MAT 415 - Introduction to Combinatorics

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1 Week of August 14th, 2016

1.1 Principle Definitions

1.1.1 Product Principle

Suppose a task can be broken into k subtasks, $t_1, t_2, \ldots, t_k$, and further suppose there are c_i ways to perform subtask t_i and each way leads to an unique result. Then the number of ways to perform the task is $c_1 \cdot c_2 \cdots c_k$.

1.1.2 Sum Principle

Suppose the objects in a counting problem can be divided into k disjoint and exhaustive cases. If there are n_i objects in the i^{th} case for $i = 1, 2, \ldots, k$ then there are $n_1 + n_2 + \ldots + n_k$ objects.

1.1.3 Bijection Principle

Two finite sets have the same cardinality if and only if there exists a bijection between them.

Example How many subsets does $\{k_1, k_2, k_3, k_4\}$ have?
Find a bijection between the binary string $b_1 b_2 b_3 b_4$ and $\{k_1, k_2, k_3, k_4\}$.

$$S \subseteq \{k_1, k_2, k_3, k_4\} \longleftrightarrow b_1 b_2 b_3 b_4 \quad \text{where} \quad b_i = \begin{cases} 0 & \text{if} \quad k_i \notin S \\ 1 & \text{if} \quad k_i \in S \end{cases}$$

There are $2^4 = 16$ possibilites for the binary string so the set has 16 subsets.

1.1.4 Quotient Principle

A *partition* of a set, S, is a division of a set into disjoint subsets whose union is S. The subsets in a set of partitions are often called blocks of the partition.
Suppose a set S has p elements. If we partition S into q blocks of size r, then $q = p/r$ and $r = p/q$.

2 Week of August 21st, 2016

2.1 Binomial Coefficients and the Quotient Principle

Let P of size p be the set of all permutations of $S = \{1, 2, \ldots, n\}$. We say that two permutations are in the same block if they have the same first k values. Then $p = n!$ and $q = k!$ (?).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ is the number of } k\text{-element subsets of an } n\text{-element set.}$$

2.2 Lattice Paths

A lattice path is a path from (a, b) to (m, n) that only takes steps of length 1 in north or east directions. Thus, the number of north steps is $n - b$ and the number of east steps is $m - a$. So the total steps required is $n - b + m - a$.

Since a step is either east or north, we can uniquely identify all lattice paths by only considering when an east step is taken. For example, between $(0, 0)$ and $(2, 1)$, we must take 2 steps to the east. Thus out of our 3 total steps the set of all unique sets of when we can take steps to the east is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. The missing member of each subset is when we take north steps and thus we have identified all unique lattice paths betweeen $(0, 0)$ and $(2, 1)$. In general, the number of unique lattice paths between (a, b) and (m, n) is:

$$\binom{n - b + m - a}{m - a} = \binom{n - b + m - a}{n - b}$$

2.2.1 Catalan Numbers

If we only consider paths between $(0,0)$ and (n,n), then we can define *good paths* as those that do not touch the line $y = x+1$ which is equivalent to those paths that do not cross $y = x$. *Bad paths* are simply paths that are not good. A bijection exists between bad paths, and all paths between $(-1,1)$ and (n,n) (see notes). Thus the number of bad paths is simply:

$$\binom{n-1+n+1}{n+1} = \binom{2n}{n+1}$$

The number of good paths is then the subtraction of the total number of paths and the number of bad paths:

$$\begin{aligned}\binom{n-0+n-0}{n-0} - \binom{2n}{n+1} = \binom{2n}{n} - \binom{2n}{n+1} &= \frac{(2n)!}{n!\; n!} - \frac{(2n)!}{(n+1)!\; (n-1)!} \\ &= \frac{(2n)!\; (n+1) - (2n)!\; n}{(n+1)!\; n!} \\ &= \frac{(2n)!}{(n+1)!\; n!}\end{aligned}$$

The values this expression generates are called the Catalan numbers.

2.2.2 The Pigeonhole Principle

Suppose that there are 7 houses and 10 people to place in these houses. Obviously not everyone can have their own house.

Corollary. A function from a set with $k+1$ elements to a set with k elements cannot be one-to-one.

Example (California Example)**.** Suppose we label boxes with three letters representing a person's initials and a day of the year representing their birthday. This generates $26 \cdot 26 \cdot 26 \cdot 365 = 6415240$ boxes. Given that there are about 38.8 million people in California, if we assign each person a box based on their initials and birthday, there must be a box shared by at least $\lceil \frac{N}{k} \rceil = \lceil 6.048 \rceil = 7$ people.

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4 Week of September 4th, 2016

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