# STP 425 - Stochastic Processes

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## Contents

# 1 Week of August 14th, 2016

**Stochastic Process** a collection of random variables

## 1.1 Basic Probability Review

### 1.1.1 Example 1.12

Compute the probability that event E occurs before event F if we repeat the experiment.

$$\begin{aligned} p &= \mathbb{P}(E) + \mathbb{P}\left((E \cup F)^C\right) \cdot p \\ &= \mathbb{P}(E) + 1 - \mathbb{P}(E \cup F) \cdot p \\ &= \mathbb{P}(E) + (1 - \mathbb{P}(E) - \mathbb{P}(F)) \cdot p \\ &= \frac{\mathbb{P}(E)}{\mathbb{P}(E) + \mathbb{P}(F)} \end{aligned}$$

### 1.1.2 Partitions

If $\{B_1, B_2, . . ., B_n\}$ is a partition of $\Omega$ and $A \subset \Omega$ then,

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\left(\bigcup_{i=1}^{n} A \cap B_i\right) \\ &= \sum_{i=1}^{n} \left(\mathbb{P}(A \cap B_i)\right) \\ &= \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i) \end{aligned}$$

### 1.1.3 Example 1.13 - Craps

Roll two dice, then if

1. the sum is 7 or 11 $\longrightarrow$ you win!
2. the sum is 2, 3, or 12 $\longrightarrow$ you lose!
3. the sum, $i$, is such that $i \in \{4, 5, 6, 8, 9, 10\}$, keep rolling until
   (a) the sum is 7 $\longrightarrow$ you lose!
   (b) the sum is $i$ $\longrightarrow$ you win!

Let $W$ be the event that you win and $D$ the sum of the two dice.

$$\begin{aligned} \mathbb{P}(W) &= \sum_{i=2}^{12} \left(\mathbb{P}(W \mid D = i)\, \mathbb{P}(D = i)\right) \\ &= \mathbb{P}(D = 7) + \mathbb{P}(D = 11) + \sum_{i \in \{4,5,6,8,9,10\}} \left(\mathbb{P}(W \mid D = i)\, \mathbb{P}(D = i)\right) \\ &= \frac{6}{36} + \frac{2}{36} + \sum_{i \in \{4,5,6,8,9,10\}} \left(\frac{\mathbb{P}(D = i)^2}{\mathbb{P}(D = i) + \mathbb{P}(D = 7)}\right) \\ &= \frac{6}{36} + \frac{2}{36} + 2\left(\frac{1}{36} + \frac{4}{90} + \frac{25}{396}\right) \\ &= \frac{4880}{9900} \approx 0.4929 \end{aligned}$$

## 2 Week of August 21st, 2016

Let $X$ represent the number of times that a coin changes between heads and tails when flipped with the probability of landing on heads for any given flip $p$. Find the $pmf$ of $X$ when $p = 1/2$. All outcomes can be expressed as having probability $(1/2)^n$.

$$X : \Omega \longrightarrow \{0, 1, 2, . . ., n - 1\}$$

$$\mathbb{P}(X = k) = 2\binom{n-1}{k}\frac{1}{2} = \binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1} \quad (?)$$

It is difficult to find $\mathbb{E}(X)$ from the $pmf$, especially when $p \neq 1/2$.

**Example** (2.4.1)**.** Find $\mathbb{E}(X)$ for a general $p$. Start by writing $X$ as a sum of simple random variables.

$$X = \sum_{i=1}^{n-1}(X_i) \text{ where } X_i = (?)\mathbb{L}\{\text{flip } i \neq \text{flip } i+1\}$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n-1}(X_i)\right) = \sum_{i=1}^{n-1}(\mathbb{E}(X_i)) = \sum_{i=1}^{n-1}(\mathbb{P}(X_i = 1)) = \sum_{i=1}^{n-1}(2p(1-p)) = 2p(p-1)(n-1)$$

#### 2.0.1 Coupon Collection Problem

There are $m$ different types of coupons. Each time you purchase a coupon it is equally likely to be of any type. Let $X$ be the number of coupons we need to purchase to get a coupon of every type and let $X_i$ be the number of coupons needed to get $i$ different coupon types. Then $X = X_m$. Notice that $X_{i+1} - X_i$ is geometric with $\frac{m-i}{m}$ (?).

$$X = X_m = (X_m - X_{m-1}) + (X_{m-1} - X_{m-2}) + \cdot\ \cdot\ \cdot + (X_1 - X_0)$$

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=0}^{m-1}(X_{i+1} - X_i)\right) \\
&= \sum_{i=0}^{m-1}(\mathbb{E}(X_{i+1} - X_i)) \\
&= \sum_{i=0}^{m-1}\left(\frac{m}{m-i}\right) \\
&= \sum_{i=0}^{m}\left(\frac{m}{i}\right) \\
&= m\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdot\ \cdot\ \cdot + \frac{1}{n}\right)
\end{aligned}$$

When $m$ is large, then this value is approximately $m \log(m)$.

