Tutorial T7

Example T7.1: Let the joint pdf of X and Y be given by

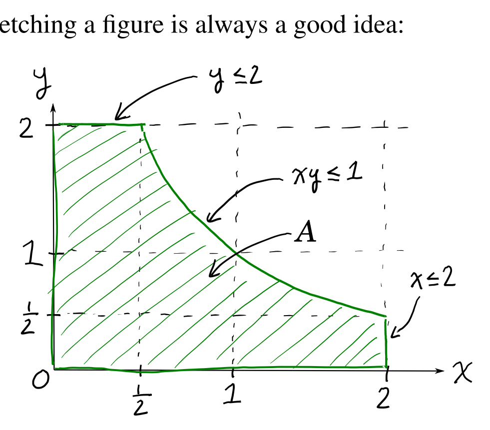
$$f_{XY}(x,y) = \begin{cases} c(1-xy) & 0 \le x \le 2, 0 \le y \le 2, xy \le 1 \\ 0 & \text{else} \end{cases}$$

a) What is c?

b) What is $P[X > 1]$?

c) What is $P[X > Y]$?

Solution: a) Sketching a figure is always a good idea:



$$\begin{aligned}
1 &= \iint_A f_{XY}(x,y)dxdy \\
&= c\int_0^{1/2}\int_0^2 (1-xy)dydx + c\int_{1/2}^2\int_0^{1/x}(1-xy)dydx \\
&= c\int_0^{1/2}\left[y - \frac{xy^2}{2}\right]_{y=0}^{y=2} dx + c\int_{1/2}^2 \left[y - \frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx \\
&= c\int_0^{1/2}[2-2x]dx + c\int_{1/2}^2\left[\frac{1}{x} - \frac{1}{2x}\right]dx \\
&= c[2x - x^2]_0^{1/2} + \frac{c}{2}[\ln x]_{1/2}^2 \\
&= c \times \frac{3}{4} + c\ln 2
\end{aligned}$$

So $c = (3/4 + \ln 2)^{-1}$.

b)

$$\begin{aligned}
P[X > 1] &= \iint_{x>1} f_{XY}(x,y)dxdy \\
&= c\int_{x=1}^2\int_{y=0}^{1/x}(1-xy)dydx \\
&= c\int_1^2\left[y - \frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx \\
&= c\int_1^2\left[\frac{1}{x} - \frac{1}{2x}\right]dx \\
&= \frac{c}{2}[\ln x]_1^2 \\
&= \frac{c}{2}\ln 2
\end{aligned}$$

c) Hard way:

$$\begin{aligned}
P[X > Y] &= \iint_{x>y} f_{XY}(x,y)dxdy \\
&= \int_0^\infty\int_0^x f_{XY}(x,y)dydx \\
&= c\int_0^1\int_0^x(1-xy)dydx + c\int_1^2\int_0^{1/x}(1-xy)dydx \\
&= \cdots
\end{aligned}$$

Easy way:

$$\begin{aligned}
1 &= P[X > Y] + P[X = Y] + P[Y > X] \\
&= P[X > Y] + P[Y > X] && \text{since } x = y \text{ has 0 area in } x-y \text{ plane} \\
&= P[X > Y] + P[X > Y] && \text{by symmetry of pdf w.r.t. } x = y
\end{aligned}$$

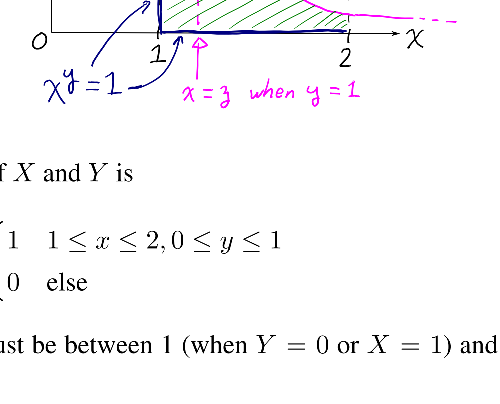
So $P[X > Y] = 1/2$.

Example T7.2: Let $X \sim U(1,2)$ and $Y \sim U(0,1)$ be independent rvs. Let

$$Z = X^Y$$

What is the pdf of Z?

