Tutorial T7

Example T7.1: Let the joint pdf of X and Y be given by
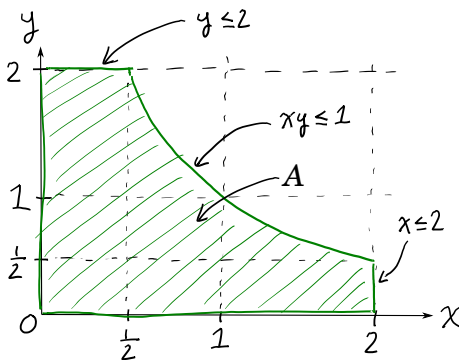
$$f_{XY}(x,y) = \begin{cases} c(1-xy) & 0 \le x \le 2, 0 \le y \le 2, xy \le 1 \\ 0 & \text{else} \end{cases}$$

a) What is c?

b) What is $P[X > 1]$?

c) What is $P[X > Y]$?

Solution: a) Sketching a figure is always a good idea:

$$1 = \iint_A f_{XY}(x,y)dxdy$$
$$= c\int_0^{1/2} \int_0^2 (1-xy)dydx + c\int_{1/2}^2 \int_0^{1/x} (1-xy)dydx$$
$$= c\int_0^{1/2} \left[y - \frac{xy^2}{2} \right]_{y=0}^{y=2} dx + c\int_{1/2}^2 \left[y - \frac{xy^2}{2} \right]_{y=0}^{y=1/x} dx$$

$$= c \int_0^{1/2} [2 - 2x]dx + c \int_{1/2}^2 \left[\frac{1}{x} - \frac{1}{2x}\right] dx$$

$$= c[2x - x^2]_0^{1/2} + \frac{c}{2} [\ln x]_{1/2}^2$$

$$= c \times \frac{3}{4} + c \ln 2$$

So $c = (3/4 + \ln 2)^{-1}$.

b)

$$P[X > 1] = \iint_{x>1} f_{XY}(x, y)dxdy$$

$$= c \int_{x=1}^2 \int_{y=0}^{1/x} (1 - xy)dydx$$

$$= c \int_1^2 \left[y - \frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx$$

$$= c \int_1^2 \left[\frac{1}{x} - \frac{1}{2x}\right] dx$$

$$= \frac{c}{2} [\ln x]_1^2$$

$$= \frac{c}{2} \ln 2$$

c) Hard way:

$$P[X > Y] = \iint_{x>y} f_{XY}(x, y)dxdy$$

$$= \int_0^\infty \int_0^x f_{XY}(x, y)dydx$$

$$= c \int_0^1 \int_0^x (1 - xy)dydx + c \int_1^2 \int_0^{1/x} (1 - xy)dydx$$

$$= \cdots$$

Easy way:

$$1 = P[X > Y] + P[X = Y] + P[Y > X]$$
$$ = P[X > Y] + P[Y > X] \qquad \text{since } x = y \text{ has 0 area in } x - y \text{ plane}$$
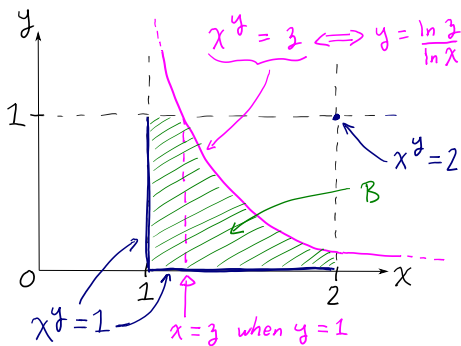$$ = P[X > Y] + P[X > Y] \qquad \text{by symmetry of pdf w.r.t. } x = y$$

So $P[X > Y] = 1/2$.

Example T7.2: Let $X \sim U(1, 2)$ and $Y \sim U(0, 1)$ be independent rvs. Let

$$Z = X^Y$$

What is the pdf of Z?

