Tutorial T11

Example T11.1: Let (X, Y) be jointly Gaussian with mean vector $\boldsymbol{\mu} = 0$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Let

$$U = X + 2Y$$
$$V = 2X + Y$$

a) What are $Cov[X, Y]$ and $E[XY]$?

c) What is the correlation $\rho(U, V)$?

Solution:

a) In Video 34, showed that $\rho(X, Y) = \rho$:

$$\rho = \rho(X, Y) = \frac{Cov[X, Y]}{\sigma_X \sigma_Y} = \frac{E[XY] - \overbrace{E[X]}^{=0}\,\overbrace{E[Y]}^{=0}}{\sigma_X \sigma_Y}$$

$$\Rightarrow Cov[X, Y] = \rho\sigma_X\sigma_Y$$
$$E[XY] = \rho\sigma_X\sigma_Y$$

b)

$$\sigma_X^2 = Var[X] = E[X^2] - (E[X])^2 = E[X^2]$$
$$\sigma_Y^2 = Var[Y] = E[Y^2] - (E[Y])^2 = E[Y^2]$$

c) We want

$$\rho(U,V) = \frac{Cov[U,V]}{\sqrt{Var[U]Var[V]}}$$

$$\begin{aligned}
Cov[U,V] &= E[UV] - E[U]E[V] \\
&= E[(X+2Y)(2X+Y)] - E[X+2Y]E[Y+2X] \\
&= E[2X^2 + 5XY + 2Y^2] - 0 \times 0 \\
&= 2E[X^2] + 5E[XY] + 2E[Y^2] \\
&= 2\sigma_X^2 + 5\rho\sigma_X\sigma_Y + 2\sigma_Y^2
\end{aligned}$$

$$\begin{aligned}
Var[U] &= E[U^2] - (E[U])^2 \\
&= E[(X+2Y)^2] - (E[X+2Y])^2 \\
&= E[X^2 + 4XY + 4Y^2] - 0^2 \\
&= \sigma_X^2 + 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2
\end{aligned}$$

$$\begin{aligned}
Var[V] &= E[V^2] - (E[V])^2 \\
&= E[(2X+Y)^2] - (E[2X+Y])^2 \\
&= E[4X^2 + 4XY + Y^2] - 0^2 \\
&= 4\sigma_X^2 + 4\rho\sigma_X\sigma_Y + \sigma_Y^2
\end{aligned}$$

Example T11.2: Let $X_1, X_2, \ldots$ be iid with $E[X_1] = 2$ and $E[X_1^2] = 5$. Let $N \sim \mathsf{Poisson}(\lambda)$ and independent of $X_1, X_2, \ldots$.

Find $Var[X_1 X_2 \cdots X_N]$.

Solution:

Approach 1: We use the Conditional Variance Formula (also called Law of Total Variance)

$$Var[X] = E[\ Var[X|Y]\ ] + Var[\ E[X|Y]\ ]$$

with

$$X = X_1 X_2 \cdots X_N$$
$$Y = N$$

i) From Tutorial 10, $E[X|Y] = E[X_1 X_2 \cdots X_N|N] = 2^N$ so

$$\begin{aligned}
Var[\ E[X|Y]\ ] &= Var[2^N] \\
&= E[(2^N)^2] - (E[2^N])^2 \\
&= E[4^N] - (E[2^N])^2 \\
&= e^{3\lambda} - (e^{\lambda})^2 && \text{[See Tutorial 10]} \\
&= e^{3\lambda} - e^{2\lambda}
\end{aligned}$$

ii)

$$\begin{aligned}
&Var[X|Y] \\
&\quad = E[X^2|Y] - (\ E[X|Y]\ )^2 \\
&\quad = E[X_1^2 X_2^2 \cdots X_N^2|N] - (\ E[X_1 X_2 \cdots X_N|N]\ )^2 \\
&\quad = E[X_1^2 X_2^2 \cdots X_N^2|N] - (2^N)^2
\end{aligned}$$

$$\begin{aligned}
&E[X_1^2 X_2^2 \cdots X_N^2|N = n] \\
&\quad = E[X_1^2 X_2^2 \cdots X_n^2|N = n] \\
&\quad = E[X_1^2 X_2^2 \cdots X_n^2] && \text{[Since } X_i\text{'s independent of } N\text{]} \\
&\quad = E[X_1^2]E[X_2^2] \cdots E[X_n^2] && \text{[By Proposition 31.1]}
\end{aligned}$$

$$= 5^n$$

$$\Rightarrow E[X_1^2 X_2^2 \cdots X_N^2 | N] = 5^N$$

$$\begin{aligned}
\Rightarrow & E[\ Var[X|Y]\ ] \\
&= E[5^N - 4^N] \\
&= e^{4\lambda} - e^{3\lambda}
\end{aligned}$$

Combining

$$Var[X] = e^{4\lambda} - e^{3\lambda} + (e^{3\lambda} - e^{2\lambda}) = e^{4\lambda} - e^{2\lambda}$$

Approach 2: More direct calculation

$$Var[X] = E[X^2] - (E[X])^2$$

$$\begin{aligned}
E[X] &= E[\ E[X|Y]\ ] \\
&= E[\ E[X_1 X_2 \cdots X_N | N]\ ] \\
&= E[\ 2^N\ ] && \text{[From Tutorial 10]} \\
&= e^{\lambda} && \text{[From Tutorial 10]}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= E[\ E[X^2|Y]\ ] \\
&= E[\ E[X_1^2 X_2^2 \cdots X_N^2 | N]\ ] \\
&= E[\ 5^N\ ] && \text{[From method 1]} \\
&= e^{4\lambda}
\end{aligned}$$

$$\Rightarrow Var[X] = e^{4\lambda} - (e^{\lambda})^2$$

Example T11.3: Compute the MGF of X where

$$f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x \geq 0 \\ 0 & \text{else,} \end{cases}$$

and use the result to compute $E[X]$ and $E[X^2]$.

Solution:

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\
&= \lambda^2 \int_0^{\infty} x e^{-\lambda x} e^{tx} dx \\
&= \lambda^2 \int_0^{\infty} x e^{(t-\lambda)x} dx \\
&= \lambda^2 \left[\frac{x e^{(t-\lambda)x}}{(t-\lambda)} - \frac{e^{(t-\lambda)x}}{(t-\lambda)^2} \right]_{x=0}^{x=\infty} && \text{[From P14b) of HW0 with } a = t - \lambda] \\
&= \frac{\lambda^2}{(t-\lambda)^2} && [\text{for } t < \lambda]
\end{aligned}$$

$$\begin{aligned}
E[X] &= M'_X(t)|_{t=0} \\
&= -2\lambda^2 (t-\lambda)^{-3}|_{t=0} \\
&= 2\lambda^{-1}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= M''_X(t)|_{t=0} \\
&= 6\lambda^2 (t-\lambda)^{-4}|_{t=0} \\
&= 6\lambda^{-2}
\end{aligned}$$