Examples of Risch Integration

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1 Logarithmic Extensions

1.1 Polynomial Part

Example 1.1:

$$\int \log(x)dx$$

The integral has integrand $f(\theta) = \theta$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta = \log(x))$. Then the integral is given by

$$\int \theta dx = A_0 + A_1\theta + A_2\theta^2 + \sum_i c_i \log(v_i), \tag{1}$$

where $A_0, A_1, A_2 \in \mathbb{Q}(x)$, $v_i \in \mathbb{Q}(x, \theta)$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of 1 we have

$$\theta = A_0' + A_1'\theta + A_1\theta' + A_2'\theta^2 + 2A_2\theta\theta' + \sum_i c_i \frac{v_i'}{v_i}.$$

Since the A_i's are independent of θ, the coefficients of θ^j $(j = 0, 1, 2)$ must be equal on both sides. Equating powers of θ we obtain the following system of differential equations.

$$0 = A_2' \tag{2}$$
$$1 = A_1' + 2A_2\theta' \tag{3}$$
$$0 = A_0' + A_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \tag{4}$$

What have we just done? We began with an integration problem and we seem to have converted it into solving a system of differential equations (DE's)! Surely this more difficult than the previous problem? The answer is no and reason is that the original problem required integrating in the field $\mathbb{Q}(x, \theta = \log(x))$, whereas the recursive integration required to solve the system of DE's is in the field $\mathbb{Q}(x)$ (with the exception of the final DE, where the possible inclusion of logarithmic terms are permitted by Liouville's theorem).

So let's solve this system of DE's, we begin with equation 3 where by recursive Risch integration we find $A_2 = a_2 \in \mathbb{Q}$ (of course in practice we do not use Risch integration, but in general this is the method used). Then we integrate equation 4 to find

$$\int 1dx = A_1 + 2a_2\theta$$

$$\Rightarrow x + a_1 = A_1 + 2a_2\theta$$

Given the constraints $A_1 \in \mathbb{Q}(x)$ and $a_1, a_2 \in \mathbb{Q}$ which we see that only $a_2 = 0$ and $A_1 = x + a_1$ is a solution. Then equation 4 becomes

$$\begin{aligned} 0 &= A_0' + A_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \\ &= A_0' + (x + a_1)\theta' + \sum_i c_i \frac{v_i'}{v_i} \\ &= A_0' + 1 + a_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \\ -1 &= A_0' + a_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \end{aligned}$$

Integrating the left hand side and using known properties of derivatives for the right hand side we find

$$\begin{aligned} \int -1dx &= A_0 + a_1\theta + \sum_i c_i \log(v_i) \\ -x + a_0 &= A_0 + a_1\theta + \sum_i c_i \log(v_i) \end{aligned}$$

From which we see the only solution to this equation is $a_1 = 0$, $c_i = 0$ $(i = 1, \cdots, n)$ and $A_0 = -x + a_0$, here a_0 is just the constant of integration which we will ignore. We have now determined A_1, A_2 and A_3 so the solution to the integral is

$$\begin{aligned} \int \log(x) &= A_0 + A_1\theta + A_2\theta^2 \\ &= A_0 + A_1 \log(x) + A_2 \log(x)^2 \\ &= x \log(x) - x. \end{aligned}$$

This is the standard result obtained from integration by parts or the substitution $u = \log(x)$. While using the Risch algorithm for this integral was more work than using a freshmen calculus method we will see that this algorithm can be easily applied to integrals which would be almost impossible to find using integration by parts or substitution.

Example 1.2:

$$\int x^5 \log(x)^2 dx$$

The integral has integrand $f(\theta) = x^5\theta^2$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta = \log(x))$. Then the general form of the integral is given by

$$\int x^5\theta^2 dx = A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3 + \sum_i c_i \log(v_i) \tag{5}$$

where $A_0, A_1, A_2, A_3 \in \mathbb{Q}(x)$, $v_i \in \mathbb{Q}(x, \theta)$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of equation 5 we have

$$x^5\theta^2 = A_0' + A_1'\theta + A_1\theta' + A_2'\theta^2 + 2A_2\theta\theta' + A_3'\theta^3 + 3A_3\theta^2\theta' + \sum_i c_i \frac{v_i'}{v_i}$$

Since the A_i's are independent of θ, the coefficients of θ^j $(j = 0, 1, 2, 3)$ must be equal on both sides. Equating powers of θ we obtain the following system of differential equations.

