Tutorial T6

Example T6.1: Suppose a normal rv X is such that

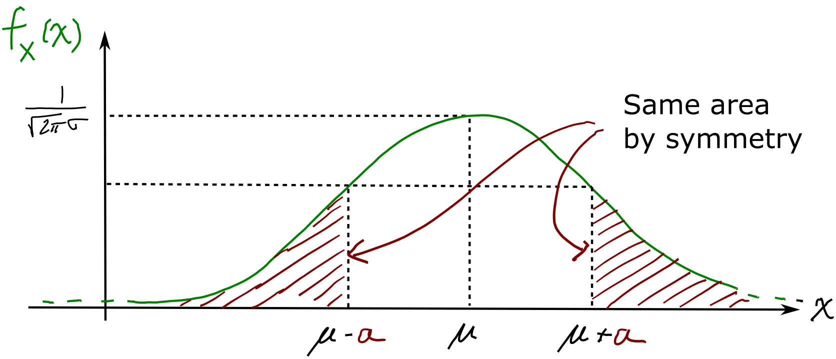
$$P[X < 5] = P[X > 15] = \frac{1}{2}P[X < 10].$$

Find the distribution of X.

Solution: First, note that since the pdf of a normal rv X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{(x-\mu)^2}{2\sigma^2}},$$

the pdf is symmetric about its mean μ.



So, therefore, if μ is the mean of X, then

$$P[X < \mu - a] = P[X > \mu + a]$$

So therefore:

$$\mu - a = 5$$
$$\mu + a = 15$$

and hence $\mu = 10$.

Also, $P[X < 10] = P[X < \mu] = 1/2$. Therefore $P[X > 15] = \frac{1}{2}P[X < 10] = 1/4$.

Finally,

$$\begin{aligned} P[X > 15] &= P\left[\frac{X-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right] \\ &= 1 - \Phi\left(\frac{15-\mu}{\sigma}\right) \\ &= 0.25 \end{aligned}$$

So, $\Phi(\frac{15-\mu}{\sigma}) = 0.75$. Using the Φ table, $\frac{15-\mu}{\sigma} = 0.67$

Since $\mu = 10$, then $\sigma = 7.46$

$X \sim \mathcal{N}(10, 7.46^2)$

Example T6.2: Let Y be a geometric rv with parameter p.

a) Show that

$$P[Y > a + b \mid Y > b] = P[Y > a]$$

b) Why is your answer in a) not surprising?

Solution: Here,

$$p_Y(k) = \begin{cases} (1-p)^{k-1}p & k \geq 1 \\ 0 & \text{else} \end{cases}$$

and

$$
\begin{aligned}
P[Y > a+b \mid Y > b] &= \frac{P[Y > a+b, Y > b]}{P[Y > b]} \\
&= \frac{P[Y > a+b]}{P[Y > b]} \\
&= \frac{1 - P[Y \le a+b]}{1 - P[Y \le b]} \\
&= \frac{1 - (1 - (1-p)^{a+b})}{1 - (1 - (1-p)^{b})} \\
&= \frac{(1-p)^{a+b}}{(1-p)^{b}} \\
&= (1-p)^{a} \\
&= 1 - (1 - (1-p)^{a}) \\
&= 1 - P[Y \le a] \\
&= P[Y > a]
\end{aligned}
$$

b) Recall that a geometric rv with parameter p is the number of independent coin flips until the first 'heads', when the prob of 'heads' is p.

$Y > b$ tells us that the first b flips are tails. Since the flips are independent, the number of flips to go after the first b tails is the same whether i) I know that I have had b tails, or ii) I reset my count of the number of tails to 0.

Example T6.3: Let $X \sim \mathcal{N}(0,1)$ and $Z = \sqrt{|X|}$. What is the pdf of Z?

Solution: Note that Z is non-negative, so $P[Z \le a] = 0$ for $a < 0$. That leaves the case of $a \ge 0$:

$$
\begin{aligned}
P[Z \le a] &= P[\sqrt{|X|} \le a] \\
&= P[|X| \le a^2]
\end{aligned}
$$

$$
\begin{aligned}
&= P[-a^2 \le X \le a^2] \\
&= 1 - P[X < -a^2] - P[X > a^2] \\
&= 1 - 2P[X < -a^2] \\
&= 1 - 2P[X \le -a^2] \\
&= 1 - 2\Phi(-a^2)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_Z(a) &= \frac{d}{da} P[Z \le a] \\
&= \frac{d}{da}\left(1 - 2\Phi(-a^2)\right) \\
&= \frac{d}{da}\left(1 - 2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a^2} e^{-u^2/2} du\right) \\
&= \frac{-2}{\sqrt{2\pi}} e^{-a^4/2} \times -2a \\
&= \frac{2\sqrt{2}a}{\sqrt{\pi}} e^{-a^4/2}
\end{aligned}
$$

Example T6.4: The time T for a server to process a job has pdf

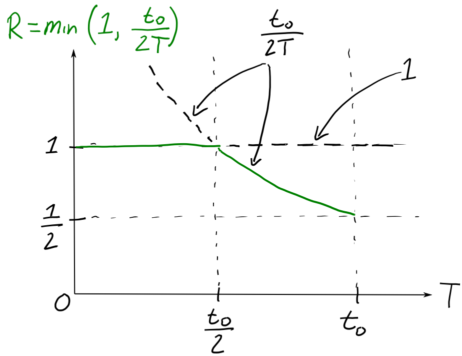
$$
f_T(t) = \begin{cases} \frac{1}{t_0} & 0 < t < t_0 \\ 0 & \text{else} \end{cases}
$$

The revenue from processing the job is $R = \min(1, \frac{t_0}{2T})$.

What is the cdf of R?

Solution: Lets first plot the relationship between T and R:

Since $R = \min(1, \frac{t_0}{2T}) \le 1$, then $F_R(r) = P[R \le r] = 1$ for $r \ge 1$.



From the graph, since $0 < T < t_0$ then $\frac{1}{2} < R \leq 1$. So $F_R(r) = P[R \leq r] = 0$ for $r \leq 1/2$.

So we now only consider that $1/2 < r < 1$. From the graph, for $1/2 < r < 1$ the events $\{R \leq r\}$ and $\{\frac{t_0}{2T} \leq r\}$ are the same:

$$
\begin{aligned}
F_R(r) &= P[R \leq r] \\
&= P\left[\frac{t_0}{2T} \leq r\right] \\
&= P\left[T \geq \frac{t_0}{2r}\right] \\
&= 1 - P\left[T < \frac{t_0}{2r}\right] \\
&= 1 - P\left[T \leq \frac{t_0}{2r}\right]
\end{aligned}
$$

Note that

$$
P[T \leq t] = \int_{-\infty}^{t} f_T(u)\, du
$$

$$= \begin{cases} 0 & t \le 0 \\ t/t_0 & 0 < t < t_0 \\ 1 & t_0 \le t \end{cases}$$

So, for $\frac{1}{2} < r < 1$ we have $1 < \frac{1}{r} < 2$ and $\frac{t_0}{2} < \frac{t_0}{2r} < t_0$. Therefore

$$\begin{aligned} F_R(r) &= 1 - P\left[T \le \frac{t_0}{2r}\right] \\ &= 1 - \frac{t_0}{2r}\frac{1}{t_0} \\ &= 1 - \frac{1}{2r} \end{aligned}$$

Combining it all:

$$F_R(r) = \begin{cases} 0 & r \le \frac{1}{2} \\ 1 - \frac{1}{2r} & \frac{1}{2} < r < 1 \\ 1 & 1 \le r \end{cases}$$