Tutorial T2

Example T2.1: Two fair four-sided die are rolled. What is the probability that both die roll the same number given that the sum is 3 or less?

Solution: Here

$$S = \{(a, b) \mid a = 1, \ldots, 4, b = 1, \ldots, 4\}$$
$$A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$B = \{(1, 1), (1, 2), (2, 1)\}$$

So

$$P[A|B] = P[AB]/P[B] = P[(1, 1)]/P[B] = 1/3.$$

Example T2.2: A pouch contains a four sided die and a six sided die. You pick one of the die at random and roll it.

a) What is the probability of rolling i, for $i = 1, 2, \ldots 6$?

b) If you roll a 3, what is the probability that you picked the 6-sided die?

Solution: Let

$A = \{$4-sided die picked$\}$

$B = \{$6-sided die picked$\}$

$E_i = \{$roll is $i\}$.

i) By law of total probability,

$$P[E_i] = P[E_i|A]P[A] + P[E_i|B]P[B]$$

$$\begin{aligned}
P[E_1] &= 1/4 \times 1/2 + 1/6 \times 1/2 \\
P[E_2] &= 1/4 \times 1/2 + 1/6 \times 1/2 \\
P[E_3] &= 1/4 \times 1/2 + 1/6 \times 1/2 \\
P[E_4] &= 1/4 \times 1/2 + 1/6 \times 1/2 \\
P[E_5] &= 0 \times 1/2 + 1/6 \times 1/2 \\
P[E_6] &= 0 \times 1/2 + 1/6 \times 1/2
\end{aligned}$$

ii)

$$\begin{aligned}
P[B|E_3] &= \frac{P[E_3B]}{P[E_3]} \\
&= \frac{P[E_3|B]P[B]}{P[E_3|B]P[B] + P[E_3|A]P[A]} \\
&= \frac{1/6 \times 1/2}{1/6 \times 1/2 + 1/4 \times 1/2} \\
&= \frac{2}{5}
\end{aligned}$$

Example T2.3: Let V be the event that a random person is vaccinated against a desease, and D the event that the random person develops the desease. The effectiveness e of a vaccine is defined to be

$$e = \frac{P[D|V^c] - P[D|V]}{P[D|V^c]} = 1 - \frac{P[D|V]}{P[D|V^c]}$$

Say 80% of the population is vaccinated and 40% of all cases of the desease occur in people who are vaccinated. What can you say about e?

[Note: See https://www.cdc.gov/csels/dsepd/ss1978/lesson3/section6.html]

Solution: We are given

$$\begin{aligned} P[V] &= 0.8 \\ P[V|D] &= 0.4 \end{aligned}$$

We want

$$\begin{aligned} e &= 1 - \frac{P[D|V]}{P[D|V^c]} \\ &= 1 - \frac{P[D|V]}{P[D|V^c]} \frac{P[V]}{P[V^c]} \frac{P[V^c]}{P[V]} \\ &= 1 - \frac{P[DV]}{P[DV^c]} \frac{P[V^c]}{P[V]} \\ &= 1 - \frac{P[V|D]P[D]}{P[V^c|D]P[D]} \frac{P[V^c]}{P[V]} \\ &= 1 - \frac{P[V|D]}{P[V^c|D]} \frac{P[V^c]}{P[V]} \\ &= 1 - \frac{0.4}{0.6} \frac{0.2}{0.8} \\ &= 1 - \frac{1}{6} = \frac{5}{6} \end{aligned}$$

In this case, being vaccinated reduces by 5/6 the probability of getting the desease compared to not being vaccinated.

Example T2.4: Alice and Bob have a pair of biased coins:

$$\begin{aligned} P[hh] &= 0.2 \qquad & P[ht] &= 0.3 \\ P[th] &= 0.3 \qquad & P[tt] &= 0.2 \end{aligned}$$

Let

$$A = \{\text{first coin is heads}\}$$
$$B = \{\text{second coin is heads}\}$$

a) Are the events $\{hh\}, \{ht\}, \{th\}, \{tt\}$ independent?

b) Are the events A and B independent?

c) Can you find two events that are independent (other than the trivial case where one of C or D is $\emptyset$ or S)?

d) Alice and Bob want to create/emulate a fair coin flip from this pair of coins. Is there a way to do so?

