Chapter II. The Optimal Control Problem

§ 1. Introduction

In this chapter we shall discuss an optimization problem that we will call "the optimal control problem." In the 1950's, motivated especially by aerospace problems, engineers became interested in the problem of controlling a system governed by a set of differential equations. In many of the problems it was natural to want to control the system so that a given performance index would be minimized. In some aerospace problems large savings in cost could be obtained with a small improvement in performance so that optimal operation became very important. As techniques were developed which were practical for computation and implementation of optimal controls the use of this theory became common in a large number of fields. References which illustrate work typical in applying optimal control to economic problems are Burmeister-Döbell [1], Pindyck [1], Shell [1].

In classical calculus of variations there are three equivalent optimization problems called the Bolza problem, the Lagrange problem, and the Mayer problem, which deal with minimizing a performance index of a system governed by a set of differential equations. See Bliss [2] for a discussion of these problems. There are two differences between these problems and the optimal control problem. The differential equations involved are of a slightly less general type in the optimal control problem and certain variables called control variables are required to lie in a closed set for the optimal control problem while for the problems of Bolza, Lagrange, and Mayer the variables of the problem are assumed to lie in an open set. In fact, the optimal control problem which we shall formulate in §3 is a Mayer problem with the added condition that the control variables are restricted to lie in a closed set. Conversely, except that the form of the differential equation involved in the optimal control problem is slightly less general, the optimal control problem includes the Bolza problem, the Lagrange problem, and the Mayer problem as special cases.

Necessary conditions for optimality for the optimal control problem were derived by Pontryagin, Boltyanskii and Gamkrelidze. It has become common terminology to call these necessary conditions "Pontryagin's Principle." It has been shown by Berkovitz [3], using constructions of Valentine [2], that Pontryagin's principle can be derived from the necessary conditions for optimality of the Bolza problem. Since there is considerable conceptional simplicity in a direct proof of Pontryagin's Principle and the optimal control problem nearly includes the Bolza, Lagrange, and Mayer problems as special cases, we will only discuss the optimal control problem.

The chapter begins by giving several examples of optimal control problems. Then in §3 the optimal control problem is formally stated. The optimality conditions of Pontryagin's principle are stated in §5. They are applied to compute extremal control laws for three of the examples in §6, §7, and §8. A number of the topics and concepts which arise in optimal control problems are discussed in §9. To motivate the proof of Pontryagin's principle and to illustrate the role of the methods of Chap. I in the proof of Pontryagin's principle, two derivations of optimality conditions are given in a special case of the optimal control problem. In this case the derivation is especially simple and conceptually clear. The proof of Pontryagin's principle in its general form is given in §13, §14, and §15.

We shall mention some notation which will be used in the remainder of this book. Vectors will be identified with column matrices. Components of a vector will be indicated by subscripts. Sequences of vectors will be indicated by superscripts. The symbol A' will denote the transpose of a matrix A. Thus $x(t)'$ is the row vector whose elements are the corresponding elements of the column vector $x(t)$. Whenever matrices are written adjacent to each other it will be understood that this indicates the product of the matrices involved. Thus $x(t)' L(t) x(t)$ is the product of the three matrices involved which agrees with the quadratic form in the components of $x(t)$

$$\sum_{i,j=1}^{n} x_i(t) L_{ij}(t) x_j(t).$$

Matrices of partial derivatives of vector functions will be indicated using a subscript which denotes the vector of variables involved in the partial differentiation. Thus if $f(x, y, z)$ is a vector function of the vector variables x, y, z, $f_y(x, y, z)$ will denote the matrix whose ij-th element is the partial derivative of the i-th component of f with respect to the variable y_j. To coincide with this notation we will make the following exception to our identification of vectors with column matrices. If, say, $f(x, y, z)$ is a scalar function of vector variables, we shall identify a vector of partial derivatives such as $f_y(x, y, z)$ with the corresponding row matrix rather than the corresponding column matrix.

§ 2. Examples

We shall begin our discussion of the optimal control problem by giving several examples of this type of problem. The following is a simple example from the aerospace field.

Example 2.1. Consider the problem of a spacecraft attempting to make a soft landing on the moon using a minimum amount of fuel. To define a simplified version of this problem, let m denote the mass, h and v denote the height and vertical velocity of the spacecraft above the moon, and u denote the thrust of the spacecraft's engine. Let M denote the mass of the spacecraft without fuel, h_0 and

v_0 the initial height and vertical velocity of the spacecraft, F the initial amount of fuel, α the maximum thrust attainable by the spacecraft's engine, k a constant, and g the gravitational acceleration of the moon.

The gravitational acceleration g may be considered constant near the moon. The equations of motion of the spacecraft are

$$\begin{aligned} \dot{h} &= v \\ \dot{v} &= -g + m^{-1} u \\ \dot{m} &= -k u. \end{aligned} \tag{2.1}$$

The thrust $u(t)$ of the spacecraft's engine is the control for the problem. Suppose the class $\mathscr{U}$ of control functions is all piecewise continuous functions $u(t)$ defined on an interval $[t_0, t_1]$ such that

$$0 \leqq u(t) \leqq \alpha. \tag{2.2}$$

It is natural to take initial time, $t_0 = 0$, and terminal time t_1 equal to the first time the spacecraft reaches the moon. End conditions which must be satisfied at the initial time and terminal time are

$$\begin{gathered} h(0) - h_0 = 0, \quad v(0) - v_0 = 0, \quad m(0) - M - F = 0 \\ h(t_1) = 0, \quad v(t_1) = 0. \end{gathered} \tag{2.3}$$

The problem is to land using a minimum amount of fuel or equivalently to minimize $-m(t_1)$ over the class $\mathscr{U}$.

Example 2.2. In the economics the following situation is called a Ramsey model for a one sector economy. The output rate $y(t)$ of the economy and the capital $K(t)$ are related through a production function $y(t) = F(K(t))$. The rate of consumption $c(t)$ is a proportion $u(t)$ of a function $G(K(t))$ of the capital. That is

$$c(t) = u(t)\, G(K(t)) \tag{2.4}$$

and

$$0 \leqq u(t) \leqq 1. \tag{2.5}$$

Then the rate of change of capital is given by

$$\dot{K}(t) = F(K(t)) - u(t)\, G(K(t)). \tag{2.6}$$

Let $H(c)$ be a utility function which represents the utility to the system of consuming at rate c. Consider the performance of the economy to be given by the integral of the utility function

$$\int_{t_0}^{t_1} H(u(t)\, G(K(t)))\, dt. \tag{2.7}$$

Formulate the problem of choosing the proportion to be consumed, $u(t)$, so the capital satisfying (2.6) changes from an initial value $K(t_0) = K_0$ to a desired terminal value $K(t_1) = K_1$ and the performance (2.7) is maximized.

Example 2.3. The following is an optimization problem called the linear regulator problem which is applied to a large number of design problems in engineering. Let $A(t)$, $M(t)$ and D be $n\times n$ matrices and $B(t)$, $n\times m$, and $N(t)$, $m\times m$, matrices of continuous functions. Let $u(t)$ be an m-dimensional piecewise continuous vector function defined on a fixed interval $[t_0, t_1]$. Let $x(t)$ be the n-dimensional vector function which is the corresponding solution of

$$\dot{x}(t)=A(t)\,x(t)+B(t)\,u(t) \tag{2.8}$$

with initial condition $x(t_0)=x_0$. Suppose $M(t)$, $N(t)$, and D are symmetric with $M(t)$, D non negative definite and $N(t)$ positive definite. Consider the problem of choosing $u(t)$ so that

$$x(t_1)'\,D\,x(t_1)+\int_{t_0}^{t_1}[x(t)'\,M(t)\,x(t)+u(t)'\,N(t)\,u(t)]\,dt \tag{2.9}$$

is minimized.

The optimal control for the linear regulator problem is a linear function of $x(t)$. (See Theorem IV.5.1.) This is particularly convenient for implementation. Because of this, controls have been designed for many nonlinear problems as well as linear problems, using the solution of the linear regulator problem. Nonlinear problems may be linearized (see Theorem 10.2, particularly formula (10.7)) and the performance criterion approximated by a quadratic criterion for the linearized equations. See Athans, [1], [2] for examples of this type of application.

Example 2.4. The simplest problem in calculus of variations formulated in I.3 can be rewritten as an optimal control problem. The optimal control problem with equations of motion,

$$\dot{x}_1=u(t)$$
$$\dot{x}_2=L(t, x_1(t), u(t))$$

control set $U=E^1$, performance index $x_2(b)$, and end conditions

$$x_1(a)=c, \quad x_2(a)=0, \quad x_1(b)=d$$

for fixed a, b, c, d is equivalent to the simple problem in calculus of variations of minimizing

$$J(x)=\int_a^b L(t, x(t), \dot{x}(t))\,dt$$

over the class of curves $x(t)$ with piecewise continuous derivatives which satisfy

$$x(a)=c, x(b)=d.$$

§ 3. Statement of the Optimal Control Problem

To state the optimal control problem we will make the following notational conventions. Let U be a closed subset of E^m; t, x, u variables respectively in

E^1, E^n, E^m; $f(t, x, u)$ a vector function

$$f\colon\ E^1 \times E^n \times E^m \to E^n$$

which is continuous and has continuous first partial derivatives with respect to the coordinates of x. Let $\phi(t_0, t_1, x_0, x_1)$ be a vector function

$$\phi\colon\ E^1 \times E^1 \times E^n \times E^n \to E^k$$

which is of class C^1.

Let $\mathscr{U}$ be a set of piecewise continuous functions $u(t)$ with values in U, each function $u(t)$ being defined on some interval $[t_0, t_1]$, which may differ for different elements of $\mathscr{U}$. A function $u(t)$ in $\mathscr{U}$ will be called a *control*. The set $\mathscr{U}$ of controls will be assumed to have the following property. If $u(t)$ defined on $t_0 \leqq t \leqq t_1$ is in $\mathscr{U}$ and for $i=1, \ldots, p$, $v_i \in U$ and $\tau_i - h_i < t \leqq \tau_i$ are non-overlapping intervals intersecting $[t_0, t_1]$ then

$$\tilde{u}(t) = \begin{cases} v_i & \text{if } \tau_i - h_i < t \leqq \tau_i \\ u(t) & \text{if } t \in [t_0, t_1] \text{ and } \notin \text{ one of the intervals } \tau_i - h_i < t \leqq \tau_i \end{cases}$$

is in $\mathscr{U}$.

For a control $u(t)$ defined on $[t_0, t_1]$ the solution $x(t)$ of the differential equation

$$\dot{x} = f\big(t, x(t), u(t)\big) \tag{3.1}$$

on the interval $[t_0, t_1]$ with initial condition $x(t_0) = x_0$ will be called the *trajectory* corresponding to the control $u(t)$ and initial condition x_0. The value of $x(t)$ at time t is called the *state* of the system at time t. Eq. (3.1) is called the *equation of motion* of the system. Frequently our discussions will involve controls and their corresponding trajectories. If $x(t)$ appears without mention in a formula it is always understood that a control $u(t)$ and initial condition x_0 have been specified and $x(t)$ is the trajectory corresponding to $u(t)$ and x_0 through Eq. (3.1).

The first component of ϕ evaluated at $\big(t_0, t_1, x(t_0), x(t_1)\big)$, where $x(t)$ is a solution of (3.1),

$$\phi_1\big(t_0, t_1, x(t_0), x(t_1)\big) \tag{3.2}$$

is the *performance index* or *performance criterion* of the system. To indicate the performance's dependence on the initial state $x_0 = x(t_0)$ and control $u(t)$ we shall denote the performance (3.2) by

$$J(x_0, u) = \phi_1\big(t_0, t_1, x(t_0), x(t_1)\big).$$

The next k-1 components of ϕ define through the equations

$$\phi_j\big(t_0, t_1, x(t_0), x(t_1)\big) = 0 \qquad j = 2, \ldots, k \tag{3.3}$$

end conditions for the trajectories of the system. A pair (x_0, u), of an initial condition x_0 and a control $u = u(t)$, will be called *feasible*[1] if there is a solution $x(t)$ of (3.1) on $[t_0, t_1]$ with initial condition $x(t_0) = x_0$ and the end conditions (3.3) are satisfied by $x(t)$. Let $\mathscr{F}$ denote the class of feasible pairs (x_0, u).

[1] Notice that a feasible pair also depends on t_0, t_1 the end points of the interval $[t_0, t_1]$ on which the control $u(t)$ is defined.

The optimal control problem is to find in the class $\mathscr{F}$ an element (x_0, u) such that the corresponding performance index (3.2) is minimized. A pair (x_0, u) of $\mathscr{F}$ which achieves this minimum will be called an *optimal initial condition* and an *optimal control.*

Notice that the trajectory $x(t)$ and the performance index (3.2) are unchanged if the corresponding control $u(t)$ is altered so as to be continuous from the left. Hence in the future we shall assume without loss of generality that $\mathscr{U}$ is a class of left continuous, piecewise continuous, control functions. For brevity the notation e will be used to denote a $(2n+2)$-tuple of end points.

$$e = \big(t_0, t_1, x(t_0), x(t_1)\big).$$

§ 4. Equivalent Problems

If U was taken to be an open subset of E^m rather than a closed subset the optimal control problem would be an optimization problem discussed in classical calculus of variations, Bliss [2], called a Mayer problem. Extending this terminology from calculus of variations we shall also call the optimal control problem defined in § 3 a *Mayer problem.*

Let $L(t, x, u)$ denote a continuous function

$$L\colon\ E^1 \times E^n \times E^m \to E^1$$

of class C^1 in (x, u). If instead of a performance index

$$J(x_0, u) = \phi_1(e) \tag{4.1}$$

the performance index is

$$J(x_0, u) = \int_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt \tag{4.2}$$

the optimization problem is called a *Lagrange problem.* If the performance index is

$$J(x_0, u) = \phi_1(e) + \int_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt \tag{4.3}$$

the problem is called a *Bolza problem.* These are again names used in classical calculus of variations, Bliss [2].

