Valuation Project 2: Finite Difference

Brian Wu / UID: bhwu3@illinois.edu / UIN: 664883556

Executive summary

Product: Autocallable Barrier Notes with Contingent Coupons due November 03, 2023
Underlying asset: S&P 500® Index
Issuer: BMO Capital Market Corp. (BMOCM)
Estimated value per security by BMOCM: $966.92
Estimated value per security for Project #2: $973.96

The product is an auto-callable security with roughly 13 months to maturity and with 13 contingent coupon payments. Then, there is an automatic redemption feature on 8 of the 13 observation dates. Finally, a trigger feature is introduced through out the life time of the product which serves as continuous barrier that could affect the payoff at maturity.

Through Crank-Nicolson Finite Difference method and sensitivity analysis on a list of volatility inputs, our product is valued around $973.96 per security with a grid of 500 price steps (max price = 2.5 * initial price), 5174 time steps, and a volatility of 0.27405.

Data collection and parameters used

For the grid design of the finite difference method, we choose 5174 as our time step and a range of 50 to 500 as our price step, where jmax is 2.5 times the initial index level. Choosing 5174 step has some upsides, it is a multiple of 398, which is the actual day count of the life time of the product, making each observation falls right on an integer time step. Also, setting the time step above 5000 could insure the stability of the grid, although Crank-Nicolson method has already taken care of this problem. Price steps with a range of 50 to 500 shows a smooth convergence of the estimated price. We also did a 100 to 1000 price step, and the result is almost identical to 500 step convergence, and thus, we think 500 steps should be good for our valuation. Lastly, as for jmax, the multiplier 2.5 is chosen because it corrects the lambda value of the barrier so that we can easily remove non-linearity error.

The risk-free rate is 0.044118 quoted from Bloomberg terminal's OIS rate after linear interpolation of the interest rate curve.

The dividend yield is 0.01845 also quoted from Bloomberg terminal.

We quoted a range of sigma for moneyness of 75 and 100, and for each of the 13 months (every observation date), which gives us 26 different sigma values. The volatility used in our valuation is 0.27405, which is one of the volatilities from 100 moneyness, because the volatilities from 75 moneyness are generally much higher that are somewhat unreasonable and makes the valuation way lower than the face value. Thus, choosing 0.27405 should be a better alternative and give us a much closer valuation to the BMOCM estimates. The implied volatility for the BMOCM estimated value, $966.92, is 0.28936.

Key terms

Initial index level, S0, is 3719.04, followed by the barrier level, also the trigger level, 2789.28, which is 75% of the initial index level.

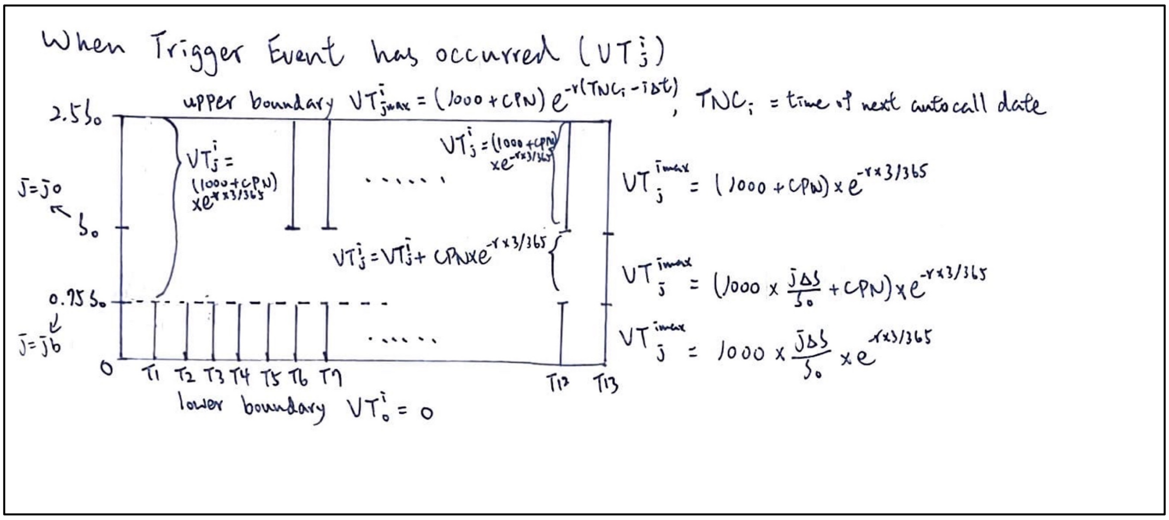
Principal amount per security is $1000, which we'll call it face value, pays a coupon of $9.67 per security on the coupon dates.

