Tutorial T10

Example T10.1: Two particles are positioned on a line with random and independent positions

$$\begin{aligned} X_1 &\sim \mathcal{N}(0,1) \\ X_2 &\sim U[-1,1] \end{aligned}$$

Find $E[(X_1+X_2)^3]$.

Solution:

$$\begin{aligned} E[(X_1+X_2)^3] &= E[X_1^3 + 3X_1^2X_2 + 3X_1^1X_2^2 + X_2^3] \\ &= E[X_1^3] + 3E[X_1^2X_2] + 3[X_1^1X_2^2] + E[X_2^3] \\ &= E[X_1^3] + 3E[X_1^2]E[X_2] + 3[X_1^1]E[X_2^2] + E[X_2^3] && \text{(a)} \\ &= E[X_1^3] + 3E[X_1^2] \times 0 + 3 \times 0 \times E[X_2^2] + E[X_2^3] \\ &= E[X_1^3] + E[X_2^3] \\ &= 0 && \text{(b)} \end{aligned}$$

(a) follows because if X_1 and X_2 are independent, then $E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)]$ (Prop. 31.1),

(b) follows from

$$\begin{aligned} E[X_2^3] &= \int_{-1}^{1} x^3 \times \frac{1}{2} dx \\ &= 0 && \text{by (a)symmetry} \end{aligned}$$

$$E[X_1^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx$$
$$= 0 \qquad \text{by (a)symmetry}$$

Example T10.2: Let $X_1, X_2, \ldots, X_n$ be independent and $\sim \mathcal{N}(0,1)$. Let

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$
$$Z = b_1 X_1 + b_2 X_2 + \cdots + b_n X_n$$

Find $Cov[Y, Z]$.

Solution:

$$\begin{aligned}
Cov[Y,Z] &= Cov\left[\sum_{k=1}^{n} a_k X_k, \sum_{m=1}^{n} b_m X_m\right] \\
&= \sum_{k=1}^{n} \sum_{m=1}^{n} Cov[a_k X_k, b_m X_m] \\
&= \sum_{k=1}^{n} \sum_{m=1}^{n} a_k Cov[X_k, b_m X_m] \\
&= \sum_{k=1}^{n} \sum_{m=1}^{n} a_k b_m Cov[X_k, X_m]
\end{aligned}$$

Since $X_1, X_2, \ldots, X_n$ are independent, $Cov[X_k, X_m] = 0$ when $k \neq m$:

$$\begin{aligned}
Cov[Y,Z] &= \sum_{k=1}^{n} \left[\sum_{m=1}^{n} a_k b_m Cov[X_k, X_m]\right] \\
&= \sum_{k=1}^{n} \left[\sum_{m=k}^{k} a_k b_m Cov[X_k, X_m]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} a_k b_k Cov[X_k, X_k] \\
&= \sum_{k=1}^{n} a_k b_k Var[X_k] \\
&= \sum_{k=1}^{n} a_k b_k
\end{aligned}
$$

Example T10.3: Let X be a rv and $Y = aX + Z$ where X and Z are uncorrelated. What is the correlation coefficient between X and Y?

Solution:

$$
\rho(X, Y) = \frac{Cov[X, Y]}{\sqrt{Var[X]\ Var[Y]}}
$$

$$
\begin{aligned}
Cov[X, Y] &= Cov[X, aX + Z] \\
&= aCov[X, X] + Cov[X, Z] \\
&= aVar[X]
\end{aligned}
$$

$$
\begin{aligned}
Var[Y] &= Var[aX + Z] \\
&= Var[aX] + Var[Z] + 2Cov[aX, Z] \\
&= a^2 Var[X] + Var[Z]
\end{aligned}
$$

$$
\begin{aligned}
\rho(X, Y) &= \frac{aVar[X]}{\sqrt{Var[X]\ (a^2 Var[X] + Var[Z])}} \\
&= \frac{a\sqrt{Var[X]}}{\sqrt{a^2 Var[X] + Var[Z]}}
\end{aligned}
$$

$$= \frac{\sqrt{a^2 Var[X]}}{\sqrt{a^2 Var[X] + Var[Z]}}$$
$$= \frac{1}{\sqrt{1 + \frac{Var[Z]}{a^2 Var[X]}}}$$

Note: If

- $Var[Z] \ll a^2 Var[X]$ then $\rho(X,Y) \approx 1$
- $Var[Z] \gg a^2 Var[X]$ then $\rho(X,Y) \approx 0$
- $Var[Z] = a^2 Var[X]$ then $\rho(X,Y) = 1/\sqrt{2}$

Example T10.4: Let $X_1, X_2, \ldots$ be iid with $E[X_1] = 2$. Let $N \sim \mathsf{Poisson}(\lambda)$ and independent of $X_1, X_2, \ldots$

Find $E[X_1 X_2 \cdots X_N]$.

Solution: We use conditional probability:

$$E[X_1 X_2 \cdots X_N] = E[\ E[X_1 X_2 \cdots X_N | N]\ ]$$

$$\begin{aligned} E[X_1 X_2 \cdots X_N | N = n] &= E[X_1 X_2 \cdots X_n | N = n] \\ &= E[X_1 X_2 \cdots X_n] \\ &= E[X_1] E[X_2] \cdots E[X_n] \\ &= 2 \times 2 \times \cdots \times 2 \\ &= 2^n \end{aligned}$$

$$\Rightarrow E[X_1 X_2 \cdots X_N | N] = 2^N$$

$$
\begin{aligned}
E[X_1 X_2 \cdots X_N] &= E[2^N] \\
&= e^{\lambda} && \text{[See below for why]}
\end{aligned}
$$

$$
\begin{aligned}
E[a^N] &= \sum_{n=0}^{\infty} a^n P[N = n] \\
&= \sum_{n=0}^{\infty} a^n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(a\lambda)^n}{n!} \\
&= e^{-\lambda} e^{a\lambda} \\
&= e^{(a-1)\lambda}
\end{aligned}
$$