The three optimization problems as formulated with performances (4.1), (4.2) and (4.3) are equivalent in that each can be formulated as one of the other forms. To see this notice that a Lagrange problem can be formulated as a Mayer problem by adding another component differential equation

$$\dot{x}_{n+1}(t) = L\big(t, x(t), u(t)\big)$$

with initial condition $x_{n+1}(t_0) = 0$ to those of (3.1). Then the performance (4.2) is given by

$$J(x_0, u) = \phi_1(e) = x_{n+1}(t_1) = \int_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt$$

as a performance of Mayer form. A Bolza problem can be converted to a Mayer problem in a similar fashion with (4.3) rewritten as $\phi_1(e)+x_{n+1}(t_1)$. A Mayer problem can be converted to a Lagrange problem by adding an extra coordinate and differential equation

$$\dot{x}_{n+1}=0$$

to (3.1) and adding an extra component $\phi_{k+1}(e)$ to the $\phi_2(e), \ldots, \phi_k(e)$ with $\phi_{k+1}(e)$ defined by

$$\phi_{k+1}(e)=x_{n+1}(t_1)-\frac{\phi_1(e)}{t_1-t_0}.$$

Then $\phi_{k+1}(e)=0$ implies

$$\int_{t_0}^{t_1} x_{n+1}(t)\,dt=\phi_1(e).$$

Thus (4.1) has been expressed in the form (4.2) with $L(t,x,u)=x_{n+1}$.

While the three optimization problems are equivalent, it is natural to suspect that a more general theorem could be established for the problem expressed in one form than another. For necessary conditions for optimality this is not the case. In Chap. III existence theorems for optimal controls are obtained for problems in both Mayer form and Bolza form. The existence theorem obtained directly for the problem in Bolza form is obtained under weaker hypotheses than the theorem which could have been obtained by converting the Bolza form to Mayer form and applying the existence theorem obtained for the problem in Mayer form.

Considerable notational simplicity is achieved in Chap. II and IV by using the optimal control problem in Mayer form. If examples arise naturally in Lagrange or Bolza form we shall assume without comment that they satisfy theorems established for problems in Mayer form by making use of the correspondence between the problems mentioned above.

§ 5. Statement of Pontryagin's Principle

The derivation of necessary conditions for optimality for the optimal control problem, that is the proof of "Pontryagin's principle", is quite long. In §11 we shall try to motivate the proof by giving a discussion of the derivation of these conditions and alternative approaches in a simple special case. The derivation of Pontryagin's principle is given in §12–15. However before doing this we shall state "Pontryagin's principle" and illustrate its application by computing controls for Examples 2.1, 2.3 and 2.4.

The conditions of Pontryagin's principle reduce the computation of an optimal control to the solution of a two point boundary problem for a set of differential equations together with a minimization side condition. In many important applications optimal controls have been computed through use of Pontryagin's principle. However in complicated examples solution of the two point boundary value problem can be difficult. Computing methods which make a direct numerical attack on the optimization problem have been devised and widely used, Kelly [1], Bryson Denham [1], McGill [1]. The theory and use of these direct numerical methods is an important part of the subject. However, we shall not discuss them in this book. See Falb DeJong [1], Dyer McReynolds [1], Polak [1], [2] for a discussion of these techniques.

Theorem 5.1. (Pontryagin's Principle). *Necessary conditions that $(x_0^*, u^*(t))$ be an optimal initial condition and optimal control for the optimal control problem are the existence of a nonzero k-dimensional vector λ with $\lambda_1 \leqq 0$ and an n-dimensional vector function $P(t)$ such that for $t \in [t_0, t_1]$:*

$$\dot{P}(t)' = -P(t)' f_x(t, x^*(t), u^*(t)); \tag{5.1}$$

for $t \in (t_0, t_1)$ and $u \in U$

$$P(t)'[f(t, x^*(t), u) - f(t, x^*(t), u^*(t))] \leqq 0; \tag{5.2}$$

$$P(t_1)' = \lambda' \phi_{x_1}(e); \tag{5.3}$$

$$P(t_0)' = -\lambda' \phi_{x_0}(e); \tag{5.4}$$

$$P(t_1)' f(t_1, x^*(t_1), u^*(t_1)) = -\lambda' \phi_{t_1}(e); \tag{5.5}$$

$$P(t_0)' f(t_0, x^*(t_0), u^*(t_0)) = \lambda' \phi_{t_0}(e). \tag{5.6}$$

If $f(t, x, u)$ has a continuous partial derivative $f_t(t, x, u)$, then the condition

$$P(t)' f(t, x^*(t), u^*(t)) = \lambda' \phi_{t_0}(t_0, t_1, x^*(t_0), x^*(t_1)) + \int_{t_0}^{t} P(s)' f_t(s, x^*(s), u^*(s))\, ds \tag{5.7}$$

holds for each $t \in [t_0, t_1]$.

The quantity

$$H(t, x, u) = P(t)' f(t, x, u) \tag{5.8}$$

is generally called the *Hamiltonian* in analogy with a corresponding quantity occuring in classical mechanics. See for instance, Goldstein [1] for a discussion of this concept in mechanics. Condition (5.2) can be expressed as

$$\max_{u \in U} \{H(t, x^*(t), u)\} = H(t, x^*(t), u^*(t)) \tag{5.9}$$

and is called Pontryagin's maximum principle. Conditions (5.3)–(5.6) are generalizations of conditions found in calculus of variations and optimal control problems Bliss [2], p. 202, Pontryagin *et al.* [1], p. 49 called *transversality conditions.* Eq. (5.1) are called the *adjoint equations.*

A control will be called an *extremal* if (5.1)–(5.7) are satisfied and the corresponding trajectory satisfies the end conditions (3.3). Since the conditions of Pontryagin's principle are necessary conditions for optimality each optimal control must be an extremal; however, since the conditions need not be sufficient for optimality there may be extremal controls which are not optimal.

It will be seen in the proof of Pontryagin's principle that condition (5.7) is implied by condition (5.2) and (5.6). Thus in checking to see if a trajectory is extremal it is not necessary to verify (5.7). Formula (5.7) implies $H(t, x^*(t), u^*(t))$ is continuous even though the control $u^*(t)$ may be discontinuous.

Remark 5.1. If the statement of Theorem 5.1 holds for $P(t)$, and $P(t)$ is replaced by $\tilde{P}(t)=\gamma P(t)$ and λ replaced by $\tilde{\lambda}=\gamma\lambda$ where γ is a positive real number, then Eq. (5.1)–(5.6) still hold. If it can be determined from other considerations that $\lambda_1<0$, then a constant γ may be selected so that $\lambda_1=-1$. Hence in the case in which $\lambda_1<0$ it is no loss of generality to assume $\lambda_1=-1$.

There are problems in which λ_1 must equal zero. These problems are called *abnormal*. In §12 abnormal problems are discussed again and an example of one is given. Suppose a control $u(t)$ is an extremal control with $\lambda_1=0$. Notice, from Eq. (5.1)–(5.7), if the original control problem is replaced by a control problem with the same equations of motion, control set, end conditons, and any other performance index $\phi_1(e)$, that the same control $u(t)$ is extremal for the new problem. Thus for abnormal problems the necessary conditions of Pontryagin's principle do not involve the performance index, but are already specified by the equations of motion, control set, and end conditions.

§ 6. Extremals for the Moon Landing Problem

Let us apply Pontryagin's Principle to compute the extremal control for the moon landing problem of §2. This problem was first solved by Miele [1], [2]. Our discussion is close to that of Meditch [1]. The computation is somewhat long and complicated, however, it illustrates well many of the features involved in the computation of extremal controls.

Referring to the statement of the moonlanding problem given in §2 let $(x_1, x_2, x_3)'=(h, v, m)'$. Fix $t_0=0$ and $x_0=(x_{01}, x_{02}, x_{03})'=(h_0, v_0, M+F)'$. Terminal values of h, v are 0, that is $x_{11}=0$, $x_{12}=0$. The vector function f of the equation of motion is

$$f(t,x,u)=\begin{pmatrix} x_2 \\ -g+x_3^{-1}u \\ -ku \end{pmatrix}. \tag{6.1}$$

Hence Eqs. (5.1) are

$$(\dot{P}_1(t), \dot{P}_2(t), \dot{P}_3(t))=-(P_1(t), P_2(t), P_3(t))\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -x_3(t)^{-2}u(t) \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2}$$

or

$$\dot{P}_1(t)=0, \quad \dot{P}_2(t)=-P_1(t), \quad \dot{P}_3(t)=P_2(t)\,x_3(t)^{-2}\,u(t). \tag{6.2}$$

Inequality (5.2) in the form (5.9) is

$$\begin{aligned}&\max_{u\in U}\{P_1(t)\,x_2(t)+P_2(t)(x_3(t)^{-1}u-g)-P_3(t)\,k\,u\}\\ &\quad=P_1(t)\,x_2(t)+P_2(t)(x_3(t)^{-1}u(t)-g)-P_3(t)\,k\,u(t).\end{aligned} \tag{6.3}$$

Defining the vector function $\phi(t_0, t_1, x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{13})$ corresponding to the performance function (3.2) and end conditions (3.3), we have

$$\phi(t_0, t_1, x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{13}) = \begin{pmatrix} -x_{13} \\ t_0 \\ x_{01}-h_0 \\ x_{02}-v_0 \\ x_{03}-M-F \\ x_{11} \\ x_{12} \end{pmatrix}. \tag{6.4}$$

Hence (5.3)–(5.7) are

$$(P_1(t_1), P_2(t_1), P_3(t_1)) = (\lambda_6, \lambda_7, -\lambda_1) \tag{6.5}$$

$$(P_1(0), P_2(0), P_3(0)) = -(\lambda_3, \lambda_4, \lambda_5) \tag{6.6}$$

$$P_1(t_1)\, x_2(t_1) + P_2(t_1)[x_3(t_1)^{-1} u(t_1) - g] - P_3(t_1)\, k\, u(t_1) = 0 \tag{6.7}$$

$$P_1(0)\, x_2(0) + P_2(0)[x_3(0)^{-1} u(0) - g] - P_3(0)\, k\, u(0) = \lambda_2 \tag{6.8}$$

$$P_1(t)\, x_2(t) + P_2(t)[x_3(t)^{-1} u(t) - g] - P_3(t)\, k\, u(t) = \text{constant} \tag{6.9}$$

for all $t \in [0, t_1]$. From (6.7) and (6.8) we see that the constant in (6.9) is λ_2, and that $\lambda_2 = 0$.

Conditions (6.3) and (2.2) imply an extremal control $u(t)$ must satisfy

$$u(t) = \begin{cases} 0 & \text{if } P_2(t)\, x_3(t)^{-1} - k\, P_3(t) < 0 \\ \alpha & \text{if } P_2(t)\, x_3(t)^{-1} - k\, P_3(t) > 0. \end{cases} \tag{6.10}$$

First we shall show that a control for which $u(t)$ is zero over an initial interval and then switches to α over an appropriately chosen terminal interval is extremal. Later it will be shown that this is the only extremal control.

On an interval $[\tau, \tilde{\tau}]$ on which $u(t) = 0$ the solutions of (2.1) are

$$\begin{aligned} h(t) &= -\frac{g(t-\tau)^2}{2} + v(\tau)(t-\tau) + h(\tau) \\ v(t) &= -g(t-\tau) + v(\tau) \\ m(t) &= m(\tau). \end{aligned} \tag{6.11}$$

On an interval $[\tau, \tilde{\tau}]$ on which $u(t) = \alpha$ solutions of (2.1) are

$$\begin{aligned} h(t) &= -\tfrac{1}{2} g(t-\tau)^2 + \frac{m(\tau) - k\,\alpha(t-\tau)}{k^2 \alpha} \ln \frac{m(\tau) - k\,\alpha(t-\tau)}{m(\tau)} + \frac{t-\tau}{k} + v(\tau)(t-\tau) + h(\tau) \\ v(t) &= -g(t-\tau) - \frac{1}{k} \ln \frac{m(\tau) - k\,\alpha(t-\tau)}{m(\tau)} + v(\tau) \\ m(t) &= -k\,\alpha(t-\tau) + m(\tau). \end{aligned} \tag{6.12}$$

Let us determine the locus of points, (h, v) which can be initial points of a segment on which $u(t) = \alpha$ and $h = 0$, $v = 0$ is satisfied at the terminal time. Setting

$h(\tilde{\tau})=0,\ v(\tilde{\tau})=0,\ h(\tau)=h,\ v(\tau)=v,\ m(\tau)=m=M+F,\ \tilde{\tau}-\tau=s$ in Eq. (6.12) gives

$$h=\tfrac{1}{2}g s^2-\frac{m-k\alpha s}{k^2\alpha}\ln\frac{m-k\alpha s}{m}-\frac{s}{k}-v s \tag{6.13}$$

$$v=g s+\frac{1}{k}\ln\frac{m-k\alpha s}{m}. \tag{6.14}$$

Substituting (6.14) in (6.13) gives

$$h=-\tfrac{1}{2}g s^2-\frac{m}{k^2\alpha}\ln\frac{m-k\alpha s}{m}-\frac{s}{k} \tag{6.15}$$

$$v=g s+\frac{1}{k}\ln\frac{m-k\alpha s}{m}. \tag{6.14}$$

This describes the initial height and velocity from which a soft landing $h=0$, $v=0$ is possible by using full thrust α for time s.

For the problem to be realistic the relation

$$\frac{\alpha}{M+F}>g \tag{6.16}$$

should hold. This implies that if thrusting is begun at maximum rate the thrust to initial mass ratio is greater than the gravitational acceleration so that braking can take place.

If the spacecraft burns fuel at rate $k\alpha$ the total amount of fuel will be burned in time $\frac{F}{k\alpha}$. Plotting (6.14), (6.15) under the condition (6.16) for $0\leqq s\leqq\frac{F}{k\alpha}$ gives the curve in the second quadrant of the (v,h)-plane indicated in Fig. II.1. This curve will be called the switching curve. From the construction of this curve, its interpretation is that if the spacecraft is at a (v,h)-point on the curve corresponding to the parameter s and if it thrusts at maximum rate α then it will arrive at $v=0$, $h=0$ in time s.

