nb}} \\ | & | & \cdots & | \end{pmatrix} \in \mathcal{M}_{n,n}(E)$$

where $u_i = ub_i$ for $i = 1 \dots n$. For any $u, v \in V$, we then have

$$(\mathbf{uv})_{\mathbf{b}} = M_{\mathbf{b}}(u) \cdot \mathbf{v_b}.$$

Suppose in addition that E is a differential field and that V is a differential extension field of E. Write $'$ for the derivation on E and V. We define the *derivation matrix* of $\mathbf{b}$ to be

$$D_{\mathbf{b}} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{d_{1b}} & \mathbf{d_{2b}} & \cdots & \mathbf{d_{nb}} \\ | & | & \cdots & | \end{pmatrix} \in \mathcal{M}_{n,n}(E)$$

where $d_i = b_i{}'$ for $i = 1 \dots n$. If we write $'$ for the pointwise derivation on E^n (i.e. $(a_1, \dots, a_n)' = (a_1{}', \dots, a_n{}')$), we then have

$$(\mathbf{u}')_{\mathbf{b}} = (\mathbf{u_b})' + D_{\mathbf{b}} \cdot \mathbf{u_b}$$

for any $u \in V$.

Let now E be a differential field of characteristic 0, and $K = E(y)$ where y is algebraic over E, and let n be the degree of the minimal polynomial for y over E. K is then a finite dimensional algebra over E of dimension n, and $\mathbf{y} = (1, y, y^2, \ldots, y^{n-1})$ is a basis for K over E. Let $f, g \in K$. For any $z \in K$ we have

$$(\mathbf{z}' + \mathbf{fz})_{\mathbf{y}} = (\mathbf{z}_{\mathbf{y}})' + (M_{\mathbf{y}}(f) + D_{\mathbf{y}}) \cdot \mathbf{z}_{\mathbf{y}}$$

hence, for any solution $z \in K$ of equation (R), $\mathbf{z}_{\mathbf{y}} \in E^n$ is a solution of

$$\mathbf{z}' + (D_{\mathbf{y}} + M_{\mathbf{y}}(f))\mathbf{z} = \mathbf{g}_{\mathbf{y}}. \qquad (S)$$

Conversely, for any solution $\mathbf{z} \in E^n$ of the system (S), $z = \mathbf{y} \cdot \mathbf{z}$ is a solution of equation (R) in K.

We now have a system of linear ordinary differential equations over E. Such a system can always be written in the form $M\mathbf{z} = \mathbf{g}_{\mathbf{y}}$ where M is an $n \times n$ matrix with entries in $E[D]$, the ring of linear ordinary differential operators over E. For a Risch differential equation, system (S) becomes

$$(D \cdot I_n + D_{\mathbf{y}} + M_{\mathbf{y}}(f))\mathbf{z} = \mathbf{g}_{\mathbf{y}} \qquad (T)$$

where I_n is the identity matrix in $\mathcal{M}_{n,n}(E[D])$.

For any m^{th}-order linear ordinary differential equation over K of the form

$$Lz = \sum_{i=0}^{m} f_i z^{(i)} = g$$

where $g, f_0, \ldots, f_m \in K$, this procedure can be used, yielding the equivalent system of coupled linear ordinary differential equations over E

$$\left(\sum_{i=0}^{m} M_{\mathbf{y}}(f_i) M_D^i\right) \mathbf{z} = \mathbf{g}_{\mathbf{y}}$$

where

$$M_D = D \cdot I_n + D_{\mathbf{y}} \quad \in \mathcal{M}_{n,n}(E[D]).$$

This is the explicit formula for what is done by Singer in the algebraic case of [12].

