ECE203 Mathematical Background

Preamble

The following are mathematical concepts and questions that are useful and will appear in ECE203. Solving these questions can help you to self-assess your preparation for this course as well as provide a quick reference for a mathematical concept that may need review.

Questions

1. Let $q \neq 1$. Find a simple expression for the sum $S = \sum_{i=0}^{n} q^i$.
   Hint: What is the product $(1-q)(1+q+q^2+\cdots+q^n)$?

   Solution: Multiplying out

   $$\begin{aligned}(1-q)(1+q+q^2+\cdots+q^n) &= (1+q+q^2+\cdots+q^n) - (q+q^2+\cdots+q^n+q^{n+1}) \\ &= 1-q^{n+1}\end{aligned}$$

   So, dividing both sides by $(1-q)$, we get

   $$1+q+q^2+\cdots+q^n = \frac{1-q^{n+1}}{1-q}$$

2. Let $-1 < q < 1$. Use the result of the previous problem to find a simple expression for the sum $S = \sum_{i=0}^{\infty} q^i$.

   Solution:

   $$\begin{aligned}S &= \sum_{i=0}^{\infty} q^i \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} q^i \\ &= \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} \\ &= \frac{1}{1-q}\end{aligned}$$

3. Let $-1 < q < 1$. Use the result of the previous problem to find a simple expression for the sum $S = \sum_{i=1}^{\infty} pq^{i-1}$.

   Solution:

   $$\begin{aligned}S &= \sum_{i=1}^{\infty} pq^{i-1} && \text{Let } n = i-1 \\ &= p\sum_{n=0}^{\infty} q^n \\ &= \frac{p}{1-q}\end{aligned}$$

4. Find a simple expression for $S = \sum_{i=1}^{\infty} ipq^{i-1}$.
Hint: First i) expand i into $i = i - 1 + 1$ and use this to expand the sum into 2 sums. Then ii) make the substitution $n = i - 1$, and iii) express the right side in terms of S. Finally, iv) solve for S.

Solution:

$$
\begin{aligned}
S &= \sum_{i=1}^{\infty} ipq^{i-1} \\
&= \sum_{i=1}^{\infty} (i-1+1)pq^{i-1} \\
&= \sum_{i=1}^{\infty} (i-1)pq^{i-1} + \sum_{i=1}^{\infty} pq^{i-1} \\
&= \sum_{i=1}^{\infty} (i-1)pq^{i-1} + \frac{p}{1-q} && \text{Let } n = i-1 \\
&= \sum_{n=0}^{\infty} npq^{n} + \frac{p}{1-q} \\
&= \sum_{n=1}^{\infty} npq^{n} + \frac{p}{1-q} && \text{since } npq^n = 0 \text{ when } n = 0. \\
&= q\sum_{n=1}^{\infty} npq^{n-1} + \frac{p}{1-q} \\
&= qS + \frac{p}{1-q}
\end{aligned}
$$

Solving for S, we get

$$S = \frac{p}{(1-q)^2}$$

5. Recall that the Taylor series of a function $f(x)$ about $x = 0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(x)$ is the nth derivative of $f(x)$. Use this to show that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = 1$$

Solution: Choosing $f(x) = e^{\lambda x}$, and computing the first few derivatives:

$$
\begin{aligned}
f^{(0)}(0) &= f(0) = 1 \\
f^{(1)}(0) &= \lambda e^{\lambda x}|_{x=0} = \lambda \\
f^{(2)}(0) &= \lambda^2 e^{\lambda x}|_{x=0} = \lambda^2 \\
f^{(3)}(0) &= \lambda^3 e^{\lambda x}|_{x=0} = \lambda^3 \\
&\vdots \\
f^{(n)}(0) &= \lambda^n e^{\lambda x}|_{x=0} = \lambda^n
\end{aligned}
$$

So

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} x^n$$

hence, for $x = 1$, we get

$$e^{\lambda} = f(1) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$$

and therefore

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1$$

6. Find a simple expression for the sum $S = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda}$.
   Hint: i) Explain why the sum can start at $n = 1$ instead of $n = 0$, ii) then simplify and factor and make the substitution $m = n - 1$, iii) apply result of previous problem.