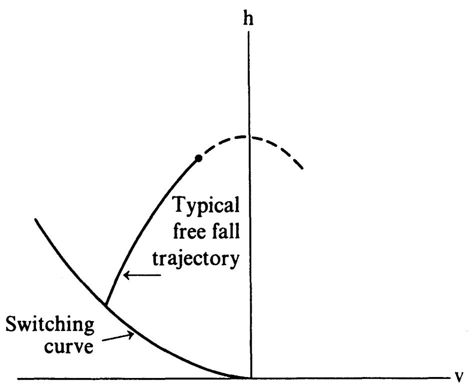

Fig. II.1. Switching curve and free fall trajectory

If the spacecraft free falls it will follow (6.11). If at $t=0$ it is at v_0, h_0 and begins a free fall trajectory, the first two equations of (6.11) imply that it follows the parabola

$$h=h_0-\frac{1}{2g}[v^2-v_0^2] \tag{6.17}$$

in the direction of decreasing v from (v_0, h_0).

Let (v_0, h_0) be a point in the (v, h) plane so that (6.17) intersects the switching curve at a (v, h) point parameterized by a value of s for which $0 \leqq s \leqq \frac{F}{k\alpha}$. Let τ denote the time the spacecraft would spend on this free fall trajectory. We shall show that the control

$$u(t)=\begin{cases} 0 & \text{if } 0 \leqq t \leqq \tau \\ \alpha & \text{if } \tau < t \leqq \tau+s \end{cases} \tag{6.18}$$

is an extremal control. This will follow if it is shown that a choice of λ can be made so that (6.2), (6.3), (6.5)–(6.9) and the end conditions are satisfied.

Refering to Fig. II.1 we see that the velocity at time τ of a spacecraft using control (6.18) is negative and is given by $v_0 - g\tau$. Conditions (6.2), (6.5) and (6.6) are satisfied by:

$$\begin{aligned} P_1(t) &= -\lambda_3 = \lambda_6 \\ P_2(t) &= \lambda_3 t - \lambda_4 = \lambda_6(t_1 - t) + \lambda_7 \\ P_3(t) &= \begin{cases} -\lambda_5, & \text{if } t_0 \leqq t \leqq \tau \\ -\lambda_5 + \int\limits_\tau^t \frac{(\lambda_3 \tilde{t} - \lambda_4)\alpha}{m(\tilde{t})^2} d\tilde{t} & \text{if } \tau \leqq t \leqq t_1. \end{cases} \end{aligned} \tag{6.19}$$

Define $r(t)$ by

$$r(t) = P_2(t)\, x_3(t)^{-1} - k P_3(t). \tag{6.20}$$

From (2.1) (6.2) and (6.6)

$$\dot{r}(t) = \lambda_3\, x_3(t)^{-1}. \tag{6.21}$$

From (6.19) and (6.10) we must have

$$r(\tau) = (\lambda_3 \tau - \lambda_4)(M+F)^{-1} + \lambda_5 k = 0. \tag{6.22}$$

From (6.8) (6.9) (6.7) and (6.18)

$$-\lambda_3 v_0 + \lambda_4 g = 0. \tag{6.23}$$

Eqs. (6.22) (6.23) have the solution

$$\lambda_3 = \frac{\lambda_5 k g (M+F)}{v_0 - g\tau}, \qquad \lambda_4 = \frac{\lambda_5 k v_0 (M+F)}{v_0 - g\tau}. \tag{6.24}$$

When (6.24) are substituted in (6.19)

$$P_3(t) = \lambda_5 \left[-1 + \frac{k(M+F)}{v_0 - g\tau} \int\limits_\tau^t \frac{(g\tilde{t} - v_0)}{(m(\tilde{t}))^2} \alpha\, d\tilde{t}\right], \quad \text{if } \tau \leqq t \leqq t_1. \tag{6.25}$$

If the choices, $\lambda_5=-1$, $\lambda_2=0$ are made; λ_3 and λ_4 given by (6.24); $P_1(t)$, $P_2(t)$, $P_3(t)$, λ_6 and λ_7 given by (6.19), we see that (6.2), (6.5), (6.6) and (6.8) are satisfied. Notice that the integrand in (6.25) is positive and the coefficient of the integral is negative because $v_0-g\,\tau<0$. Thus if $P_3(t_1)=-\lambda_1$ then $\lambda_1<0$. The choice $\lambda_5=-1$ and (6.24) imply $\lambda_3>0$, hence by (6.21), $r(t)$ is strictly increasing. By (6.22), $r(\tau)=0$, thus (6.10) and (6.3) are satisfied. As mentioned in § 5, (6.9) and consequently (6.7) are implied by (6.2) and (6.8). Thus we have verified that (6.18) is an extremal control.

Next it will be shown that (6.18) is the only extremal control law. To begin this it will be shown that an extremal control law can switch at most once from zero to α or from α to zero. If $\lambda_3\neq 0$ this will follow from (6.10) and (6.21) since in this case (6.20) implies $r(t)$ is strictly monotonic. If $\lambda_3=0$, (6.21) implies $r(t)$ is constant. The assertion will follow again from (6.10) if $r(t)\neq 0$, since in this case (6.10) implies $u(t)$ is constant. Thus we need only show that $\lambda_3=0$ and $r(t)=0$ cannot both occur.

If $\lambda_3=0$, by (6.2) and (6.6) $P_1(t)=0$. If $P_1(t)=0$ and $r(t)=0$, from (6.9) $P_2(t)=0$. If $P_2(t)=0$ and $r(t)=0$, (6.19) implies $P_3(t)=0$. Thus $(P_1(t), P_2(t), P_3(t))$ is the zero vector. This implies through (6.2) that this vector is identically zero and through (6.5)–(6.8) that $\lambda=0$, contradicting $\lambda\neq 0$.

Next we will show that if v_0 and h_0 are above the switching curve a control for which $u(t)=\alpha$ and then $u(t)=0$ cannot satisfy the end conditions. If there were such a control the segment of the trajectory on which $u(t)=0$ is of the form (6.11). This trajectory has $v(t)=0$ only at the time

$$t=\frac{v(\tau)}{g}+\tau.$$

If $h(t)=0$ at this time the relation

$$h(\tau)=-\frac{v(\tau)^2}{2g}$$

must be satisfied. That is the initial height of this segment, and in fact the entire segment, is below the surface of the moon. It is not difficult to show, that for initial height and velocity above the switching curve, the trajectory of the form (6.12), on which $u(t)=\alpha$, lies above the surface of the moon. Thus these two segments do not meet to form a single trajectory satisfying the end conditions. Thus if the initial height and velocity are above the switching curve the control of the form (6.18) is the only extremal for which the end conditions are satisfied.

Notice that if initially the spacecraft was at a height and velocity (v_0, h_0) below the switching curve a soft landing is not possible, because even by using full thrust for the entire trajectory the spacecraft will impact the moon with a nonzero velocity.

If the initial height and velocity are above the switching curve (6.14), (6.15), but such that the parabola (6.17) intersects this curve at an (h, s) corresponding to a value of s greater than $F/k\alpha$, a soft landing cannot be achieved. This can be inferred from the following argument. The existence theorems given in Chap. III will apply to this problem. They assert that if it is possible to have a soft landing at all, then there is a control which makes a soft landing using minimum fuel.

Such a control must be an extremal and hence of the form (6.18). For initial conditions for which the parabola (6.17) intersects the switching curve at a point corresponding to a value of s greater than $F/k\alpha$ the spacecraft using a control of the type (6.18) runs out of fuel before reaching the moon. Thus there is a final interval of zero thrust contradicting that the control has the form (6.18).

§ 7. Extremals for the Linear Regulator Problem

The linear regulator problem is formulated in Bolza form. As in §4 reduce it to Mayer form by introducing an extra coordinate x_{n+1} and differential equation

$$\dot{x}_{n+1}(t)=x(t)'M(t)x(t)+u(t)'N(t)u(t).$$

Then the performance $\phi_1(e)$ becomes

$$\phi_1(e)=x(t_1)'Dx(t_1)+x_{n+1}(t_1).$$

To write out the conditions of Pontryagin's principle let

$$P(t)'=(\tilde{P}(t)', P_{n+1}(t))$$

where $P_{n+1}(t)$ is a scalar function and $\tilde{P}(t)$ is an n-dimensional vector function. Using formulas for differentiating quadratic forms, see Problem [16], the adjoint Eqs. (5.1) are

$$\begin{aligned}\dot{\tilde{P}}(t)'&=-\tilde{P}(t)'A(t)-2P_{n+1}(t)x(t)'M(t)\\ \dot{P}_{n+1}(t)&=0.\end{aligned}\tag{7.1}$$

The transversality conditions (5.3)–(5.6) are

$$\begin{aligned}\tilde{P}(t_1)'&=2\lambda_1 x(t_1)'D\\ P_{n+1}(t_1)&=\lambda_1\end{aligned}\tag{7.2}$$

$$\begin{aligned}\tilde{P}(t_0)'&=-(\lambda_4,\dots,\lambda_{n+3})\\ P_{n+1}(t_0)&=-\lambda_{n+4}\end{aligned}\tag{7.3}$$

$$\begin{aligned}H(t_0,x(t_0),u(t_0))&=\lambda_2\\ H(t_1,x(t_1),u(t_1))&=-\lambda_3.\end{aligned}\tag{7.4}$$

Notice that λ_1 cannot be zero, for if it was, by (7.2) $P(t_1)=0$ and hence by (7.1) $P(t)\equiv 0$. This would imply by (7.2)–(7.4) that $\lambda=0$ which is a contradiction. Since the equations of Pontryagin's principle are homogeneous in λ and $\lambda_1 \neq 0$ we may divide these equations by an appropriate positive number to obtain $\lambda_1=-\frac{1}{2}$. With this normalization

$$P_{n+1}(t)=-\tfrac{1}{2}.\tag{7.5}$$

The Hamiltonian is

$$H(t,x,u)=\tilde{P}(t)'[A(t)x+B(t)u]-\tfrac{1}{2}[x'M(t)x+u'N(t)u]. \tag{7.6}$$

The unique u which maximizes $H(t,x(t),u)$ can be found by standard methods to be

$$u(t)=N(t)^{-1}B(t)'\tilde{P}(t). \tag{7.7}$$

Substituting (7.5) into (7.1) and (7.7) into (2.8) replaces the problem of finding extremal controls for the linear regulator problem to finding solutions $x(t)$, $\tilde{P}(t)$ of the system

$$\begin{aligned}\dot{x}(t)&=A(t)x(t)+B(t)N(t)^{-1}B(t)'\tilde{P}(t)\\ \dot{\tilde{P}}(t)&=-A(t)'\tilde{P}(t)+M(t)x(t)\end{aligned} \tag{7.8}$$

with initial condition for $x(t)$ and terminal condition for $\tilde{P}(t)$ given by

$$\begin{aligned}x(t_0)&=x_0\\ \tilde{P}(t_1)&=-Dx(t_1).\end{aligned} \tag{7.9}$$

Given a solution of (7.8) with boundary conditions (7.9) the extremal control is given by (7.7).

Later in Chap. IV a different form is obtained for the optimal control law. See Problems IV.4 and IV.5 for the connection between these two representations of the optimal control.

§ 8. Extremals for the Simplest Problem in Calculus of Variations

Referring to Example 2.4, for this problem Eq. (5.1) is

$$(\dot{P}_1(t),\dot{P}_2(t))=-(P_1(t),P_2(t))\begin{pmatrix}0 & 0\\ L_x(t,x_1(t),u(t)) & 0\end{pmatrix} \tag{8.1}$$

or

$$\dot{P}_1(t)=-P_2(t)L_x(t,x_1(t),u(t)),\qquad \dot{P}_2(t)=0. \tag{8.2}$$

Inequality (5.2) is

$$P_1(t)u+P_2(t)L(t,x_1(t),u)-[P_1(t)u(t)+P_2(t)L(t,x_1(t),u(t))]\leqq 0. \tag{8.3}$$

The vector function ϕ giving the performance and end conditions is

$$\phi(t_0,t_1,x_{01},x_{02},x_{11},x_{12})=\begin{pmatrix}x_{12}\\ t_0-a\\ t_1-b\\ x_{01}-c\\ x_{02}\\ x_{11}-d\end{pmatrix}. \tag{8.4}$$

Hence conditions (5.3)–(5.6) are

$$(8.5)\qquad \begin{aligned} &(P_1(t_1), P_2(t_1)) = (\lambda_6, \lambda_1)\\ &(P_1(t_0), P_2(t_0)) = -(\lambda_4, \lambda_5)\\ &H(t_1, x(t_1), u(t_1)) = -\lambda_3\\ &H(t_0, x(t_0), u(t_0)) = \lambda_2. \end{aligned}$$

Condition (5.7) is

$$(8.6)\qquad P_1(t)u(t) + P_2(t)L(t, x_1(t), u(t)) = \lambda_2 + \int_{t_0}^{t} P_2(s)L_t(s, x_1(s), u(s))\,ds.$$

Notice since (8.3) implies $H(t, x(t), u)$ has a maximum at $u = u(t)$, the control set is all of E^1, and $L(t, x, u)$ is differentiable in u we must have that

$$(8.7)\qquad 0 = \frac{\partial}{\partial u} H(t, x(t), u(t)) = P_1(t) + P_2(t)L_u(t, x_1(t), u(t)).$$

Eq. (8.2) implies $P_2(t)$ is a constant P_2. From (8.7) if $P_2 = 0$, $P_1(t) \equiv 0$. This would imply by (8.5) that $\lambda = 0$ contradicting $\lambda \neq 0$. Hence $P_2 \neq 0$. By Remark 5.1 we may assume $P_2 = -1$. Hence from (8.7)

$$(8.8)\qquad P_1(t) = L_u(t, x_1(t), u(t)).$$

Putting $P_2(t) = -1$ in (8.2) and integrating using the end condition (8.5) we have

$$(8.9)\qquad P_1(t) = -\lambda_4 + \int_{t_0}^{t} L_x(s, x(s), u(s))\,ds.$$

Combining (8.8) and (8.9) and identifying $x(t) = x_1(t)$, $\dot{x}(t) = u(t)$ we obtain the Euler equation I (3.5)

$$(8.10)\qquad L_{\dot{x}}(t, x(t), \dot{x}(t)) - \int_{t_0}^{t} L_x(s, x(s), \dot{x}(s))\,ds = -\lambda_4.$$

Formula (8.6) can be similarly rewritten as

$$L_{\dot{x}}\dot{x} - L = \lambda_2 - \int_{t_0}^{t} L_t(s, x(s), \dot{x}(s))\,ds.$$

This is a generalization of Lemma I.4.1. Similarly formula (8.3) may be rewritten as the inequality

$$L(t, x(t), u) - L(t, x(t), \dot{x}(t)) - L_{\dot{x}}(t, x(t), \dot{x}(t))(u - \dot{x}(t)) \geq 0$$

for each u in E^1. This is a necessary condition for optimality in classical calculus of variations called the *Weierstrass condition.*

§ 9. General Features of the Moon Landing Problem

We shall discuss a number of topics and concepts important for optimal control problems in general by illustrating how they arise in Example 2.1.

