Tutorial T9

Example T9.1: Let R and Θ be two random variables with joint pdf $f_{R\Theta}(r,\theta)$ and consider the change of variables

$$
\begin{aligned}
X &= R\cos\Theta \\
Y &= R\sin\Theta.
\end{aligned}
$$

Find $f_{R\Theta}(r,\theta)$ in terms of $f_{XY}(x,y)$.

Solution: [See Ex 7b of Section 6.7 for a different tedious approach.]

Here, we have the system of equations

$$
\begin{aligned}
x &= g_1(r,\theta) = r\cos\theta \\
y &= g_2(r,\theta) = r\sin\theta
\end{aligned}
$$

Also note that

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} = h_1(x,y) \\
\theta &= h_2(x,y)
\end{aligned}
$$

where $h_2(x,y)$ is the angle of the vector (x,y), i.e., $h_2(x,y) = \arctan(y/x)$ when $x>0, y>0$.

Computing the Jacobian

$$
\begin{aligned}
J(r,\theta) &= \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta
\end{aligned}
$$

$$= r$$

So, the pdf $f_{X,Y}(x, y)$ and $f_{R\Theta}(r, \theta)$ are related by

$$f_{XY}(x, y) = f_{R\Theta}(r, \theta)\ |J(r, \theta)|^{-1} = f_{R\Theta}(r, \theta)/r \tag{T9.1}$$

or equivalently:

$$f_{R\Theta}(r, \theta) = f_{XY}(x, y)r \tag{T9.2}$$
$$= f_{XY}(r\cos\theta, r\sin\theta)r \tag{T9.3}$$

So, to compute the probability that $(R, \Theta) \in A$:

$$\begin{aligned} P[(R, \Theta) \in A] &= \iint_{(r,\theta)\in A} f_{R\Theta}(r, \theta) dr d\theta \\ &= \iint_{(r,\theta)\in A} f_{XY}(r\cos\theta, r\sin\theta) r dr d\theta \end{aligned}$$

Example T9.2: Two particles are positioned on a line with random and independent positions

$$X_1 \sim \mathcal{N}(0, 1)$$
$$X_2 \sim U[-1, 1]$$

a) Find $E[X_1 - X_2]$.

b) Find $E[(X_1 - X_2)^2]$.

c) Find $E[(X_1 + X_2)^2]$.

Solution:

$$a) \quad E[X_1 - X_2] = E[X_1] - E[X_2] = 0 - 0 = 0$$

$$\begin{aligned} b) \quad & E[(X_1 - X_2)^2] \\ & = E[X_1^2 - 2X_1X_2 + X_2^2] \\ & = E[X_1^2] - 2E[X_1X_2] + E[X_2^2] \\ & = 1 - 2E[X_1]E[X_2] + \frac{(1-(-1))^2}{12} \\ & = 1 - 2 \times 0 \times 0 + \frac{1}{3} \\ & = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} c) \quad & E[(X_1 + X_2)^2] \\ & = E[X_1^2 + 2X_1X_2 + X_2^2] \\ & = E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\ & = 1 + 2E[X_1]E[X_2] + \frac{1}{3} \\ & = \frac{4}{3} \end{aligned}$$

Example T9.3: You flip n 4-sided dice with outcomes $X_1, X_2, \ldots, X_n$. Let

$$Y = X_1 + X_2 + \cdots + X_n$$

What is the mean and variance of Y?

Solution: The hard way is to try to compute the pmf of Y first, and then use the pmf to compute $E[Y]$, etc. Instead, we use properties of expectations:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \end{aligned}$$

$$= n \times \frac{5}{2}$$

$$\begin{aligned}
E[Y^2] &= E\left[(X_1 + X_2 + \cdots + X_n)^2\right] \\
&= E\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{m=1}^{n} X_m\right)\right] \\
&= E\left[\sum_{k=1}^{n}\sum_{m=1}^{n} X_k X_m\right] \\
&= E\left[\sum_{k=1}^{n} X_k^2 + \sum\sum_{k \neq m} X_k X_m\right] \\
&= \sum_{k=1}^{n} E[X_k^2] + \sum\sum_{k \neq m} E[X_k X_m] \\
&= \sum_{k=1}^{n}\left(\frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4}\right) + \sum\sum_{k \neq m} \frac{5}{2}\frac{5}{2} \\
&= n\frac{30}{4} + n(n-1)\frac{25}{4}
\end{aligned}$$

$$\begin{aligned}
Var[Y] = E[Y^2] - E[Y]^2 & \\
&= n\frac{30}{4} + n(n-1)\frac{25}{4} - n^2\frac{25}{4} \\
&= n\frac{5}{4}
\end{aligned}$$

Note: Since the X_k are independent, in Video 31 we will see that

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] = nVar[X_1] = n\frac{5}{4}$$

which is a much simpler solution.