$$0 = A_3' \tag{6}$$
$$x^5 = A_2' + 3A_3\theta' \tag{7}$$
$$0 = A_1' + 2A_2\theta' \tag{8}$$
$$0 = A_0' + A_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \tag{9}$$

From equation 7 we have $A_3 = a_3 \in \mathbb{Q}$. Integrating equation 8 we find

$$\int x^5 dx = A_2 + 3a_3\theta$$
$$\Rightarrow \frac{x^6}{6} + a_2 = A_2 + 3a_3\theta$$

Given the constraints that $a_2 \in \mathbb{Q}$ and $A_2 \in \mathbb{Q}(x)$ we find $a_3 = 0$ and $A_2 = x^6/6 + a_2$. Now we inspect equation 9 and find

$$\begin{aligned} 0 &= A_1' + 2(\frac{x^6}{6} + a_2)\theta' \\ &= A_1' + \frac{x^6}{3}\theta' + 2a_2\theta' \\ \Rightarrow -\frac{x^6}{3}\theta' &= A_1' + 2a_2\theta' \\ \Rightarrow -\frac{x^5}{3} &= A_1' + 2a_2\theta' \end{aligned}$$

Integrating the left hand side and using known properties of derivatives for the right hand side we find

$$-\frac{x^6}{18} = A_1 + 2a_2\theta$$

From which we find $a_2 = 0$ and $A_1 = -x^6/18$. Finally we inspect equation 9 and find

$$\begin{aligned} 0 &= A_0' - \frac{x^6}{18}\theta' + \sum_i c_i \frac{v_i'}{v_i} \\ \Rightarrow \frac{x^5}{18} &= A_0' + \sum_i c_i \frac{v_i'}{v_i} \end{aligned}$$

Integrating the left hand side and using known properties of derivatives for the right hand side we find

$$\begin{aligned} \int \frac{x^5}{18} dx &= A_0 + \sum_i c_i \log(v_i) \\ \Rightarrow \frac{x^6}{108} + a_0 &= A_0 + \sum_i c_i \log(v_i) \end{aligned}$$

From which we find $c_i = 0$ $(i = 1, \cdots, n)$ and $A_0 = x^6/108 + a_0$, ignoring the constant of integration we have $A_0 = x^6/108$. Then the solution to the integral is given by

$$\begin{aligned}\int x^5 \log(x)^2 dx &= A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3 + \sum_i c_i \log(v_i) \\ &= \frac{x^6}{108} - \frac{x^6}{18}\log(x) + \frac{1}{6}\log(x)^2.\end{aligned}$$

Example 1.3:

$$\int \frac{\log(x)\log(\log(x)) + 1}{\log(x)} dx$$

This integral has integrand $f(\theta_2) = (\theta_1\theta_2 + 1)/\theta_1$, where we have constructed the elementary field extensions $\mathbb{Q}(x, \theta_1 = \log(x), \theta_2 = \log(\theta_1))$. Then the general form of the integral is given by

$$\int \frac{\theta_1\theta_2 + 1}{\theta_1} dx = A_0 + A_1\theta_2 + A_2\theta_2^2 + \sum_i c_i \log(v_i), \tag{10}$$

where $A_0, A_1, A_2 \in \mathbb{Q}(x, \theta_1)$, $v_i \in \mathbb{Q}(x, \theta_1, \theta_2)$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of 10 we have

$$\theta_2 + \frac{1}{\theta_1} = A_0' + A_1'\theta_2 + A_1\theta_2' + A_2'\theta_2^2 + 2A_2\theta_2\theta_2' + \sum_i c_i \frac{v_i'}{v_i}.$$

Since the A_i's are independent of θ_2, the coefficients of θ_2^j $(j = 0, 1, 2)$ must be equal on both sides. Equating powers of θ_2 we obtain the following system of differential equations.

$$0 = A_2' \tag{11}$$

$$1 = A_1' + A_2\theta_2' \tag{12}$$

$$\frac{1}{\theta_1} = A_0' + A_1\theta_2' + \sum_i c_i \frac{v_i'}{v_i} \tag{13}$$

From 12 we have $A_2 = a_2 \in \mathbb{Q}$. Integrating both sides of 13 using a recursive call to the Risch algorithm we find

$$\begin{aligned}\int 1 dx &= A_1 + 2A_2\theta_2 \\ \Rightarrow \qquad x + a_1 &= A_1 + 2a_2\theta_2\end{aligned}$$