Example: Computing

$$\int \left(\frac{5x^4 + 2x - 2}{x^2}(1 + \frac{1}{\sqrt{x^3+1}}) + \frac{x}{\sqrt{x^3+1}}\right) e^{x\sqrt{x^3+1}} dx \qquad (1)$$

reduces to finding a solution in $\mathbf{Q}(x, \sqrt{x^3+1})$ of

$$z' + \frac{5x^3+2}{2\sqrt{x^3+1}} z = \frac{5x^4+2x-2}{x^2}(1 + \frac{1}{\sqrt{x^3+1}}) + \frac{x}{\sqrt{x^3+1}} \qquad (2)$$

Setting $E = \mathbf{Q}(x)$, $y = \sqrt{x^3+1}$ a root of $Y^2 - x^3 - 1 \in E[Y]$, and $K = E(y)$, we find that equation (2) is reduced to

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} + (D_{\mathbf{y}} + M_{\mathbf{y}}(\frac{5x^3+2}{2y})) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (5x^4+2x-2)/x^2 \\ (5x^3 - 4x^2 + 4x - 2)/(x^4 - x^3 + x^2) \end{pmatrix} \qquad (3)$$

where

$$D_{\mathbf{y}} = \frac{3x^2}{2x^3+2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$M_{\mathbf{y}}(\frac{5x^3+2}{2y}) = \frac{5x^3+2}{2} \begin{pmatrix} 0 & 1 \\ 1/(x^3+1) & 0 \end{pmatrix}.$$

3 Uncoupling the linear system

We now have a system of the form $M\mathbf{z} = \mathbf{g}_{\mathbf{y}}$ where $M \in \mathcal{M}_{n,n}(E[D])$. While $E[D]$ is in general a noncommutative ring, it has a left and right euclidean division algorithms, so we can compute left and right gcd's and lcm's in $E[D]$ ([8], ch.III, §9). Thus, we can diagonalize M using elementary row and column operations ([8], ch.III, §11), which yields an equivalent diagonal system of uncoupled linear ordinary differential equations of arbitrary order. The resulting system is equivalent to the initial one since all the elementary operations are invertible.

Example: In the previous example, the extended matrix over $\mathbf{Q}(x)[D]$ for the linear system (3) is

$$\left(\begin{array}{cc|c} D & \frac{5x^3+2}{2} & \frac{5x^4+2x-2}{x^2} \\ \frac{5x^3+2}{2x^3+2} & D + \frac{3x^2}{2x^3+2} & \frac{5x^3-4x^2+4x-2}{x^4-x^3+x^2} \end{array}\right).$$

Computing a left lcm of D and $(5x^3+2)/(2x^3+2)$ we find that $D = \mathrm{llcm}(D, (5x^3+2)/(2x^3+2)) = AD = B(5x^3+2)/(2x^3+2)$ where $A = 1$ and

$$B = \frac{2x^3+2}{5x^3+2} D - \frac{18x^2}{5x^3+2^2}.$$

So, multiplying the second row on the left by B and substracting from the first, we get the following matrix:

$$\left(\begin{array}{cc|c} 0 & L_2 & g_2 \\ \frac{5x^3+2}{2x^3+2} & D + \frac{3x^2}{2x^3+2} & \frac{5x^3-4x^2+4x-2}{x^4-x^3+x^2} \end{array}\right)$$

where

$$L_2 = -\frac{2x^3+2}{5x^3+2} D^2 - \frac{15x^5 - 12x^2}{25x^6 + 20x^3 + 4} D + \frac{125x^9 + 150x^6 + 30x^4 + 60x^3 - 24x + 8}{50x^6 + 40x^3 + 8}$$

and

$$g_2 = \frac{125x^{11} + 150x^8 + 20x^6 + 60x^5 - 104x^3 + 8x^2 - 16}{25x^9 + 20x^6 + 4x^3}.$$