Notice that the requirement that the spacecraft make a soft landing, that is have velocity $v=0$ at the first time $h=0$, restricts trajectories to those which satisfy $h \geqq 0$. This is a constraint on the state of the system. Generally we shall not consider problems with state space constraints in this book. In this problem the state space constraint was taken into account by allowing the controls to violate the constraint and determining that in this wider class the optimum satisfied the constraint $h>0$ except at the terminal time. In some control problems with state space constraints part of the optimal trajectory will satisfy the constraint as an equality. In this case some controls of the type $\tilde{u}(t)$ defined in §3 may violate the state space constraints. Since the class of controls satisfying the state space constraints may not be closed under adding controls of the form $\tilde{u}(t)$ other conditions arise which are necessary for a control to be optimal. See Pontryagin *et. al.* [1] for a discussion of these conditions.

Notice that the terminal condition $v=0$ at the first time that $h=0$ could only be satisfied for certain initial conditions. Referring to Fig. II.2 these initial conditions are the portion of the region $h>0$ below and on the parabola (6.17) which intersects the switching curve at the point (v, h) parameterized by $s \leqq F/k\alpha$. This set is called the set of initial conditions from which the terminal conditions are reachable or more briefly the *reachable set*.

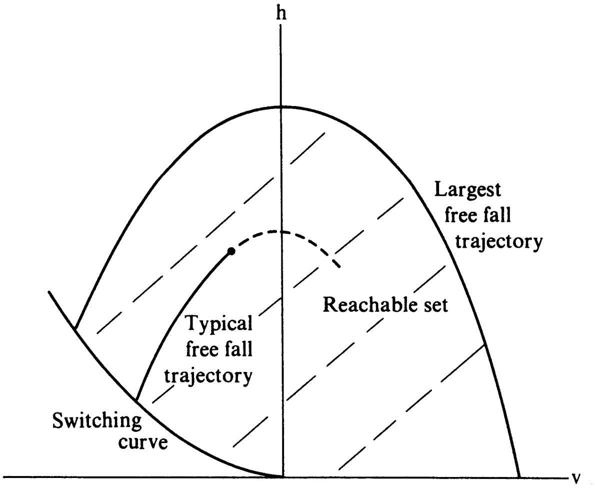

Fig. II.2. Reachable set. For moon landing problem

Actually in determining the extremal controls (6.18) we did not determine extremal controls for a single optimization problem but for a whole family of optimization problems. This follows because for each initial (v, h) in the reachable set, the controls (6.18) are extremal for the optimization problem with this (v, h) as initial conditions. The control (6.18) could be expressed as a function of (v, h) rather than as a function of t as in (6.18). The control is zero if (v, h) is a point of the reachable set above the switching curve and α if (v, h) lies on the switching curve. A control expressed in this form is said to be synthesized in *feedback form* or more briefly a *feedback control.* It is called feedback form because the control is expressed as a function of the state of the system and the values of the state are thought of as being observed and fed back into the control. The control in feedback form is extremal for any of the optimization problems with initial conditions in the reachable set. Controls in feedback form will play an important role in the sufficiency conditions for optimality given in Chap. IV. They are closely related to a concept called a "field of extremals" in classical calculus of variations.

In the computation of the extremal control law (6.18) an argument was given to show that the terms of the Hamiltonian which multiplied the control $u(t)$ were not zero. In the case in which the Hamiltonian can be zero on an interval for every value of u in the control set U the optimization problem is called *singular*.

In this case the maximization of the Hamiltonian gives no information about the value of the control. The possibility of the control problem being singular is quite common in optimal control problems. Even when the problem is not singular, often a lengthy argument showing that singularity does not occur may be required. In some problems portions of optimal trajectories are singular, see Problems (9), (10). In the problem of the motion of a rocket in an inverse square field, a singular extremal called Lawden's spiral (after its discoverer Derick Lawden [1]) occurs. A great deal of work was done to determine if Lawden's spiral could be a portion of an optimal trajectory, Kelley [2], Gurley [1], Robbins [1], Kopp and Moyer [1], Goh [2]. Further optimality conditions than those of §5 which are necessary for a singular arc to be a portion of an optimal trajectory are given in Johnson-Gibson [1], Kelley *et al.* [1], Goh [1], [2], Gabasov [1], Gabsov-Kirillova [1], Jacobson [1].

§ 10. Summary of Preliminary Results

Before beginning a discussion of optimality conditions for the optimal control problem, we will state some results from differential equations which are needed in the discussion of these conditions.

Theorem 10.1. *Let $A(t)$ be an $n\times n$ matrix, $G(t)$ an n-dimensional vector of piecewise continuous functions defined on an interval $[t_0, t_1]$, and y_0 an n-dimensional vector. Then if $\tau\in[t_0, t_1]$ there is a unique piecewise continuously differentiable solution of the vector differential equation*

$$\dot{y}=A(t)\,y+G(t) \tag{10.1}$$

on the interval $[t_0, t_1]$ which satisfies the condition

$$y(\tau)=y_0. \tag{10.2}$$

The system of vector differential equations

$$\dot{P}=-A(t)'P \tag{10.3}$$

is called the system of differential equations adjoint to Eqs. (10.1) or more briefly the *adjoint equations*. By Theorem 10.1 it has a unique solution for a given boundary condition. If $y(t)$ is a solution of (10.1) and $P(t)$ a solution of (10.3),

$$\frac{d}{dt} P(t)' y(t) = -P(t)' A(t) y(t) + P(t)' A(t) y(t) + P(t)' G(t). \tag{10.4}$$

For any two times τ_1 and τ_2 an integration from τ_1 to τ_2 of (10.4) establishes the important formula

$$P(\tau_2)' y(\tau_2) - P(\tau_1)' y(\tau_1) = \int_{\tau_1}^{\tau_2} P(t)' G(t)\, dt. \tag{10.5}$$

The next theorems express the dependence of corresponding trajectories on the parameters of various mappings.

Theorem 10.2. *Let $f_u(t, x, u)$ be continuous. For $0 \leqq \varepsilon \leqq \eta$, let $x^\varepsilon(t)$ be the solutions of* (3.1) *corresponding to controls $u(t) + \varepsilon\, v(t)$ with the same initial condition $x^\varepsilon(t_0) = x_0$. Then*

$$x^\varepsilon(t) = x(t) + \varepsilon\, \delta x(t) + o(t, \varepsilon) \tag{10.6}$$

where $\delta x(t)$ is the solution of

$$\delta \dot{x}(t) = f_x(t, x(t), u(t))\, \delta x(t) + f_u(t, x(t), u(t))\, v(t) \tag{10.7}$$

with initial condition

$$\delta x(t_0) = 0. \tag{10.8}$$

Theorem 10.3. *For a real variable ε, let $x^\varepsilon(t)$ be the solution of* (3.1) *on $[t_0, t_1]$ corresponding to the control $u(t)$ with initial condition*

$$x^\varepsilon(t_0) = x_0 + \varepsilon\, y_0 + o(\varepsilon). \tag{10.9}$$

Then:

$$x^\varepsilon(t) = x(t) + \varepsilon\, \delta x(t) + o(t, \varepsilon) \tag{10.10}$$

where $\delta x(t)$ is the solution of

$$\delta \dot{x}(t) = f_x(t, x(t), u(t))\, \delta x(t) \tag{10.11}$$

on $[t_0, t_1]$ with initial condition

$$\delta x(t_0) = y_0. \tag{10.12}$$

An n-dimensional version of Theorem 10.3 is given by Theorem 10.4.

Theorem 10.4. *Let ρ denote a vector variable in E^n, M an $n \times n$ matrix, and $x^\rho(t)$ the solution of* (3.1) *on $[t_0, t_1]$ corresponding to the control $u(t)$ with initial condition*

$$x^\rho(t_0) = x_0 + M\rho + o(\rho). \tag{10.13}$$

Then

$$x^\rho(t) = x(t) + \delta x(t)\, \rho + o(t, \rho) \tag{10.14}$$

where $\delta x(t)$ is the $n \times n$ matrix which is the solution of

$$\delta \dot{x}(t) = f_x(t, x(t), u(t))\, \delta x(t) \tag{10.15}$$

on $[t_0, t_1]$ with initial condition

$$\delta x(t_0) = M. \tag{10.16}$$

§ 11. The Free Terminal Point Problem

To motivate the proof of Pontryagin's Principle, we will carry out two derivations of necessary conditions for optimality in the special case of the problem of optimal control in which the initial time and state and the final time are fixed and there are no conditions on the final state. This problem is called the *free terminal point problem.* The derivation of Pontryagin's principle in this case is especially simple and conceptually clear. These derivations illustrate the role the adjoint equations play in evaluating directional derivatives of the mappings involved. They also compare results obtained using the two different families of mappings.

Let t_0 and t_1 denote the fixed initial and terminal times. In these derivations the space $\mathscr{V}$ described in Chap. I will be taken to be the space of all left continuous piecewise continuous functions defined on $[t_0, t_1]$ with values in E^m. Call this the space of control functions. The subset $\mathscr{K}$ will be the set of control functions u such that $u(t) \in U$ for each $t \in [t_0, t_1]$ and (x_0, u) is feasible pair for the fixed initial state x_0. Being a feasible pair in this case merely requires that (3.1) with control u and initial condition x_0 have a solution on the entire interval $[t_0, t_1]$.

There are many ways of constructing mappings ζ into $\mathscr{K}$ as discussed in Chap. I, § 2. We will consider two such mappings. First let us consider a family of mappings similar to those used in Chap. I. If U is convex and $u(t)$ and $u(t)+v(t)$ are in U for each $t \in [t_0, t_1]$ then $u(t) + \varepsilon\, v(t)$ is in U for each $t \in [t_0, t_1]$ and $0 \leqq \varepsilon \leqq 1$. Theorems on continuous dependence of solutions of differential equations on parameters imply $(x_0, u + \varepsilon v)$ is a feasible pair for $0 \leqq \varepsilon \leqq \eta$ for some $\eta > 0$. Therefore, we may construct the mapping

$$\zeta\colon\ 0 \leqq \varepsilon \leqq \eta \to \mathscr{K} \tag{11.1}$$

into the controls u such that (x_0, u) is feasible, by defining

$$\zeta(\varepsilon) = u + \varepsilon\, v. \tag{11.2}$$

Let x^ε denote the solution of (3.1) with initial condition $x^\varepsilon(t_0) = x_0$ corresponding to $\zeta(\varepsilon)$ and x the similar solution corresponding to u. Then x^ε for small ε, remains in a weak neighborhood of x as defined in Remark I.5.2. Consequently (11.2) is called a *weak variation* of the control.

Another family of mappings is the following. Suppose u is an element of $\mathscr{K}$, v a fixed vector in U and $\tau \in (t_0, t_1]$. For this fixed v, small enough η, and $0 < \varepsilon < \eta$ existence theorems for differential equations imply there will be a solution x^ε of

$$\dot{x} = f(t, x, v) \tag{11.3}$$

on the interval $\tau-\varepsilon \leqq t \leqq \tau$ with the condition

$$x^{\varepsilon}(\tau-\varepsilon)=x(\tau-\varepsilon). \tag{11.4}$$

The theorem for continuous dependence of solutions of differential equations on initial conditions implies that for small enough η, (3.1) has a solution on $[\tau, t_1]$ with initial condition $x^{\varepsilon}(\tau)$. Define the control u^{ε} by

$$u^{\varepsilon}(t)=\begin{cases} v & \text{if } \tau-\varepsilon<t\leqq\tau \\ u(t) & \text{for other values of } [t_0, t_1]. \end{cases} \tag{11.5}$$

The previous remarks imply that for small enough η we may define a mapping

$$\zeta\colon\ 0 \leqq \varepsilon \leqq \eta \to \mathscr{K}$$

into the controls which are feasible for x_0 by defining

$$\zeta(\varepsilon)=u^{\varepsilon}. \tag{11.6}$$

For the mapping (11.6), x^{ε} for small ε need not be contained in a weak neighborhood of x. However x^{ε}, for small ε, is contained in a strong neighborhood of x. (See Theorem 11.3.) Thus (11.5) is called a *strong variation* of the control.

We will carry out two derivations of necessary conditions for optimality, one using weak variations, the other strong variations of the control. The derivation using strong variations will obtain Pontryagin's principle for the free endpoint problem. The one using weak variations will achieve a slightly weaker result.

