ECE203 Mathematical Background

Preamble

The following are mathematical concepts and questions that are useful and will appear in ECE203. Solving these questions can help you to self-assess your preparation for this course as well as provide a quick reference for a mathematical concept that may need review.

Questions

1. Let $q \neq 1$. Find a simple expression for the sum $S = \sum_{i=0}^{n} q^i$.
   Hint: What is the product $(1-q)(1+q+q^2+\cdots+q^n)$?

2. Let $-1 < q < 1$. Use the result of the previous problem to find a simple expression for the sum $S = \sum_{i=0}^{\infty} q^i$.

3. Let $-1 < q < 1$. Use the result of the previous problem to find a simple expression for the sum $S = \sum_{i=1}^{\infty} pq^{i-1}$.

4. Find a simple expression for $S = \sum_{i=1}^{\infty} ipq^{i-1}$.
   Hint: First i) expand i into $i = i - 1 + 1$ and use this to expand the sum into 2 sums. Then ii) make the substitution $n = i - 1$, and iii) express the right side in terms of S. Finally, iv) solve for S.

5. Recall that the Taylor series of a function $f(x)$ about $x = 0$ is

   $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

   where $f^{(n)}(x)$ is the nth derivative of $f(x)$. Use this to show that

   $$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = 1$$

6. Find a simple expression for the sum $S = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda}$.
   Hint: i) Explain why the sum can start at $n = 1$ instead of $n = 0$, ii) then simplify and factor and make the substitution $m = n - 1$, iii) apply result of previous problem.

7. For $a > 0$, find a simple expression for the sum $S = \sum_{n=0}^{\infty} e^{-na}$.

8. Expand $S = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j$ and verify that $S = (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$.

9. Show that in general $S = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j = (\sum_{i=1}^{n} a_i)(\sum_{i=1}^{m} b_j)$.

10. Suppose $a_{ij} = a_{ji}$. Expand $S = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}$ and verify that
    $S = a_{11} + a_{22} + a_{33} + 2[a_{12} + a_{13} + a_{23}]$

11. Suppose $a_{ij} = a_{ji}$. Expand $S = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$ and verify that $S = \sum_{i=1}^{n} a_{ii} + 2 \sum_{i<j} a_{ij}$ where the sum over $i < j$ means the sum over all pairs (i, j) such that $1 \leq i < j \leq n$.

12. Compute the derivative of

    (a) $f(x) = x \ln x$
    (b) $f(x) = xe^{ax}$
    (c) $f(x) = e^{1+x^2}$
    (d) $f(x) = (x + \ln x)^n$
    (e) $f(x) = (x^2 e^{bx} + c)^n$
    (f) $f(x) = \int_0^x y^2 dy$
    (g) $f(x) = \int_{x^2}^x y^2 dy$
    (h) $f(x) = \int_{x^2}^x (y + x)^2 dy$

13. Assume that a function $F(t)$ has derivative $f(t)$, i.e.,

$$\frac{d}{dt} F(t) = f(t).$$

    Express the following integrals in terms of $F(.)$. If this is not possible, explain why. If the integral does not make sense, explain why.

    (a) $\int_u^v f(t)\, dt$
    (b) $\int_u^v f(t - u)\, dt$
    (c) $\int_u^{f(v)-F(u)} f(t - u)\, dt$
    (d) $\int_v^{2v} f(t - u)\, du$
    (e) $\int_u^v f(ut - u)\, dt$
    (f) $\int_0^v f(ut - u)\, du$
    (g) $\int_u^v t f(t)\, dt$
    (h) $\int_u^t f(t)\, dt$

14. Compute the following indefinite integrals

    (a) $\int e^{ax} dx$
    (b) $\int xe^{ax} dx$
    (c) $\int x^2 e^{ax} dx$
    (d) $\int x^3 e^{ax} dx$
    (e) $\int \ln x dx$
    (f) $\int x \ln x dx$

15. Let A be the area given by

$$A = \{(x, y) \in \mathbb{R}^2 | x < 1, y < 1, x + y > 1\}$$

    Consider the integral

$$I = \iint_A f(x, y) dx dy$$

    Find the limits of integration for this integral when we compute it as an interated integral with x integrated in the inner integral, i.e., find a, b, c and d in terms of x and y:

$$I = \int_a^b \int_c^d f(x, y) dx dy$$

16. Express the integral in the previous problem in the form

$$I = \int_a^b \int_c^d f(x,y)dydx$$

i.e., with y as the inner integration variable.

17. Consider

$$I = \int_0^1 \int_{x^2}^x g(x,y)dydx$$

Find the limits of integration when we switch the order of integration to $dxdy$, i.e., find a, b, c and d in terms of x and y such that

$$I = \int_a^b \int_c^d g(x,y)dxdy$$

18. With a, b, c and d as fixed constants, explain why

$$\int_a^b \int_c^d f(x)g(y)dxdy = \int_a^b g(y)dy \int_c^d f(x)dx$$