Computing a left lcm of L_2 and $D + 3x^2/(2x^3+2)$ we find P and Q such that $PL_2 = Q(D + 3x^2/(2x^3+2))$, so multiplying the second row on the left by Q and substracting P times the first row, we get the two uncoupled equations $L_2(z_2) = g_2$ and $L_1(z_1) = g_1$ where

$$L_1 = -\frac{2}{5x^3+2}D^2 + \frac{15x^5+24x^2}{25x^9+45x^6+24x^3+4}D + \frac{5x^3+2}{2x^3+2} \tag{4}$$

and

$$g_1 = \frac{125x^{11}+150x^8-70x^6+60x^5-104x^3+8x^2-16}{25x^{12}+45x^9+24x^6+4x^3}. \tag{5}$$

We note that in this example, one can use the fact that there are nonzero entries of degree 0 in D in the matrix of the system in order to eliminate one of the variables: from the second row, one gets

$$z_1 = \frac{1}{5x^3+2}\left(\frac{10x^4+2x^3+4x-4}{x^2} - 2(x^3+1)z_2{}' - 3x^2 z_2\right)$$

and, substituting in the first equation, we get $L_2(z_2) = g_2$.

In general, for systems arising from Risch differential equations, the nondiagonal entries in the initial matrix all have degree 0 in D. Thus, either the system is already diagonal, or at least one of the variables can be eliminated before the diagonalization. This means in particular that for curves of degree 2 (i.e. generated by square roots), the system can always be decoupled as above by substitution without using the arithmetic in $E[D]$.

4 Bounding the denominator

The problem is now reduced to solving a single linear ordinary differential equation over a differential field E. We restrict E to be of the form $E = k(\theta)$ where k is a differential field of characteristic 0 and θ is a *Liouvillian monomial* over E, i.e. θ is transcendental over k, k and $k(\theta)$ have the same subfield of constants, and either $\theta' \in k$, or $\theta'/\theta \in k$. After normalizing the equation, we are looking for the solutions in $k(\theta)$ of

$$z^{(n)} + f_{n-1}z^{(n-1)} + \ldots + f_1 z' + f_0 z = g \tag{E}$$

where $g, f_0, \ldots, f_{n-1} \in k(\theta)$. If $\theta' \in k$, write g as B/E and each f_i as A_i/D_i where $B, E, A_i, D_i \in k[\theta]$, E and the D_i's are monic, and $(B, E) = (A_i, D_i) = (1)$. If $\theta'/\theta \in k$, write g as $B/E + C/\theta^e$ and each f_i as $A_i/D_i + C_i/\theta^{e_i}$ where B, E, A_i, D_i are as before and in addition, $(E, \theta) = (D_i, \theta) = (1)$. Let $D = \operatorname{lcm}(D_0, \ldots, D_{n-1})$. While the traditional algorithms require looking at each singularity of the equation separately (thus factoring D over k), it is possible to find a denominator for any solution of (E) in $k(\theta)$ using only rational operations, by computing a balanced factorisation of D.

Recall that for $P, Q \in k[\theta] \setminus \{0\}$, $\deg(P) > 0$, the *order* of Q at P, denoted $\nu_P(Q)$ is the unique integer m such that $P^m \mid Q$ and $P^{m+1} \nmid Q$. Furthermore, for $A, B \in k[\theta] \setminus \{0\}$, if $(A, B) = (1)$, we define $\nu_P(A/B)$ to be $\nu_P(A) - \nu_P(B)$. This allows ν_P to be well-defined on $k(\theta) \setminus \{0\}$. By convention, $\nu_P(0) = +\infty$.

