Tutorial T1

Example T1.1: Describe the sample spaces of the following random experiments:

a) Draw a ball from an urn that contains balls numbered from 1 to 10.

b) Draw a card from a deck that only contains the following 4 cards i) 2 of hearts, ii) the three of spades, iii) the four of diamonds, and iv) the three of clubs.

c) Measure the lifetime of a memory chip.

d) Measure the value of two 1kΩ resistors with tolerances of $\pm 10\%$.

Solution:

a) $S = \{1, 2, \ldots, 10\}$.

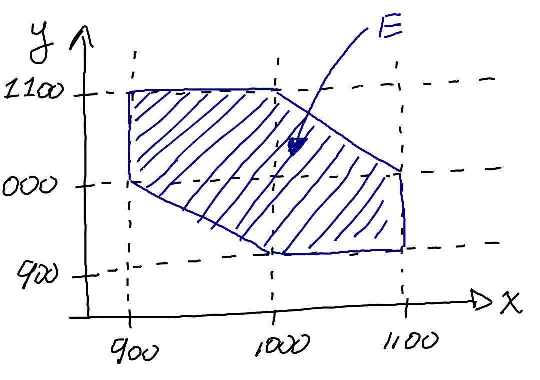
b) $S = \{(2, \heartsuit), (3, \spadesuit), (4, \diamondsuit), (3, \clubsuit)\}$

c) $S = \{x \in \mathbb{R} \mid x \geq 0\}$

d) $S = \{(x, y) \in \mathbb{R}^2 \mid 900 \leq x \leq 1100, 900 \leq y \leq 1100\}$

Example T1.2: In Example T1.1d, sketch the event that the sum of the two resistors is within 5% of 2kΩ.

Solution: This event is $E = \{(x, y) \in S \mid 1900 \leq x + y \leq 2100\}$ and sketched in the figure below.



Example T1.3: Let A, B and C be three events. Show that $P[AB^cC^c] \geq P[A] - P[AB] - P[AC]$.

Solution:

$$\begin{aligned} A &= A(B \cup B^c)(C \cup C^c) \\ &= ABC \cup AB^cC \cup ABC^c \cup AB^cC^c \end{aligned}$$

and each of these four sets are disjoint. So

$$P[A] = P[ABC] + P[AB^cC] + P[ABC^c] + P[AB^cC^c]$$

and hence

$$\begin{aligned} P[AB^cC^c] &= P[A] - P[ABC] - P[AB^cC] - P[ABC^c] \\ &= P[A] - (P[ABC] + P[AB^cC]) - P[ABC^c] \\ &= P[A] - P[ABC \cup AB^cC] - P[ABC^c] \\ &= P[A] - P[AC] - P[ABC^c] \\ &\geq P[A] - P[AC] - P[AB] \end{aligned}$$

where we used the fact that $P[AB] \geq P[ABC^c]$.

Example T1.4: An urn has 10 balls numbered from 1 to 10.

If we draw 3 balls at random, what is the probability that 1 is even and 2 are odd?

Solution: a) Order matters.

Let S_1 be all possible ways to pick 3 balls in order.
Let $E_1 \subset S_1$ be the event that 1 ball is even and 2 are odd.

Then $|S_1| = 10 \times 9 \times 8 = 720$ possible outcomes.

Case 1: 1st ball is even; there are $5 \times 5 \times (5 - 1) = 100$ ways.

Case 2: 2nd ball is even; there are $5 \times 5 \times (5 - 1) = 100$ ways.

Case 3: 3rd ball is even; there are $5 \times (5 - 1) \times 5 = 100$ ways.

$$\Rightarrow \qquad \text{prob} = \frac{3 \times 100}{720} = 5/12$$

b) Order does not matter

Let S_2 be all possible ways to pick 3 balls disregarding order.
Let $E_2 \subset S_2$ be the event that 1 ball is even and 2 are odd.

For each unordered $s_2 \in S_2$, there are 6 ways that the 3 balls can be ordered, and 6 ordered outcomes in S_1.

e.g., $\{2, 5, 6\} \in S_2$ corresponds to
$\{(2, 5, 6), (2, 6, 5), (5, 2, 6), (5, 6, 2), (6, 2, 5), (6, 5, 2)\} \in S_1$.

For each unordered $x_2 \in E_2$, there are 6 ways that the 3 balls can be ordered, and 6 ordered outcomes in E_1.

So

$$\begin{aligned} |S_2| &= \frac{|S_1|}{6} = \frac{720}{6} = \binom{10}{3} \\ |E_2| &= \frac{|E_1|}{6} = \frac{300}{6} = \binom{5}{2}\binom{5}{1} \end{aligned}$$

and therefore

$$P[E_2] = \frac{|E_2|}{|S_2|} = \frac{|E_1|/6}{|S_1|/6} = \frac{|E_1|}{|S_1|} = P[E_1] = 5/12$$

So here, "whether order matters or not" does not matter.

The reason is that the event "one ball is even and 2 are odd" does not depend on order. So we are free to create a sample space that is unordered (S_2), or ordered (S_1), and the only difference is that there are 6 ordered sequences for each unordered sequence.

If the event had been "one ball is even, 2 are odd, and first was odd", we would have no choice but to pick an ordered sample space.

Example T1.5: Show that $P[A \cup B \cup C] = P[A] + P[A^cB] + P[A^cB^cC]$

Solution: We first show that $A \cup B = P[A] + P[A^cB]$. Now

$$\begin{aligned} A \cup B &= A \cup (\ SB\ ) \\ &= A \cup (\ (A \cup A^c)B\ ) \\ &= A \cup (\ AB \cup A^cB\ ) \\ &= (\ A \cup AB\ ) \cup A^cB \\ &= A \cup A^cB \end{aligned}$$

Since A and A^cB are disjoint, then

$$P[A \cup B] = P[A \cup A^cB] = P[A] + P[A^cB] \qquad (*)$$

To get the desired result, we apply $(*)$ twice:

$$\begin{aligned} P[A \cup B \cup C] &= P[A \cup (B \cup C)] \\ &= P[A] + P[A^c(B \cup C)] \\ &= P[A] + P[A^cB \cup A^cC] \\ &= P[A] + P[A^cB] + P[(A^cB)^cA^cC] \\ &= P[A] + P[A^cB] + P[(A \cup B^c)A^cC] \\ &= P[A] + P[A^cB] + P[AA^cC \cup A^cB^cC] \\ &= P[A] + P[A^cB] + P[\emptyset \cup A^cB^cC] \\ &= P[A] + P[A^cB] + P[A^cB^cC] \end{aligned}$$