ECE 203: Probability and Statistics I
University of Waterloo

Homework No. 1 Solutions

Problem 1:
a) $S = \{(r,r),(r,g),(r,b),(g,r),(g,g),(g,b),(b,r),(b,g),(b,b)\}$
b) $S = \{(r,g),(r,b),(g,r),(g,b),(b,r),(b,g)\}$

Problem 2: This problem has a subtle difficulty. There are two types of sequences of outcomes: those where a 6 occurs at some point, and those where a 6 never occurs.
Suppose that the first 6 occurs on roll n. Then we can denote this (sequence) outcome by $(x_1,\ldots,x_{n-1})$ where $x_1,\ldots,x_{n-1} \in \{1,2,3,4,5\}$. So let

$$E_n = \{(x_1,\ldots,x_{n-1}) | x_1,\ldots,x_{n-1} \in \{1,2,3,4,5\}\}.$$

Now, suppose a 6 never occurs. Then we have an infinite sequence $x_1, x_2, \ldots$, where $x_i \in \{1,2,3,4,5\}$ for all $i \geq 1$. So let

$$F = \{(x_1,x_2,\ldots) | x_i \in \{1,2,3,4,5\} \ \forall i \geq 1\}.$$

Then

$$S = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup F$$

Then $(\cup_{n=1}^{\infty} E_n)^c = F$ is the event that 6 never appears.

Problem 3:
$EF = \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1)\}$
$E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1.
$FG = \{(1,4),(4,1)\}$.
EF^c is the event that neither of the dice lands on 1 and the sum is odd. So $EF^c = \{(2,3),(2,5),(3,2),(3,4),$ $(3,6),(4,3),(4,5),(5,2),(5,4),(5,6),(6,3),(6,5)\}$
$EFG = FG$.

Problem 5:
a) $2^5 = 32$
b) $W = \{(1,1,1,1,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),(1,1,1,0,0),(1,1,0,1,0),(1,1,0,0,1),(1,1,0,0,0),$ $(1,0,1,1,1),(0,1,1,1,1),(1,0,1,1,0),(0,1,1,1,0)(0,0,1,1,1)(0,0,1,1,0),(1,0,1,0,1)\}$
c) $2^3 = 8$
d) $AW = \{(1,1,1,0,0),(1,1,0,0,0)\}$

Problem 6:
a) $S = \{(1,g),(0,g),(1,f),(0,f),(1,s),(0,s)\}$
b) $A = \{(1,s),(0,s)\}$
c) $B = \{(0,g),(0,f),(0,s)\}$
d) $\{(1,s),(0,s),(1,g),(1,f)\}$

Problem 9:
Choose a customer at random. Let A denote the event that the customer carries an American Express card and V the event they carry a VISA card.

$P[A \cup V] = P[A] + P[V] - P[AV] = 0.24 + 0.61 - 0.11 = 0.74$

Problem 10: Let R and N denote the events that the randomly chosen student is wearing a ring and wearing a necklace.

a) By DeMorgan $P[R \cup N] = P[(R^c \cap N^c)^c] = 1 - P[R^c \cap N^c] = 1 - 0.6 = 0.4$

b) $0.4 = P[R \cup N] = P[R] + P[N] - P[RN] = 0.2 + 0.3 - P[RN]$
Thus, $P[RN] = 0.1$

Problem 12:
a) $P[S \cup F \cup G] = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $P[S^c F^c G^c] = 1 - P[S \cup F \cup G] = 1/2$.

b) We want $P[SF^cG^c \cup S^cFG^c \cup S^cF^cG]$
Since $P[SF] = 12/100$ and $P[SFG] = 2/100$ then $P[SFG^c] = 10/100$.
Since $P[FG] = 6/100$ and $P[SFG] = 2/100$ then $P[S^cFG] = 4/100$.
Since $P[SG] = 4/100$ and $P[SFG] = 2/100$ then $P[SF^cG] = 2/100$.

Then $P[SF^cG^c] = P[S] - P[SFG] - P[SF^cG] - P[SFG^c] = (28 - 2 - 2 - 10)/100 = 14/100$
Then $P[S^cFG^c] = P[F] - P[SFG] - P[S^cFG] - P[SFG^c] = (26 - 2 - 4 - 10)/100 = 10/100$
Then $P[S^cF^cG] = P[G] - P[SFG] - P[S^cFG] - P[SF^cG] = (16 - 2 - 4 - 2)/100 = 8/100$

So $P[SF^cG^c \cup S^cFG^c \cup S^cF^cG] = (14 + 10 + 8)/100 = 32/100$.

c) 50 students are not taking any courses, and 50 are taking at least 1 course.
The probability of picking 2 students that are not taking any courses are

$$\frac{\binom{50}{2}}{\binom{100}{2}}$$

So the probability that at least one is taking a course is

$$1 - \frac{\binom{50}{2}}{\binom{100}{2}}$$

Problem 14:
$P[M \cup W \cup G] = P[M] + P[W] + P[G] - P[MW] - P[MG] - P[WG] + P[MWG] = 0.312 + 0.470 + 0.525 - 0.086 - 0.042 - 0.147 + .025 = 1.057$ and this is greater than 1!

