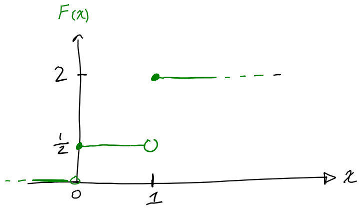
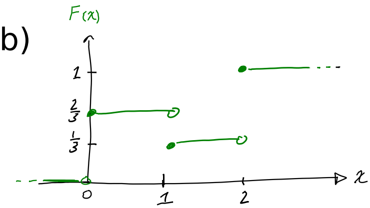
Tutorial T4

Example T4.1: Consider the following functions $F(x)$. Can $F(x)$ be the CDF of a random variable. If so, why? If not, why not?

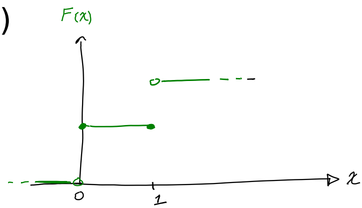
a)



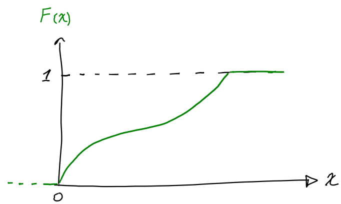
b)



c)



d)



Solution: a) Not a CDF. This is because $\lim_{x\to\infty} F(x) = 2 \neq 1$.

b) Not a CDF. This is because $F(x)$ is not non-decreasing.

c) Not a CDF. This is because $F(x)$ must be continuous from the right, but $F(1) \neq \lim_{x\downarrow 1} F(x)$.

d) This is a CDF. This is because $F(x)$ satisfies
i) $\lim_{x\to-\infty} F(x) = 0$,
ii) $\lim_{x\to\infty} F(x) = 1$,
iii) $F(x)$ is continuous from the right, and has limits from the left (in fact, this $F(x)$ is continuous everywhere).

Example T4.2: Let N be the number of calls received at a call center over a period of t seconds.

a) What distribution do you think would be a good model for N and why?

b) Now assume that N is Poisson with parameter $\lambda = \alpha t$ where α is the average number of calls per second. Say $\alpha = 0.2$ calls/sec.

i) What is the probability of more than 3 calls over a period of 10 seconds?

ii) What is the probability of exactly 12 calls in one minute?

Solution:

a) There are large number of potential callers, but each has a small probability of calling over the period of t seconds, and the product of these two is a moderate number. So a Poisson distribution is likely a good model.

b) i) Here, N is Poisson with parameter $\lambda = 0.2 \times 10 = 2.0$.

$$\begin{aligned}
P[N > 3] &= 1 - P[N = 0] - P[N = 1] - P[N = 2] - P[N = 3] \\
&= 1 - \frac{\lambda^0}{0!}e^{-\lambda} - \frac{\lambda^1}{1!}e^{-\lambda} - \frac{\lambda^2}{2!}e^{-\lambda} - \frac{\lambda^3}{3!}e^{-\lambda} \\
&= 1 - \frac{1}{1}e^{-2} - \frac{2}{1}e^{-2} - \frac{2^2}{2}e^{-2} - \frac{2^3}{6}e^{-2} \\
&= 1 - \frac{19}{3}e^{-2}
\end{aligned}$$

ii) Here, N is Poisson with parameter $\lambda = 0.2 \times 60 = 12$.

$$\begin{aligned}
P[N = 12] &= \frac{12^{12}}{12!}e^{-12} \\
&\approx 0.11437
\end{aligned}$$

Example T4.3: You have an urn with 3 balls: one red, one green and one blue. You keep drawing balls until you draw the green one. Let X be the number of draws.

What is the PMF of X? What is $E[X]$?

Solution: We recognize this as a geometric random variable. If we draw the green ball, we have a "success", and this occurs with probability 1/3 at each draw. Otherwise we have a "failure" and this occurs with proability 2/3 at each draw.

$$X \sim \mathsf{Geometric}(1/3)$$

If $X \sim \mathsf{Geometric}(p)$ then $E[X] = 1/p$ so $E[X] = 3$.

Example T4.4: A person types an incorrect word with probability p, independently of any other word.

a) What is the distribution of the number of error-free words between errors?

b) If you want to be 99% sure there are 100 correct words (or more) between incorrect ones, what should p be?

Solution:

a) If there are exactly n error-free words, then we have n error-free words followed by an error. This has probability:

$$P[N = n] = \begin{cases} (1-p)^n p & n \geq 0 \\ 0 & \text{else} \end{cases}$$

(This is not exactly the same as a Geometric rv.)

b) We want

$$\sum_{n \geq 100} P[N = n] \geq 0.99$$

which is equivalent to

$$\sum_{n=0}^{99} P[N = n] \leq 0.01$$

or

$$\begin{aligned} 0.01 &\geq p \sum_{n=0}^{99} (1-p)^n \\ &= p \frac{1-(1-p)^{100}}{1-(1-p)} \\ &= 1-(1-p)^{100} \end{aligned}$$

So we want $(1-p)^{100} \geq 0.99$ or equivalently $1-p \geq (0.99)^{1/100}$ or

$p \leq 1-(0.99)^{1/100} \approx 0.005$