# Tutorial T11

**Example T11.1:** Let $(X, Y)$ be jointly Gaussian with mean vector $\boldsymbol{\mu} = 0$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Let

$$U = X + 2Y$$
$$V = 2X + Y$$

a) What are $Cov[X, Y]$ and $E[XY]$?

c) What is the correlation $\rho(U, V)$?

*Solution:*

a) In Video 34, showed that $\rho(X, Y) = \rho$:

$$\rho = \rho(X, Y) = \frac{Cov[X, Y]}{\sigma_X\sigma_Y} = \frac{E[XY] - \overbrace{E[X]}^{=0}\overbrace{E[Y]}^{=0}}{\sigma_X\sigma_Y}$$

$$\Rightarrow Cov[X, Y] = \rho\sigma_X\sigma_Y$$
$$E[XY] = \rho\sigma_X\sigma_Y$$

b)

$$\sigma_X^2 = Var[X] = E[X^2] - (E[X])^2 = E[X^2]$$
$$\sigma_Y^2 = Var[Y] = E[Y^2] - (E[Y])^2 = E[Y^2]$$

c) We want

$$\rho(U, V) = \frac{Cov[U, V]}{\sqrt{Var[U]Var[V]}}$$

$$\begin{aligned} Cov[U, V] &= E[UV] - E[U]E[V] \\ &= E[(X + 2Y)(2X + Y)] - E[X + 2Y]E[Y + 2X] \\ &= E[2X^2 + 5XY + 2Y^2] - 0 \times 0 \\ &= 2E[X^2] + 5E[XY] + 2E[Y^2] \\ &= 2\sigma_X^2 + 5\rho\sigma_X\sigma_Y + 2\sigma_Y^2 \end{aligned}$$

$$\begin{aligned} Var[U] &= E[U^2] - (E[U])^2 \\ &= E[(X + 2Y)^2] - (E[X + 2Y])^2 \\ &= E[X^2 + 4XY + 4Y^2] - 0^2 \\ &= \sigma_X^2 + 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2 \end{aligned}$$

$$\begin{aligned} Var[V] &= E[V^2] - (E[V])^2 \\ &= E[(2X + Y)^2] - (E[2X + Y])^2 \\ &= E[4X^2 + 4XY + Y^2] - 0^2 \\ &= 4\sigma_X^2 + 4\rho\sigma_X\sigma_Y + \sigma_Y^2 \end{aligned}$$

**Example T11.2:** Let $X_1, X_2, \ldots$ be iid with $E[X_1] = 2$ and $E[X_1^2] = 5$. Let $N \sim Poisson(\lambda)$ and independent of $X_1, X_2, \ldots$.
Find $Var[X_1 X_2 \cdots X_N]$.

*Solution:*

Approach 1: We use the Conditional Variance Formula (also called Law of Total Variance)

$$Var[X] = E[\, Var[X|Y]\,] + Var[\, E[X|Y]\,]$$

with

$$X = X_1 X_2 \cdots X_N$$
$$Y = N$$

i) From Tutorial 10, $E[X|Y] = E[X_1 X_2 \cdots X_N|N] = 2^N$ so

$$\begin{aligned} Var[\, E[X|Y]\,] &= Var[2^N] \\ &= E[(2^N)^2] - (E[2^N])^2 \\ &= E[4^N] - (E[2^N])^2 \\ &= e^{3\lambda} - (e^\lambda)^2 \qquad \text{[See Tutorial 10]} \\ &= e^{3\lambda} - e^{2\lambda} \end{aligned}$$

ii)

$$\begin{aligned} Var[X|Y] \\ &= E[X^2|Y] - (\, E[X|Y]\,)^2 \\ &= E[X_1^2 X_2^2 \cdots X_N^2|N] - (\, E[X_1 X_2 \cdots X_N|N]\,)^2 \\ &= E[X_1^2 X_2^2 \cdots X_N^2|N] - (2^N)^2 \end{aligned}$$

$$\begin{aligned} E[X_1^2 X_2^2 \cdots X_N^2|N = n] \\ &= E[X_1^2 X_2^2 \cdots X_n^2|N = n] \\ &= E[X_1^2 X_2^2 \cdots X_n^2] \qquad \text{[Since } X_i\text{'s independent of } N\text{]} \\ &= E[X_1^2]E[X_2^2] \cdots E[X_n^2] \qquad \text{[By Proposition 31.1 ]} \\ &= 5^n \end{aligned}$$

$$\Rightarrow E[X_1^2 X_2^2 \cdots X_N^2|N] = 5^N$$

$$\begin{aligned} \Rightarrow E[\, Var[X|Y]\,] \\ &= E[5^N - 4^N] \\ &= e^{4\lambda} - e^{3\lambda} \end{aligned}$$

Combining

$$Var[X] = e^{4\lambda} - e^{3\lambda} + (e^{3\lambda} - e^{2\lambda}) = e^{4\lambda} - e^{2\lambda}$$

Approach 2: More direct calculation

$$Var[X] = E[X^2] - (E[X])^2$$

$$\begin{aligned} E[X] &= E[\, E[X|Y]\,] \\ &= E[\, E[X_1 X_2 \cdots X_N|N]\,] \\ &= E[\, 2^N\,] \qquad \text{[From Tutorial 10]} \\ &= e^\lambda \qquad \text{[From Tutorial 10]} \end{aligned}$$

$$\begin{aligned} E[X^2] &= E[\, E[X^2|Y]\,] \\ &= E[\, E[X_1^2 X_2^2 \cdots X_N^2|N]\,] \\ &= E[\, 5^N\,] \qquad \text{[From method 1]} \\ &= e^{4\lambda} \end{aligned}$$

$$\Rightarrow Var[X] = e^{4\lambda} - (e^\lambda)^2$$

**Example T11.3:** Compute the MGF of $X$ where

$$f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x \geq 0 \\ 0 & \text{else,} \end{cases}$$

and use the result to compute $E[X]$ and $E[X^2]$.

*Solution:*

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \lambda^2 \int_0^\infty x e^{-\lambda x} e^{tx} dx \\ &= \lambda^2 \int_0^\infty x e^{(t-\lambda)x} dx \\ &= \lambda^2 \left[ \frac{xe^{(t-\lambda)x}}{(t-\lambda)} - \frac{e^{(t-\lambda)x}}{(t-\lambda)^2} \right]_{x=0}^{x=\infty} \qquad \text{[From (P14b) of HW0 with } a = t - \lambda\text{]} \\ &= \frac{\lambda^2}{(t-\lambda)^2} \qquad \text{[for } t < \lambda\text{]} \end{aligned}$$

$$\begin{aligned} E[X] &= M_X'(t)|_{t=0} \\ &= -2\lambda^2(t-\lambda)^{-3}|_{t=0} \\ &= 2\lambda^{-1} \end{aligned}$$

$$\begin{aligned} E[X^2] &= M_X''(t)|_{t=0} \\ &= 6\lambda^2(t-\lambda)^{-4}|_{t=0} \\ &= 6\lambda^{-2} \end{aligned}$$