ECE 208, Spring 2020
Assignment No. 1
Due 5:00 p.m., Thursday, May 28
10 points in total

Submission: Submissions are to be in pdf format. No late submissions accepted. Submit to the appropriate dropbox in Learn.

1. (10 points)

Definition 2.15 Let $A \in \mathscr{F}$ be a formula and let $\mathscr{P}_A$ be the set of atoms appearing in A. An *interpretation* for A is a total function $\mathscr{I}_A : \mathscr{P}_A \to \{T, F\}$ that assigns a truth value T or F to each atom in A.

Definition 2.16 Let $\mathscr{I}_A$ be an interpretation for $A \in \mathscr{F}$. Then $v_{\mathscr{I}_A}(A')$, the *truth value of subformula A' under $\mathscr{I}_A$* is defined recursively on the structure of A, according to the truth tables for the respective operators.

For example, for any subformulas A_1 and A_2 of A,

$$\begin{aligned} v_{\mathscr{I}_A}(\neg A_1) = T &\text{ if and only if } v_{\mathscr{I}_A}(A_1) = F\ ; \\ v_{\mathscr{I}_A}(A_1 \vee A_2) = T &\text{ if and only if either } v_{\mathscr{I}_A}(A_1) = T \text{ or } v_{\mathscr{I}_A}(A_2) = T\ ; \text{ and} \\ v_{\mathscr{I}_A}(A_1 \wedge A_2) = T &\text{ if and only if both } v_{\mathscr{I}_A}(A_1) = T \text{ and } v_{\mathscr{I}_A}(A_2) = T\ . \end{aligned}$$

For any two interpretations $\mathscr{I}_A$ and $\mathscr{I}'_A$ for a formula A, write

$$\mathscr{I}_A \leq \mathscr{I}'_A \text{ if and only if, for any atom } p \in \mathscr{P}_A\ , \mathscr{I}_A(p) = T \text{ implies } \mathscr{I}_A(p') = T\ .$$

In other words, every atom that is assigned the truth value T by $\mathscr{I}_A$ is also assigned T by $\mathscr{I}'_A$ (and $\mathscr{I}'_A$ may assign other atoms the truth value T as well).

Similarly, write

$$v_{\mathscr{I}_A} \preceq v_{\mathscr{I}'_A} \text{ if and only if, for any subformula } A' \text{ of } A\ , v_{\mathscr{I}_A}(A') = T \text{ implies } v_{\mathscr{I}'_A}(A') = T\ ;$$

in other words, any subformula that comes out to be true under $\mathscr{I}_A$ is also true under $\mathscr{I}'_A$.

(a) Show (by structural induction) that, if A contains no operators other than $\vee$ and $\wedge$, then $\mathscr{I}_A \leq \mathscr{I}'_A$ implies $v_{\mathscr{I}_A} \preceq v_{\mathscr{I}'_A}$.

Solution:

Suppose that $\mathscr{I}_A \leq \mathscr{I}'_A$.

In the base case, A' is an atom, so

$$v_{\mathscr{I}_A}(A') = T \text{ iff } \mathscr{I}_A(A') = T \implies \mathscr{I}'_A(A') = T \text{ iff } v_{\mathscr{I}'_A}(A') = T\ .$$

For the induction step,

$$\begin{aligned} v_{\mathscr{I}_A}(F_1 \vee F_2) = T &\text{ iff } v_{\mathscr{I}_A}(F_1) = T \text{ or } v_{\mathscr{I}_A}(F_2) = T \ , \\ &\text{ so } v_{\mathscr{I}'_A}(F_1) = T \text{ or } v_{\mathscr{I}'_A}(F_2) = T \ , \qquad \text{ind. hyp.} \\ &\text{ iff } v_{\mathscr{I}'_A}(F_1 \vee F_2) = T \ . \end{aligned}$$

$$\begin{aligned} v_{\mathscr{I}_A}(F_1 \wedge F_2) = T &\text{ iff } v_{\mathscr{I}_A}(F_1) = T \text{ and } v_{\mathscr{I}_A}(F_2) = T \ , \\ &\text{ so } v_{\mathscr{I}'_A}(F_1) = T \text{ and } v_{\mathscr{I}'_A}(F_2) = T \ , \qquad \text{ind. hyp.} \\ &\text{ iff } v_{\mathscr{I}'_A}(F_1 \wedge F_2) = T \ . \end{aligned}$$

(b) Say that a formula A is *isotone* if $\mathscr{I}_A \leq \mathscr{I}'_A$ implies $v_{\mathscr{I}_A} \preceq v_{\mathscr{I}'_A}$ (roughly speaking, "the truer atoms are, the truer the formula is"); and say that A is *antitone* if $\mathscr{I}_A \leq \mathscr{I}'_A$ implies $v_{\mathscr{I}'_A} \preceq v_{\mathscr{I}_A}$ ("the truer the atoms are, the falser the formula is").

For instance, the formulas considered in part (a) are isotone, and a formula such as $\neg p$ is antitone.