From which we find $a_2 = 0$ and $A_1 = x + a_1$. Finally we inspect 13 and find

$$\begin{aligned}\frac{1}{\theta_1} &= A_0' + A_1\theta_2' + \sum_i c_i \frac{v_i'}{v_i} \\ &= A_0' + \frac{x + a_1}{x\theta_1} + \sum_i c_i \frac{v_i'}{v_i}\end{aligned}$$

which has a solution when $a_1 = 0$, all the $c_i = 0$ $(i = 1, \cdots, n)$ and $A_0' = 0$ so $A_0 = a_0 \in \mathbb{Q}$. Then the integral is given by

$$\begin{aligned}\int \frac{\log(x)\log(\log(x)) + 1}{\log(x)} dx &= A_0 + A_1\theta_2 + A_2\theta_2^2 + \sum_i c_i \log(v_i) \\ &= x\theta_2 \\ &= x\log(\log(x)).\end{aligned}$$

Example 1.4:

$$\int \frac{(x-1)\log(x)^3 + (x+1)\log(x)^2 - 1}{x\log(x)} dx$$

This integral has integrand $f(\theta) = \left(\frac{x-1}{x}\right)\theta^2 + \left(\frac{x+1}{x}\right)\theta - \frac{1}{x\theta}$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta = \log(x))$. Then the general form of the solution is given by

$$\int \left(\frac{x-1}{x}\right)\theta^2 + \left(\frac{x+1}{x}\right)\theta - \frac{1}{x\theta} dx = A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3 + \sum_i c_i \log(v_i) \tag{14}$$

where $A_0, A_1, A_2, A_3 \in \mathbb{Q}(x)$, $v_i \in \mathbb{Q}(x, \theta = \log(x))$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of equation 14 we find

$$\left(\frac{x-1}{x}\right)\theta^2 + \left(\frac{x+1}{x}\right)\theta - \frac{1}{x\theta} = A_0' + A_1'\theta + A_1\theta' + A_2'\theta^2 + 2A_2\theta\theta' + A_3'\theta^3 + 3A_3\theta^2\theta' + \sum_i c_i \frac{v_i'}{v_i}$$

Since the A_i's are independent of θ, the coefficients of θ^j $(j = 0, 1, 2, 3)$ must be equal on both sides. Equating powers of θ we obtain the following system of differential equations.

$$0 = A_3' \tag{15}$$

$$\frac{x-1}{x} = A_2' + 3A_3\theta' \tag{16}$$

$$\frac{x+1}{x} = A_1' + 2A_2\theta' \tag{17}$$

$$-\frac{1}{x\theta} = A_0' + A_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \tag{18}$$

From equation 16 we have $A_3 = a_3 \in \mathbb{Q}$. Integrating the left hand side of equation 17 and using known properties of derivatives for the right hand side we find

$$\begin{aligned}\int \frac{x-1}{x} dx &= A_2 + 3a_3\theta \\ \Rightarrow x - \log(x) + a_2 &= A_2 + 3a_3\theta\end{aligned}$$

From which the only solution is $a_3 = -1/3$ and $A_2 = x + a_2$. Inspecting equation 18 we find

$$\begin{aligned}\frac{x+1}{x} &= A_1' + 2A_2\theta' \\ &= A_1' + 2\frac{x + a_2}{x} \\ \Rightarrow \frac{1}{x} - 1 &= A_1' + \frac{2a_2}{x}\end{aligned}$$

Integrating as before we find

$$\int \frac{1}{x} - 1 dx = A_1 + 2a_2 \log(x)$$
$$\Rightarrow \log(x) - x + a_1 = A_1 + 2a_2 \log(x)$$

Here the only solution is $a_2 = 1/2$ and $A_1 = a_1 - x$. Finally, inspecting equation 18 we find

$$-\frac{1}{x\theta} = A_0' + \frac{a_1}{x} - 1 + \sum_i c_i \frac{v_i'}{v_i}$$
$$\Rightarrow 1 - \frac{1}{x\theta} = A_0' + \frac{a_1}{x} + \sum_i c_i \frac{v_i'}{v_i}$$

Integrating again we find

$$\int 1 - \frac{1}{x\theta} dx = A_0 + a_1 \log(x) + \sum_i c_i \log(v_i)$$
$$\Rightarrow x - \log(\log(x)) + a_0 = A_0 + a_1 \log(x) + \sum_i c_i \log(v_i)$$

