MAT 472 - Intermediate Real Analysis I

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1.1 The Completeness Property of Real Numbers

$\mathbb{R}$ and $\mathbb{Q}$ are ordered fields.

- $|\mathbb{Q}| < |\mathbb{R}|$
- $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable (cardinality)
- $x^2 - 2 = 0$ has solutions in $\mathbb{R}$ but none in $\mathbb{Q}$
- $\mathbb{R}$ has no gaps (order)

1.1.1 Axiom of Completeness

Every non-empty subset of $\mathbb{R}$ that has an upper bound has a least upper bound. Along with the greatest lower bound, these bounds can be classified using the following definitions:

Upper Bound Let the set S be a non-empty subset of $\mathbb{R}$ and suppose $b \in \mathbb{R}$. b is an upper bound of the set S if $\forall x \in S,\ x \leq b$.

Lower Bound Let the set S be a non-empty subset of $\mathbb{R}$ and suppose $b \in \mathbb{R}$. b is a lower bound of the set S if $\forall x \in S,\ x \geq b$.

Supremum Let B be the set of upper bounds of the non-empty subset, S, of $\mathbb{R}$ and suppose $b \in B$. If $\forall x \in B,\ b \leq x$, then b is the least upper bound of S, or *supremum* of S. This can be denoted as $\sup S = b$.

Infinum Let B be the set of lower bounds of the non-empty subset, S, of $\mathbb{R}$ and suppose $b \in B$. If $\forall x \in B,\ b \geq x$, then b is the greatest lower bound of S, or *infinum* of S. This can be denoted as $\inf S = b$.

Example $A = [0,\ 2] \quad \longrightarrow \quad \sup A = 2$
$A = [0,\ 2) \quad \longrightarrow \quad \sup A = 2$

Maximum If the supremum of a set is also a member of the set, then the supremum is also the maximum.

Example Let $A = \{1/n \mid n \in \mathbb{N}\}$, then $\max A = \sup A = 1$ and $\inf A = 0$, however there is no minimum.

There are non-empty bounded subsets of $\mathbb{Q}$ with no supremum (or infinum) in $\mathbb{Q}$.

Example Let $S = \{r \in \mathbb{Q} \mid r^2 < 2\}$. There does not exist any $b \in \mathbb{Q}$ such that $b \leq c$ where $c \in$ UB$\{S\}$ and $c \in \mathbb{Q}$.

Proof S is non-empty and bounded above by 2. Suppose $x \in \mathbb{Q}$ and $x > 2$. Then $0 \leq 2 < x$ so $0 \leq 2 \cdot 2 < 2x$ and $0 \leq 2x < x^2$. Then $2 < 4 < x^2$ and thus $x \notin S$.

QED

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