On observation dates, check if the coupon is payable. Starting from the sixth observation date, check if the security is callable. Lastly, trigger event monitors if the index level fall below the trigger level during the life time of the product which could affect the payoff at maturity.

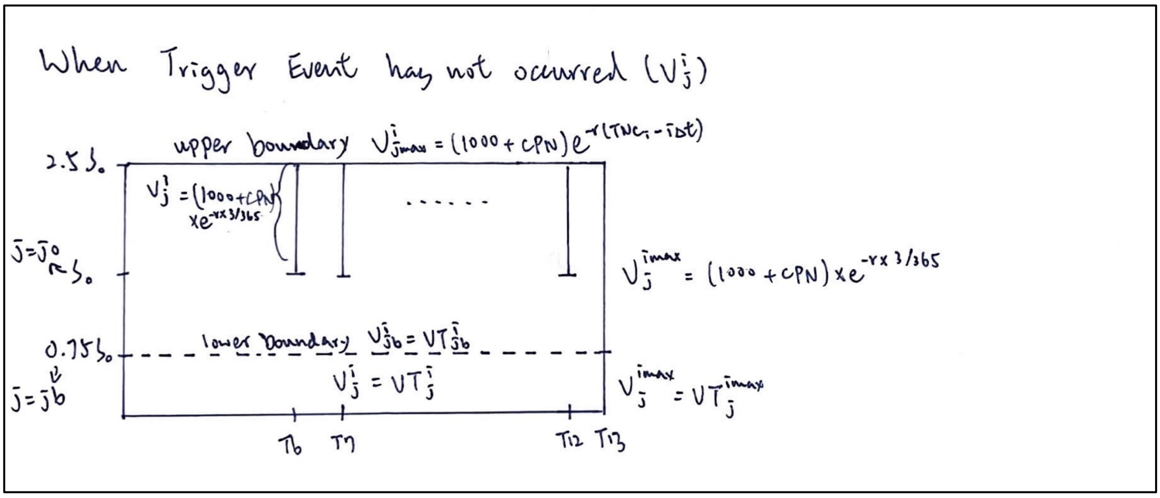
Methodology

We construct two grids: 1) When trigger event has occurred 2) When trigger event has not occurred. We will call the former VT grid and the latter V grid, and we first build VT grid and then use part of VT grid to complete V grid.

Starting with VT grid, we first determine the terminal boundary, which is the payoff at maturity, where there are three different scenarios. Then, work back through every time step by setting upper and lower boundaries, and then apply Crank-Nicolson Finite Difference method to get the all the values at that time step. On observation dates, adjust the upper boundary to the value at S0, the values above the boundary will be face value plus coupon; whereas the values between the boundaries are solved by Crank-Nicolson Finite Difference method, then adjust the values above barrier level by adding coupon payments.



For the V grid, same procedure as VT grid, we first determine the terminal boundary, where we have only two scenarios, and the payoff below barrier level can be replicated directly from VT grid. Working back through the time steps, we adjust the lower boundary to the barrier level, which the value can be replicated from VT grid, for values below the lower boundary, replicate directly from VT grid, and for values above, use the same finite difference method. On observation dates, same as VT grid, set new upper boundary and the values above will be the face value plus coupon; values between two boundaries can be solved by finite difference method and then add coupon payments; values below lower boundary can be replicated from VT grid.