   Solution:

$$
\begin{aligned}
S &= \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \sum_{n=1}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} \\
&= \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} && \text{Let } m = n - 1 \\
&= \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} \\
&= \lambda && \text{By result of previous problem.}
\end{aligned}
$$

7. For $a > 0$, find a simple expression for the sum $S = \sum_{n=0}^{\infty} e^{-na}$.

Solution:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} e^{-na} \\ &= \sum_{n=0}^{\infty} (e^{-a})^n \\ &= \frac{1}{1-e^{-a}} && \text{Since } 0 < e^{-a} < 1 \end{aligned}$$

8. Expand $S = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j$ and verify that $S = (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$.

Solution:

$$\begin{aligned} S &= a_1b_1 + a_1b_2 + a_1b_3 + a_2b_1 + a_2b_2 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3 \\ &= a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 + b_3) + a_3(b_1 + b_2b_3) \\ &= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \end{aligned}$$

9. Show that in general $S = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j = (\sum_{i=1}^{n} a_i)(\sum_{i=1}^{m} b_j)$.

Solution:

$$\begin{aligned} S &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \\ &= \sum_{i=1}^{n} (a_ib_1 + a_ib_2 + \cdots a_ib_m) \\ &= \sum_{i=1}^{n} [a_i(b_1 + b_2 + \cdots b_m)] \\ &= a_1(b_1 + b_2 + \cdots b_m) + a_2(b_1 + b_2 + \cdots b_m) + \cdots + a_n(b_1 + b_2 + \cdots b_m) \\ &= (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots b_m) \\ &= (\sum_{i=1}^{n} a_i)(\sum_{i=1}^{m} b_j) \end{aligned}$$

10. Suppose $a_{ij} = a_{ji}$. Expand $S = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}$ and verify that $S = a_{11} + a_{22} + a_{33} + 2[a_{12} + a_{13} + a_{23}]$

Solution:

$$\begin{aligned} S &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \\ &= a_{11} + a_{22} + a_{33} + [a_{12} + a_{13} + a_{23}] + [a_{21} + a_{31} + a_{32}] \\ &= a_{11} + a_{22} + a_{33} + 2[a_{12} + a_{13} + a_{23}] && \text{where we used } a_{ij} = a_{ji} \end{aligned}$$

11. Suppose $a_{ij} = a_{ji}$. Expand $S = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$ and verify that $S = \sum_{i=1}^n a_{ii} + 2\sum\limits_{i<j} a_{ij}$ where the sum over $i < j$ means the sum over all pairs (i, j) such that $1 \leq i < j \leq n$.

Solution: Let

$$
\begin{aligned}
B &= \{(i,j) \in \mathbb{Z}^2 | 1 \leq i \leq n, 1 \leq j \leq n\} \\
C &= \{(i,j) \in A | i = j\} \\
D &= \{(i,j) \in A | i < j\} \\
E &= \{(i,j) \in A | i > j\}
\end{aligned}
$$

Then $B = C \cup D \cup E$, and each (i, j) pair in B belongs to only one of C, D or E. So

$$
\begin{aligned}
S &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} \\
&= \sum_{(i,j)\in B} a_{ij} \\
&= \sum_{(i,j)\in C} a_{ij} + \sum_{(i,j)\in D} a_{ij} + \sum_{(i,j)\in E} a_{ij} \\
&= \sum_{i=1}^n a_{ii} + \sum_{(i,j)\in D} a_{ij} + \sum_{(i,j)\in E} a_{ij} \\
&=^{(a)} \sum_{i=1}^n a_{ii} + \sum_{(i,j)\in D} a_{ij} + \sum_{(i,j)\in D} a_{ji} \\
&=^{(b)} \sum_{i=1}^n a_{ii} + \sum_{(i,j)\in D} a_{ij} + \sum_{(i,j)\in D} a_{ij} \\
&= \sum_{i=1}^n a_{ii} + 2 \sum_{(i,j)\in D} a_{ij} \\
&= \sum_{i=1}^n a_{ii} + 2 \sum_{i<j} a_{ij}
\end{aligned}
$$

where (a) follows since we can get the elements of E by swapping the coordinates of the elements of D, (b) follows since $a_{ij} = a_{ji}$.