Let a formula A contain only the operators $\neg$, $\vee$ and $\wedge$. Show that A is isotone (respectively, antitone) if, along every path from the root to a leaf of A, the number of nodes labelled by $\neg$ is even (respectively, odd).

Hint: prove both of those facts in a single induction.

Solution:

Suppose that, along every path from the root of A to a leaf, the number of nodes labelled by $\neg$ has the same parity. We prove by structural induction that the result holds for every subformula A' of A.

Suppose that $\mathscr{I}_A \leq \mathscr{I}'_A$.

In the base case, A' is an atom.

$$v_{\mathscr{I}_A}(A') = T \text{ iff } \mathscr{I}_A(A') = T \implies \mathscr{I}'_A(A') = T \text{ iff } v_{\mathscr{I}'_A}(A') = T \ .$$

Thus, because the interpretations $\mathscr{I}_A \leq \mathscr{I}'_A$ are otherwise arbitrary, A' is isotone, and the base case holds.

For the induction step, suppose first that, along every path from the root of A' to a leaf, the number of nodes labelled by $\neg$ is even. Then, if A' is of the form $F_1 \vee F_2$ or $F_1 \wedge F_2$, the same is true of the subformulas F_1 and F_2, so, by inductive hypothesis, those subformulas are both isotone.

$$\begin{aligned} v_{\mathscr{I}_A}(F_1 \vee F_2) = T &\text{ iff } v_{\mathscr{I}_A}(F_1) = T \text{ or } v_{\mathscr{I}_A}(F_2) = T \ , \\ &\text{ so } v_{\mathscr{I}'_A}(F_1) = T \text{ or } v_{\mathscr{I}'_A}(F_2) = T \ , \qquad \text{ind. hyp.} \\ &\text{ iff } v_{\mathscr{I}'_A}(F_1 \vee F_2) = T \ . \end{aligned}$$

$$\begin{aligned} v_{\mathscr{I}_A}(F_1 \wedge F_2) = T &\text{ iff } v_{\mathscr{I}_A}(F_1) = T \text{ and } v_{\mathscr{I}_A}(F_2) = T \ , \\ &\text{ so } v_{\mathscr{I}'_A}(F_1) = T \text{ and } v_{\mathscr{I}'_A}(F_2) = T \ , \qquad \text{ind. hyp.} \\ &\text{ iff } v_{\mathscr{I}'_A}(F_1 \wedge F_2) = T \ . \end{aligned}$$

Now, if A' has the form $\neg F_1$, then the number of negations along any path from the root of F_1 to a leaf is odd, so, by inductive hypothesis, F_1 is antitone. We therefore have,

$$\begin{aligned} v_{\mathscr{I}_A}(\neg F_1) = T \text{ iff } & v_{\mathscr{I}_A}(F_1) = F & \\ \text{so } & v_{\mathscr{I}'_A}(F_1) = F & \text{ind. hyp.} \\ \text{iff } & v_{\mathscr{I}'_A}(\neg F_1) = T \ . & \end{aligned}$$

Now suppose that, along every path from the root of A' to a leaf, the number of nodes labelled by $\neg$ is odd. Then, if A' is of the form $F_1 \vee F_2$ or $F_1 \wedge F_2$, the same is true of the subformulas F_1 and F_2, so, by inductive hypothesis, those subformulas are both antitone:

$$\begin{aligned} v_{\mathscr{I}'_A}(F_1 \vee F_2) = T \text{ iff } & v_{\mathscr{I}'_A}(F_1) = T \text{ or } v_{\mathscr{I}'_A}(F_2) = T \ , & \\ \text{so } & v_{\mathscr{I}_A}(F_1) = T \text{ or } v_{\mathscr{I}_A}(F_2) = T \ , & \text{ind. hyp.} \\ \text{iff } & v_{\mathscr{I}_A}(F_1 \vee F_2) = T \ . & \end{aligned}$$

$$\begin{aligned} v_{\mathscr{I}'_A}(F_1 \wedge F_2) = T \text{ iff } & v_{\mathscr{I}'_A}(F_1) = T \text{ and } v_{\mathscr{I}'_A}(F_2) = T \ , & \\ \text{so } & v_{\mathscr{I}_A}(F_1) = T \text{ and } v_{\mathscr{I}_A}(F_2) = T \ , & \text{ind. hyp.} \\ \text{iff } & v_{\mathscr{I}_A}(F_1 \wedge F_2) = T \ . & \end{aligned}$$

Finally, if A' has the form $\neg F_1$, then the number of negations along any path from the root of F_1 to a leaf is even, so, by inductive hypothesis, F_1 is isotone. We therefore have,

$$\begin{aligned} v_{\mathscr{I}'_A}(\neg F_1) = T \text{ iff } & v_{\mathscr{I}'_A}(F_1) = F & \\ \text{so } & v_{\mathscr{I}_A}(F_1) = F & \text{ind. hyp.} \\ \text{iff } & v_{\mathscr{I}_A}(\neg F_1) = T \ . & \end{aligned}$$

Because the interpretations $\mathscr{I}_A \leq \mathscr{I}'_A$ are arbitrary, the result holds for A'. This completes the induction.