Definition 1 *Let $A, B \in k[\theta]$. We say that A is* balanced w.r.t. B *if $\nu_P(B) = \nu_Q(B)$ for any two irreducible factors P and Q of A in $k[\theta]$. Let $\mathcal{S} \subseteq k[\theta]$. We say that $A = A_1{}^{e_1} \cdots A_m{}^{e_m}$ is a* balanced factorisation of A *w.r.t $\mathcal{S}$ if each A_i is squarefree and balanced w.r.t B for any $B \in \mathcal{S}$, and $(A_i, A_j) = (1)$ for $i \neq j$.*

For any $A \in k[\theta] \setminus \{0\}$ and any finite $\mathcal{S} \subset k[\theta]$, the following algorithm, which is a slightly modified presentation of Abramov's algorithm ([2, 3]), will compute a balanced factorisation of A w.r.t. $\mathcal{S}$.

balanced_fact$(A, \mathcal{S})$
INPUT:

- $A \in k[\theta] \setminus \{0\}$ monic,
- $\mathcal{S} \subset k[\theta]$ finite.

OUTPUT: $A = A_1{}^{e_1} \cdots A_m{}^{e_m}$, a balanced factorisation of A w.r.t $\mathcal{S}$.

- Let $A = A_1 A_2{}^2 \cdots A_m{}^m$ be a squarefree factorisation
- **for** $i = 1 \ldots m$ **do**
  $A_{i1} \cdots A_{im_i} \leftarrow$ **balanced_fact_sqfr**$(A_i, \mathcal{S})$
- return $\prod_{i=1}^m \prod_{j=1}^{m_i} A_{ij}{}^i$

balanced_fact_sqfr$(A, \mathcal{S})$
INPUT:

- $A \in k[\theta] \setminus \{0\}$ monic squarefree,
- $\mathcal{S} \subset k[\theta]$ finite.

OUTPUT: $A = A_1 \cdots A_m$, a balanced factorisation of A w.r.t $\mathcal{S}$.

- **if** $\mathcal{S} = \emptyset$ **then** return A
- Choose $B \in \mathcal{S}$.
- $\mathcal{S} \leftarrow \mathcal{S} \setminus \{B\}$
- $A_1 \cdots A_m \leftarrow$ **balanced_fact_sqfr1**(A, B)
- **for** $i = 1 \ldots m$ **do**
  $A_{i1} \cdots A_{im_i} \leftarrow$ **balanced_fact_sqfr**$(A_i, \mathcal{S})$
- return $\prod_{i=1}^m \prod_{j=1}^{m_i} A_{ij}$

balanced_fact_sqfr1(A, B)
INPUT:

- $A \in k[\theta] \setminus \{0\}$ monic squarefree,
- $B \in k[\theta]$

OUTPUT: $A = A_1 \cdots A_m$, a balanced factorisation of A w.r.t B.

- **if** $\deg(B) = 0$ **then** return A
- $G \leftarrow \gcd(A, B)$
- **if** $\deg(G) = 0$ **then** return A
- $G_1 \cdots G_m \leftarrow$ **balanced_fact_sqfr1**$(G, B/G^{\nu_G(B)})$
- return $(A/G)G_1 \cdots G_m$

Let $P \in k[\theta]$ be squarefree and $A, B \in k[\theta]$. If $(B, P) = (1)$ then B is invertible in $k[\theta]/(P)$ and B^{-1} can be computed by the extended Euclidean algorithm. In that case, we define $\phi_P(A/B)$ to be the remainder of AB^{-1} modulo P.

For convenience, we write equation (E) as $\sum_{i=0}^{n} f_i z^{(i)} = g$ where $f_n = 1$. Let t be an indeterminate. For each squarefree factor C in a balanced factorisation of D w.r.t. $\{A_0, \ldots, A_{n-1}, D_0, \ldots, D_{n-1}\}$ we need to compute the *associated indicial equation* (in $k[t]$) defined by:

$$\chi(C) = resultant_\theta(C, \sum_{\substack{0 \le i \le n \\ i - \nu_C(f_i) = \mu(C)}} \phi_C(\frac{f_i C'^i}{C^{\nu_C(f_i)}}) \prod_{j=0}^{i-1}(t-j))$$

where $\mu(C) = \max_{0 \le i \le n}(i - \nu_C(f_i))$. We can prove ([6]) that if C is squarefree and balanced w.r.t. the A_i's and D_i's, then $\chi(C)$ is well-defined and is not identically 0.