Notice, since the quantities $(t_0, t_1, x(t_0))$ are fixed, for the free terminal point problem we may as well assume the performance index

$$\phi_1(t_0, t_1, x(t_0), x(t_1))$$

is of the form $\phi_1(x(t_1))$. Since the other components of the vector ϕ defined in § 3 will not play a role in our discussion of the free terminal point problem we will also drop the subscript one and write the performance index as

$$\phi(x(t_1)).$$

In the notation of Chap. I.2 the performance index determines the real valued function of the control

$$J(u)=\phi(x(t_1)).$$

The following theorem evaluates the directional derivative of the mapping $J(\zeta(\varepsilon))$ when $\zeta(\varepsilon)$ is the control corresponding to a weak variation of the type (11.2).

Theorem 11.1. *Let U be convex and $f_u(t, x, u)$ be continuous. Let $P(t)$ be the solution of the adjoint equations of Eq.* (10.7), *that is,*

$$\dot{P}(t)'=-P(t)' f_x(t, x(t), u(t)), \tag{11.7}$$

with boundary condition

$$P(t_1)'=-\phi_x(x(t_1)). \tag{11.8}$$

Then

$$\delta J(u,v) = -\int_{t_0}^{t_1} P(t)' f_u(t, x(t), u(t))\, v(t)\, dt. \tag{11.9}$$

Proof. If $\delta x(t)$ is the solution of (10.7) with initial condition (10.8), formula (10.5) implies

$$\phi_x(x(t_1))\,\delta x(t_1) = -\int_{t_0}^{t_1} P(t)' f_u(t, x(t), u(t))\, v(t)\, dt. \tag{11.10}$$

Theorem 10.2 and the chain rule for differentiation imply that

$$\delta J(u,v) = \phi_x(x(t_1))\,\delta x(t_1). \tag{11.11}$$

Thus (11.11) and (11.10) imply (11.9). □

Theorem 11.2. *Under the hypotheses of Theorem* 11.1, *a necessary condition that a control u be optimal for the free terminal point problem is that for each fixed* $t \in (t_0, t_1]$

$$P(t)' f_u(t, x(t), u(t))\, v \leqq 0 \tag{11.12}$$

for each $v \in U$ *such that* $u(t) + v \in U$.

Proof. Since u may be assumed left continuous there is some interval $t_0 < t - \delta < s \leqq t$ on which $u(s)$ is continuous. Define a function w on $[t_0, t_1]$ by

$$w(s) = \begin{cases} u(t) + v - u(s) & \text{if } t - \delta < s \leqq t \\ 0 & \text{otherwise} \end{cases} \tag{11.13}$$

Considerations similar to those of (11.3) through (11.6) imply for δ small enough $u(s) + w(s) \in \mathscr{K}$. Hence for δ small enough by Theorem 11.1 and Theorem I.2.2

$$\delta J(u,w) = -\int_{t-\delta}^{t} P(s)' f_u(s, x(s), u(s))\,[u(t) + v - u(s)]\, ds \geqq 0. \tag{11.14}$$

Since the integrand is left continuous dividing (11.14) by δ and taking the limit as δ approaches zero implies (11.12). □

Next we will consider the analogs of Theorems 11.1 and 11.2 when strong variations of the control are considered.

Theorem 11.3. *If* x^ε *are solutions of* (3.1) *corresponding to controls* u^ε *defined by* (11.5) *with the same initial condition* $x^\varepsilon(t_0) = x_0$ *then*

$$x^\varepsilon(t) = x(t) + \varepsilon\,\delta x(t) + o(t, \varepsilon) \tag{11.15}$$

where $\delta x(t)$ *is the solution of*

$$\delta x(t) = \begin{cases} 0 & \text{if } t_0 \leqq t < \tau \\ [f(\tau, x(\tau), v) - f(\tau, x(\tau), u(\tau))] + \int_{\tau}^{t} f_x(s, x(s), u(s))\,\delta x(s)\, ds & \\ & \text{if } \tau \leqq t \leqq t_1 \end{cases} \tag{11.16}$$

Proof. From (11.5) $u^\varepsilon(t)=u(t)$ if $t_0 \leqq t \leqq \tau-\varepsilon$. Hence $x^\varepsilon(t)=x(t)$ if $t_0 \leqq t \leqq \tau-\varepsilon$. Thus (11.16) is true with $\delta x(t)=0$ if $t_0 \leqq t \leqq \tau-\varepsilon$.

We have if $\tau-\varepsilon \leqq t \leqq \tau$ that

$$x^\varepsilon(t)=x(t)-\int_{\tau-\varepsilon}^{t} f(s, x(s), u(s))\, ds+\int_{\tau-\varepsilon}^{t} f(s, x^\varepsilon(s), v)\, ds. \tag{11.17}$$

Let for $\eta>0$

$$A=\{(\varepsilon, t)\colon 0 \leqq \varepsilon \leqq \eta,\ \tau-\varepsilon \leqq t \leqq \tau\}. \tag{11.18}$$

The theorem of continuous dependence of solutions of differential equations on initial conditions implies for some $\eta>0$ that the mapping

$$(\varepsilon, t) \to x^\varepsilon(t)$$

of A into E^n is continuous. Hence

$$|f(s, x(s), u(s))-f(s, x^\varepsilon(s), v)|$$

is bounded on A. Thus (11.17) implies

$$x^\varepsilon(t)=x(t)+O(t, \varepsilon) \quad \text{if } \tau-\varepsilon \leqq t \leqq \tau. \tag{11.19}$$

Applying (11.19) in (11.17) and using the left continuity of $f(t, x(t), u(t))$ gives

$$x^\varepsilon(\tau)=x(\tau)+[f(\tau, x(\tau), v)-f(\tau, x(\tau), u(\tau))]\varepsilon+o(\tau, \varepsilon). \tag{11.20}$$

Now an application of Theorem 10.3 and (11.20) completes the derivation of (11.16). $\square$

Theorem 11.4. *For* $0 \leqq \varepsilon \leqq \eta$, *let*

$$J(u^\varepsilon)=\phi(x^\varepsilon(t_1)) \tag{11.21}$$

where $u^\varepsilon(t)$ *is the control defined by* (11.5). *Let* $P(t)$ *be the solution of* (11.7) *with boundary condition* (11.8). *Then*

$$\frac{d}{d\varepsilon} J(u^\varepsilon)\big|_{\varepsilon=0}=-P(\tau)'[f(\tau, x(\tau), v)-f(\tau, x(\tau), u(\tau))]. \tag{11.22}$$

Proof. Formulas (10.5), (11.7), (11.8) and (11.16) imply

$$-P(\tau)'[f(\tau, x(\tau), v)-f(\tau, x(\tau), u(\tau))]=\phi_x(x(t_1))\,\delta x(t_1) \tag{11.23}$$

where $\delta x(t_1)$ is the solution of (11.16). Formulas (11.21), (11.15), and the chain rule for differentiation imply (11.22). $\square$

Formula I (2.5) and Theorem 11.4 imply Theorem 11.5.

Theorem 11.5. (Pontryagin's Principle for the Free Terminal Point Problem). *A necessary condition for optimality of a control u for the Free Terminal Point*

Problem is that

$$P(t)'\big[f\big(t, x(t), v\big) - f\big(t, x(t), u(t)\big)\big] \leqq 0 \tag{11.24}$$

for each $v \in U$ *and* $t \in (t_0, t_1]$, *where* $P(t)'$ *is the solution of*

$$\dot{P}(t)' = -P(t)' f_x\big(t, x(t), u(t)\big) \tag{11.25}$$

with boundary condition

$$P(t_1)' = -\phi_x\big(x(t_1)\big). \tag{11.26}$$

It is natural to ask if the conditions of Theorem 11.2 or 11.5 are sufficient conditions for optimality as well as necessary ones. They are not sufficient in general. However Theorem I.2.3 implies that the conditions of Theorem 11.2 are sufficient when $\mathscr{K}$ is convex and the mapping

$$J(u) = \phi\big(x(t_1)\big) \tag{11.27}$$

is a convex function on $\mathscr{K}$. Theorem 11.6 discusses a situation in which this happens.

Let U be convex. Let the equations of motion of the system be given by the linear differential equation

$$\dot{x} = A(t)\, x(t) + B(t)\, u(t) \tag{11.28}$$

where $A(t)$ and $B(t)$ are appropriate dimensional matrices of continuous functions. Let the performance index be given by

$$J(u) = \int_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt + \psi\big(x(t_1)\big), \tag{11.29}$$

where L is a continuous real valued function continuously differentiable and convex in (x, u) and ψ is a continuously differentiable convex function of x. Under these assumptions the conditions of Theorem 11.2 can be rewritten.

Theorem 11.6. *A necessary and sufficient condition for optimality of a control* $u(t)$ *for the free terminal point problem with system Eq.* (11.28) *and performance index* (11.29) *is that for* $t \in (t_0, t_1]$

$$-L_u\big(t, x(t), u(t)\big) v + \tilde{P}(t)' B(t) v \leqq 0 \tag{11.30}$$

for each $v \in U$ *such that* $u(t) + v \in U$, *where* $\tilde{P}(t)$ *is the solution of*

$$\dot{\tilde{P}}(t)' = -\tilde{P}(t)' A(t) + L_x\big(t, x(t), u(t)\big) \tag{11.31}$$

$$\tilde{P}(t_1)' = -\psi_x\big(x(t_1)\big). \tag{11.32}$$

Moreover if in addition $L(t, x, u)$ *is strictly convex in* (x, u) *for each fixed* t, *the optimal control* $u(t)$ *is unique.*

Proof. As mentioned the necessity part is just a restatement of Theorem 11.2 using the technique of § 4 to rewrite the problem as one with performance index of the Mayer type (4.1). Since (11.28) is a linear system of differential equations, the set $\mathscr{K}$ of controls such that $u(t)\in U$ and $(x_0, u(t))$ is a feasible pair is all piecewise continuous functions such that $u(t)\in U$. This is a convex set. Let $u^0(t)$ and $u^1(t)$ be controls in $\mathscr{K}$ and $x^0(t)$ and $x^1(t)$ corresponding solutions of (11.28) with initial condition $x(t_0)=x_0$. If $0<\alpha<1$, the convexity of $L(t,x,u)$ and $\psi(x)$ implies

$$
\begin{aligned}
\alpha J(u^0)+(1-\alpha)J(u^1) &= \alpha\left[\int_{t_0}^{t_1} L(t,x^0(t),u^0(t))\,dt+\psi(x^0(t_1))\right]\\
&\quad+(1-\alpha)\left[\int_{t_0}^{t_1} L(t,x^1(t),u^1(t))\,dt+\psi(x^1(t_1))\right]\\
&\geqq \int_{t_0}^{t_1} L(t,\alpha x^0(t)+(1-\alpha)x^1(t),\alpha u^0(t)+(1-\alpha)u^1(t))\,dt\\
&\quad+\psi(\alpha x^0(t_1)+(1-\alpha)x^1(t_1))=J(\alpha u^0+(1-\alpha)u^1).
\end{aligned}
\tag{11.33}
$$

The last equality of (11.33) follows because $\alpha x^0(t)+(1-\alpha)x^1(t)$ is the solution (11.28) corresponding to $\alpha u^0(t)+(1-\alpha)u^1(t)$. Thus $J(u)$ is a convex function defined on a convex set $\mathscr{K}$. Theorem 11.1 and condition (11.30) imply for each $v(t)$ which satisfies $u(t)+v(t)\in U$ for each $t\in[t_0,t_1]$ that

$$\delta J(u,v)=\int_{t_0}^{t_1}\left[L_u(t,x(t),u(t))\,v(t)-\tilde{P}(t)'B(t)\,v(t)\right]dt\geqq 0.$$

Hence by Theorem I.2.3 $J(u)$ has a minimum at $u=u(t)$.

To show uniqueness of the minimum, note that if $L(t,x,u)$ is strictly convex the inequality in (11.33) is strict. Thus $J(u)$ is a strictly convex function on $\mathscr{K}$ and by Theorem I.2.4 the minimum is unique. □

§ 12. Preliminary Discussion of the Proof of Pontryagin's Principle

In trying to establish necessary conditions for the general problem of optimal control, it might seem natural to consider the performance index as a function

$$J(x_0,u(t))=\phi_1(t_0,t_1,x(t_0),x(t_1))$$

on the space $\mathscr{F}$ defined in § 3, and to consider the optimization problem as one of minimizing this function on this space. To proceed as in Chap. I, mappings into the space $\mathscr{F}$

$$\zeta\colon 0\leqq\varepsilon\leqq\eta\to\mathscr{F}$$

such that the composition of these mappings with the mapping J taking $\mathscr{F}$ into the performance index would be constructed. A problem occurs in that there are examples, one is given below, in which the space $\mathscr{F}$ is a single element. While this is no difficulty in determining the optimum which must be the single element, it hinders the development of the theory in that it precludes the construction of mappings of the type given by (11.2) or (11.5) into the space $\mathscr{F}$.

The following is an example in which $\mathscr{F}$ has only one element. Consider the optimal control problem with scalar differential equation

$$\dot{x} = u^2,$$

terminal conditions,

$$x(0)=0, \qquad x(1)=0,$$

control set

$$U=\{u: \ -1 \leqq u \leqq 1\},$$

and performance criterion

$$\int_0^1 L(x(t), u(t))\, dt.$$

The only control whose corresponding trajectory satisfies the terminal conditions is

$$u(t) \equiv 0 \qquad \text{if } 0 \leqq t \leqq 1.$$

Hence this is the only element of $\mathscr{F}$.

In deriving an optimality theory for the Bolza problem in classical calculus of variations, a concept called normality was introduced and assumed. This condition assured that there were enough nontrival mappings into the space $\mathscr{F}$ so that conditions for optimality could be developed. Problems such as the example just given are abnormal. The need for making normality assumptions to derive optimality conditions for the Bolza problem was removed in a paper by McShane [6]. Similar constructions were carried out by Pontryagin, Boltyanskii and Gamkrelidze [1] in their proof of Pontryagin's principle. We shall give a proof of Pontryagin's Principle which combines features of proofs by McShane [6]; Pontryagin, Boltyanskii, and Gamkrelidze [1]; Halkin [2]; and Hestenes [1].