12. Compute the derivative of

    (a) $f(x) = x \ln x$
    (b) $f(x) = xe^{ax}$
    (c) $f(x) = e^{1+x^2}$
    (d) $f(x) = (x + \ln x)^n$
    (e) $f(x) = (x^2 e^{bx} + c)^n$
    (f) $f(x) = \int_0^x y^2 dy$

(g) $f(x) = \int_{x^2}^{x} y^2 dy$

(h) $f(x) = \int_{x^2}^{x} (y+x)^2 dy$

Solution:

(a) Applying the product rule

$$f'(x) = (x)' \ln x + x(\ln x)' = \ln x + x/x = 1 + \ln x$$

(b) Applying the product rule

$$f'(x) = (x)' e^{ax} + x(e^{ax})' = e^{ax} + axe^{ax}$$

(c) Applying the chain rule

$$f'(x) = (e^{1+x^2})' = (1+x^2)' e^{1+x^2} = 2xe^{1+x^2}$$

(d) Applying the chain rule

$$f'(x) = ((x+\ln x)^n)' = n(x+\ln x)^{n-1}(x+\ln x)' = n(x+\ln x)^{n-1}(1+1/x)$$

(e) Applying the chain rule and product rule

$$\begin{aligned} f'(x) &= ((x^2 e^{bx} + c)^n)' \\ &= n(x^2 e^{bx} + c)^{n-1}(x^2 e^{bx} + c)' \\ &= n(x^2 e^{bx} + c)^{n-1}(x^2 e^{bx})' \\ &= n(x^2 e^{bx} + c)^{n-1}((x^2)' e^{bx} + x^2(e^{bx})') \\ &= n(x^2 e^{bx} + c)^{n-1}(2xe^{bx} + bx^2 e^{bx}) \end{aligned}$$

(f) Background: Consider the integral

$$g(x) = \int_{a(x)}^{b(x)} f(x,y) dy$$

When x increases, there are three sources of change in the value of the integral: i) the lower limit of integration $a(x)$ can change, the upper limit of integration $b(x)$ can change, and iii) the integrand $f(x,y)$ can change. The derivative must account for these 3 sources. Strictly speaking, this problem requires Leibniz's rule, which states that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,y) dy = b'(x) f(x, b(x)) - a'(x) f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y) dy$$

and each term on the right side above accounts for the three sources of change i), ii) and iii).

Now, back to finding the derivative of $\int_0^x y^2 dy$. Applying Leibniz's rule, we get

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\int_0^x y^2 dy \\
&= \left[\frac{d}{dx}\,x\right] \times [y^2]_{y=x} - \left[\frac{d}{dx}\,0\right] \times [x^2] - \int_0^x \frac{\partial}{\partial x} y^2 dy \\
&= 1 \times [x^2] - 0 + \int_0^x 0 dy \\
&= x^2
\end{aligned}$$

which for this simple problem, you could have obtained by inspection.

(g) Applying Leibniz's rule, we get

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\int_{x^2}^x y^2 dy \\
&= \left[\frac{d}{dx}\,x\right] \times [y^2]_{y=x} - \left[\frac{d}{dx}\,x^2\right] \times [y^2]_{y=x^2} - \int_{x^2}^x \frac{\partial}{\partial x} y^2 dy \\
&= 1 \times [x^2] - 2x \times x^4 + \int_{x^2}^x 0 dy \\
&= x^2 - 2x^5
\end{aligned}$$

An alternate way to get this is by computing the integral:

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\int_{x^2}^x y^2 dy \\
&= \frac{d}{dx}[y^3/3]_{x^2}^x \\
&= \frac{d}{dx}[x^3/3 - x^6/3] \\
&= x^2 - 2x^5
\end{aligned}$$

(h) Applying Leibniz's rule, we get

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\int_{x^2}^x (y+x)^2 dy \\
&= \left[\frac{d}{dx}\,x\right] \times [(x+y)^2]_{y=x} - \left[\frac{d}{dx}\,x^2\right] \times [(x+y)^2]_{y=x^2} - \int_{x^2}^x \frac{\partial}{\partial x}(x+y)^2 dy \\
&= 1 \times (x+x)^2 - 2x \times (x+x^2)^2 - \int_{x^2}^x 2(x+y) dy \\
&= 1 \times (x+x)^2 - 2x \times (x+x^2)^2 - 2x\int_{x^2}^x 1\ dy - \int_{x^2}^x 2y\ dy \\
&= 4x^2 - 2x(x+x^2)^2 - 2x[y]_{x^2}^x + [y^2]_{x^2}^x \\
&= 4x^2 - 2x(x+x^2)^2 - 2x(x-x^2) + x^2 - x^4
\end{aligned}$$

13. Assume that a function $F(t)$ has derivative $f(t)$, i.e.,

$$\frac{d}{dt}F(t) = f(t).$$

Express the following integrals in terms of $F(.)$. If this is not possible, explain why. If the integral does not make sense, explain why.

