Tutorial T6

Example T6.1: Suppose a normal rv X is such that

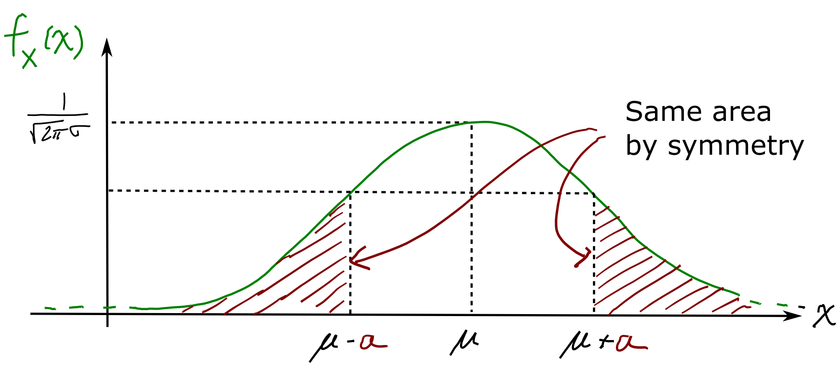
$$P[X<5]=P[X>15]=\frac{1}{2}P[X<10].$$

Find the distribution of X.

Solution: First, note that since the pdf of a normal rv X is

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{(x-\mu)^2}{2\sigma^2}},$$

the pdf is symmetric about its mean μ.



So, therefore, if μ is the mean of X, then

$$P[X<\mu-a]=P[X>\mu+a]$$

So therefore:

$\mu-a=5$
$\mu+a=15$

and hence $\mu=10$.

Also, $P[X<10]=P[X<\mu]=1/2$. Therefore $P[X>15]=\frac{1}{2}P[X<10]=1/4$.

Finally,

$$\begin{aligned}P[X>15]&=P\left[\frac{X-\mu}{\sigma}>\frac{15-\mu}{\sigma}\right]\\&=1-\Phi\left(\frac{15-\mu}{\sigma}\right)\\&=0.25\end{aligned}$$

So, $\Phi(\frac{15-\mu}{\sigma})=0.75$. Using the Φ table, $\frac{15-\mu}{\sigma}=0.67$

Since $\mu=10$, then $\sigma=7.46$

$X\sim\mathcal{N}(10,7.46^2)$

Example T6.2: Let Y be a geometric rv with parameter p.

a) Show that

$$P[Y>a+b\mid Y>b]=P[Y>a]$$

b) Why is your answer in a) not surprising?

Solution: Here,

$$p_Y(k)=\begin{cases}(1-p)^{k-1}p & k\geq 1\\ 0 & \text{else}\end{cases}$$

and

$$\begin{aligned}P[Y>a+b\mid Y>b]&=\frac{P[Y>a+b,Y>b]}{P[Y>b]}\\&=\frac{P[Y>a+b]}{P[Y>b]}\\&=\frac{1-P[Y\leq a+b]}{1-P[Y\leq b]}\\&=\frac{1-(1-(1-p)^{a+b})}{1-(1-(1-p)^b)}\\&=\frac{(1-p)^{a+b}}{(1-p)^b}\\&=(1-p)^a\\&=1-(1-(1-p)^a)\\&=1-P[Y\leq a]\\&=P[Y>a]\end{aligned}$$

b) Recall that a geometric rv with parameter p is the number of independent coin flips until the first 'heads', when the prob of 'heads' is p.

$Y>b$ tells us that the first b flips are tails. Since the flips are independent, the number of flips to go after the first b tails is the same whether i) I know that I have had b tails, or ii) I reset my count of the number of tails to 0.

Example T6.3: Let $X\sim\mathcal{N}(0,1)$ and $Z=\sqrt{|X|}$. What is the pdf of Z?