#### 2.0.2 Problem 1.32

Suppose there are $n$ people and $n$ hats. Each person selects a hat at random. Find the probability of $A =$ nobody gets their hat. Let $A_i =$ person $i$ gets their hat. Then,

$$\begin{aligned}
A &= A_1^C \cap A_2^C \cap \cdot\ \cdot\ \cdot \cap A_n^C \\
&= (A_1 \cup A_2 \cup \cdot\ \cdot\ \cdot \cup A_n)^C
\end{aligned}$$

Using the inclusion-exclusion identity we get:

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i)\right) \\
&= 1 - \sum_{i=1}^{n} \left( (-1)^{i+1} \binom{n}{i} \mathbb{P}(A_1 \cap A_2 \cap \cdot\ \cdot\ \cdot \cap A_n) \right) \\
&= 1 - \sum_{i=1}^{n} \left( (-1)^{i+1} \binom{n}{i} \frac{(n-i)!}{n!} \right) \\
&= 1 - \sum_{i=1}^{n} \left( (-1)^{i+1} \binom{n}{i} \frac{1}{i!} \right) \\
&= 1 + \sum_{i=1}^{n} \left( (-1)^{i} \binom{n}{i} \frac{1}{i!} \right) \\
&= \sum_{i=0}^{n} \left( (-1)^{i} \binom{n}{i} \frac{1}{i!} \right) \\
\lim_{n\to\infty} \left( \sum_{i=0}^{n} \left( (-1)^{i} \binom{n}{i} \frac{1}{i!} \right) \right) &= e^{-1}
\end{aligned}$$

### 2.0.3 Problem 2.72

Same story as in Problem 1.32.

$$\mathbb{P}(X=0) = \mathbb{P}(A) = \sum_{i=0}^{n} \left( (-1)^{i} \frac{1}{i!} \right)$$

Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. We do not know the *pmf*. Define

$$X_i = \begin{cases} 0 & \text{if person } i \text{ does not get their hat} \\ 1 & \text{otherwise} \end{cases}$$

and let

$$X = X_1 + X_2 + \cdot\ \cdot\ \cdot + X_n\ .$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} (X_i)\right) = \sum_{i=1}^{n} (\mathbb{E}(X_i)) = n\ \mathbb{E}(X_1) = n\ \mathbb{P}(X_1 = 1) = n\ \mathbb{P}(A_1) = n\ \frac{1}{n} = 1$$

$$\begin{aligned}
\mathbb{E}\left(X^2\right) = \mathbb{E}\left(\left(\sum_{i=1}^{n} (X_i)\right)^2\right) = \mathbb{E}\left(\sum_{i=1}^{n} \left(X_i^2\right) + \sum_{i\neq j} (X_i\ X_j)\right) &= \sum_{i=1}^{n} \left(\mathbb{E}\left(X_i^2\right)\right) + \sum_{i\neq j} (X_i\ X_j) \\
&= n\ \frac{1}{n} + \sum_{i\neq j} (\mathbb{P}(X_i = X_j = 1)) \\
&= 1 + (n^2 - n)\mathbb{P}(X_1 = X_2 = 1) \\
&= 1 + \frac{(n^2 - n)}{n(n-1)} \\
&= 2
\end{aligned}$$

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) = \mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2 = 2 - 1 = 1$$

## 3 Week of August 28th, 2016

## 4 Week of September 4th, 2016

## 5 Week of September 11th, 2016

## 6 Week of September 18th, 2016

## 7 Week of September 25th, 2016

## 8 Week of October 2nd, 2016

## 9 Week of October 9th, 2016

## 10 Week of October 16th, 2016

## 11 Week of October 23rd, 2016

## 12 Week of October 30th, 2016

## 13 Week of November 6th, 2016

## 14 Week of November 13th, 2016

## 15 Week of November 20th, 2016

## 16 Week of November 27th, 2016