Solution:

The joint pdf of X and Y is

$$f_{XY}(x, y) = \begin{cases} 1 & 1 \leq x \leq 2, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Note that Z must be between 1 (when $Y = 0$ or $X = 1$) and 2 (when $X = 2, Y = 1$). So

$$F_Z(z) = \begin{cases} 0 & z \leq 1 \\ 1 & z \geq 2 \end{cases}$$

This leaves the case that $1 < z < 2$:

$$\begin{aligned} P[Z \leq z] &= P[X^Y \leq z] \\ &= P[Y \ln X \leq \ln z] \\ &= P[Y \leq \frac{\ln z}{\ln X}] \end{aligned}$$

Let

$$A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2, 0 \leq y \leq 1\}$$
$$B = \left\{(x, y) \in A \mid y \leq \frac{\ln z}{\ln x}\right\}$$

$$\begin{aligned} P[Z \leq z] &= \iint_{(x,y) \in B} f_{XY}(x, y) dx dy \\ &= \int_1^z \int_0^1 1 dy dx + \int_z^2 \int_0^{\frac{\ln z}{\ln x}} 1 dy dx \\ &= z - 1 + \int_z^2 \frac{\ln z}{\ln x} dx \end{aligned}$$

This integral can't be simplified further. To get the pdf,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 & z \leq 1 \\ 0 & z \geq 2 \end{cases}$$

For $1 < z < 2$:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_Z(z) \\
&= \frac{d}{dz}\left[z - 1 + \int_z^2 \frac{\ln z}{\ln x}dx\right] \\
&= 1 + \frac{d}{dz}\int_z^2 \frac{\ln z}{\ln x}dx \\
&= 1 - \frac{\ln z}{\ln z} \times \frac{d}{dz}z + \int_z^2 \frac{d}{dz}\frac{\ln z}{\ln x}dx \qquad \text{by Leibniz's rule} \\
&= \frac{1}{z}\int_z^2 \frac{1}{\ln x}dx
\end{aligned}
$$

Example T7.3: A particle is known to be in a cylinder. The pdf of the random location (R, Θ, Z) in cylindrical coordinates is

$$
f(r, \theta, z) = \begin{cases} c(r - r^2)z & 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1 \\ 0 & \text{else} \end{cases}
$$

a) What is the constant c?

b) What is $P[Z < 1/2]$?

Solution: We find c by requiring that f integrate to 1:

$$
\begin{aligned}
1 &= \iiint\limits_{\substack{0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z \le 1}} f(r, \theta, z)drd\theta dz \\
&= \int_0^1 \int_0^1 \int_0^{2\pi} c(r - r^2)z \ d\theta dz dr \\
&= \int_0^1 \int_0^1 2\pi c(r - r^2)z \ dz dr
\end{aligned}
$$

$$= \int_0^1 2\pi c(r - r^2) \left. \frac{z^2}{2} \right|_{z=0}^{z=1} dr$$

$$= \int_0^1 \pi c(r - r^2) dr$$

$$= \pi c \left[\frac{r^2}{2} - \frac{r^3}{3} \right]_{r=0}^{r=1}$$

$$= \pi c/6$$

So $c = 6/\pi$.

b)

$$P[Z < 1/2] = \iiint\limits_{z<1/2} f(r, \theta, z) dr d\theta dz$$

$$= \int_0^1 \int_0^{1/2} \int_0^{2\pi} c(r - r^2) z \ d\theta dz dr$$

$$= \int_0^1 \int_0^{1/2} 2\pi c(r - r^2) z \ dz dr$$

$$= \int_0^1 2\pi c(r - r^2) \left. \frac{z^2}{2} \right|_{z=0}^{z=1/2} dr$$

$$= \int_0^1 \frac{\pi}{4} c(r - r^2) dr$$

$$= \frac{\pi}{4} c \left[\frac{r^2}{2} - \frac{r^3}{3} \right]_0^1$$

$$= \frac{\pi}{4} \frac{c}{6}$$

$$= \frac{1}{4}$$

Question: why didn't we use $r \ dr d\theta dz$ instead of $dr d\theta dz$ as you were taught

in calculus?

Answer: If I instead had:

$$f(u, v, w) = \begin{cases} c(u - u^2)w & 0 \leq u \leq 1, 0 \leq v \leq 2\pi, 0 \leq w \leq 1 \\ 0 & \text{else} \end{cases}$$

you wouldn't think twice before writing

$$1 = \iiint\limits_{\substack{0 \leq v \leq 2\pi \\ 0 \leq u \leq 1 \\ 0 \leq w \leq 1}} f(r, \theta, z) du dv dw.$$

Longer answer: the r in $r \, dr d\theta dz$ is already in the pdf $f(r, \theta, z)$, so we don't need to add it. We will see this when we study "change of variables".