# Tutorial T10

**Example T10.1:** Two particles are positioned on a line with random and independent positions

$$X_1 \sim \mathcal{N}(0,1)$$
$$X_2 \sim U[-1,1]$$

Find $E[(X_1+X_2)^3]$.

*Solution:*

$$\begin{aligned}
E[(X_1+X_2)^3] &= E[X_1^3 + 3X_1^2X_2 + 3X_1^1X_2^2 + X_2^3] \\
&= E[X_1^3] + 3E[X_1^2X_2] + 3[X_1^1X_2^2] + E[X_2^3] \\
&= E[X_1^3] + 3E[X_1^2]E[X_2] + 3[X_1^1]E[X_2^2] + E[X_2^3] && \text{(a)} \\
&= E[X_1^3] + 3E[X_1^2] \times 0 + 3 \times 0 \times E[X_2^2] + E[X_2^3] \\
&= E[X_1^3] + E[X_2^3] \\
&= 0 && \text{(b)}
\end{aligned}$$

(a) follows because if $X_1$ and $X_2$ are independent, then $E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)]$ (Prop. 31.1),

(b) follows from

$$\begin{aligned}
E[X_2^3] &= \int_{-1}^{1} x^3 \times \frac{1}{2} dx \\
&= 0 && \text{by (a)symmetry}
\end{aligned}$$

$$\begin{aligned}
E[X_1^3] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx \\
&= 0 && \text{by (a)symmetry}
\end{aligned}$$

**Example T10.2:** Let $X_1, X_2, \ldots, X_n$ be independent and $\sim \mathcal{N}(0,1)$. Let

$$Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$$
$$Z = b_1X_1 + b_2X_2 + \cdots + b_nX_n$$

Find $Cov[Y,Z]$.

*Solution:*

$$\begin{aligned}
Cov[Y,Z] &= Cov\left[\sum_{k=1}^{n} a_kX_k, \sum_{m=1}^{n} b_mX_m\right] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} Cov[a_kX_k, b_mX_m] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} a_kCov[X_k, b_mX_m] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} a_kb_mCov[X_k, X_m]
\end{aligned}$$

Since $X_1, X_2, \ldots, X_n$ are independent, $Cov[X_k, X_m] = 0$ when $k \neq m$:

$$\begin{aligned}
Cov[Y,Z] &= \sum_{k=1}^{n}\left[\sum_{m=1}^{n} a_kb_mCov[X_k, X_m]\right] \\
&= \sum_{k=1}^{n}\left[\sum_{m=k}^{k} a_kb_mCov[X_k, X_m]\right] \\
&= \sum_{k=1}^{n} a_kb_kCov[X_k, X_k] \\
&= \sum_{k=1}^{n} a_kb_kVar[X_k] \\
&= \sum_{k=1}^{n} a_kb_k
\end{aligned}$$

**Example T10.3:** Let $X$ be a rv and $Y = aX + Z$ where $X$ and $Z$ are uncorrelated. What is the correlation coefficient between $X$ and $Y$?

*Solution:*

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}}$$

$$\begin{aligned}
Cov[X,Y] &= Cov[X, aX+Z] \\
&= aCov[X,X] + Cov[X,Z] \\
&= aVar[X]
\end{aligned}$$

$$\begin{aligned}
Var[Y] &= Var[aX+Z] \\
&= Var[aX] + Var[Z] + 2Cov[aX, Z] \\
&= a^2Var[X] + Var[Z]
\end{aligned}$$

$$\begin{aligned}
\rho(X,Y) &= \frac{aVar[X]}{\sqrt{Var[X]\,(a^2Var[X] + Var[Z])}} \\
&= \frac{a\sqrt{Var[X]}}{\sqrt{a^2Var[X] + Var[Z]}} \\
&= \frac{\sqrt{a^2Var[X]}}{\sqrt{a^2Var[X] + Var[Z]}} \\
&= \frac{1}{\sqrt{1 + \frac{Var[Z]}{a^2Var[X]}}}
\end{aligned}$$

Note: If

- $Var[Z] \ll a^2Var[X]$ then $\rho(X,Y) \approx 1$
- $Var[Z] \gg a^2Var[X]$ then $\rho(X,Y) \approx 0$
- $Var[Z] = a^2Var[X]$ then $\rho(X,Y) = 1/\sqrt{2}$

**Example T10.4:** Let $X_1, X_2, \ldots$ be iid with $E[X_1] = 2$. Let $N \sim$ Poisson($\lambda$) and independent of $X_1, X_2, \ldots$

Find $E[X_1X_2\cdots X_N]$.

*Solution:* We use conditional probability:

$$E[X_1X_2\cdots X_N] = E[\;E[X_1X_2\cdots X_N|N]\;]$$

$$\begin{aligned}
E[X_1X_2\cdots X_N|N=n] &= E[X_1X_2\cdots X_n|N=n] \\
&= E[X_1X_2\cdots X_n] \\
&= E[X_1]E[X_2]\cdots E[X_n] \\
&= 2 \times 2 \times \cdots \times 2 \\
&= 2^n
\end{aligned}$$

$$\Rightarrow E[X_1X_2\cdots X_N|N] = 2^N$$

$$\begin{aligned}
E[X_1X_2\cdots X_N] &= E[2^N] \\
&= e^{\lambda} && \text{[See below for why]}
\end{aligned}$$

$$\begin{aligned}
E[a^N] &= \sum_{n=0}^{\infty} a^n P[N=n] \\
&= \sum_{n=0}^{\infty} a^n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(a\lambda)^n}{n!} \\
&= e^{-\lambda} e^{a\lambda} \\
&= e^{(a-1)\lambda}
\end{aligned}$$