Tutorial T8

Example T8.1: A particle is known to be in a cylinder. The pdf of the random location (R, Θ, Z) in cylindrical coordinates is

$$f(r, \theta, z) = \begin{cases} c(1-r)rz & 0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

Are R, Θ and Z independent?

Solution:

Yes. This is because $f(r, \theta, z) = a(r)b(\theta)d(z)$ where

$$a(r) = \begin{cases} r - r^2 & 0 \le r \le 1 \\ 0 & \text{else} \end{cases}$$

$$b(\theta) = \begin{cases} c & 0 \le \theta \le 2\pi \\ 0 & \text{else} \end{cases}$$

$$c(z) = \begin{cases} z & 0 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

Example T8.2: Let X and Y be the outcomes of rolling a 4-sided and a 6-sided die. Let $Z = X + Y$. What is the pmf of Z?

Solution:

$$p_X(k) = \begin{cases} 1/4 & k = 1, \ldots, 4 \\ 0 & \text{else} \end{cases}$$

$$p_Y(k) = \begin{cases} 1/6 & k = 1, \ldots, 6 \\ 0 & \text{else} \end{cases}$$

$$
\begin{aligned}
P[Z = n] &= P[X + Y = n] \\
&= P[\cup_{k=-\infty}^{\infty} \{X = k, Y = n - k\}] \\
&= \sum_{k=-\infty}^{\infty} P[X = k, Y = n - k] \\
&= \sum_{k=-\infty}^{\infty} P[X = k]P[Y = n - k] \\
&= \sum_{k=1}^{4} P[X = k]P[Y = n - k] \qquad \text{[since } X \in \{1, 2, 3, 4\}] \\
&= P[X = 1]P[Y = n - 1] \\
&\quad + P[X = 2]P[Y = n - 2] \\
&\quad + P[X = 3]P[Y = n - 3] \\
&\quad + P[X = 4]P[Y = n - 4] \\
&= \frac{1}{4}P[Y = n - 1] + \frac{1}{4}P[Y = n - 2] \\
&\quad + \frac{1}{4}P[Y = n - 3] + \frac{1}{4}P[Y = n - 4]
\end{aligned}
$$

If $n \le 1$, then sum is 0.

If $n = 2$, then sum is $\frac{1}{4}\frac{1}{6} = \frac{1}{24}$

If $n = 3$, then sum is $\frac{2}{24}$

If $n = 4$, then sum is $\frac{3}{24}$

If $n = 5$, then sum is $\frac{4}{24}$

If $n = 6$, then sum is $\frac{4}{24}$

If $n = 7$, then sum is $\frac{4}{24}$

If $n = 8$, then sum is $\frac{3}{24}$

If $n = 9$, then sum is $\frac{2}{24}$

If $n = 10$, then sum is $\frac{1}{24}$

If $n \ge 11$, then sum is 0

Example T8.3: Let $X_1, \ldots, X_n$ be independent with $X_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$.

Then, we know (Proposition 26.1) that:
$X = X_1 + \cdots + X_n$ is $\sim \mathcal{N}(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$.

a) If $Z_1 = a_1X_1 + b_1, \ldots, Z_n = a_nX_n + b_n$, show that $Z_1, \ldots, Z_n$ are independent. [Assume $a_1 > 0, \ldots, a_n > 0$].

b) What is the pdf of $Z = (a_1X_1 + b_1) + \cdots + (a_nX_n + b_n)$?