Algorithms

Applying Crank-Nicolson method could be computing power intensive, we thus introduce LU decomposition method, which enables us to solve the matrix equations systematically without having to compute the zeros in the matrix. Starting by introducing new parameters, α_j and S_j, to Crank-Nicolson method:

$$\begin{cases} \alpha_{j-1} V_{j-1}^i + c_{j-1} V_j^i = S_{j-1} \\ a_j V_{j-1}^i + b_j V_j^i + c_j V_{j+1}^i = d_j^i \end{cases}, \quad \text{where } \alpha_0 = b_0 \ \& \ S_0 = d_0^i$$

Eliminating V_j^i and rearrange gives:

$$\left(b_j - \frac{a_j c_{j-1}}{\alpha_{j-1}} \right) V_j^i + c_j V_{j+1}^i = d_j^i - \frac{a_j S_{j-1}}{\alpha_{j-1}}$$

Which can be rewrite into the form:

$$\alpha_j V_j^i + c_j V_{j+1}^i = S_j$$

The last set of simultaneous equations are given by:

$$a_{jmax} V_{jmax-1}^i + b_{jmax} V_{jmax}^i = d_{jmax}^i$$

By the same process as before and then get V_{jmax}^i :

$$\left(b_{jmax} - \frac{a_{jmax} c_{jmax-1}}{\alpha_{jmax-1}} \right) V_{jmax}^i = d_{jmax}^i - \frac{a_{jmax} S_{jmax-1}}{\alpha_{jmax-1}}$$

$$\Rightarrow \alpha_{jmax} V_{jmax}^i = S_{jmax}$$

$$\Rightarrow V_{jmax}^i = \frac{S_{jmax}}{\alpha_{jmax}}$$

Then work backward from jmax calculating V_j^i :

$$V_j^i = \frac{1}{\alpha_j} \left(S_j - c_j V_{j+1}^i \right)$$

Calculate the finite difference values backward in time step i. On observation dates, the upper boundary will be adjusted to j0 (price step of S0). The values above the upper boundary will be equal to the value when auto-redemption occurs, and the values below the upper boundary will be equal to the values we get from finite difference method, except for the values above jb (price step of barrier level) will be the value from finite difference method plus coupon payments.

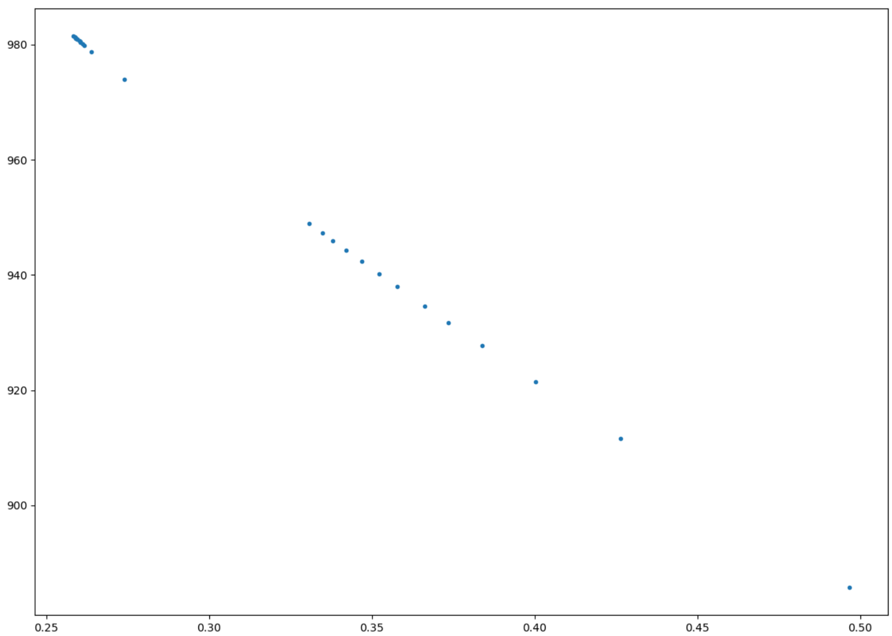
$$\begin{cases} V_j^i = e^{-r \times 3/365} \times (1000 + CPN), & \text{if } j \geq j0 \\ V_j^i = V_j^i + e^{-r \times 3/365} \times CPN, & \text{if } j0 > j > jb \\ V_j^i = V_j^i, & \text{if } j \leq jb \end{cases}$$