The basic idea in overcoming the difficulty mentioned above will be to consider the set $\mathscr{G}$ of pairs (x_0, u) such that $u \in \mathscr{U}$ and there is a solution of (3.1) on the interval $[t_0, t_1]$ corresponding to the control u with initial condition $x(t_0)=x_0$. Trajectories corresponding to pairs of $\mathscr{G}$ do not necessarily satisfy the end conditions $\phi_2(e)=0, \ldots, \phi_k(e)=0$. On the set $\mathscr{G}$ consider the mapping

$$\mathscr{J}: \mathscr{G} \to E^k$$

defined by

$$\mathscr{J}(x_0, u) = \phi(e).$$

Let the components of $\mathscr{J}$ be denoted by $\mathscr{J}=(J_1, \ldots, J_k)$. In terms of the mapping $\mathscr{J}$ the problem of optimal control becomes the abstract nonlinear problem of finding the element of $\mathscr{G}$ such that the first component $J_1(x_0, u)$ of $\mathscr{J}$ is minimized subject to the satisfaction of the conditions $J_i(x_0, u)=0$, $i=2, \ldots, k$. Mappings of the type (11.2) or (11.5) into the space $\mathscr{G}$ can be constructed. In § 14 a mapping more general than, but analogous to, those of (11.5) will be constructed.

The proof of Pontryagin's Principle will be broken up into three parts. First optimality conditions will be deduced for an abstract nonlinear programming problem (§ 13). Then it will be shown that the hypotheses of this problem are satisfied for the problem of optimal control (§ 14). The final part interprets the conditions obtained in the programming problem to obtain Pontryagin's Principle (§ 15).

§ 13. A Multiplier Rule for an Abstract Nonlinear Programming Problem

Let $\mathscr{S}$ be a set and $\mathscr{J}(s)$ a mapping

$$\mathscr{J}: \mathscr{S} \to E^k. \tag{13.1}$$

Consider the problem of finding an element s^* of $\mathscr{S}$ that minimizes $J_1(s)$ on the set $\mathscr{S} \cap \{s\colon J_i(s)=0,\ i=2, \ldots, k\}$.

To develop conditions for optimality for this problem we will need a notion of differentiation slightly stronger than could be obtained by considering derivatives of $\mathscr{J}$ along curves mapping into the set $\mathscr{S}$. Roughly what we will need will be a continuous directional derivative in a k-dimensional cone of directions.

Let e^i, $i=1, \ldots, k$ denote the standard basis vectors of E^k; i.e., $e^1=(1,0,\ldots,0)$, etc. Let C denote the convex hull of $\{0, e^1, \ldots, e^k\}$. We shall call C the standard k-dimensional simplex of E^k. Let $\eta\, C$ denote the convex hull of $\{0, \eta\, e^1, \ldots, \eta\, e^k\}$. Instead of curves mapping into the space $\mathscr{S}$, we shall consider mappings $\zeta(\rho)$ defined on $\eta\, C$ for some $\eta>0$, that is,

$$\zeta\colon \eta\, C \to \mathscr{S}. \tag{13.2}$$

We shall say that a $k \times k$ matrix M is the *conical differential of* $\mathscr{J}$ *at* s *along* ζ if $\zeta(0)=s$ and for ρ in $\eta\, C$

$$\mathscr{J}(\zeta(\rho)) = \mathscr{J}(s) + M\rho + o(\rho). \tag{13.3}$$

Let D be a convex cone with vertex zero contained in E^k. The set D will be called a *cone of variations of* $\mathscr{J}$ *at* s if for each linearly independent set $\{d^1, \ldots, d^k\}$ of vectors in D there is for some $\eta>0$ a mapping $\zeta\colon \eta\, C \to \mathscr{S}$ such that the composition mapping $\mathscr{J}(\zeta(\rho))$ is continuous on $\eta\, C$ and the matrix M whose columns are $d^1, \ldots, d^k$ is the conical differential of $\mathscr{J}$ at s along ζ.

Theorem 13.1. (Abstract Multiplier Rule). *If s^* is a solution to the abstract nonlinear programming problem and if D is a cone of variations of $\mathscr{J}$ at s^*, then there is a nonzero vector λ in E^k such that $\lambda_1 \leqq 0$ and $\lambda' d \leqq 0$ for all d in D.*

Proof. Let

$$L = \{y \in E^k\colon y_1 < 0,\ y_i = 0,\ i=2, \ldots, k\}. \tag{13.4}$$

The conditions of the theorem are equivalent to the statement: There exists a hyperplane $\lambda' y = 0$ such that $\lambda' y \geqq 0$, $y \in L$; $\lambda' y \leqq 0$, $y \in D$. Thus we must show the existence of such a hyperplane separating L and D.

We shall suppose L and D are not separated and show this contradicts the optimality of s^*. If L and D are not separated the vector $(-1, 0, \ldots, 0)$ which generates L must be interior to D. If it were not the separation theorem for convex sets would imply that there was a hyperplane separating L and D. Thus there must be a sphere about $(-1, 0, \ldots, 0)$ contained in D. Let y_1 denote the first coordinate of a vector $(y_1, \ldots, y_k)$ in E^k. Consider the hyperplane $y_1 = -1$. In this hyperplane place a suitable scaled and translated standard $k-1$ dimensional simplex so that it is contained in the intersection of the sphere and the hyperplane and $(-1, 0, \dots, 0)$ is in the interior of the simplex in the relative topology of the hyperplane. Draw k vectors $d_1, \dots, d_k$ from the origin of E_k to the vertices of this simplex. Then these k vectors are linearly independent, contained in the interior of D, and all have first coordinate -1.

Let M denote the matrix whose columns are $\{d_1, \dots, d_k\}$. Let Δ denote the convex hull of $\{0, d_1, \dots, d_k\}$. Since D is a cone of variations of $\mathscr{J}$ at s^* there is for some $\gamma > 0$ a mapping $\zeta \colon \gamma C \to \mathscr{S}$ such that

$$\mathscr{J}(\zeta(\rho)) = \mathscr{J}(s^*) + M\rho + o(\rho). \tag{13.5}$$

The definition of M and Δ imply that the mapping $M\rho$ maps γC into $\gamma\Delta$. Since the d^i are linearly independent, M^{-1} exists. The mapping defined for $y \in \gamma\Delta$ by

$$\psi(y) = \mathscr{J}(\zeta(M^{-1}y)) - \mathscr{J}(s^*) \tag{13.6}$$

maps $\gamma\Delta$ into E^k. Since M^{-1} maps $\gamma\Delta$ into γC, Eq. (13.5) implies for $y \in \gamma\Delta$

$$|\psi(y) - y| \leqq |o(M^{-1}y)|. \tag{13.7}$$

Let $a = (-\frac{1}{2}, 0, \dots, 0)$. By the construction of Δ for some ε, $0 < \varepsilon < 1$, a spherical neighborhood $N(a, \varepsilon)$ of radius ε of a is interior to Δ. Choose an ε so this is true. Since

$$\lim_{|M^{-1}y| \downarrow 0} \frac{|o(M^{-1}y)|}{|M^{-1}y|} = 0 \tag{13.8}$$

and for $\eta > 0$, $M^{-1}y$ maps $\eta\Delta$ into ηC there is an η, $0 < \eta < \gamma$ so that if $y \in \eta\Delta$

$$|o(M^{-1}y)| < \varepsilon\eta. \tag{13.9}$$

Choose such an η.

Let π denote the projection of E^k on E^{k-1} defined by

$$\pi(y_1, \dots, y_k) = (y_2, \dots, y_k). \tag{13.10}$$

We have $\pi(a) = 0$. Since $N(a, \varepsilon)$ is contained in Δ, the $\eta\varepsilon$ neighborhood of zero, $N(0, \eta\varepsilon)$ in E^{k-1} is contained in $\pi(\eta\Delta)$. Define

$$\chi \colon \pi(\eta\Delta) \to E^{k-1}$$

by $\chi(y_2, \dots, y_k) = (y_2, \dots, y_k) - \pi(\psi(-\eta, y_2, \dots, y_k))$. Now if $(y_2, \dots, y_k) \in \pi(\eta\Delta)$, (13.7) and (13.9) imply

$$\begin{aligned} |\chi(y_2, \dots, y_k)| &= |\pi(\psi(-\eta, y_2, \dots, y_k)) - \pi(-\eta, y_2, \dots, y_k)| \\ &\leqq |\psi(-\eta, y_2, \dots, y_k) - (-\eta, y_2, \dots, y_k)| < \eta\varepsilon. \end{aligned}$$

Hence since $\pi(\eta\Delta)$ contains the $\eta\varepsilon$ neighborhood of zero in E^{k-1} we have

$$\chi(\pi(\eta\Delta)) \subset \pi(\eta\Delta).$$

Since χ is clearly continuous, Brouwer's fixed point theorem, Hurewicz Wallman [1], implies that χ must have a fixed point on $\pi(\eta\Delta)$. Thus there exists a point

$(\bar{y}_2, \ldots, \bar{y}_k)$ such that

$$\chi(\bar{y}_2, \ldots, \bar{y}_k) - (\bar{y}_2, \ldots, \bar{y}_k) = -\pi(\psi(-\eta, \bar{y}_2, \ldots, \bar{y}_k)) = 0.$$

Since $|\psi(-\eta, \bar{y}_2, \ldots, \bar{y}_k) - (-\eta, \bar{y}_2, \ldots, \bar{y}_k)| < \eta\,\varepsilon < \eta$ and $\pi(\psi(-\eta, \bar{y}_2, \ldots, \bar{y}_k)) = 0$, we must have

$$\psi(-\eta, \bar{y}_2, \ldots, \bar{y}_k) = (y_1, 0, \ldots, 0) \quad \text{with } y_1 < 0.$$

But this and the definition of ψ imply

$$J_1(\zeta(M^{-1}(-\eta, \bar{y}_2, \ldots, \bar{y}_k))) - J_1(s^*) = y_1 < 0$$

while

$$J_i(\zeta(M^{-1}(-\eta, \bar{y}_2, \ldots, \bar{y}_k))) - J_i(s^*) = 0 \qquad i = 2, \ldots, k$$

which contradicts the optimality of s^*. □

§14. A Cone of Variations for the Problem of Optimal Control

Recall the set $\mathcal{G}$ and the mapping $\mathcal{J}$ defined in §12.

Theorem 14.1. *The convex cone generated by the vectors* (i)–(iv) *listed below is a cone of variations of the mapping* $\mathcal{J}(x_0, u) = \phi(e)$ *at* (x_0, u).

(i) $$\phi_{x_0}(e)\, y + \phi_{x_1}(e)\, \delta x(t_1)$$

where y is an element of E^n and $\delta x(t)$ is the solution

(14.1) $$\delta \dot{x}(t) = f_x(t, x(t), u(t))\, \delta x(t)$$

with

(14.2) $$\delta x(t_0) = y.$$

(ii) $$\phi_{x_1}(e)\, \delta x(t_1)$$

where $\delta x(t)$ is a solution of (14.1) *with boundary condition given for some $\tau \in (t_0, t_1]$ and $u \in U$ by*

(14.3) $$\delta x(\tau) = f(\tau, x(\tau), u) - f(\tau, x(\tau), u(\tau)).$$

(iii) $$\pm[\phi_{t_0}(e) + \phi_{x_0}(e) f(t_0, x(t_0), u(t_0))].$$

(iv) $$\pm[\phi_{t_1}(e) + \phi_{x_1}(e) f(t_1, x(t_1), u(t_1))].$$

In the rest of this section we prove this theorem. To begin its proof we must construct mappings $\zeta\colon \eta\, C \to \mathcal{G}$ so that $\mathcal{J}(\zeta(\rho))$ have appropriate conical differentials. Consider a vector d in the convex cone generated by vectors of the form (i)–(iv). Each vector will be a positive linear combination of the vectors (i)–(iv). Thus each such vector will have a representation of the form (14.4) in terms of a vector of E^n, pairs of times and control values (τ_i, u^i), $i = 2, \ldots, j$, and numbers a_i, $i=0,\ldots,j+1$ such that a_i, $i=1,\ldots,j$ are nonnegative:

$$\begin{aligned}(14.4)\qquad d=&a_0[\phi_{t_0}+\phi_{x_0}f(t_0,x(t_0),u(t_0))]+a_1[\phi_{x_0}y+\phi_{x_1}\delta x(t_1)^1]\\ &+\sum_{i=2}^{j}a_i\phi_{x_1}\delta x(t_1)^i+a_{j+1}[\phi_{t_1}+\phi_{x_1}f(t_1,x(t_1),u(t_1))].\end{aligned}$$

In (14.4), $\delta x(t)^1$ is the solution of (14.1) with boundary condition (14.2) and $\delta x(t)^i$, $i=2,\ldots,j$, is the solution of (14.1) with boundary condition of (14.3) corresponding to τ_i and u^i. In (14.4), (τ_i, u^i) and (τ_m, u^m) are distinct pairs in the representation if $\tau_i=\tau_m$ but $u^i \neq u^m$.

Notice that given k such vectors $d^1,\ldots,d^k$, we may assume each of the vectors is represented in terms of the same set of quantities $\delta x(t_1)^i$, corresponding to (τ_i, u^i) $i=2,\ldots,j$. If this were not the case it could be accomplished by considering the union of the set of these quantities appearing in the representation of all the vectors. Then a given vector could be represented in terms of this set using a coefficient $a_i=0$ if a particular $\delta x(t_1)^i$ was not involved in its previous representation. Thus given k such vectors we may arrange the numbers appearing in their representation into a $(j+1)\times k$ matrix A whose columns are the numbers a_i in the representation of each vector. Let a^i denote the row matrix which is i-th row of this matrix.