Solution:



The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} 1 & 1 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Note that Z must be between 1 (when $Y = 0$ or $X = 1$) and 2 (when $X = 2, Y = 1$). So

$$F_Z(z) = \begin{cases} 0 & z \le 1 \\ 1 & z \ge 2 \end{cases}$$

This leaves the case that $1 < z < 2$:

$$\begin{aligned}
P[Z \le z] &= P[X^Y \le z] \\
&= P[Y \ln X \le \ln z] \\
&= P[Y \le \frac{\ln z}{\ln X}]
\end{aligned}$$

Let

$$\begin{aligned}
A &= \{(x,y) \in \mathbb{R}^2 | \, 1 \le x \le 2, 0 \le y \le 1\} \\
B &= \left\{(x,y) \in A \mid y \le \frac{\ln z}{\ln x}\right\}
\end{aligned}$$

$$\begin{aligned}
P[Z \le z] &= \iint_{(x,y)\in B} f_{XY}(x,y)dxdy \\
&= \int_1^z\int_0^1 1dydx + \int_z^2\int_0^{\frac{\ln z}{\ln x}} 1dydx \\
&= z - 1 + \int_z^2 \frac{\ln z}{\ln x}dx
\end{aligned}$$

This integral can't be simplified further. To get the pdf,

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} 0 & z \le 1 \\ 0 & z \ge 2 \end{cases}$$

For $1 < z < 2$:

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_Z(z) \\
&= \frac{d}{dz}\left[z - 1 + \int_z^2 \frac{\ln z}{\ln x}dx\right] \\
&= 1 + \frac{d}{dz}\int_z^2\frac{\ln z}{\ln x}dx \\
&= 1 - \frac{\ln z}{\ln z} \times \frac{d}{dz}z + \int_z^2 \frac{d}{dz}\frac{\ln z}{\ln x}dx && \text{by Leibniz's rule} \\
&= \frac{1}{z}\int_z^2 \frac{1}{\ln x}dx
\end{aligned}$$

Example T7.3: A particle is known to be in a cylinder. The pdf of the random location (R, Θ, Z) in cylindrical coordinates is

$$f(r,\theta,z) = \begin{cases} c(r - r^2)z & 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

a) What is the constant c?

b) What is $P[Z < 1/2]$?

Solution: We find c by requiring that f integrate to 1:

$$\begin{aligned}
1 &= \iiint\limits_{\substack{0\le\theta\le2\pi \\ 0\le r\le 1 \\ 0\le z\le 1}} f(r,\theta,z)drd\theta dz \\
&= \int_0^1\int_0^1\int_0^{2\pi} c(r-r^2)z \; d\theta dzdr \\
&= \int_0^1\int_0^1 2\pi c(r-r^2)z \; dzdr \\
&= \int_0^1 2\pi c(r-r^2)\left.\frac{z^2}{2}\right|_{z=0}^{z=1} dr \\
&= \int_0^1 \pi c(r-r^2)dr \\
&= \pi c\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_{r=0}^{r=1} \\
&= \pi c/6
\end{aligned}$$

So $c = 6/\pi$.

b)

$$\begin{aligned}
P[Z < 1/2] &= \iiint\limits_{z<1/2} f(r,\theta,z)drd\theta dz \\
&= \int_0^1\int_0^{1/2}\int_0^{2\pi} c(r-r^2)z \; d\theta dzdr \\
&= \int_0^1\int_0^{1/2} 2\pi c(r-r^2)z \; dzdr \\
&= \int_0^1 2\pi c(r-r^2)\left.\frac{z^2}{2}\right|_{z=0}^{z=1/2} dr \\
&= \int_0^1 \frac{\pi}{4}c(r-r^2)dr \\
&= \frac{\pi}{4}c\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_0^1 \\
&= \frac{\pi}{4}\frac{c}{6} \\
&= \frac{1}{4}
\end{aligned}$$

Question: why didn't we use $r\, drd\theta dz$ instead of $drd\theta dz$ as you were taught in calculus?

Answer: If I instead had:

$$f(u,v,w) = \begin{cases} c(u-u^2)w & 0 \le u \le 1, 0 \le v \le 2\pi, 0 \le w \le 1 \\ 0 & \text{else} \end{cases}$$

you wouldn't think twice before writing

$$1 = \iiint\limits_{\substack{0\le v\le2\pi \\ 0\le u\le 1 \\ 0\le w\le 1}} f(r,\theta,z)dudvdw.$$

Longer answer: the r in $r\, drd\theta dz$ is already in the pdf $f(r,\theta,z)$, so we don't need to add it. We will see this when we study "change of variables".