Problem 19: There are 36 possible outcomes.
4 result in red showing up on both dice, 4 result in black showing up on both dice, 1 results in yellow showing up on both dice and 1 results in white showing up on both dice.
Hence, the probability is $(4 + 4 + 1 + 1)/36$.

Problem 23: The answer is $5/12$, which can be seen as follows:

$1 = P[\text{first die is higher}] + P[\text{second die is higher}] + P[\text{die are the same}]$

By symmetry, $P[\text{first die is higher}] = P[\text{second die is higher}]$. Also, $P[\text{die are the same}] = 6/36$.
So

$$\begin{aligned}1 &= P[\text{first die is higher}] + P[\text{second die is higher}] + P[\text{die are the same}]\\ &= 2P[\text{first die is higher}] + 1/6\end{aligned}$$

and hence $P[\text{first die is higher}] = 5/12$.

Another way of solving is to list all the outcomes for which the second is higher. There are 5 outcomes when the first lands on 1, 4 outcomes when the first lands on 2, 3 outcomes when the first lands on 3, 2 outcomes when the first lands on 4, and 1 outcomes when the first lands on 5.

Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

Problem 24:
There is 1 way (i.e. $\{(1, 1)\}$) to get a sum of 2, so $P[\text{sum is } 2] = 1/36$
There are 2 ways (i.e. $\{(1, 2), (2, 1)\}$) to get a sum of 3, so $P[\text{sum is } 3] = 2/36$
$\vdots$
There are 6 ways (i.e. $\{(1, 6), \ldots, (6, 1)\}$) to get a sum of 7, so $P[\text{sum is } 6] = 6/36$
There are 5 ways (i.e. $\{(2, 6), \ldots, (6, 2)\}$) to get a sum of 8, so $P[\text{sum is } 5] = 5/36$
$\vdots$
There is 1 way (i.e. $\{(6, 6)\}$) to get a sum of 12, so $P[\text{sum is } 12] = 1/36$

Problem 36:
a) There are $\binom{52}{2}$ possible unique pairs of cards, and $\binom{4}{2}$ unique pairs of aces. So $\binom{4}{2}/\binom{52}{2}$.
b) Fix a value of the card. Then there are $\binom{4}{2}$ unique pairs where both cards have that value. There are 13 choices for the value of the card. So $13 \times \binom{4}{2}/\binom{52}{2}$.

Theoretical Exercises

Problem 3:

$$F = F(E \cup E^c) = FE \cup FE^c$$

Since $F = FE \cup FE^c$, then

$$\begin{aligned} E \cup F &= E \cup FE \cup FE^c \\ &= E \cup FE^c \end{aligned}$$

since $E \cup FE = E$.

Problem 10: First, note that

$$\begin{aligned} P[EF] &= P[EF(G \cup G^c] \\ &= P[EFG] + P[EFG^c] \end{aligned}$$

And likewise

$$\begin{aligned} P[EG] &= P[EFG] + P[EF^cG] \\ P[FG] &= P[EFG] + P[E^cFG] \end{aligned}$$

By the inclusion-exclusion principle and using the results just above:

$$\begin{aligned} P[E \cup F \cup G] &= P[E] + P[F] + P[G] - P[EF] - P[EG] - P[FG] + P[EFG] \\ &= P[E] + P[F] + P[G] \\ &\quad\; - (P[EFG] + P[EFG^c]) - (P[EFG] + P[EF^cG]) - (P[EFG] + P[E^cFG]) \\ &\quad\; + P[EFG] \\ &= P[E] + P[F] + P[G] - P[EFG^c] - P[EF^cG] - P[E^cFG] - 2P[EFG] \end{aligned}$$

Problem 11:

$1 \geq P[E \cup F] = P[E] + P[F] - P[EF]$

So, solving for $P[EF]$:

$P[EF] \geq P[E] + P[F] - 1$

Problem 12: We want the probability that E occurs (and F does not occur), or that F occurs (and E does not occur). Hence

$$\begin{aligned} P[EF^c \cup E^cF] &= P[EF^c] + P[E^cF] \\ &= P[E] - P[EF] + P[F] - P[EF] \end{aligned}$$

where we used the fact that $P[AB^c] = P[A] - P[AB]$ since $A = AB^c \cup AB$ and AB^c and Ab are disjoint.

Problem 13: This follows from

$$P[E] = P[E(F \cup F^c)] = P[EF] + P[EF^c]$$

where we used the fact that EF and EF^c are disjoint.