Solution: Note that Z is non-negative, so $P[Z\leq a]=0$ for $a<0$. That leaves the case of $a\geq 0$:

$$\begin{aligned}P[Z\leq a]&=P[\sqrt{|X|}\leq a]\\&=P[|X|\leq a^2]\\&=P[-a^2\leq X\leq a^2]\\&=1-P[X<-a^2]-P[X>a^2]\\&=1-2P[X<-a^2]\\&=1-2P[X\leq -a^2]\\&=1-2\Phi(-a^2)\end{aligned}$$

Therefore

$$\begin{aligned}f_Z(a)&=\frac{d}{da}P[Z\leq a]\\&=\frac{d}{da}\left(1-2\Phi(-a^2)\right)\\&=\frac{d}{da}\left(1-2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a^2}e^{-u^2/2}du\right)\\&=\frac{-2}{\sqrt{2\pi}}e^{-a^4/2}\times -2a\\&=\frac{2\sqrt{2}a}{\sqrt{\pi}}e^{-a^4/2}\end{aligned}$$

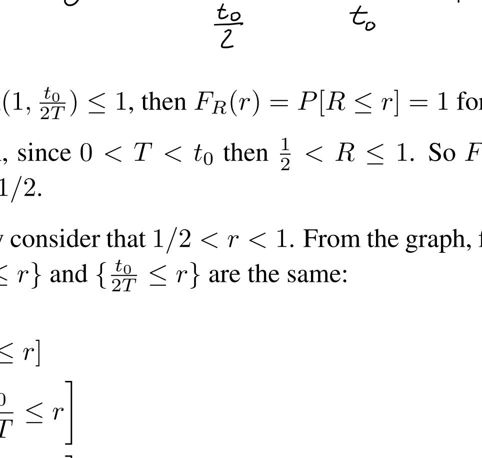
Example T6.4: The time T for a server to process a job has pdf

$$f_T(t)=\begin{cases}\frac{1}{t_0} & 0<t<t_0\\ 0 & \text{else}\end{cases}$$

The revenue from processing the job is $R=\min(1,\frac{t_0}{2T})$.

What is the cdf of R?

Solution: Lets first plot the relationship between T and R:



Since $R=\min(1,\frac{t_0}{2T})\leq 1$, then $F_R(r)=P[R\leq r]=1$ for $r\geq 1$.

From the graph, since $0<T<t_0$ then $\frac{1}{2}<R\leq 1$. So $F_R(r)=P[R\leq r]=0$ for $r\leq 1/2$.

So we now only consider that $1/2<r<1$. From the graph, for $1/2<r<1$ the events $\{R\leq r\}$ and $\{\frac{t_0}{2T}\leq r\}$ are the same:

$$\begin{aligned}F_R(r)&=P[R\leq r]\\&=P\left[\frac{t_0}{2T}\leq r\right]\\&=P\left[T\geq\frac{t_0}{2r}\right]\\&=1-P\left[T<\frac{t_0}{2r}\right]\\&=1-P\left[T\leq\frac{t_0}{2r}\right]\end{aligned}$$

Note that

$$\begin{aligned}P[T\leq t]&=\int_{-\infty}^{t}f_T(u)\,du\\&=\begin{cases}0 & t\leq 0\\ t/t_0 & 0<t<t_0\\ 1 & t_0\leq t\end{cases}\end{aligned}$$

So, for $\frac{1}{2}<r<1$ we have $1<\frac{1}{r}<2$ and $\frac{t_0}{2}<\frac{t_0}{2r}<t_0$. Therefore

$$\begin{aligned}F_R(r)&=1-P\left[T\leq\frac{t_0}{2r}\right]\\&=1-\frac{t_0}{2r}\frac{1}{t_0}\\&=1-\frac{1}{2r}\end{aligned}$$

Combining it all:

$$F_R(r)=\begin{cases}0 & r\leq\frac{1}{2}\\ 1-\frac{1}{2r} & \frac{1}{2}<r<1\\ 1 & 1\leq r\end{cases}$$