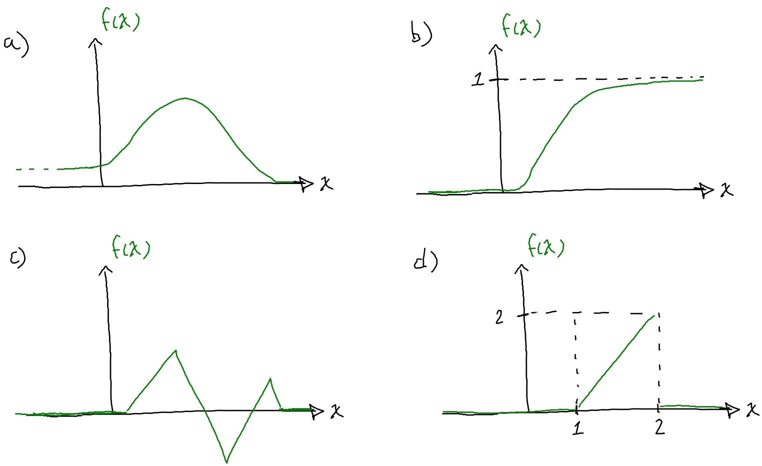
Tutorial T5

Example T5.1: Consider the following functions $f(x)$. Can $f(x)$ be the pdf of a random variable. If so, why? If not, why not?



Solution: a) Not a pdf. This is because the pdf should integrate to 1:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

However, because $f(x)$ goes to a positive constant as $x \to -\infty$, the area under the curve is infinite.

b) Not a pdf. Similar to a), the area under the curve is infinite, but should be 1. Note that this is a valid CDF though.

c) Not a pdf. While there is not enough information provided to tell if $\int_{-\infty}^{\infty} f(x)dx = 1$ or not, even if it is equal to 1, this is not a valid pdf. We can argue this two ways:

First, let $a < x < b$ be such that over this interval $f(x) < 0$. Then

$$P[X \in (a,b)] = \int_a^b f(x)dx < 0 \tag{T5.1}$$

so we would have the probablity $P[X \in (a,b)]$ be -ve, which is not possible. So $f(x)$ is not a valid pdf.

A second way to see the problem is that a pdf can be found by differentiating the CDF

$$f_X(x) = \frac{d}{dx}F_X(x)$$

and since the cdf $F_X(x)$ is non-decreasing, then we should have $f_X(x) \geq 0$.

d) This $f(x)$ is non-negative, and integrates to 1, so it is a valid pdf.

Example T5.2: Suppose I have two functions $g(x)$ and $h(x)$ such that $g(x) \geq h(x)$ for all x. Let X be a r.v. with pdf $f_X(x)$. Show that

$E[g(X)] \geq E[h(X)]$

Solution:

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{\infty} f_X(x)g(x)dx \\ &\geq \int_{-\infty}^{\infty} f_X(x)h(x)dx \qquad \text{since } f_X(x) \geq 0 \text{ and } g(x) \geq h(x) \\ &= E[h(X)] \end{aligned}$$

Example T5.3: A rv X has a Cauchy distribution with mean 0 and scale parameter $\gamma > 0$ if its pdf is

$$f(x) = \frac{1}{\pi\gamma}\frac{1}{x^2+\gamma^2}$$

a) Find $E[X^2]$

b) Find $E[X]$

c) Find $Var[X]$

Solution: a)

$$\begin{aligned} E[X^2] &= \frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\frac{x^2}{x^2+\gamma^2}\,dx \\ &= \infty && \text{since } \frac{x^2}{x^2+\gamma^2} \to 1 \text{ as } x \to \pm\infty \end{aligned}$$

b)

$$\begin{aligned} E[X] &= \frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\frac{x}{x^2+\gamma^2}\,dx \\ &= \frac{1}{\pi\gamma}\int_{0}^{\infty}\frac{x}{x^2+\gamma^2} + \frac{1}{\pi\gamma}\int_{-\infty}^{0}\frac{x}{x^2+\gamma^2}\,dx \\ &= \infty - \infty \\ &= \text{undefined} \end{aligned}$$

c)

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= \text{undefined} \end{aligned}$$

The Cauchy distribution is an example of a distribution that behaves poorly.

Example T5.4: Let the rv T be uniform on (a, b). Find the nth central moment $\mu_n = E[\ (T - E[T])^n\ ]$.

Solution:

$$f_T(t) = \begin{cases} \frac{1}{b-a} & a < t < b \\ 0 & \text{else} \end{cases}$$

First

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} t f_T(t)\, dt \\ &= \int_a^b \frac{t}{b-a}\, dt \\ &= \left.\frac{t^2}{2(b-a)}\right|_a^b \\ &= \left.\frac{b^2-a^2}{2(b-a)}\right|_a^b \\ &= \frac{a+b}{2} \end{aligned}$$

So

$$\begin{aligned} \mu_n &= E\left[\left(T - \frac{a+b}{2}\right)^n\right] \\ &= \int_{-\infty}^{\infty} \left(t - \frac{a+b}{2}\right)^n f_T(t) dt \\ &= \int_a^b \left(t - \frac{a+b}{2}\right)^n \frac{1}{b-a} dt \end{aligned}$$

Let $u = t - \dfrac{a+b}{2} \Rightarrow \quad du = dt$

$t = b \Rightarrow \quad u = \dfrac{b-a}{2}$

$t = a \Rightarrow \quad u = -\dfrac{b-a}{2}$

$$
\begin{aligned}
&= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} u^n \frac{1}{b-a} du \\
&= \left[\frac{u^{n+1}}{n+1}\right]_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{1}{b-a} \\
&= \frac{1}{b-a}\left[\frac{(\frac{b-a}{2})^{n+1}}{n+1} - \frac{(-\frac{b-a}{2})^{n+1}}{n+1}\right]
\end{aligned}
$$

When $n+1$ is even, this is zero. When $n+1$ is odd:

$$
\begin{aligned}
\mu_n &= \frac{2}{b-a}\frac{(\frac{b-a}{2})^{n+1}}{n+1} \\
&= \frac{(\frac{b-a}{2})^{n}}{n+1}
\end{aligned}
$$

So

$$
\mu_n = \begin{cases} \frac{1}{n+1}\left(\frac{b-a}{2}\right)^n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
$$

and we note that $\mu_2 = \frac{(b-a)^2}{12} = Var[T]$ as expected.