Here the only solution is $a_1 = 0$, $A_0 = x + a_0$ (a_0 is the constant of integration), $c_1 = -1$ and $v_1 = \log(x)$ (recall that the v_i's can take elements from the same field as the original integral) with all other $c_i, v_i = 0$. The integral $1/(x\theta)$ was calculated using techniques from the rational part of the logarithmic extension which were/are covered in XXXX. So the complete solution is given by

$$\int \frac{(x-1)\log(x)^3 + (x+1)\log(x)^2 - 1}{x \log(x)} dx =$$
$$x - x\log(x) + \left(x + \frac{1}{2}\right) \log(x)^2 - \frac{1}{3}\log(x)^2 - \log(\log(x))$$

Example 1.5:

$$\int \frac{\log(x)^3 + (x+1)\log(x) - x - 1}{(x+1)\log(x)^2} dx$$

So far we have used the Risch algorithm to find closed form solutions to integrals, we can also easily show that an integral has no elementary solution (an otherwise difficult task!). We need to know a couple of results from Risch (XXXX) to proceed.

- If at any step in solving the system of DE's the integral to be computed is not elementary then the original integral is not elementary.
- If at any step, except the last, the result of recursive integration introduces any terms that are not in $\mathbb{Q}_{n-1}$ (where the original integrand is in the field $\mathbb{Q}_n$) then the original integral is not elementary. At the last step new logarithmic extensions may appear.

The integral has integrand $f(\theta) = \left(\frac{1}{x+1}\right)\theta + \frac{1}{\theta} - \frac{1}{\theta^2}$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta = \log(x))$. Then the general form of the solution is given by

$$\int \left(\frac{1}{x+1}\right)\theta + \frac{1}{\theta} - \frac{1}{\theta^2} dx = A_0 + A_1\theta + A_2\theta^2 + \sum_i c_i \log(v_i) \tag{19}$$

where $A_0, A_1, A_1 \in \mathbb{Q}(x)$, $v_i \in \mathbb{Q}(x, \theta = \log(x))$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of 19 we find

$$\left(\frac{1}{x+1}\right)\theta + \frac{1}{\theta} - \frac{1}{\theta^2} = A_0' + A_1'\theta + A_1\theta' + A_2'\theta^2 + 2A_2\theta\theta' + \sum_i c_i \frac{v_i'}{v_i}.$$

Equating powers of θ we obtain the following system of differential equations.

$$0 = A_2' \tag{20}$$

$$\frac{1}{x+1} = A_1' + 2A_2\theta' \tag{21}$$

$$\frac{1}{\theta} - \frac{1}{\theta^2} = A_0' + A_1\theta' + \sum_i c_i \frac{v_i'}{v_i} \tag{22}$$

From equation 21 we find $A_2 = a_2 \in \mathbb{Q}$. Integrating the left hand side of equation 22 and using known properties of derivatives for the right hand side we find

$$\begin{aligned} \int \frac{1}{x+1} dx &= A_1 + 2A_2\theta \\ \Rightarrow \log(x+1) + a_1 &= A_1 + 2A_2\theta \end{aligned}$$

Now we must find a constant a_1 and a rational function A_1 such that the equation is satisfied. This is clearly impossible since $\log(x)$ and $\log(x+1)$ are linearly independent functions, thus the original integral is not elementary (we also know the original integral is not elementary because integrating $1/(x+1)$ introduced terms from $\mathbb{Q}(x, \log(x+1))$ and this recursive integration can only result in function from $\mathbb{Q}(x)$).

1.2 Rational Part

Example 1.1:

$$\int \frac{1}{x\log(x)} dx$$

We had to calculate this integral as a part of example XXXX, we proceed as folllows. Extend the field of integration to $\mathbb{Q}(x, \theta = \log(x))$ and $\log(x)$ is monomial over $\mathbb{Q}(x)$. Then the degree of the numerator is less than the degree of the denominator (with respect to θ) and the denominator is squarefree, so the Rothstein-Trager algorithm is applicable. The general form of the integral is given by

$$\int \frac{1}{x\theta} dx = \sum_i c_i \log(v_i).$$

Define the numerator and denominator of the integral to be

$$
\begin{aligned}
a &= 1 \\
b &= x\theta.
\end{aligned}
$$

Then $b' = 1 + \theta$ and

$$
\begin{aligned}
R(z) &= \operatorname{res}_\theta(a - zb', b) \\
&= x(z - 1).
\end{aligned}
$$