Let the times τ_i, $i=2,\ldots,j$ in the representation (14.4) be given in nondecreasing order. Let s_i be the largest integer so that $\tau_i=\tau_m$ if $i\leqq m\leqq s_i$. Define for $\rho\in C$ the function $l_i(\rho)$ by

$$(14.5)\qquad l_i(\rho)=\sum_{m=i}^{s_i}a^m\rho.$$

If $\rho\in\eta C$ and η is small enough the intervals

$$(14.6)\qquad \tau_i-l_i(\rho)<t\leqq\tau_i-l_{i+1}(\rho)\qquad \text{if } i\neq s_i$$

$$(14.7)\qquad \tau_i-l_i(\rho)<t\leqq\tau_i\qquad \text{if } i=s_i$$

and the two intervals which have respective endpoints

$$(14.8)\qquad t_0,\ t_0+a^0\rho$$

and

$$(14.9)\qquad t_1,\ t_1+a^{j+1}\rho$$

are nonoverlapping. For a given control u defined on $[t_0,t_1]$ and $\eta>0$ such that the intervals (14.6)–(14.9) are nonoverlapping define for $\rho\in\eta C$ a control u^ρ on $[t_0+a^0\rho,\ t_1+a^{j+1}\rho]$ by

$$(14.10)\qquad u^\rho(t)=\begin{cases}u(t_0) & \text{if } t_0+a^0\rho\leqq t\leqq t_0\\ u^i & \text{if } \tau_i-l_i(\rho)<t\leqq\tau_i-l_{i+1}(\rho)\\ u(t_1) & \text{if } t_1<t\leqq t_1+a^{j+1}\rho\\ u(t) & \text{otherwise.}\end{cases}$$

The vectors $a_{11} y^1, \dots, a_{1k} y^k$ which appear in the representation of $d^1, \dots, d^k$ can be arranged into a matrix N whose columns are these vectors. Define the mapping

$$\zeta(\rho) = (x_0 + N\rho, u^\rho). \tag{14.11}$$

We must show that this mapping maps ηC, for η small enough, into $\mathscr{G}$. The control u^ρ is of the form $\tilde{u}$ described in § 3 and hence the assumption made there implies $u^\rho \in \mathscr{U}$. The existence theorem for differential equations implies that Eq. (3.1) corresponding to u^ρ has a solution in some small interval adjacent to a point at which a boundary condition is specified. The theorem for continuous dependence of solutions of differential equations on initial conditions will imply that there is a solution of (3.1) for both u^ρ and u on an interval $\tau_i \leqq t \leqq b$ on which $u^\rho(t) \equiv u(t)$ provided that $x^\rho(\tau_i)$ is close enough to $x(\tau_i)$. Hence by using these two theorems successively, it can be seen that if η is small enough and $\rho \in \eta C$ that there is a solution of (3.1) on the interval $[t_0 + a^0\rho, t_1 + a^{j+1}\rho]$ (or on $[t_0, t_1 + a^{j+1}\rho]$ if $a^0\rho > 0$) which satisfies the boundary condition

$$x^\rho(t_0) = x_0 + N\rho. \tag{14.12}$$

Thus for small enough η, ζ does map ηC into the set $\mathscr{G}$. Let us choose such an $\eta > 0$.

The theorem on continuous dependence of solutions on initial conditions can be used in a manner similar to that sketched above to show that the mapping

$$\rho \to \big(x^\rho(t_0 + a^0\rho), x^\rho(t_1 + a^{j+1}\rho)\big) \tag{14.13}$$

is a continuous mapping of ηC into E^{2n}. It can be used further to show that on $[t_0, t_1]$, $x^\rho(t)$ converges uniformly to $x(t)$ as $|\rho|$ approaches zero.

From the preceeding remarks it follows that

$$\mathscr{J}(x_0 + N\rho, u^\rho) = \phi\big(t_0 + a^0\rho, t_1 + a^{j+1}\rho, x^\rho(t_0 + a^0\rho), x^\rho(t_1 + a^{j+1}\rho)\big) \tag{14.14}$$

is a continuous function of ρ on ηC. Therefore, we may conclude that D is a cone of variations if we can conclude that $\mathscr{J}(x_0, u)$ has the appropriate conical differential at (x_0, u) along ζ. The next two lemmas will enable us to establish this.

Lemma 14.1. *Let the times τ_i, $i = 2, \dots, j$ defined in the representation of the vectors $d^1, \dots, d^k$ be ordered by magnitude. Define $\tau_1 = t_0$ and $\tau_{j+1} = t_1$. Then if $\tau_i \leqq t \leqq \tau_{i+1}$*

$$x^\rho(t) = x(t) + \delta x(t)\,\rho + o(t, \rho) \tag{14.15}$$

where

$$\delta x(t) = \sum_{m=1}^{i} \delta x(t)^m \tag{14.16}$$

where $\delta x(t)^1$ is the $n \times n$ matrix which is the solution of

$$\delta\dot{x}(t) = f_x\big(t, x(t), u(t)\big)\,\delta x(t) \tag{14.17}$$

with initial condition

$$\delta x(t_0) = N \tag{14.18}$$

and $\delta(t)^m$ *for* $m \geqq 1$ *is the* $n \times n$ *matrix which is the solution of* (14.17) *with boundary condition*

$$\delta x(\tau_m) = [f(\tau_m, x(\tau_m), u^m) - f(\tau_m, x(\tau_m), u(\tau_m))]\, a^m. \tag{14.19}$$

Proof. We will prove Lemma 14.1 by induction on the number of distinct times of $\tau_1, \ldots, \tau_{j+1}$ which are $\leqq t$. If there is one such time, $\tau_1 = t_0$, we have by the definition of the initial condition $x^\rho(t_0) = x_0 + N\rho$, hence the theorem follows from Theorem 10.4. Suppose the lemma has been established when there are s distinct times $\leqq t$ and we wish to prove it when there are $s+1$. Let τ_i be the largest element of $\tau_1, \ldots, \tau_{j+1}$ which is $\leqq t$. Suppose q is the smallest integer so that $\tau_q = \tau_i$. Now[1]

$$x^\rho(\tau_i) = x^\rho(\tau_i - l_q(\rho)) + \sum_{m=q}^{i} \int_{\tau_i - l_m(\rho)}^{\tau_i - l_{m+1}(\rho)} f(t, x^\rho(t), u^m)\, dt \tag{14.20}$$

and

$$x(\tau_i) = x(\tau_i - l_q(\rho)) + \sum_{m=q}^{i} \int_{\tau_i - l_m(\rho)}^{\tau_i - l_{m+1}(\rho)} f(t, x(t), u(t))\, dt. \tag{14.21}$$

There are fewer than s distinct times of the set of τ_k's which are $\leqq \tau_i - l_q(\rho)$, hence by the induction hypothesis

$$x^\rho(\tau_i - l_q(\rho)) = x(\tau_i - l_q(\rho)) + \delta x(\tau_i - l_q(\rho))\,\rho + o(\tau_i - l_q(\rho), \rho) \tag{14.22}$$

where

$$\delta x(\tau_i - l_q(\rho)) = \sum_{j=1}^{q-1} \delta x(\tau_i - l_q(\rho))^j. \tag{14.23}$$

Since the $\delta x(t)^j$ are continuous and $l_q(\rho)$ approaches zero as $|\rho|$ approaches zero

$$\sum_{j=1}^{q-1} \delta x(\tau_i - l_q(\rho))^j \rho + o(\tau_i - l_q(\rho), \rho) = \sum_{j=1}^{q-1} \delta x(\tau_i)^j \rho + o(\tau_j, \rho). \tag{14.24}$$

Since $x^\rho(t) = x(t) + O(t, \rho)$ and the integrands in (14.20) and (14.21) are left continuous,

$$\begin{aligned} &\sum_{m=q+1}^{i} \int_{\tau_i - l_m(\rho)}^{\tau_i - l_{m+1}(\rho)} [f(t, x^\rho(t), u^m) - f(t, x(t), u(t))]\, dt \\ &\quad = \sum_{m=q}^{i} [f(\tau_m, x(\tau_m), u^m) - f(\tau_m, x(\tau_m)\, u(\tau_m))]\, a^m \rho + o(\tau, \rho). \end{aligned} \tag{14.25}$$

Hence combining (14.22)–(14.25) gives

$$\begin{aligned} x^\rho(\tau_i) &= x(\tau_i) + \sum_{j=1}^{q-1} \delta x(\tau_i)^j \rho \\ &\quad + \sum_{m=q}^{i} [f(\tau_i, x(\tau_i), u^m) - f(\tau_i, x(\tau_i), u(\tau_i))]\, a^m \rho + o(\tau_i, \rho). \end{aligned} \tag{14.26}$$

---

[1] If $i = s_i$, interpret $l_{i+1}(\rho)$ as zero in (14.20), (14.21) and (14.25).

The lemma now follows from (14.26) by applying Theorem 10.4. □

Lemma 14.2. *If* $\delta x(t_1)$ *is as defined in Lemma* 14.1 *then*

(14.27) $$x^\rho(t_0+a^0\rho)=x_0(t_0)+N\rho+f(t_0,x(t_0),u(t_0))\,a^0\rho+o(\rho)$$

and

(14.28) $$x^\rho(t_1+a^{j+1}\rho)=x(t_1)+\delta x(t_1)\,\rho+f(t_1,x(t_1),u(t_1))\,a^{j+1}\rho+o(\rho).$$

The proof of Lemma 14.2 uses arguments similar to parts of Lemma 14.1 and is left for the reader.

It now follows, applying Lemma 14.2 and the chain rule for differentiation to the vector function $\phi(t_0+a^0\rho, t_1+a^{j+1}\rho, x^\rho(t_0+a^0\rho), x^\rho(t_1+a^{j+1}\rho))$, and using the representation (14.4) for d^i and formulas (14.16) through (14.19) that $\mathscr{J}(x_0, u)$ has a conical differential at (x_0, u) along ζ and that the matrix M in this differential is the matrix whose columns are the vectors $d^1, \ldots, d^k$ of the type given in (14.4). This and the remarks preceding Lemma 14.1 establish Theorem 14.1.

§ 15. Verification of Pontryagin's Principle

We are now in a position to deduce the assertions of § 5, the Pontryagin Principle, from the results of § 13, and § 14. For brevity we write $u^*=u$, $x^*=x$. Theorems 13.1 and 14.1 imply the existence of a nonzero vector λ such that $\lambda_1 \leqq 0$ and

(15.1) $$\lambda' d \leqq 0$$

for vectors d of the convex cone D generated by the vectors (i)–(iv) of Theorem 14.1. Let $P(t)$ be the solution of (5.1) with boundary condition (5.3). Applying formula (10.5) with $G(t)=0$, $y(t)=\delta x(t)$ which is the solution of (14.1) with boundary condition (14.3), gives

(15.2) $$P(\tau)'[f(\tau,x(\tau),u)-f(\tau,x(\tau),u(\tau))]-\lambda'\phi_{x_1}(t_0,t_1,x(t_0),x(t_1))\,\delta x(t_1)=0.$$

Hence (15.1) for the vector (ii) of Theorem 14.1 and (15.2) imply (5.2).

Applying (10.5) with $y(t)=\delta x(t)$ which is the solution of (14.1) with boundary condition (14.2) gives

(15.3) $$P(t_0)'\,y-\lambda'\phi_{x_1}(t_0,t_1,x(t_0),x(t_1))\,\delta x(t_1)=0.$$

Now (15.1) implies for the vector (i) of Theorem 14.1 that

(15.4) $$\lambda'\phi_{x_0}(t_0,t_1,x(t_0),x(t_1))\,y+\lambda'\phi_{x_1}(t_0,t_1,x(t_0),x(t_1))\,\delta x(t_1)\leqq 0.$$

Adding (15.3) and (15.4) gives

(15.5) $$[P(t_0)'+\lambda'\phi_{x_0}(t_0,t_1,x(t_0),x(t_1))]\,y\leqq 0.$$

Since this must hold for each $y\in E^n$, the quantity in brackets must be zero which establishes (5.4).

For the vectors (iv) of Theorem 14.1 we must have

$$(15.6)\quad \pm\lambda'\left[\phi_{t_1}(t_0, t_1, x(t_0), x(t_1)) + \phi_{x_1}(t_0, t_1, x(t_0), x(t_1)) f(t_1, x(t_1), u(t_1))\right] \leqq 0.$$

Hence (15.6) and (5.3) imply (5.5). A similar argument using (iii) of Theorem 14.1. and (5.4) establishes (5.6). Condition (5.7) is implied by (5.2) and (5.6). This result is asserted in Corollary 15.1. Lemma 15.1 is a general result of this type.

Lemma 15.1. *Let $h(t, u)$ be a continuous function defined on $E^1 \times E^m$ which has a continuous partial derivative with respect to t. Let U be a closed subset of E^m. Let $u(t)$ be a left continuous piecewise continuous function defined on $[t_0, t_1]$ with values in U. If*

$$(15.7)\qquad \max_{u\in U} h(t, u) = h(t, u(t))$$

for each $t \in [t_0, t_1]$ then $h(t, u(t))$ is piecewise continuously differentiable on $[t_0, t_1]$ and

$$(15.8)\qquad h(t, u(t)) = \int_{t_0}^{t} h_t(s, u(s))\, ds + h(t_0, u(t_0)).$$

Proof. First we will show that (15.7) implies $h(t, u(t))$ is continuous. Since it is piecewise continuous and left continuous, this will follow if we show it is right continuous at each interior point of $[t_0, t_1]$. For such a point (15.7) implies

$$h(t, u(t+\tau)) \leqq h(t, u(t)) \quad \text{and} \quad h(t+\tau, u(t)) \leqq h(t+\tau, u(t+\tau)).$$

Taking limits as τ decreases to zero implies

$$h(t, u(t)^+) \leqq h(t, u(t)) \leqq h(t, u(t)^+).$$

Thus $h(t, u(t))$ is continuous.