Solution:

a) No, they are not. If I know hh has occured, I know that ht has not occured. So hh and ht are not independent. So the 4 events are not independent. Specifically:

$$P[hh]P[ht] = 0.2 \times 0.3 \neq 0 = P[\{hh\} \cap \{ht\}]$$

b) No.

$$P[A] = 0.2 + 0.3$$
$$P[B] = 0.2 + 0.3$$

but $P[A]P[B] = 0.5 \times 0.5 \neq 0.2 = P[AB] = P[hh]$.

c) Let $C = \{ht, hh\}$ and $D = \{hh, tt\}$. Then

$$P[C]P[D] = 0.5 \times 0.4 = 0.2 = P[hh] = P[CD]$$

Also, note that

$$P[C|D] = P[CD]/P[D] = 0.2/0.4 = 0.5 = P[C]$$
$$P[D|C] = P[CD]/P[C] = 0.2/0.5 = 0.4 = P[D]$$

d) This requires a bit of creative thinking:

Let $F = \{hh, tt\}$. Alice and Bob flip the pair of coins until event F occurs. Once it occurs, if both biased coins are heads, it is declared that the fair coin is heads. Otherwise both coins are tails, and the fair coin is declared to be tails.

$$P[h] = P[hh|F] = \frac{P[\{hh\}F]}{P[F]} = \frac{P[hh]}{P[\{hh, tt\}]} = \frac{0.2}{0.2 + 0.2} = 0.5$$

Example T2.5: Suppose you have a hypothesis H. So either H or H^c occurs, e.g.

$$H = \{\text{there is a plane in the sky}\}$$
$$H = \{\text{the person who is browsing this website is male}\}$$

You now observe event E_1 occurs and know prior $P[H]$ and both $P[E_1|H]$ and $P[E_1|H^c]$.

a) What is posterior $P[H|E_1]$ in terms of what you know?

b) Now suppose you observe that E_2 also occured. What is posterior $P[H|E_1E_2]$?

c) Suppose that E_1 and E_2 are i) conditionally independent given H, and ii) conditionally independent given H^c. Express the posterior $P[H|E_1E_2]$ in terms of $P[H|E_1]$.

Solution: a) Deriving Baye's rule from first principles:

$$\begin{aligned} P[H|E_1] &= \frac{P[HE_1]}{P[E_1]} \\ &= \frac{P[E_1|H]P[H]}{P[E_1|H]P[H] + P[E_1|H^c]P[H^c]} \end{aligned}$$

b) This is the same as a), except we replace E_1 with E_2E_1:

$$P[H|E_2E_1] = \frac{P[E_2E_1|H]P[H]}{P[E_2E_1|H]P[H] + P[E_2E_1|H^c]P[H^c]}$$

c) We are told

$$\begin{aligned} P[E_2E_1|H] &= P[E_2|H]P[E_1|H] \\ P[E_2E_1|H^c] &= P[E_2|H^c]P[E_1|H^c] \end{aligned}$$

So

$$\begin{aligned} P[H|E_2E_1] &= \frac{P[E_2E_1|H]P[H]}{P[E_2E_1|H]P[H] + P[E_2E_1|H^c]P[H^c]} \\ &= \frac{P[E_2|H]P[E_1|H]P[H]}{P[E_2|H]P[E_1|H]P[H] + P[E_2|H^c]P[E_1|H^c]P[H^c]} \\ &= \frac{P[E_2|H]\frac{P[E_1|H]P[H]}{P[E_1]}}{P[E_2|H]\frac{P[E_1|H]P[H]}{P[E_1]} + P[E_2|H^c]\frac{P[E_1|H^c]P[H^c]}{P[E_1]}} \\ &= \frac{P[E_2|H]P[H|E_1]}{P[E_2|H]P[H|E_1] + P[E_2|H^c]P[H^c|E_1]} \end{aligned}$$

Basically, same as Baye's rule, but:

- $P[E_1|H]$ replaced with $P[E_2|H]$ (since we want to incorporate effect of event E_2)

- $P[H]$ replaced with $P[H|E_1]$

This technique lets you update in an 'online' manner the probability of H as observations $E_1, E_2, E_3, \ldots$ are collected. You just update your last posterior probability $P[H|E_1 \cdots E_{n-1}]$ by applying Baye's rule with $P[H|E_1 \cdots E_{n-1}]$ as your prior and $P[E_n|H]$ and $P[E_n|H^c]$.

This technique is used in many applications:

- your cellphone when estimating the bits sent by the base-station
- machine learning
- statistics