Tutorial T3

Example T3.1: Let the PMF of X be given by

$$p_X(x) = ax^2 \qquad \text{for } x = -2, -1, 0, 1, 2, 3$$

a) Find a.

b) What is $P[-1 < X \leq 2]$?

c) What is $E[X]$?

d) What is $Var[X]$?

Solution: a) We have that

$$\begin{aligned}
p_X(-2) &= 4a \\
p_X(-1) &= a \\
p_X(0) &= 0 \\
p_X(1) &= a \\
p_X(2) &= 4a \\
p_X(3) &= 9a.
\end{aligned}$$

These must sum to 1. So $4a + a + 0 + a + 4a + 9a = 19a = 1 \rightarrow a = 1/19$.

b) $P[-1 < X \leq 2] = P[X \in \{0, 1, 2\}] = P_X(0) + P_X(1) + P_X(2) = 5/19$.

c)

$$\begin{aligned}
E[X] &= \sum_x p_X(x)x \\
&= 4a \times -2 + a \times -1 + 0 \times 0 + a \times 1 + 4a \times 2 + 9a \times 3 \\
&= 27/19
\end{aligned}$$

d)

$$
\begin{aligned}
E[X^2] &= \sum_x p_X(x)x^2 \\
&= 4a \times 4 + a \times 1 + 0 \times 0 + a \times 1 + 4a \times 4 + 9a \times 9 \\
&= 115/19 \\
Var[X] &= E[X^2] - (E[X])^2 = 115/19 - (27/19)^2
\end{aligned}
$$

Example T3.2: The nth central moment of a r.v. X is defined by

$$\mu_{X,n} = E[\ (X - E[X])^n\ ]$$

a) Find $\mu_{X,0}$, $\mu_{X,1}$ and $\mu_{X,2}$.

b) Let $Y = aX + b$. What is the n-th central moment $\mu_{Y,n}$ of Y?

Solution: a)

$$
\begin{aligned}
\mu_{X,0} &= E[\ (X - E[X])^0\ ] = E[1] = 1 \\
\mu_{X,1} &= E[\ (X - E[X])^1\ ] = E[X] - E[X] = 0 \\
\mu_{X,2} &= E[\ (X - E[X])^2\ ] = Var[X]
\end{aligned}
$$

b)

$$
\begin{aligned}
\mu_{Y,n} &= E[\ (Y - E[Y])^n\ ] \\
&= E[\ (aX + b - E[aX + b])^n\ ] \\
&= E[\ (aX + b - aE[X] - b)^n\ ] \\
&= E[\ (aX - aE[X])^n\ ] \\
&= a^n E[\ (X - E[X])^n\ ] \\
&= a^n \mu_{X,n}
\end{aligned}
$$

Example T3.3: Let $a < b$ be integers. Let X be a r.v. whose outcomes are the integers in the interval $[a, b]$, and X takes each outcome with equal probability.

a) What is the PMF of X?

b) What is the PMF of $Y = e^X$?

c) What is $E[Y]$?

Solution:

a) There are $b - a + 1$ integer outcomes in the interval $[a, b]$. So each of these has probability $1/(b - a + 1)$, i.e.,

$$p_X(x) = \begin{cases} \frac{1}{b-a+1} & x \in \{a, a+1, \dots, b\} \\ 0 & \text{else} \end{cases}$$

b) $Y = e^X$ takes $b - a + 1$ distinct values since X does. So

$$p_Y(y) = \begin{cases} \frac{1}{b-a+1} & y \in \{e^a, e^{a+1}, \dots, e^b\} \\ 0 & \text{else} \end{cases}$$

c)

$$\begin{aligned} E[Y] &= E[e^X] \\ &= \sum_{x=a}^{b} p_X(x) e^x \\ &= \frac{1}{b-a+1} \sum_{x=a}^{b} e^x \\ &= \frac{1}{b-a+1} (e^a + e^{a+1} + \cdots + e^b) \end{aligned}$$

$$= \frac{e^a}{b-a+1}(1 + e + \cdots + e^{b-a})$$
$$= \frac{e^a}{b-a+1} \frac{1-e^{b-a+1}}{1-e}$$

Example T3.4: Consider the random variable X with PMF below. What is the PMF of $Y = |X - 1|$?

$$\begin{aligned}
p_X(-2) &= 0.1\\
p_X(-1) &= 0.2\\
p_X(0) &= 0.1\\
p_X(1) &= 0.1\\
p_X(2) &= 0.3\\
p_X(3) &= 0.2
\end{aligned}$$

Solution: Since $Y = |X - 1|$ then Y can take outcomes $0, 1, 2$ and 3.

$$\begin{aligned}
P[Y = 0] &= P[|X - 1| = 0] = P[X = 1] = 0.1\\
P[Y = 1] &= P[|X - 1| = 1] = P[\{X = 0\} \cup \{X = 2\}] = 0.1 + 0.3\\
P[Y = 2] &= P[|X - 1| = 2] = P[\{X = -1\} \cup \{X = 3\}] = 0.2 + 0.2\\
P[Y = 3] &= P[|X - 1| = 3] = P[\{X = -2\} \cup \{X = 4\}] = 0.1
\end{aligned}$$