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Differential algebra

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In mathematics, **differential rings**, **differential fields**, and **differential algebras** are rings, fields, and algebras equipped with finitely many derivations, which are unary functions that are linear and satisfy the Leibniz product rule. A natural example of a differential field is the field of rational functions in one variable over the complex numbers, $\mathbb{C}(t)$, where the derivation is differentiation with respect to t.

Differential algebra refers also to the area of mathematics consisting in the study of these algebraic objects and their use in the algebraic study of differential equations. Differential algebra was introduced by Joseph Ritt in 1950.[1]

Contents

Open problems

The biggest open problems in the field include the Kolchin catenary conjecture, the Ritt problem, and the Jacobi bound problem. All of these deal with the structure of differential ideals in differential rings.

Differential ring

A *differential ring* is a ring R equipped with one or more *derivations*, which are homomorphisms of additive groups

$$\partial\colon R \to R$$

such that each derivation ∂ satisfies the Leibniz product rule

$$\partial(r_1 r_2) = (\partial r_1) r_2 + r_1 (\partial r_2),$$

for every $r_1, r_2 \in R$. Note that the ring could be noncommutative, so the somewhat standard $d(xy) = xdy + ydx$ form of the product rule in commutative settings may be false. If $M\colon R \times R \to R$ is multiplication on the ring, the product rule is the identity

$$\partial \circ M = M \circ (\partial \times \mathrm{id}) + M \circ (\mathrm{id} \times \partial).$$

where $f \times g$ means the function which maps a pair (x, y) to the pair $(f(x), g(y))$.

Note that a differential ring is a (not necessarily graded) $\mathbb{Z}$-differential algebra.

Differential field

A differential field is a commutative field K equipped with derivations.

The well-known formula for differentiating fractions

$$\partial\left(\frac{u}{v}\right) = \frac{\partial(u)\, v - u\, \partial(v)}{v^2}$$

follows from the product rule. Indeed, we must have

$$\partial\left(\frac{u}{v} \times v\right) = \partial(u)$$

By the product rule,

$$\partial\left(\frac{u}{v}\right) v + \frac{u}{v}\, \partial(v) = \partial(u).$$

Solving with respect to $\partial(u/v)$, we obtain the sought identity.

If K is a differential field then *the field of constants* of K is $k = \{u \in K : \partial(u) = 0\}$.

A differential algebra over a field K is a K-algebra A wherein the derivation(s) commutes with the scalar multiplication. That is, for all $k \in K$ and $x \in A$,

$$\partial(kx) = k\partial x.$$

If $\eta : K \to Z(A)$ is the ring homomorphism to the center of A defining scalar multiplication on the algebra, one has

$$\partial \circ M \circ (\eta \times \mathrm{Id}) = M \circ (\eta \times \partial).$$

As above, the derivation must obey the Leibniz rule over the algebra multiplication, and must be linear over addition. Thus, for all $a, b \in K$ and $x, y \in A$

$$\partial(xy) = (\partial x)y + x(\partial y)$$

and

$$\partial(ax + by) = a\,\partial x + b\,\partial y.$$

Derivation on a Lie algebra

A derivation on a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz rule:

$$D([a, b]) = [a, D(b)] + [D(a), b].$$

For any $a \in \mathfrak{g}$, $\mathrm{ad}(a)$ is a derivation on $\mathfrak{g}$, which follows from the Jacobi identity. Any such derivation is called an **inner derivation**. This derivation extends to the universal enveloping algebra of the Lie algebra.

Examples

If A is a unital algebra, then $\partial(1) = 0$ since $\partial(1) = \partial(1 \times 1) = \partial(1) + \partial(1)$. For example, in a differential field of characteristic zero K, the rationals are always a subfield of the field of constants of K.

Any ring is a differential ring with respect to the trivial derivation which maps any ring element to zero.

The field $\mathbb{Q}(t)$ has a unique structure as a differential field, determined by setting $\partial(t) = 1$: the field axioms along with the axioms for derivations ensure that the derivation is differentiation with respect to t. For example, by commutativity of multiplication and the Leibniz law one has that $\partial\left(u^2\right) = u\partial(u) + \partial(u)u = 2u\partial(u)$.

The differential field $\mathbb{Q}(t)$ fails to have a solution to the differential equation

$$\partial(u) = u$$

but expands to a larger differential field including the function e^t which does have a solution to this equation. A differential field with solutions to all systems of differential equations is called a differentially closed field. Such fields exist, although they do not appear as natural algebraic or geometric objects. All differential fields (of bounded cardinality) embed into a large differentially closed field. Differential fields are the objects of study in differential Galois theory.

Naturally occurring examples of derivations are partial derivatives, Lie derivatives, the Pincherle derivative, and the commutator with respect to an element of an algebra.

Weyl Algebra

Every differential ring (R, ∂) has a naturally associated Weyl algebra $R[\partial]$, which is a noncommutative ring where $r \in R$ and ∂ satisfy the relation $\partial r = r\partial + \partial(r)$.

Such $R[\partial]$ modules are called D-modules. In particular R itself is a $R[\partial]$-module. All ∂-ideals in R are $R[\partial]$-submodule.

For a differential rings R there is an embedding of the Weyl algebra in the ring of pseudodifferential operators $R((\partial^{-1}))$ as the finite tails of these infinite series.

Ring of pseudo-differential operators

In this ring we work with $\xi = \partial^{-1}$ which is a stand-in for the integral operator. Differential rings and differential algebras are often studied by means of the ring of pseudo-differential operators on them.

This is the set of formal infinite sums

$$\left\{ \sum_{n \ll \infty} r_n \xi^n \mid r_n \in R \right\},$$

where $n \ll \infty$ means that the sum runs on all integers that are not greater than a fixed (finite) value.

This set is made a ring with the multiplication defined by linearly extending the following formula for "monomials":

$$\left(r\xi^m\right)\left(s\xi^n\right) = \sum_{k=0}^{\infty} r\left(\partial^k s\right)\binom{m}{k}\xi^{m+n-k},$$

where $\binom{m}{k} = \frac{m(m-1)\ldots(m-k+1)}{k!}$ is the binomial coefficient. (If $m > 0,$ the sum is finite, as the terms with $k > m$ are all equal to zero.) In particular, one has

$$\xi^{-1} s = \sum_{k=0}^{\infty} (-1)^k \left(\partial^k s\right) \xi^{-1-k}$$

for $r = 1, m = -1,$ and $n = 0,$ and using the identity $\binom{-1}{k} = (-1)^k.$

See also

- Arithmetic derivative – Function defined on integers in number theory
- Difference algebra
- Differential algebraic geometry
- Differential calculus over commutative algebras – part of commutative algebra
- Differential Galois theory – Study of Galois symmetry groups of differential fields
- Differentially closed field

- Differential graded algebra – differential associative algebra with integer grading in which the differential has grading +1 (cohomological convention) or −1 (homological convention) − a differential algebra with an additional grading.
- D-module – module over a sheaf of differential operators − an algebraic structure with several differential operators acting on it.
- Hardy field
- Kähler differential – Differential form in commutative algebra
- Liouville's theorem (differential algebra) – Says when antiderivatives of elementary functions can expressed as elementary functions
- Picard–Vessiot theory – Study of differential field extensions induced by linear differential equations

References

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External links

- David Marker's home page (http://www.math.uic.edu/~marker/) has several online surveys discussing differential fields.