Solution: a) Since $X_1, \ldots, X_n$ are independent:

$$F_{X_1, \ldots, X_n}(x_1, \ldots x_n) = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)$$

Let $Y \sim \mathcal{N}(0, 1)$. Since $X_k \sim \mathcal{N}(\mu_1, \sigma_1^2)$:

$$
\begin{aligned}
F_{X_k}(x_k) &= P[X_k \le x_k] \\
&= P\left[\frac{X_k - \mu_k}{\sigma_k} \le \frac{x_k - \mu_k}{\sigma_k}\right] \\
&= P\left[Y \le \frac{x_k - \mu_k}{\sigma_k}\right] \\
&= \Phi\left(\frac{x_k - \mu_k}{\sigma_k}\right)
\end{aligned}
$$

So

$$F_{X_1, \ldots, X_n}(x_1, \ldots x_n) = \Phi\left(\frac{x_1 - \mu_1}{\sigma_1}\right) \times \cdots \times \Phi\left(\frac{x_n - \mu_n}{\sigma_n}\right)$$

$$
\begin{aligned}
F_{Z_1, \ldots, Z_n}(z_1, \ldots z_n) &= P[Z_1 \le z_1, \ldots, Z_n \le z_n] \\
&= P[a_1X_1 + b_1 \le z_1, \ldots, a_nX_n + b_n \le z_n] \\
&= P\left[X_1 \le \frac{z_1 - b_1}{a_1}, \ldots, X_n \le \frac{z_n - b_n}{a_n}\right] \\
&= F_{X_1, \ldots, X_n}\left(\frac{z_1 - b_1}{a_1}, \ldots, \frac{z_n - b_n}{a_n}\right) \\
&= \Phi\left(\frac{\frac{z_1 - b_1}{a_1} - \mu_1}{\sigma_1}\right) \times \cdots \times \Phi\left(\frac{\frac{z_n - b_n}{a_n} - \mu_n}{\sigma_n}\right) \\
&= \Phi\left(\frac{z_1 - b_1 - a_1\mu_1}{a_1\sigma_1}\right) \times \cdots \times \Phi\left(\frac{z_n - b_n - a_n\mu_n}{a_n\sigma_n}\right)
\end{aligned}
$$

So $Z_1, \ldots, Z_n$ are independent with $Z_k \sim \mathcal{N}(b_k + a_k\mu_k, a_k^2\sigma_k^2)$.

b) Since $Z = Z_1 + \cdots Z_n$ and Z_k's are Normal and independent, we apply Proposition 26.1: Z is Normal with

$$\mu_Z = E[Z_1] + \cdots + E[Z_n] = \sum_{k=1}^{n} b_k + a_k\mu_k$$

$$\sigma_Z^2 = Var[Z_1] + \cdots + Var[Z_n] = \sum_{k=1}^{n} a_k^2\sigma_k^2$$

Example T8.4: Let X and Y be the grade that a randomly chosen student has in ECE203 and ECE207. Suppose $(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 80 \\ 75 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 100 & 40 \\ 40 & 64 \end{bmatrix}$$

a) If you pick a random student, what is the pdf of their grade in ECE203?

b) If the random student gets 79 in ECE207, what is the pdf of their grade in ECE203?

Solution: a)

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

So we identify: $\mu_X = 80$, $\mu_Y = 75$, $\sigma_X = 10$, $\sigma_Y = 8$ and $\rho = 0.5$ since $40 = 0.5 \times 10 \times 8$.

So $X \sim \mathcal{N}(80, 100)$.

b) From Video 28, we know that $f_{X|Y}(x|y)$ is the pdf of a normal with

$$\mu_{X|Y=y} = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

$$\sigma_{X|Y=y}^2 = \sigma_X^2(1 - \rho^2).$$

So given $Y = y$ where $y = 79$, X is Normal with mean $80 + 0.5\frac{10}{8}(79 - 75) = 82.5$ and variance $100(1 - 0.5^2) = 75$.

Note: if $\rho = 0$, then regardless of what the known value of Y is, $X \sim \mathcal{N}(80, 100)$. This makes sense since if $\rho = 0$, then X and Y are independent.

Note: if $\rho = 1$, then $\sigma_{X|Y=y}^2 = 0$ and

$$\mu_{X|Y=y} = \mu_X + \sigma_X\frac{y - \mu_Y}{\sigma_Y}$$