(a) $\int_u^v f(t)\ dt$

(b) $\int_u^v f(t-u)\ dt$

(c) $\int_u^{f(v)-F(u)} f(t-u)\ dt$

(d) $\int_v^{2v} f(t-u)\ du$

(e) $\int_u^v f(ut-u)\ dt$

(f) $\int_0^v f(ut-u)\ du$

(g) $\int_u^v tf(t)\ dt$

(h) $\int_u^t f(t)\ dt$

Solution:

(a) $\int_u^v f(t)\ dt = F(t)|_u^v = F(v) - F(u)$

(b) Let $w = t - u$. Then $dw = dt$, and hence

$$\int_u^v f(t-u)\ dt = \int_0^{v-u} f(w)\ dw = F(v-u) - F(0),$$

which in hindsight, was obvious, as the original expression is the area under the curve $f(t)$ between $t = 0$ and $t = v - u$.

(c) Using the same change of variable as in the previous part,

$$\int_u^{f(v)-F(u)} f(t-u)\ dt = \int_0^{f(v)-F(u)-u} f(w)\ dw = F(f(v) - F(u) - u) - F(0).$$

Without knowing anything more about $f(.)$ or $F(.)$, it is impossible to go further.

(d) The variable of integration is now du. Using $w = t - u$ again, we now have $dw = -du$. Hence

$$\int_v^{2v} f(t-u)\ du = \int_{t-v}^{t-2v} f(w)\ (-dw) = -F(t-2v) + F(t-v)$$

(e) Using the change of variable $w = ut - u$, we have $dw = u\ dt$. Hence

$$\int_u^v f(ut-u)\ dt = \int_{u^2-u}^{uv-u} f(w)\ (dw/u) = F(uv-u)/u - F(u^2-u)/u.$$

(f) Again, the variable of integration is now du, hence with $w = ut - u$, we have $dw = (t-1)du$. So

$$\int_0^v f(ut-u)\ du = \int_0^{vt-v} f(w)\ dw/(t-1) = F(vt-v)/(t-1) - F(0)/(t-1)$$

(g) Integrating by parts, we get:

$$\begin{aligned}\int_u^v tf(t)\ dt &= tF(t)|_u^v - \int_u^v F(t)\ dt \\ &= vF(v) - uF(u) - \int_u^v F(t)\ dt\end{aligned}$$

At this point, it depends on whether $\int_u^v F(t)\ dt$ is considered an expression in terms of $F(.)$. If we don't, then the answer can't be expressed in terms of $F(.)$.

(h) Here, the variable of integration is t, and one of the limits of integration is also t. Thus, the integral is not properly posed, and does not strictly make sense. What is most likely intended by such an expression is $\int_u^t f(w)\,dw$, which would be properly posed.

14. Compute the following indefinite integrals

(a) $\int e^{ax}dx$

(b) $\int xe^{ax}dx$

(c) $\int x^2e^{ax}dx$

(d) $\int x^3e^{ax}dx$

(e) $\int \ln x dx$

(f) $\int x\ln x dx$

Solution:

(a) Taking the long approach:

$$\begin{aligned}\int e^{ax}dx &= \frac{1}{a}\int e^u du \qquad\qquad u = ax, du = adx\\ &= \frac{1}{a}e^u\end{aligned}$$

It is customary to add a $+C$ since adding any constant still results in an anti-derivative.

(b) We integrate by parts. It is better to derive integration by parts as needed than to try to memorize it. Recall that $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$ so integrating all three terms, we get $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$.

$$\begin{aligned}\int xe^{ax}dx &= u(x)v(x) - \int u'(x)v(x)dx \qquad u(x) = x, v'(x) = e^{ax}, u'(x) = 1, v(x) = e^{ax}/a\\ &= xe^{ax}/a - \int e^{ax}/adx\\ &= xe^{ax}/a - e^{ax}/a^2\end{aligned}$$

(c) We integrate by parts.

$$\begin{aligned}\int x^2e^{ax}dx &= u(x)v(x) - \int u'(x)v(x)dx \qquad u(x) = x^2, v'(x) = e^{ax}, u'(x) = 2x, v(x) = e^{ax}/a\\ &= x^2e^{ax}/a - \int 2xe^{ax}/a\ dx\\ &= x^2e^{ax}/a - 2\int xe^{ax}/a\ dx\\ &= x^2e^{ax}/a - 2(x/a^2)e^{ax} + (2/a^3)e^{ax}\end{aligned}$$

where we used the result from the previous problem in the last step. We could have also gone ahead and integrated by parts a second time.

