Sum of Squares in Lean

Using Lean 3.0

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Theorem Definition

[1/2] Statement

The two statements are equivalent:

- **a)** If a sum of squares is 0, then all the elements of that sum are 0.
- **b)** -1 is not a sum of squares in $\mathbb{R}$

Notation: List = Set, but allows multiplicity

Note: We do not want to show that **a)** or **b)** is actually true!

[2/2] Statement

So now we would like to formalize the statements using mathematical notations

a) If a sum of squares is **0**, then all the elements of that sum are **0**.

$\forall$ List $L \in \mathbb{R}$, $x_i \in L$: $\sum_{i...n} x_i^2 = 0 \Rightarrow x_i = 0 \ \forall x_i \in L$

b) **-1** is not a sum of squares in $\mathbb{R}$

$\forall$ lists $L \in \mathbb{R}$, $x_i \in L$: $\sum_{i...n} x_i^2 \neq -1$

And we can write those together in an equivalence relation as follows : **a)** $\leftrightarrow$ **b)**

$\forall$ List $L \in \mathbb{R}$, $x_i \in L$: $\sum_{i...n} x_i^2 = 0 \Rightarrow x_i = 0 \ \forall x_i \in L \leftrightarrow \sum_{i...n} x_i^2 \neq -1$

Propositional Logic - Example

[1/1] Propositional Logic - Example

Given the following statements

Q : The weather is nice
P : I'm at the zoo

We can express the relation between **Q** and **P** like this:

$Q \to P$ (If the weather is nice, I'm at the zoo)

The **contraposition** of this would be:

$Q \to P = \neg P \to \neg Q$ (If I'm not at the zoo, the weather is bad)

(And *not* $Q \to P = \neg Q \to \neg P$ (If the weather is bad, I'm not at the zoo))

[1/4] Proof

We will start the proof of **a)** ↔ **b)** by proving the '→' - direction: **b)** → **a)**.
We do a proof by contraposition, meaning we show that ¬**a)** → ¬**b)** is true.

First, let's negate the two statements:

a) $\neg(\forall$ List $L \in \mathbb{R} : x_i \in L : \sum_{i \ldots n} x_i^2 = 0 \rightarrow \forall x_i \in L : x_i = 0)$

$\exists$ List $L \in \mathbb{R} : x_i \in L : \sum_{i \ldots n} x_i^2 = 0 \rightarrow \exists x_i \in L : x_i \neq 0$ $= \neg$a)

b) $\neg(\forall$ List $L \in \mathbb{R} : x_i \in L : \sum_{i \ldots n} x_i^2 \neq -1)$

$\exists$ List $L \in \mathbb{R} : x_i \in L : \sum_{i \ldots n} x_i^2 = -1$ $= \neg$b)

[2/4] Proof

We show **¬a)** → **¬b)**:

Assume **¬a)**.

Let **L** be a list of **ℝ** whose sum of squares is equal to **0** and without loss of generality, let $x_1 \neq 0$.

Then :

$$\sum_{i=1}^{n} x^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 0$$

We can now divide all the terms by x_1^2 and continue with the equality, because division of 0 by any number is still equal to zero.

[3/4] Proof

So now we have:

$$\frac{\sum_{i=1}^{n} x^2}{x_1^2} = \frac{x_1^2}{x_1^2} + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2}$$

$$= 1 + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = 0$$

By adding (-1) to both sides, we get :

$$\frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = -1$$

Which confirms our initial assumption **¬a)**

Thus we have proven **b)** → **a)**

[4/4] Proof

We will now proceed to show the '←' - direction : a)→ b) of a) ↔ b).

$$\exists \quad \text{List } L \in \mathbb{R} : x_i \in L : \quad \sum_{i \ldots n} x_i^2 = 0 \rightarrow \exists\ x_i \in L : x_i \neq 0 \qquad = \neg a)$$

$$\exists \quad \text{List } L \in \mathbb{R} : x_i \in L : \quad \sum_{i \ldots n} x_i^2 = -1 \qquad = \neg b)$$

We will prove the contraposition ¬b) → ¬a).

Assuming ¬b), we want to show ¬a).

From our assumption of ¬b), let L be that list.

By appending 1 to L, the sum of squares of L must be now 0.

Contraposition in C

[1/1] Contraposition in $\mathbb{C}$

The statement we've just proven talks specifically about field.
We would like to show that the contraposition of the statement HOLDS in $\mathbb{C}$.

Choose a list **L1** := {i} (or **-i**, **1/i**, **1/-i** }.
This obviously has a sum of squares of -1, satisfying **¬b)**.
We can then show **¬a)** by constructing
L2 := **L1** appended with **1**.
L2 is a list with **some** non-zero elements but the sum of squares is **0**, satisfying **¬a)**.
Therefore **¬b)** implies **¬a)**.
The same is also true for the other direction.

Ty for your time 🤡

Any Questions?

Resources for further reading:

Just look at the mathlib sources lol no need for actual documentation in Lean

https://leanprover-community.github.io/mathlib_docs/

Logical Proposition

it is sometimes more feasible to prove the opposite of what a statement says in order to prove the original implication.
If we negate the original statement, we get:

¬(**P** (It's a nice weather) → **Q** (I go to the zoo))

$\neg Q \rightarrow \neg P$

(I don't go to the zoo) → (It's not a nice weather)

And this implication now says :

If I don't go to the zoo (it must mean \ imply that) it's not a nice weather.

This also makes sense, because it's the exact opposite scenario of what the original statement said! So by proving that $\neg Q \rightarrow \neg P$, we can safely say that $P \rightarrow Q$.

Useful math symbols

$\sum$
$\exists$
$\forall$
$\in$
$\mathbb{N}$
$\mathbb{R}$
$\rightarrow$
$\leftrightarrow$
$\neq$
$\exists$
$\neg$
$\leftarrow$

Proof

Assuming not b), we proceed :
Let L be a subset of R whose sum of squares is equal to 0 and without loss of generality, let x_1 != 0. Then :

$$\sum_{i=1}^{n} x^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\frac{\sum_{i=1}^{n} x^2}{x_1^2} = \frac{x_1^2}{x_1^2} + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2}$$
$$= 1 + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = 0$$

$$\frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = -1$$

Proof

We will start the proof by proving the 'right' direction of the equivalence, a) → b).

First let's negate the entire statement :

¬ a) $\exists L \in \mathbb{R}: \forall x_i \in L : \sum_{i...n} x^2 = -1$

¬ **a)** $\forall$ **subsets** $L \in \mathbb{R}$ **and elements** $x_i \in L : \sum_{i...n} x^2 = 0 \Rightarrow x_i = 0 \ \forall i \in \mathbb{N}$

$\exists L \in \mathbb{R}$ **and** $x \in L : \sum_{i...n} x^2 = -1 \rightarrow \exists L \in \mathbb{R}$ **and elements** $x_i \in L : \sum_{i...n} x^2 \neq 0 \Rightarrow \exists x_i \ 0 \ \forall i \in \mathbb{N}$

(There's a list L of x's in R whose sum of square = -1 means that there's a list L in R such that the sum of squares is equal to 0 and there's an element not equal to 0 in it)

So assuming that the sum of squares in R is equal to 1 we proceed :

Let L be a list in R with $x_1 \mathrel{!=} 0$.

$$\sum_{i...n} x^2 = x_1^2 + ... + x_n^2 = 1/x_1^2 * (x_1^2 + x_2^2 + ... + x_n^2) = 1 + 1/x_1^2 * (x_2^2 + ... + x_n^2)$$