## Tutorial T3

**Example T3.1:** Let the PMF of $X$ be given by

$$p_X(x) = ax^2 \qquad \text{for } x = -2, -1, 0, 1, 2, 3$$

a) Find $a$.

b) What is $P[-1 < X \leq 2]$?

c) What is $E[X]$?

d) What is $Var[X]$?

*Solution:* a) We have that

$$\begin{aligned}
p_X(-2) &= 4a \\
p_X(-1) &= a \\
p_X(0) &= 0 \\
p_X(1) &= a \\
p_X(2) &= 4a \\
p_X(3) &= 9a.
\end{aligned}$$

These must sum to 1. So $4a + a + 0 + a + 4a + 9a = 19a = 1 \to a = 1/19$.

b) $P[-1 < X \leq 2] = P[X \in \{0, 1, 2\}] = P_X(0) + P_X(1) + P_X(2) = 5/19$.

c)

$$\begin{aligned}
E[X] &= \sum_x p_X(x)x \\
&= 4a \times -2 + a \times -1 + 0 \times 0 + a \times 1 + 4a \times 2 + 9a \times 3 \\
&= 27/19
\end{aligned}$$

d)

$$\begin{aligned}
E[X^2] &= \sum_x p_X(x)x^2 \\
&= 4a \times 4 + a \times 1 + 0 \times 0 + a \times 1 + 4a \times 4 + 9a \times 9 \\
&= 115/19 \\
Var[X] &= E[X^2] - (E[X])^2 = 115/19 - (27/19)^2
\end{aligned}$$

**Example T3.2:** The $n$th central moment of a r.v. $X$ is defined by

$$\mu_{X,n} = E[\,(X - E[X])^n\,]$$

a) Find $\mu_{X,0}$, $\mu_{X,1}$ and $\mu_{X,2}$.

b) Let $Y = aX + b$. What is the $n$-th central moment $\mu_{Y,n}$ of $Y$?

*Solution:* a)

$$\begin{aligned}
\mu_{X,0} &= E[\,(X - E[X])^0\,] = E[1] = 1 \\
\mu_{X,1} &= E[\,(X - E[X])^1\,] = E[X] - E[X] = 0 \\
\mu_{X,2} &= E[\,(X - E[X])^2\,] = Var[X]
\end{aligned}$$

b)

$$\begin{aligned}
\mu_{Y,n} &= E[\,(Y - E[Y])^n\,] \\
&= E[\,(aX + b - E[aX + b])^n\,] \\
&= E[\,(aX + b - aE[X] - b)^n\,] \\
&= E[\,(aX - aE[X])^n\,] \\
&= a^n E[\,(X - E[X])^n\,] \\
&= a^n \mu_{X,n}
\end{aligned}$$

**Example T3.3:** Let $a < b$ be integers. Let $X$ be a r.v. whose outcomes are the integers in the interval $[a, b]$, and $X$ takes each outcome with equal probability.

a) What is the PMF of $X$?

b) What is the PMF of $Y = e^X$?

c) What is $E[Y]$?

*Solution:*

a) There are $b - a + 1$ integer outcomes in the interval $[a, b]$. So each of these has probability $1/(b - a + 1)$, i.e.,

$$p_X(x) = \begin{cases} \frac{1}{b-a+1} & x \in \{a, a+1, \ldots, b\} \\ 0 & \text{else} \end{cases}$$

b) $Y = e^X$ takes $b - a + 1$ distinct values since $X$ does. So

$$p_Y(y) = \begin{cases} \frac{1}{b-a+1} & y \in \{e^a, e^{a+1}, \ldots, e^b\} \\ 0 & \text{else} \end{cases}$$

c)

$$\begin{aligned}
E[Y] &= E[e^X] \\
&= \sum_{x=a}^{b} p_X(x)e^x \\
&= \frac{1}{b-a+1} \sum_{x=a}^{b} e^x \\
&= \frac{1}{b-a+1}(e^a + e^{a+1} + \cdots + e^b) \\
&= \frac{e^a}{b-a+1}(1 + e + \cdots + e^{b-a}) \\
&= \frac{e^a}{b-a+1} \frac{1 - e^{b-a+1}}{1 - e}
\end{aligned}$$

**Example T3.4:** Consider the random variable $X$ with PMF below. What is the PMF of $Y = |X - 1|$?

$$\begin{aligned}
p_X(-2) &= 0.1 \\
p_X(-1) &= 0.2 \\
p_X(0) &= 0.1 \\
p_X(1) &= 0.1 \\
p_X(2) &= 0.3 \\
p_X(3) &= 0.2
\end{aligned}$$

*Solution:* Since $Y = |X - 1|$ then $Y$ can take outcomes $0, 1, 2$ and $3$.

$$\begin{aligned}
P[Y = 0] &= P[|X - 1| = 0] = P[X = 1] = 0.1 \\
P[Y = 1] &= P[|X - 1| = 1] = P[\{X = 0\} \cup \{X = 2\}] = 0.1 + 0.3 \\
P[Y = 2] &= P[|X - 1| = 2] = P[\{X = -1\} \cup \{X = 3\}] = 0.2 + 0.2 \\
P[Y = 3] &= P[|X - 1| = 3] = P[\{X = -2\} \cup \{X = 4\}] = 0.1
\end{aligned}$$