(d)

$$\begin{aligned}
\int x^3 e^{ax} dx &= u(x)v(x) - \int u'(x)v(x)dx && u(x) = x^3, v'(x) = e^{ax} \\
&= x^3 e^{ax}/a - \int 3x^2 e^{ax}/a \, dx && u'(x) = 3x^2, v(x) = e^{ax}/a \\
&= x^2 e^{ax}/a - 3 \int x^2 e^{ax}/a \, dx \\
&= x^3 e^{ax}/a - (3/a)[x^2 e^{ax}/a - 2(x/a^2)e^{ax} + (2/a^3)e^{ax}] \\
&= x^3 e^{ax}/a - 3x^2 e^{ax}/a^2 + 6(x/a^3)e^{ax} - (6/a^4)e^{ax}
\end{aligned}$$

(e)

$$\begin{aligned}
\int \ln x dx &= u(x)v(x) - \int u'(x)v(x)dx \qquad u(x) = \ln x, v'(x) = 1, u'(x) = 1/x, v(x) = x \\
&= x \ln x - \int x/x dx \\
&= x \ln x - x
\end{aligned}$$

(f)

$$\begin{aligned}
\int x \ln x dx &= u(x)v(x) - \int u'(x)v(x)dx \qquad u(x) = \ln x, v'(x) = x, u'(x) = 1/x, v(x) = x^2/2 \\
&= (x^2/2) \ln x - \int (x^2/2)/x \, dx \\
&= (x^2/2) \ln x - x^2/4
\end{aligned}$$

15. Let A be the area given by

$$A = \{(x, y) \in \mathbb{R}^2 | x < 1, y < 1, x + y > 1\}$$

Consider the integral

$$I = \iint\limits_A f(x, y)dxdy$$

Find the limits of integration for this integral when we compute it as an interated integral with x integrated in the inner integral, i.e., find a, b, c and d in terms of x and y:

$$I = \int_a^b \int_c^d f(x, y)dxdy$$

Solution: If we plot the area A, it is triangle formed by the vertices $(0, 1), (1, 1)$ and $(1, 0)$. Since we are integrating with respect to x first, we must express the limits of integration for x in terms of y. The diagonal line from $(0, 1)$ to $(1, 0)$ has formula $x = 1 - y$.

So, from the region A, x goes from $1 - y$ to 1. And the y goes from 0 to 1. So

$$I = \int_0^1 \int_{1-y}^1 f(x, y)dxdy$$

16. Express the integral in the previous problem in the form

$$I = \int_a^b \int_c^d f(x,y)dydx$$

i.e., with y as the inner integration variable.

Solution: The area A remains the same as before. So the limits for y need to be expressed in terms of x. Plotting the region, y goes from $1-x$ to 1, and x goes from 0 to 1, so

$$I = \int_0^1 \int_{1-x}^1 f(x,y)dydx$$

17. Consider

$$I = \int_0^1 \int_{x^2}^x g(x,y)dydx$$

Find the limits of integration when we switch the order of integration to $dxdy$, i.e., find a, b, c and d in terms of x and y such that

$$I = \int_a^b \int_c^d g(x,y)dxdy$$

Solution: The region over which the first integral integrates is the region

$$A = \{(x,y) \in \mathbb{R}^2 | x^2 < y < x, 0 < x < 1\}$$

This is the region between $y = x^2$ and $y = x$ where $0 < x < 1$.

This can equivalently be described as the region between $x = y$ and $x = \sqrt{y}$ when $0 < y < 1$. So

$$I = \int_0^1 \int_y^{\sqrt{y}} g(x,y)dxdy$$

18. With a, b, c and d as fixed constants, explain why

$$\int_a^b \int_c^d f(x)g(y)dxdy = \int_a^b g(y)dy \int_c^d f(x)dx$$

Solution:

$$\begin{aligned}
\int_a^b \int_c^d f(x)g(y)dxdy &= \int_a^b g(y) \int_c^d f(x)dx\ dy && \text{since } g(y) \text{ is constant with respect to } x \\
&= \int_a^b g(y) \left[\int_c^d f(x)dx\right] dy \\
&= \int_a^b g(y)dy \left[\int_c^d f(x)dx\right] && \text{since the term in the [ ] is a constant} \\
&= \int_a^b g(y)dy \int_c^d f(x)dx
\end{aligned}$$