Since the one root of the resultant equation is a constant the integral is elementary! Then $c_1 = 1$ and the corresponding v_1 is found via a gcd computation

$$
\begin{aligned}
v_1 &= \gcd(a - c_1 b', b) \\
&= \theta.
\end{aligned}
$$

Then the integral is given by

$$
\begin{aligned}
\int \frac{1}{x \log(x)} dx &= \sum_i c_i \log(v_i) \\
&= \log(\theta) \\
&= \log(\log(x)).
\end{aligned}
$$

Example 1.2:

$$
\int \frac{1}{x \log(x)^4 - x} dx
$$

The integral has integrand $f(\theta) = 1/(x\theta^4 - x)$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta = \log(x))$. Then the integral is given by

$$
\int \frac{1}{x\theta^4 - x} dx = \sum_i c_i \log(v_i).
$$

We now are faced with an integral where the degree of the numerator (with respect to θ) is less than the degree of the denominator and the denominator is squarefree. Thus the Rothstein-Trager algorithm is applicable. Define

$$
\begin{aligned}
a &= 1 \\
b &= x\theta^4 - x.
\end{aligned}
$$

Then $b' = \theta^4 + 4\theta^3 - 1$ and

$$
\begin{aligned}
R(z) &= \operatorname{res}_\theta\left(a - zb', b\right) \\
&= x^4 - 256x^4 z^4
\end{aligned}
$$

From inspection we see that the roots of $R(z)$ are constants so the integral is elementary. Computing these roots we find the c_i's

$$\{c_1, c_2, c_3, c_4\} = \left\{-\frac{1}{4}, -\frac{i}{4}, \frac{i}{4}, \frac{1}{4}\right\}.$$

The corresponding v_i's are found by computing

$$v_i = \gcd(a - c_i b', b)$$

for each c_i $(i = 1, 2, 3, 4)$, which are found to be

$$\{v_1, v_2, v_3, v_4\} = \left\{\frac{\theta}{4} + \frac{1}{4}, \frac{\theta}{4} + \frac{i}{4}, \frac{\theta}{4} - \frac{i}{4}, \frac{\theta}{4} - \frac{1}{4}\right\}.$$

Then the integral is given by

$$\begin{aligned}\int \frac{1}{x\theta^4 - x} dx &= \sum_i c_i \log(v_i) \\ &= \frac{1}{4}\log\left(\frac{\theta}{4} - \frac{1}{4}\right) + \frac{1}{4} i \log\left(\frac{\theta}{4} - \frac{i}{4}\right) - \frac{1}{4} i \log\left(\frac{\theta}{4} + \frac{i}{4}\right) - \frac{1}{4}\log\left(\frac{\theta}{4} + \frac{1}{4}\right).\end{aligned}$$

Using the Rioboo [XXXX] algorithm for converting complex logarithms into real arctangents we find

$$\int \frac{1}{x\theta^4 - x} dx = \frac{1}{4}\log\left(\frac{\theta - 1}{\theta + 1}\right) - \frac{1}{2}\tan^{-1}(\theta).$$

So the original integral is given by

$$\int \frac{1}{x\log(x)^4 - x} dx = \frac{1}{4}\log\left(\frac{\log(x) - 1}{\log(x) + 1}\right) - \frac{1}{2}\tan^{-1}(\log(x))$$

and expresses the solution in the minimal algebraic extension field.

2 Exponential Extensions

Example 2.1:

$$\int \left(1 - \frac{1}{x\log(x)^2}\right) \exp\left(\frac{1}{\log(x)} + x\right) dx$$

The integral has integrand $f(\theta_2) = (1 - 1/(x\theta_1^2))\theta_2$, where we have constructed the elementary field extension $\mathbb{Q}(x, \theta_1 = \log(x), \theta_2 = \exp(x + 1/\theta_1))$. We have $\deg(f(\theta_2)) = 1$, then it follows that the solution will have degree 1, since differentiating an exponential does not decrease its degree. So the general form of the solution is given by

$$\int \left(1 - \frac{1}{x\theta_1^2}\right) \theta_2 dx = A_0 + A_1\theta_2 + \sum_i c_i \log(v_i). \tag{23}$$

where $A_0, A_1 \in \mathbb{Q}(x, \theta_1)$, $v_i \in \mathbb{Q}(x, \theta_1, \theta_2)$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of 23 we find

$$\begin{aligned}\left(1 - \frac{1}{x\theta_1^2}\right)\theta_2 &= A_0' + A_1'\theta_2 + A_1\theta_2' + \sum_i c_i \frac{v_i'}{v_i} \\ &= A_0' + A_1'\theta_2 + \left(1 - \frac{1}{x\theta_1^2}\right)\theta_2 A_1 + \sum_i c_i \frac{v_i'}{v_i}.\end{aligned}$$