Sensitivity analysis

Selecting a range of sigma, for moneyness of 75 and 100 and for each of the 13 months, which gives us 26 different sigma values. Using the same grid with 500 price steps to calculate arrange of security values, and the closest the value to $966.92, BMOCM estimates, is 0.27405 sigma. The value is fairly close to the expectation, and thus it's chosen to be the input parameter for our valuation project.

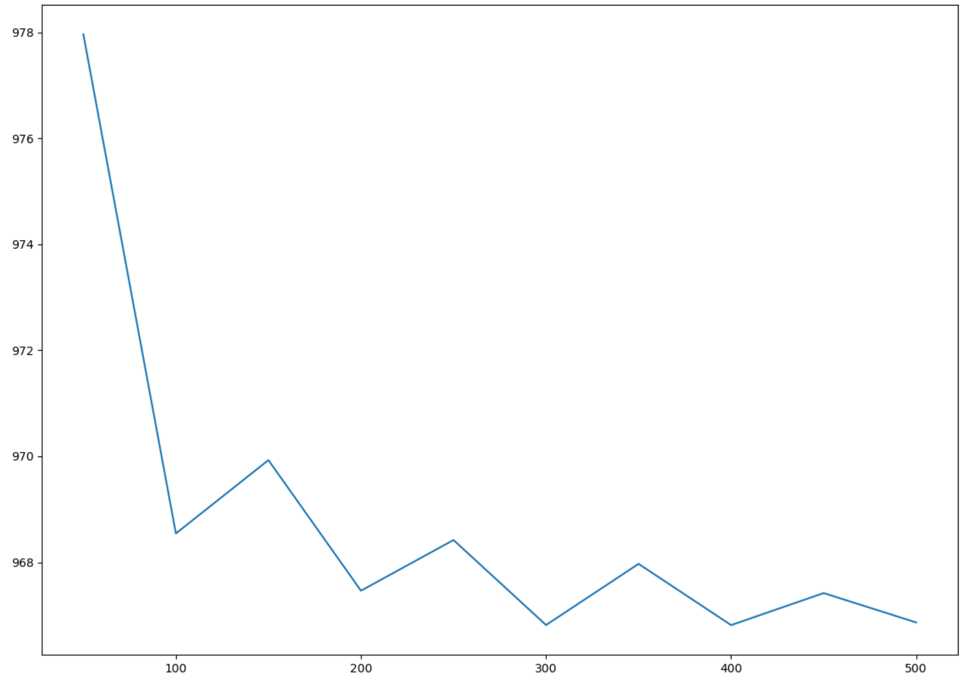
S_steps	t_steps	CN	Barrier Lambda	Sigma
500	5174	973.9694496229790	0.0	0.27405
500	5174	978.7778121889740	0.0	0.26391
500	5174	979.8459108177930	0.0	0.26169
500	5174	981.4779141694900	0.0	0.25832
500	5174	980.9292594698610	0.0	0.25945
500	5174	980.1500312929310	0.0	0.26106
500	5174	980.4884673977190	0.0	0.26036
500	5174	980.4497610472110	0.0	0.26044
500	5174	980.6191545538370	0.0	0.26009
500	5174	981.1087492478280	0.0	0.25908
500	5174	981.1572864494650	0.0	0.25898
500	5174	980.9535058397110	0.0	0.2594
500	5174	981.3223971059210	0.0	0.25864
500	5174	885.774891863147	0.0	0.49656
500	5174	911.5580701874110	0.0	0.42627
500	5174	921.4038445178370	0.0	0.40022
500	5174	927.7422736756900	0.0	0.38375
500	5174	931.7637835482110	0.0	0.37344
500	5174	934.6434690659660	0.0	0.36613
500	5174	937.9885140059280	0.0	0.35772
500	5174	940.2525721355260	0.0	0.35208
500	5174	942.3193194347980	0.0	0.34697
500	5174	944.2993768584830	0.0	0.34211
500	5174	945.9854159767960	0.0	0.338
500	5174	947.2561599133340	0.0	0.33492
500	5174	948.9910683695390	0.0	0.33074