We also need to be able to find a lower bound on the negative integer roots of any $P \in k[t]$. When k is a finitely generated extension of the rational numbers $\mathbf{Q}$, such a bound can be computed easily: we first replace P by its squarefree part, then we replace all the transcendentals by arbitrary integers, and eliminate the algebraics by taking norms. This yields a polynomial $\tilde{P} \in \mathbf{Z}[t]$ such that any integer root of P is also a root of $\tilde{P}$. If $(\tilde{P}, t) = (1)$, any integer root of $\tilde{P}$ must divide the constant coefficient of $\tilde{P}$ so we get a finite number of candidates. If the constant coefficient is large, we can either factor $\tilde{P}$ over the integers and look at the linear factors, or repeat this procedure with another choice of integers for the transcendentals, thereby obtaining another polynomial in $\mathbf{Z}[t]$, and then take the gcd of the constant coefficients (or of the two polynomials and factor it over the integers). We write $\beta(P)$ for a lower bound on the negative integer roots of $P \in k[t]$. $\beta(P) = 0$ if P has no such roots.

We can now give a complete algorithm for finding the denominator of any solution of equation (E) in $k(\theta)$. Letting B, E, the A_i's and D_i's be as defined above:

- $D \leftarrow \mathrm{lcm}(D_0, \ldots, D_{n-1})$
- $G \leftarrow E/\gcd(E, dE/d\theta)$
- $H \leftarrow G/\gcd(G, D/\gcd(D, dD/d\theta))$
- **if** $H^{n+1} \nmid E$ **then** return "no solution"
- ${C_1}^{e_1} \cdots {C_q}^{e_q} \leftarrow$ **balanced_fact**$(D, \{A_0, D_0, \ldots, A_{n-1}, D_{n-1}\})$
- $H_1 \cdots H_m \leftarrow$ **balanced_fact_sqfr1**(H, E)
- **for** $i = 1 \ldots q$ **do** $d_i \leftarrow \max(0, -\beta(\chi(C_i)), -\nu_{C_i}(g) - \mu(C_i))$
- **for** $i = 1 \ldots m$ **do** $q_i \leftarrow \max(0, \nu_{H_i}(E) - n)$
- return ${C_1}^{d_1} \cdots {C_q}^{d_q} {H_1}^{q_1} \cdots {H_m}^{q_m}$

The following Theorem ([6]) justifies that algorithm.

Theorem 1 *If the above algorithm returns "no solution" then equation (E) has no solution in $k(\theta)$. Otherwise, it returns $T \in k[\theta]$ such that for any solution $z \in k(\theta)$ of equation (E), either*

(i) $zT \in k[\theta]$ if $\theta' \in k$, or

(ii) $zT \in k[\theta, \theta^{-1}]$ if $\theta'/\theta \in k$.

Example: After normalizing the equation $L_1(z_1) = g_1$ where L_1 and g_1 are given by (4) and (5), we get the equation

$$z_1'' - \frac{15x^5 + 24x^2}{10x^6 + 14x^3 + 4} z_1' - \frac{25x^6 - 20x^3 - 4}{4x^3 + 4} z_1 = - \frac{125x^{11} + 150x^8 - 70x^6 + 60x^5 + 104x^3 - 8x^2 + 16}{10x^9 + 14x^6 + 4x^3}. \tag{6}$$

