Tutorial T9

Example T9.1: Let R and Θ be two random variables with joint pdf $f_{R\Theta}(r, \theta)$ and consider the change of variables

$$\begin{aligned} X &= R\cos\Theta \\ Y &= R\sin\Theta. \end{aligned}$$

Find $f_{R\Theta}(r, \theta)$ in terms of $f_{XY}(x, y)$.

Solution: [See Ex 7b of Section 6.7 for a different tedious approach.]

Here, we have the system of equations

$$\begin{aligned} x &= g_1(r, \theta) = r\cos\theta \\ y &= g_2(r, \theta) = r\sin\theta \end{aligned}$$

Also note that

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = h_1(x, y) \\ \theta &= h_2(x, y) \end{aligned}$$

where $h_2(x, y)$ is the angle of the vector (x, y), i.e., $h_2(x, y) = \arctan(y/x)$ when $x > 0, y > 0$.

Computing the Jacobian

$$\begin{aligned} J(r, \theta) &= \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

So, the pdf $f_{X,Y}(x, y)$ and $f_{R\Theta}(r, \theta)$ are related by

$$f_{XY}(x, y) = f_{R\Theta}(r, \theta)\,|J(r, \theta)|^{-1} = f_{R\Theta}(r, \theta)/r \qquad \text{(T9.1)}$$

or equivalently:

$$\begin{aligned} f_{R\Theta}(r, \theta) &= f_{XY}(x, y)r && \text{(T9.2)} \\ &= f_{XY}(r\cos\theta, r\sin\theta)r && \text{(T9.3)} \end{aligned}$$

So, to compute the probability that $(R, \Theta) \in A$:

$$\begin{aligned} P[(R, \Theta) \in A] &= \iint_{(r,\theta)\in A} f_{R\Theta}(r, \theta)\,dr\,d\theta \\ &= \iint_{(r,\theta)\in A} f_{XY}(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \end{aligned}$$

Example T9.2: Two particles are positioned on a line with random and independent positions

$$\begin{aligned} X_1 &\sim \mathcal{N}(0, 1) \\ X_2 &\sim U[-1, 1] \end{aligned}$$

a) Find $E[X_1 - X_2]$.

b) Find $E[(X_1 - X_2)^2]$.

c) Find $E[(X_1 + X_2)^2]$.

Solution:

$$\begin{aligned} &a) \quad E[X_1 - X_2] = E[X_1] - E[X_2] = 0 - 0 = 0 \\ &b) \quad E[(X_1 - X_2)^2] \\ &\quad = E[X_1^2 - 2X_1X_2 + X_2^2] \\ &\quad = E[X_1^2] - 2E[X_1X_2] + E[X_2^2] \\ &\quad = 1 - 2E[X_1]E[X_2] + \frac{(1 - (-1))^2}{12} \\ &\quad = 1 - 2 \times 0 \times 0 + \frac{1}{3} \\ &\quad = \frac{4}{3} \\ &c) \quad E[(X_1 + X_2)^2] \\ &\quad = E[X_1^2 + 2X_1X_2 + X_2^2] \\ &\quad = E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\ &\quad = 1 + 2E[X_1]E[X_2] + \frac{1}{3} \\ &\quad = \frac{4}{3} \end{aligned}$$

Example T9.3: You flip n 4-sided dice with outcomes $X_1, X_2, \ldots, X_n$. Let

$$Y = X_1 + X_2 + \cdots + X_n$$

What is the mean and variance of Y?

Solution: The hard way is to try to compute the pmf of Y first, and then use the pmf to compute $E[Y]$, etc. Instead, we use properties of expectations:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= n \times \frac{5}{2} \end{aligned}$$

$$\begin{aligned} E[Y^2] &= E\left[(X_1 + X_2 + \cdots + X_n)^2\right] \\ &= E\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{m=1}^{n} X_m\right)\right] \\ &= E\left[\sum_{k=1}^{n}\sum_{m=1}^{n} X_kX_m\right] \\ &= E\left[\sum_{k=1}^{n} X_k^2 + \sum\sum_{k\neq m} X_kX_m\right] \\ &= \sum_{k=1}^{n} E[X_k^2] + \sum\sum_{k\neq m} E[X_kX_m] \\ &= \sum_{k=1}^{n}\left(\frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4}\right) + \sum\sum_{k\neq m}\frac{5}{2}\frac{5}{2} \\ &= n\frac{30}{4} + n(n-1)\frac{25}{4} \end{aligned}$$

$$\begin{aligned} Var[Y] &= E[Y^2] - E[Y]^2 \\ &= n\frac{30}{4} + n(n-1)\frac{25}{4} - n^2\frac{25}{4} \\ &= n\frac{5}{4} \end{aligned}$$

Note: Since the X_k are independent, in Video 31 we will see that

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] = nVar[X_1] = n\frac{5}{4}$$

which is a much simpler solution.