Further, when we plot the sigma values against the estimated price, it yields an interesting result where the security values and sigma values are almost perfectly linear. (x-axis: sigma, y-axis: price)

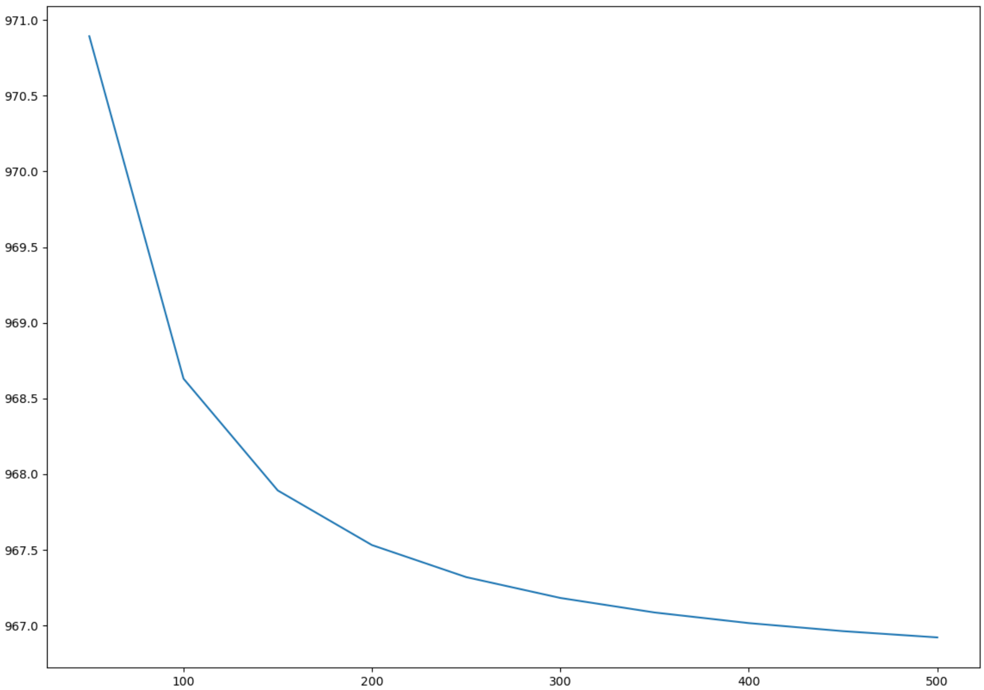


Non-linearity error

At first, we chose $jmax = 3 * S_0$ as our grid setting that gives a result of barrier lambda values oscillating between 0 and 0.5, which generates visible non-linearity error on our price chart. (x-axis: price steps, y-axis: price)



Correcting the lambda value by setting $jmax = 2.5 * S_0$ makes all lambda value equal to 0 and eliminates non-linearity error.



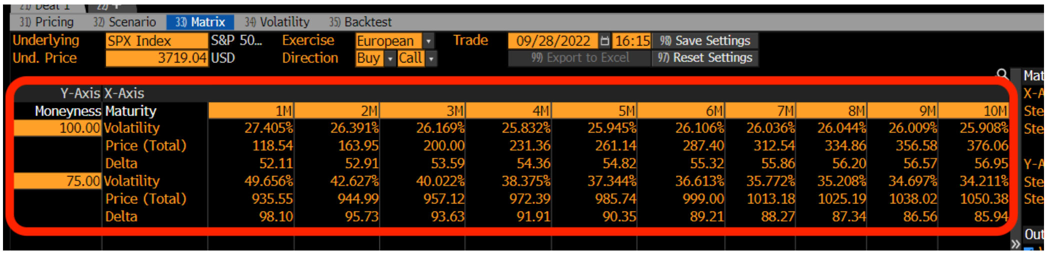
Conclusion

The estimated value for this project is $973.96. We think the volatility quoted for all observation dates should be relevant when doing product valuation, but after sensitivity test, we found that some of the inputs make the value way lower than the face value which is very unlikely and unreasonable. Removing the non-linearity error is quite a big deal since it makes the convergence smoother, giving us more confidence in the accuracy of the valuation method.