We get $D = x^6 + 7x^3/5 + 2/5$, $E = x^9 + 7x^6/5 + 2x^3/5$, $G = x^7 + 7x^4/5 + 2x/5$, and $H = x$. After checking that $H^3 \mid E$, a balanced factorisation of D w.r.t. $\{A_0, A_1, D_0, D_1\}$ is $D = C_1 C_2 = (x^3 + 2/5)(x^3 + 1)$, and a balanced factorisation of H w.r.t. E is $H = H_1 = x$. We find that $\mu(x^3 + 2/5) = \mu(x^3 + 1) = 2$, and that the indicial equations are

$$\chi(x^3 + 2/5) = \frac{11664}{625}(t^6 - 6t^5 + 12t^4 - 8t^3)$$

and

$$\chi(x^3 + 1) = \frac{729}{8}(8t^6 - 12t^5 + 6t^4 - t^3).$$

Their squarefree parts are $t^2 - 2t$ and $2t^2 - t$ respectively, so we find that $\beta(x^3+2/5) = \beta(x^3+1) = 0$. We also have $\nu_{x^3+2/5}(g) = \nu_{x^3+1}(g) = -1$, so $d_1 = d_2 = 0$. Finally, $\nu_x(E) = 3$, so $q_1 = 1$, so the denominator returned by the algorithm is $T = x$. Hence, any solution of (6) in $\mathbf{Q}(x)$ must be of the form $z_1 = P/x$ where $P \in \mathbf{Q}[x]$.

5 Finding the numerator

We now further restrict E to be of the form $k(x)$, where k is the constant subfield of E, and $x \in E$ is a Liouvillian monomial over k such that $x' = 1$ (i.e. $' = d/dx$). This is the case we get when we start with a Risch differential equation whose coefficients are pure algebraic functions. By the algorithm of the previous section, we have computed $T \in k[x]$ such that if $z \in k(x)$ is a solution of equation (E), then $z = P/T$ for some $P \in k[x]$.

Recall that for $A, B \in k[x] \setminus \{0\}$, the *order of A/B at infinity*, denoted $\nu_\infty(A/B)$ is defined by $\deg(B) - \deg(A)$. By convention, $\nu_\infty(0) = +\infty$. Furthermore, if $\deg(B) \geq \deg(A)$, then the *value of A/B at infinity*, denoted $\phi_\infty(A/B)$ is defined by 0 if $\deg(B) > \deg(A)$, and the leading coefficient of A divided by the leading coefficient of B if $\deg(B) = \deg(A)$.

Let t be an indeterminate. We first compute the *indicial equation at infinity* (in $k[t]$) defined as follows:

$$\chi(\infty) = \sum_{\substack{0 \leq i \leq n \\ -i-\nu_\infty(f_i) = \mu(\infty)}} (-1)^i \phi_\infty(f_i x^{\nu_\infty(f_i)}) \prod_{j=0}^{i-1}(t+j)$$

where $\mu(\infty) = \max_{0 \leq i \leq n}(-i - \nu_\infty(f_i))$.

The following Theorem ([1, 3, 6]) gives an upper bound on the degree of the numerator of any solution of equation (E) in $k(x)$.

Theorem 2 *Let T be as in Theorem 1. Then, for any solution $z \in k(x)$ of equation (E), there exists $P \in k[x]$ such that $z = P/T$ and*

$$\deg(P) \leq \deg(T) + \max(n-1, -\beta(\chi(\infty)), -\mu(\infty) - \nu_\infty(g)).$$

We compute the upper bound N on $\deg(P)$ given by the Theorem, set $P = a_N x^N + \dots a_1 x + a_0$, where the a_i's are undetermined, and replace z by P/T in equation (E). This yields a system Δ of linear equations over k for the a_i's. If Δ has no solution in k^N, then equation (E) has no solution in $k(x)$, so equation (R) has no solution in $k(x, y)$. Otherwise, $z = (\sum_{i=0}^N a_i x^i)/T$ is a solution of equation (E) for any solution $(a_0, \dots, a_N)$ of Δ. A slightly different and more efficient approach can be found in [1, 3].

