

Tutorial T2

Example T2.1: Two fair four-sided die are rolled. What is the probability that both die roll the same number given that the sum is 3 or less?

Solution: Here

$$S = \{(a, b) \mid a = 1, \dots, 4, b = 1, \dots, 4\}$$

$$A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$B = \{(1, 1), (1, 2), (2, 1)\}$$

So

$$P[A|B] = P[AB]/P[B] = P[(1, 1)]/P[B] = 1/3.$$

Example T2.2: A pouch contains a four sided die and a six sided die. You pick one of the die at random and roll it.

a) What is the probability of rolling i , for $i = 1, 2, \dots, 6$?

b) If you roll a 3, what is the probability that you picked the 6-sided die?

Solution: Let

$$A = \{4\text{-sided die picked}\}$$

$$B = \{6\text{-sided die picked}\}$$

$$E_i = \{\text{roll is } i\}.$$

i) By law of total probability,

$$P[E_i] = P[E_i|A]P[A] + P[E_i|B]P[B]$$

$$P[E_1] = 1/4 \times 1/2 + 1/6 \times 1/2$$

$$P[E_2] = 1/4 \times 1/2 + 1/6 \times 1/2$$

$$P[E_3] = 1/4 \times 1/2 + 1/6 \times 1/2$$

$$P[E_4] = 1/4 \times 1/2 + 1/6 \times 1/2$$

$$P[E_5] = 0 \times 1/2 + 1/6 \times 1/2$$

$$P[E_6] = 0 \times 1/2 + 1/6 \times 1/2$$

ii)

$$\begin{aligned} P[B|E_3] &= \frac{P[E_3|B]}{P[E_3]} \\ &= \frac{P[E_3|B]P[B]}{P[E_3|B]P[B] + P[E_3|A]P[A]} \\ &= \frac{1/6 \times 1/2}{1/6 \times 1/2 + 1/4 \times 1/2} \\ &= \frac{2}{5} \end{aligned}$$

Example T2.3: Let V be the event that a random person is vaccinated against a disease, and D the event that the random person develops the disease. The effectiveness e of a vaccine is defined to be

$$e = \frac{P[D|V^c] - P[D|V]}{P[D|V^c]} = 1 - \frac{P[D|V]}{P[D|V^c]}$$

Say 80% of the population is vaccinated and 40% of all cases of the disease occur in people who are vaccinated. What can you say about e ?

[Note: See <https://www.cdc.gov/csels/dsepd/ss1978/lesson3/section6.html>]

Solution: We are given

$$P[V] = 0.8$$

$$P[V|D] = 0.4$$

We want

$$e = 1 - \frac{P[D|V]}{P[D|V^c]}$$

$$= 1 - \frac{P[D|V]}{P[D|V^c]} \frac{P[V]}{P[V^c]} \frac{P[V^c]}{P[V]}$$

$$= 1 - \frac{P[D|V]}{P[D|V^c]} \frac{P[V^c]}{P[V]}$$

$$= 1 - \frac{P[V|D]P[D]}{P[V^c|D]P[D]} \frac{P[V^c]}{P[V]}$$

$$= 1 - \frac{P[V|D]}{P[V^c|D]} \frac{P[V^c]}{P[V]}$$

$$= 1 - \frac{0.4}{0.6} \frac{0.2}{0.8}$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

In this case, being vaccinated reduces by $5/6$ the probability of getting the disease compared to not being vaccinated.

Example T2.4: Alice and Bob have a pair of biased coins:

$$P[hh] = 0.2 \quad P[ht] = 0.3$$

$$P[th] = 0.3 \quad P[tt] = 0.2$$

Let

$$A = \{\text{first coin is heads}\}$$

$$B = \{\text{second coin is heads}\}$$

a) Are the events $\{hh\}, \{ht\}, \{th\}, \{tt\}$ independent?

b) Are the events A and B independent?

c) Can you find two events that are independent (other than the trivial case where one of C or D is \emptyset or S)?

d) Alice and Bob want to create/emulate a fair coin flip from this pair of coins. Is there a way to do so?

Solution: a) Deriving Baye's rule from first principles:

a) No, they are not. If I know hh has occurred, I know that ht has not occurred. So hh and ht are not independent. So the 4 events are not independent. Specifically:

$$P[hh]P[ht] = 0.2 \times 0.3 \neq 0 = P[\{hh\} \cap \{ht\}]$$

b) No.

$$P[A] = 0.2 + 0.3$$

$$P[B] = 0.2 + 0.3$$

but $P[A]P[B] = 0.5 \times 0.5 \neq 0.2 = P[AB] = P[hh]$.

c) Let $C = \{ht, hh\}$ and $D = \{hh, tt\}$. Then

$$P[C]P[D] = 0.5 \times 0.4 = 0.2 = P[hh] = P[CD]$$

Also, note that

$$P[C|D] = P[CD]/P[D] = 0.2/0.4 = 0.5 = P[C]$$

$$P[D|C] = P[CD]/P[C] = 0.2/0.5 = 0.4 = P[D]$$

d) This requires a bit of creative thinking:

Let $F = \{hh, tt\}$. Alice and Bob flip the pair of coins until event F occurs. Once it occurs, if both biased coins are heads, it is declared that the fair coin is heads. Otherwise both coins are tails, and the fair coin is declared to be tails.

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