MAT 472 - Intermediate Real Analysis I

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Contents

1 Week of August 21st, 2016

1.1 The Completeness Property of Real Numbers

$\mathbb{R}$ and $\mathbb{Q}$ are ordered fields.

- $|\ \mathbb{Q}\ | \ < \ |\ \mathbb{R}\ |$
- $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable (cardinality)
- $x^2 - 2 = 0$ has solutions in $\mathbb{R}$ but none in $\mathbb{Q}$
- $\mathbb{R}$ has no gaps (order)

1.1.1 Axiom of Completeness

Every non-empty subset of $\mathbb{R}$ that has an upper bound has a least upper bound. Along with the greatest lower bound, these bounds can be classified using the following definitions:

Definition 1 (Upper Bound)**.** Let the set S be a non-empty subset of $\mathbb{R}$ and suppose $b \in \mathbb{R}$. b is an upper bound of the set S if $\forall x \in S,\ x \leq b$.

Definition 2 (Lower Bound)**.** Let the set S be a non-empty subset of $\mathbb{R}$ and suppose $b \in \mathbb{R}$. b is a lower bound of the set S if $\forall x \in S,\ x \geq b$.

Definition 3 (Supremum)**.** Let B be the set of upper bounds of the non-empty subset, S, of $\mathbb{R}$ and suppose $b \in B$. If $\forall x \in B,\ b \leq x$, then b is the least upper bound of S, or *supremum* of S. This can be denoted as $\sup\ S = b$.

Definition 4 (Infinum)**.** Let B be the set of lower bounds of the non-empty subset, S, of $\mathbb{R}$ and suppose $b \in B$. If $\forall x \in B,\ b \geq x$, then b is the greatest lower bound of S, or *infinum* of S. This can be denoted as $\inf\ S = b$.

Example. $A = [0,\ 2] \quad \longrightarrow \quad \sup\ A = 2$
$A = [0,\ 2) \quad \longrightarrow \quad \sup\ A = 2$

Definition 5 (Maximum)**.** If the supremum of a set is also a member of the set, then the supremum is also the maximum.

2 Week of August 28th, 2016

2.1 Cardinality

Definition 6 (Finite)**.** A set, S, is finite if $S = \emptyset$ or if there exists a bijection, $f : \{1, 2, . . ., n\} \longrightarrow S$ for some $n \in \mathbb{N}$.

Definition 7 (Infinite)**.** A set, S, is infinite if it is not finite.

For sets A and B, $A \sim B$ if there exists a bijection, $f : A \longrightarrow B$. $\sim$ is an equivalence relation (symmetric, reflexive, and transitive).

Definition 8 (Finite)**.** A set, S, is finite if $S = \emptyset$ or if $S \sim \{1, 2, . . ., n\}$ for some $n \in \mathbb{N}$.

Definition 9 (Countable)**.** A set, S, is countable if $S \sim \mathbb{N}$.

Definition 10 (Uncountable)**.** A set, S, is uncountable if it is infinite but not countable.

Theorem. $\mathbb{Q}$ is countable, but $\mathbb{R}$ is uncountable.

Proof. $\mathbb{R}$ is uncountable.
Suppose it is not, and so $\mathbb{R}$ is either countable or finite. It is not finite, but this will be left unproven. So then there must exist a function, $f : \mathbb{N} \longrightarrow \mathbb{R}$ that is one-to-one and onto. Choose a closed and bounded interval, I, that doesn't contain $f(1)$. Recursively construct a closed and bounded interval, $I_n,\ n \in \mathbb{N}$ with $I_n \geq I_{n+1}$ and $f(n) \in I_n\ \forall n$. By the nested interval property, $\bigcup_{n\in\mathbb{N}}(I_n) \neq \emptyset$. Now choose $x \in \bigcup_{n\in\mathbb{N}}(I_n)$ and note that $f(x) \neq x\ \forall n$. This is a contradiction and thus $\mathbb{R}$ is uncountable. □

Theorem. Suppose A_n is countable $\forall n \in \mathbb{N}$. Then,

i) $\bigcup_{n=1}^{N}(A_n)$ is countable $\forall N \in \mathbb{N}$ (finite union of countable sets is countable), and

ii) $\bigcup_{n\in\mathbb{N}}(A_n)$ is also countable (countable union of countable sets is countable).

Proof. (?)

i) First, we know that $A_1 \cup A_2$ is countable. Equivalently, $\{\text{evens}\} \cup \{\text{odds}\} \sim \mathbb{N}$ or $f : A_1 \longrightarrow \mathbb{N} \longrightarrow \{\text{evens}\}$ and $g : A_2 \longrightarrow \mathbb{N} \longrightarrow \{\text{odds}\}$. Thus, if $h : A_1 \cup A_2 \longrightarrow \mathbb{N}$, then $h(x) = \{f^{\sim}(f(x))\ x \in A_1, g^{\sim}(g(x))\ x \in A_2\}$.

ii) $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

□

3 Week of September 4th, 2016

4 Week of September 11th, 2016

5 Week of September 18th, 2016

6 Week of September 25th, 2016

7 Week of October 2nd, 2016

8 Week of October 9th, 2016

9 Week of October 16th, 2016

10 Week of October 23rd, 2016

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12 Week of November 6th, 2016

13 Week of November 13th, 2016

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15 Week of November 27th, 2016