Example:
For equation (6), we find that $\mu(\infty) = 3$, and $\chi(\infty) = -25/4$, so $\beta(\chi(\infty)) = 0$. We also have $\nu_\infty(g) = -2$ and $T = x$, so the upper bound given by Theorem 2 is $N = \deg(T) + 1 = 2$. Setting $z_1 = (a_2 x^2 + a_1 x + a_0)/x$ and plugging into equation (E) we get a system for a_0, a_1, and a_2 which has the unique solution $(a_0, a_1, a_2) = (2, 0, 0)$. Hence the only solution in $\mathbf{Q}(x)$ of (6) is $z_1 = 2/x$. Using the same algorithm, we find that the only solution in $\mathbf{Q}(x)$ of $L_2(z_2) = g_2$ is $z_2 = 2/x$, so the only solution of (2) in $\mathbf{Q}(x, \sqrt{x^3+1})$ is

$$z = \frac{2}{x}(1 + \sqrt{x^3+1}),$$

so the integral given by (1) is $(2 + 2\sqrt{x^3+1})e^{x\sqrt{x^3+1}}/x$.

6 Conclusions

We have presented an effective rational algorithm for solving equation (R) when the coefficients are pure algebraic functions. This algorithm has been implemented in the computer algebra systems Maple and Scratchpad, and can be readily implemented in other systems. In conjunction with the existing "rational" algorithms for integration ([4, 11, 13]), one can now effectively integrate a transcendental elementary function over an algebraic curve. In the mixed elementary function case, our algorithm finds the denominator of any solution, thus simplifying the algorithms of [5, 9], to a point where Puiseux expansions are required only around infinity. That step is now the last computational bottleneck to a complete effective integration algorithm for elementary functions.

References

[1] S.A. Abramov, *Problems of Computer Algebra involved in the Search for Polynomial Solutions of Linear Differential and Difference Equations* (in russian), MGU Num. Math and Cybernetics, vol. 15, No.3, pp. 56–60, 1989.

[2] S.A. Abramov, *Rational Solutions of Linear Differential and Difference Equations with Polynomial Coefficients* (in russian), Journal of Computational Mathematics and Mathematical Physics, vol. 29, No.11, pp. 1611–1620, 1989.

[3] S.A. Abramov & K.Yu. Kvashenko, *Fast Algorithms for the Search of the Rational Solutions of Linear Differential Equations with Polynomial Coefficients*, these proceedings.

[4] M.Bronstein, *The Transcendental Risch Differential Equation*, Journal of Symbolic Computation, vol. 9, No.1, pp. 49–60, 1990.

[5] M.Bronstein, *Integration of Elementary Functions* Journal of Symbolic Computation, vol. 9, No.2, pp. 117–173, 1990.

[6] M.Bronstein, *On Solutions of Linear Ordinary Differential Equations in their Coefficient Field*, Research Report 152, Informatik, ETH Zürich, 1991.

[7] J.H. Davenport, *Intégration Algorithmique des Fonctions Elémentairement Transcendantes sur une Courbe Algébrique*, Annales de l'Institut Fourier, vol. 34, fasc.2, pp. 271–276, 1984.

[8] E.G.C. Poole, Introduction to the Theory of Linear Differential Equations, Dover Publications Inc., New York, 1960.

[9] R. Risch, *On the Integration of Elementary Functions which are built up using Algebraic Operations*, Report SP-2801/002/00, System Development Corp., Santa Monica, CA., 1968.

[10] R. Risch, *The Problem of Integration In Finite Terms*, Trans. Amer. Math. Soc., vol. 139, pp. 167–189, 1969.

[11] M. Rothstein, *A New Algorithm for the Integration of Exponential and Logarithmic Functions* in "Proceedings 1977 MACSYMA Users Conference", NASA Pub. CP-2012, pp. 263–274, 1977.

[12] M.F. Singer, *Liouvillian Solutions of Linear Differential Equations with Liouvillian Coefficients*, to appear in the Journal of Symbolic Computation.

[13] B. Trager, Integration of Algebraic Functions, Ph.D. thesis, Dpt. of EECS, Massachusetts Institute of Technology, 1984.