Considerations for this project might be the choice of volatility input. We should be able to develop a better way of choosing the volatility input. Considering the fact that sigma value has a much greater impact to the security value (the weird result from sensitivity analysis) than we thought, we should be more careful in choosing such parameters.

Supplement data

Volatility source



31) Pricing 32) Scenario 33) Matrix 34) Volatility 35) Backtest

Underlying: SPX Index S&P 50... | Exercise: European | Trade: 09/28/2022 16:15 | 98) Save Settings
Und. Price: 3719.04 USD | Direction: Buy Call | 99) Export to Excel | 97) Reset Settings

Y-Axis Moneyness	X-Axis Maturity	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M
100.00	Volatility	27.405%	26.391%	26.169%	25.832%	25.945%	26.106%	26.036%	26.044%	26.009%	25.908%
	Price (Total)	118.54	163.95	200.00	231.36	261.14	287.40	312.54	334.86	356.58	376.06
	Delta	52.11	52.91	53.59	54.36	54.82	55.32	55.86	56.20	56.57	56.95
75.00	Volatility	49.656%	42.627%	40.022%	38.375%	37.344%	36.613%	35.772%	35.208%	34.697%	34.211%
	Price (Total)	935.55	944.99	957.12	972.39	985.74	999.00	1013.18	1025.19	1038.02	1050.38
	Delta	98.10	95.73	93.63	91.91	90.35	89.21	88.27	87.34	86.56	85.94

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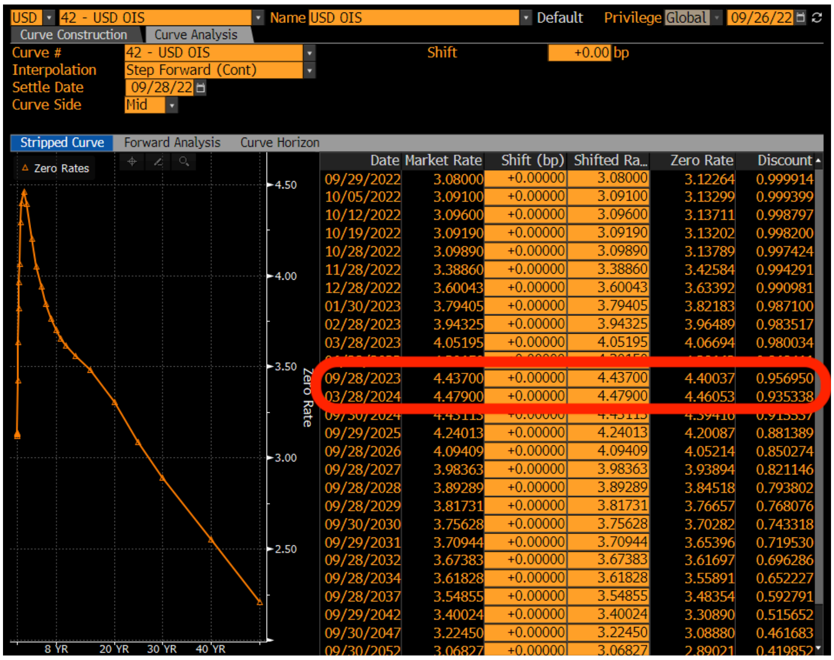
Y-Axis Moneyness	X-Axis Maturity	11M	12M	13M
100.00	Volatility	25.898%	25.940%	25.864%
	Price (Total)	395.49	414.77	432.23
	Delta	57.24	57.51	57.85
75.00	Volatility	33.800%	33.492%	33.074%
	Price (Total)	1061.68	1073.07	1084.30
	Delta	85.30	84.70	84.33

Risk-free rate



Dividend yields