Equating powers of θ_2 on both sides we find

$$0 \quad = \quad A_0' \tag{24}$$

$$1 - \frac{1}{x\theta_1^2} \quad = \quad A_1' + \left(1 - \frac{1}{x\theta_1^2}\right) A_1 + \sum_i c_i \frac{v_i'}{v_i}. \tag{25}$$

From 25, we have $A_0 = a_0 \in \mathbb{Q}$ (this is the constant of integration!). From 25 we have $0 = \sum_i c_i v_i'/v_i$ implying $c_i = 0$ for all i, and

$$1 - \frac{1}{x\theta_1^2} = A_1' + \left(1 - \frac{1}{x\theta_1^2}\right) A_1$$

which is a Risch differential equation (RDE). These are, in general, very difficult to solve and we will see harder RDE's in other examples, but here we have the trivial solution $A_1 = 1$. So the solution (ignoring the constant of integration) to the integral is

$$\begin{aligned}\int \left(1 - \frac{1}{x\log(x)^2}\right)\exp\left(\frac{1}{\log(x)} + x\right) dx &= A_0 + A_1\theta_2 + \sum_i c_i \log(v_i) \\ &= \theta_2 \\ &= \exp\left(x + \frac{1}{\log(x)}\right).\end{aligned}$$

Example 2.2:

$$\int \frac{e^{1/\log(x)}}{x\log(x)^3} dx$$

The integral has integrand $f(\theta_2) = \theta_2/(x\theta_1^3)$, where we have constructed the elementary field extension $\mathbb{Q}\left(x, \theta_1 = \log(x), \theta_2 = e^{1/\theta_1}\right)$. Then the general form of the solution is given by

$$\int \frac{\theta_2}{x\theta_1^3} dx = A_0 + A_1\theta_2 + \sum_i c_i \log(v_i) \tag{26}$$

where $A_0, A_1 \in \mathbb{Q}(x, \theta_1)$, $v_i \in \mathbb{Q}\left(x, \theta_1, \theta_2\right)$ and $c_i \in \mathbb{R}$ are undetermined. Differentiating both sides of 26 we find

$$\begin{aligned}\frac{\theta_2}{x\theta_1^3} &= A_0' + A_1'\theta_2 + A_1\theta_2' + \sum_i c_i \frac{v_i'}{v_i} \\ &= A_0' + A_1'\theta_2 - \left(\frac{\theta_2}{x\log(x)^2}\right) A_1 + \sum_i c_i \frac{v_i'}{v_i}\end{aligned}$$

Equating coefficients of powers of θ_2 on both sides yields the system of differential equations

$$\begin{aligned} 0 &= A_0' && (27) \\ \frac{1}{x\theta_1^3} &= A_1' - \frac{A_1}{x\log(x)^2} + \sum_i c_i \frac{v_i'}{v_i} && (28) \end{aligned}$$

From equation 28 we have $A_0 = \int 0dx = a_0 \in \mathbb{Q}$ (again, this is the constant of integration). Inspecting equation 28 we see that $0 = \sum_i c_i v_i'/v_i$, implying that $c_i = 0$ for all i. From 28 we also have

$$\frac{1}{x\theta_1^3} = A_1' - \frac{A_1}{x\log(x)^2}$$

which is a RDE. Unlike the RDE we saw found in example 2.1, the solution is not obvious. [GIVE A COMPLETE DERIVATION OF THE SOLUTION!!] So we have the solution $A_1 = (\theta_1 - 1)/\theta_1$. So the solution to our integral is given by

$$\begin{aligned} \int \frac{e^{1/\log(x)}}{x\log(x)^3}dx &= A_0 + A_1\theta_2 + \sum_i c_i \log(v_i) \\ &= A_1 e^{1/\log(x)} \\ &= \left(\frac{\log(x)-1}{\log(x)}\right) e^{1/\log(x)} \end{aligned}$$