Let t be a point of continuity of $u(t)$ and consider the difference

$$d(\tau) = h(t+\tau, u(t+\tau)) - h(t, u(t)).$$

From (15.7)

$$h(t+\tau, u(t)) - h(t, u(t)) \leqq d(\tau) \leqq h(t+\tau, u(t+\tau)) - h(t, u(t+\tau)).$$

By the theorem of the mean there are θ_1, θ_2 between zero and one such that

$$h_t(t+\theta_1\tau, u(t)) \leqq \frac{d(\tau)}{\tau} \leqq h_t(t+\theta_2\tau, u(t+\tau)).$$

Since t is a point of continuity of $u(t)$ taking limits as τ approaches zero we have

$$h_t(t, u(t)) = \frac{d}{dt} h(t, u(t)).$$

Thus $h(t, u(t))$ has a piecewise continuous derivative $h_t(t, u(t))$.
From this and the continuity of $h(t, u(t))$ we conclude that (15.8) holds. □

Corollary 15.1. *If $f(t, x, u)$ and $u(t)$ are as in Theorem* 5.1, *condition* (5.7) *of the statement of Theorem* 5.1 *is implied by conditions* (5.1), (5.2) *and* (5.6).

Proof. Apply Lemma 15.1 with

$$h(t, u) = P(t)' f(t, x(t), u).$$

Then

$$\begin{aligned} h_t(t, u(t)) &= -P(t)' f_x(t, x(t), u(t)) f(t, x(t), u(t)) + P(t)' f_t(t, x(t), u(t)) \\ &\quad + P(t)' f_x(t, x(t), u(t)) f(t, x(t), u(t)) = P(t) f_t(t, x(t), u(t)). \end{aligned}$$

Hence (5.7) follows directly from (15.8) and (5.6). □

Problems—Chapter II

(1) Formulate as an optimal control problem the problem of putting a satellite into a planar elliptical orbit about the earth by means of a rocket using a minimum amount of fuel. Use simplified point mass equations to represent the equations of motion of the rocket.

(2) Let the end conditions (3.3) in the statement of the optimal control problem be replaced by the requirements that t_0 be a fixed time, x_0 a fixed initial state, and that the terminal time t_1 and state $x(t_1)$ satisfy equations

$$\phi_i(t_1, x(t_1)) = 0 \qquad i = 2, \ldots, l.$$

Let the performance function ϕ_1 be a function $\phi_1(t_1, x_1)$ of only the terminal time and terminal state.

Show that for this optimal control problem necessary conditions for optimality are given by changing conditions (5.3), (5.5) and of Pontryagin's principle to the statement: There is a nonzero l-dimensional vector λ with $\lambda_1 \leqq 0$ such that

(5.3)′ $$P(t_1)' = \lambda' \phi_x(t_1, x^*(t_1))$$

(5.5)′ $$P(t_1)' f(t_1, x^*(t_1), u^*(t_1)) = -\lambda' \phi_t(t_1, x^*(t_1))$$

(5.7)′
$$P(t)' f(t, x^*(t), u^*(t)) = -\lambda' \phi_t(t_1, x^*(t_1)) - \int_t^{t_1} P(s)' f_t(s, x^*(s), u^*(s))\, ds$$

and omitting conditions (5.4) and (5.6).

(3) For the optimal control problem with equations of motion:

$$\dot{x} = f(t, x, u)$$

initial conditions:

$$x(t_0) = x_0,$$

in which t_0 and x_0 are fixed, terminal conditions:

$$\phi_i(t_1, x(t_1)) = 0 \qquad i = 2, \ldots, l$$

and performance criterion:

$$\int_{t_0}^{t_1} L(t, x(t), u(t))\, dt,$$

show that necessary conditions for $u^*(t)$ to be an optimal control are:

There is a nonzero l dimensional vector λ such that $\lambda_1 \leqq 0$ and a n-dimensional vector function $P(t)$ such that for $t \in [t_0, t_1]$

$$\dot{P}(t)' = -\lambda_1 L_x(t, x^*(t), u^*(t)) - P(t)' f_x(t, x^*(t), u^*(t)).$$

For $t \in [t_0, t_1]$ and $v \in U$

$$\lambda_1 [L(t, x^*(t), v) - L(t, x^*(t), u^*(t)] + P(t)'[f(t, x^*(t), v) - f(t, x(t), u^*(t))] \leqq 0$$

$$P'(t_1) = \tilde{\lambda}' \phi_x(t_1, x^*(t_1))$$

$$\lambda_1 L(t_1, x^*(t_1), u^*(t_1)) + P(t_1)' f(t_1, x^*(t_1), u^*(t_1)) = -\tilde{\lambda}' \phi_t(t_1, x^*(t_1))$$

where $\tilde{\lambda}' = (\lambda_2, \ldots, \lambda_l)$ and $\phi(t, x)' = (\phi_2(t, x), \ldots, \phi_l(t, x))$.

(4) Find an extremal control for the optimization problem with:
Equation of motion:

$$\dot{x} = x + u$$

in which x and u are scalars,
Control set:

$$U = \{u \colon -\infty < u < \infty\}$$

Initial condition:

$$x(0) = 1;$$

Terminal condition:

$$t_1 = 1$$

and performance index:

$$\frac{1}{2} \int_0^1 [x(t)^2 + u(t)^2] \, dt.$$

(5) The equations of motion in rectangular coordinates of a vehicle moving with an acceleration of magnitude one are:

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{u} = \cos \beta$$

$$\dot{v} = \sin \beta.$$

Consider the control problem of steering the vehicle from

$$x(0) = 0, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 0$$

to

$$y(T) = 1, \quad v(T) = 0$$

in a fixed time T while maximizing $u(T)$. Compute extremal controls for this problem.

(6) Show that if the matrix

$$(\phi_{t_0}(t_0, t_1, x_0, x_1), -\phi_{x_0}(t_0, t_1, x_0, x_1), -\phi_{t_1}(t_0, t_1, x_0, x_1), \phi_{x_1}(t_0, t_1, x_0, x_1))$$

has rank k at $e=(t_0, t_1, x(t_0), x(t_1))$, then the condition $\lambda \neq 0$ of Theorem 5.1 is equivalent to $P(t) \neq 0$ for $t \in [t_0, t_1]$.

(7) An alternate model for a one sector economy to that given in Example 2.2 takes into account population growth. A production function $f(k)$ gives the rate of increase of capital per worker due to production. If the population has growth rate constant β, there is a rate of decrease $-\beta k$ of capital per worker due to population growth. A fraction u of new production is retained in the economy and the remaining fraction $1-u$ is consumed. Hence the equation

$$\dot{k} = u f(k) - \beta k$$

governs the capital per worker in the economy. Let $h(c)$ denote the utility to the economy of consuming at rate c. Consider the problem of choosing a savings plan $u(t)$ to increase the capital per worker from k_1 to k_2 in the fixed time T, while maximizing

$$\int_0^T h([1-u(t)]\, f(k(t)))\, dt.$$

For $f(k)=\alpha k$, $h(c)=c$ compute an extremal control law. Notice that $0 \leqq u(t) \leqq 1$ must be satisfied.

(8) Let $P_i(t)$ denote the probability distribution of a finite state Markov process $\xi(t)$. That is, $\xi(t)$ has states $\{1, \ldots, n\}$ and

$$P_i(t) = Pr\{\xi(t)=i\}.$$

For a finite state Markov process there is a matrix $(a_{ij}(t))$ such that

$$a_{ij}(t) \geqq 0, \quad i \neq j$$

$$a_{ii}(t) = -\sum_{j \neq i} a_{ij}(t)$$

and $P_j(t)$ is the solution of

$$\dot{P}_j(t) = \sum_{i=1}^{n} P_i(t)\, a_{ij}(t).$$

The quantity $a_{ij}(t)\,\Delta t$ is approximately the conditional rate at which jumps from i to j take place at time t.

Consider such a process which is controllable in the following sense. The components of the matrix are continuous functions $a_{ij}(t, u)$ of time and a control variable u. A class $\mathscr{U}$ of control functions

$$\mathscr{U} = \{u(t, i)\}$$

of time and the current state whose values lie in the closed control set U is given. The controlled process then satisfies the equation

$$\dot{P}_j(t) = \sum_{i=1}^{n} P_i(t)\, a_{ij}(t, u(t, i)).$$

Let f_j be a function defined on $\{1, \ldots, n\}$, T a fixed time, and $P_i(0)$ a given initial probability distribution. Consider the problem of choosing the control $u(t, i)$ in $\mathscr{U}$

so that the expectation

$$E\{f_{\xi(T)}\}$$

is a minimum.

Apply Pontryagin's principle to this problem to show that necessary conditions for optimality are:

There are solutions $\phi_i(t)$ of

$$\dot{\phi}_i(t) = -\sum_{j=1}^{n} a_{ij}(t, u(t, i))\, \phi_j(t)$$

with $\phi_i(T) = -f_i$ and

$$\max_{v\in U} \sum_j P_i(t)\, a_{ij}(t, v)\, \phi_j(t) = \sum_j P_i(t)\, a_{ij}(t, u(t, i))\, \phi_j(t), \qquad i=1, \ldots, n, \;\; t\in(0, T].$$

(9) Consider an optimal control problem in which u is a scalar control and

$$f(t, x, u) = a(x) + b(x)\, u$$

where $a(x)$ and $b(x)$ are C^2 vector functions. If

$$P(t)'\, b(x(t)) = 0$$

on a time interval $\alpha \leqq t \leqq \beta$, the Hamiltonion does not depend on u and the problem is *singular*. Show that under these conditions

$$P(t)'\, q(x(t)) = 0 \qquad \alpha \leqq t \leqq \beta$$

where $q(x) = b_x(x)\, a(x) - a_x(x)\, b(x)$. Show further that if

$$P(t)'\, [q_x(x(t))\, b(x(t)) - b_x(x(t))\, q(x(t))] \neq 0$$

that

$$u(t) = -\frac{P(t)'\, [q_x(x(t))\, a(x(t)) - a_x(x(t))\, q(x(t))]}{P(t)'\, [q_x(x(t))\, b(x(t)) - b_x(x(t))\, q(x(t))]}.$$

(10) The equations of motion of a vehicle with velocity of magnitude one for which the angular rate of the velocity is controlled are

$$\dot{x} = \cos\theta$$

$$\dot{y} = \sin\theta$$

$$\dot{\theta} = u.$$

Assume

$$|u| \leqq 1.$$

Consider the problem of transfering the vehicle from initial conditions

$$x(0) = 4, \qquad y(0) = 0, \qquad \theta(0) = \frac{\pi}{2}$$

to

$$x(t_1) = 0, \qquad y(t_1) = 0$$

in minimum time t_1. Compute an extremal control law for this problem.

(11) Let $H(u)$ be a function defined on a closed convex subset C of E^n. Let $H(u)$ have a tangent plane at u_0. Show that a necessary condition that $H(u)$ have

a maximum at u_0 is that

$$H_u(u_0)\,v \leqq 0$$

for each $v \in E^n$ such that $u_0 + v \in C$. Apply this result to compare Theorem 11.2 and Theorem 11.5.

(12) A concept analogous to a cone of variations is the concept of *derived set.* Let Y be a subset of E^p. A set $D \subset E^p$ will be said to be a derived set for Y at y_0 if for every finite subset $d_1, \ldots, d_n$ of D, for some $\delta > 0$, there is a continuous mapping $y(\varepsilon)$ of $\{\varepsilon\colon 0 \leqq \varepsilon_i \leqq \delta;\ i = 1, \ldots, n\} \subset E^n$ into Y such that

$$y(\varepsilon) = y_0 + \sum_{i=1}^{n} \varepsilon_i d_i + o(\varepsilon).$$

Show that the convex cone generated by a derived set for Y at y_0 is a derived set for Y at y_0.

(13) Carry out explicitly the computation of the conical differential of the performance $J(x_0, u)$ discussed at the end of § 14.

(14) In the notation of Chap. I, let

$$J(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt, \qquad I(x) = \int_{t_0}^{t_1} M(t, x(t), \dot{x}(t))\, dt.$$

Given a real number a, consider the problem of minimizing $J(x)$ among all piecewise C^1 functions $x(\cdot)$ on $[t_0, t_1]$ which satisfy $x(t_0) = x(t_1)$ and $I(x) = a$. Suppose that x^* minimizes in this problem, and that x^* is not an extremal for $I(x)$ according to the definition in Sect. I.3.

(a) Show that there exists a scalar μ such that x^* is an extremal for $J + \mu I$ (you can use Theorem 13.1 or give a direct proof).

(b) Apply (a) to the classical isoperimetric problem in the plane E^2, in which

$$J(x) = \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt$$

$$I(x) = \tfrac{1}{2} \int_{t_0}^{t_1} (x_1 \dot{x}_2 - x_2 \dot{x}_1)\, dt$$

correspond respectively to length, area. Show that any minimizing x^* describes a circle. *Hint.* Introduce arc length as a parameter.

(15) Consider the characteristic value problem

$$\frac{d^2 X}{ds^2} + \lambda\, U(s) X = 0, \qquad 0 \leqq s \leqq 1,$$

$$X(0) = X(1) = 0,$$

where U is piecewise continuous, $0 \leqq U(s) \leqq M$. Let $\lambda_1 = \lambda_1(U)$ be the smallest characteristic value.

(a) From Pontryagin's principle show that $\lambda_1(U)$ is minimum when $U(s) \equiv M$. *Hint.* The substitution $t = \sqrt{\lambda}\, s$ reduces this to the problem of minimum time t_1

such that $x(t_1)=0$, where

$$\ddot{x}+ux=0, \quad x(0)=0, \qquad \dot{x}(0)=1, \quad 0 \leqq u(t) \leqq M.$$

(b) What can you say about the higher characteristic values?

(c) In part (a) add the constraint $c=\int_0^1 U(s)\,ds$ where $0 \leqq c \leqq M$. Find the optimal U.

(16) Let x be a vector of variables in E^n and A be an $n \times n$ symmetric matrix. Show that the gradient of the quadratic form $x'Ax$ with respect to the variables x is given by $2x'A$.