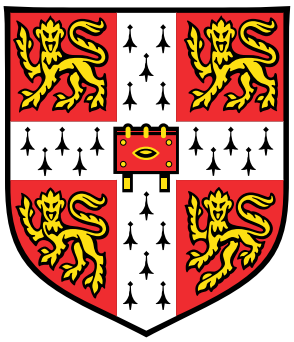
Compressive Sensing in Video Encoding



Brian Azizi

Department of Physics
Centre for Scientific Computing
University of Cambridge

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I would like to dedicate this thesis to my loving parents ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text. This dissertation contains less than 65,000 words including appendices, bibliography, footnotes, tables and equations and has less than 150 figures.

Brian Azizi
August 2016

Acknowledgements

And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

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Nomenclature

Roman Symbols

M	Number of basis functions
N	Number of training examples
$\boldsymbol{w}$	RVM weights vector
$\boldsymbol{x}^{(i)}$	ith input vector
$y^{(i)}$	ith target

Greek Symbols

$\boldsymbol{\phi}_j$	jth basis vector
$\phi_j(\cdot)$	jth basis function

Chapter 1

Introduction

There are three parts: A signal processing framework called *Compressive Sensing (CS)*, a pre-processing step in form of a basis transformation based on discrete wavelet transforms and a Machine Learning algorithm called *Sparse Bayesian Learning*.

The key notion that ties in these three areas is the notion of *sparsity*.

The ℓ_p -norm of a vector $\boldsymbol{z}\mathbb{R}^n$ is defined as follows for $p > 0$:

$$||\boldsymbol{z}||_p \equiv \left(\sum_{i=1}^{n} |z_i|^p \right)^{\frac{1}{p}}. \tag{1.1}$$

If $p = 0$ we can define the ℓ_0 "norm" of $\boldsymbol{z}$ to be the number of its non-zero entries:

$$||\boldsymbol{z}||_0 \equiv \sum_{i=1}^{n} \mathbb{1}\{z_j \neq 0\} \tag{1.2}$$

Our contributions are three-fold:

1. An extension of the MSCE-algorithm in [4] to video data.
2. An extension to MSCE to deal with additional sensing mechanisms.
3. A parameter study and performance comparison of different wavelet representations, sensing modalities,

Background

Thesis Organization

Chapter 2

Background

In this chapter, we will introduce the theory of *Compressive Sensing* (also known as *Compressed Sensing, Compressive Sampling* or simply, *CS*). CS is a framework within signal processing that allows for acquiring signals (i.e. measure or *sense*) directly in a *compressed* format.

To motivate the discussion, we will first review the conventional approach to signal acquisition and compression.

2.1 Coventional Signal Processing

2.1.1 Signal Acquisition

In order to work with information within analog signals (continuous streams of data) such as sounds, images or video, we rely on reducing the analog signals to digital (discrete) signals that can be processed with computers. This digitization is done by taking discrete measurements of the analog signal at certain points in time or space, a process known as *sampling*.

Conventional approaches to sampling are based on the *Shannon/Nyquist Sampling Theorem* [5]: When sampling a band-limited signal uniformly, we are able to *perfectly reconstruct* the signal from its samples if the sampling rate is at least twice the bandwidth of the signal.

Consider an analog signal $x(t)$ that varies with time, such as an audio wave. Let f be the highest frequency present in $x(t)$.

In order to digitise $x(t)$, we measure x at discrete points in time $t^{(0)}, \cdots, t^{(n)}$ and store the samples $x^{(i)} \equiv x(t^{(i)})$. We sample x uniformly, measuring a sample every T_s seconds, so that $t^{(i)} = iT_s$. The sampling rate is therefore $f_s = 1/T_s$.

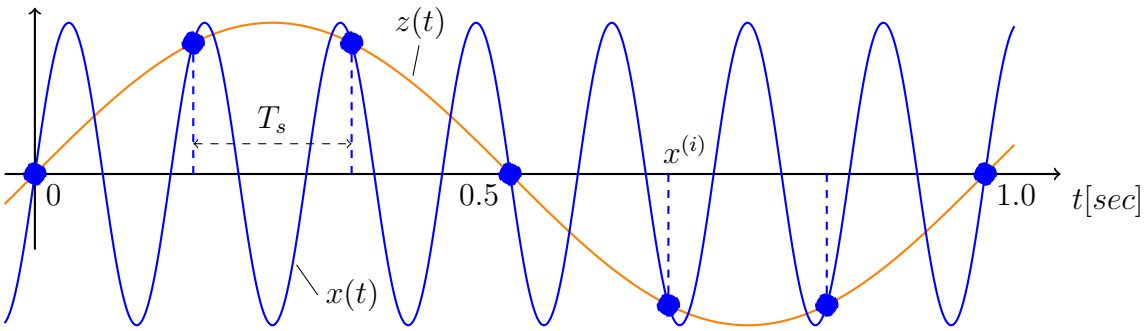


Fig. 2.1 Illustration of the Shannon/Nyquist Sampling Theorem. The orange curve is the original signal $x(t)$ which is a sinusoid with frequency $f = 7$ Hz. The blue points are discrete samples $x^{(i)}$ taken from $x(t)$ at a sampling rate $f_s = 7$ Hz, which is below the Nyquist Rate $2f = 14$ Hz. Thus, aliasing occurs and interpolation algorithms will reconstruct an alias $z(t)$ of $x(t)$.

Suppose we wish to reconstruct $x(t)$ by interpolating the samples. There is an infinite number of continuous functions that fit this set of samples. However, it can be shown that only one of them has a bandwidth of no more than $f_s/2$. Thus, if $f < f_s/2$ (the *Nyquist Criterion*), then $x(t)$ is the unique function that will be approximated by interpolation algorithm such as the *Whittaker-Shannon interpolation formula* [5].

In Figure 2.1, we have a sinusoidal signal $x(t)$ with frequency f. The sampling rate is $f_s = f$ and therefore below the signal's *Nyquist rate* $2f$. Thus, we are unable to reconstruct $x(t)$ from the samples. Instead, we will reconstruct an *alias* $z(t)$ which, in this case, is another sinusoid with frequency $f/7$. The original signal x is lost.

We have illustrated Nyquist sampling in the 1-dimensional case. The same principles hold for higher dimensional signals such as images and videos.

For signals that vary with space, the sampling rate is governed by the desired spatial resolution. In order to recover the finer details (the high-frequency components) of an image, we require higher pixel density (i.e. a larger number of pixels per centimeter (ppcm)).

Nyquist sampling underlies almost all signal acquisition protocols that are found in practice. It is the basis of medical imaging, audio and video recording and radio receivers.

2.1.2 Signal Compression

The sampling theorem imposes a lower bound on the sampling rate above which we are able to perfectly reconstruct the desired signal. This lower bound is often very high and we end up with a very large number of measurements. Storage and transfer

of such signals becomes prohibitavely expensive as the size of the signal grows. Thus, a need for *data compression* arises.

We will discuss a particular type compression algorithm known as *transform coding*. It is the standard compression method for "natural" and manmade signals such as audio, photos, and video and is the basis of many common signal formats such as JPEG (images), MPEG (video) and MP3 (audio).

Let $\boldsymbol{v}$ by a real-valued digital signal of length M, $\boldsymbol{v} \in \mathbb{R}^M$. Without loss of generality, $\boldsymbol{v}$ is assumed to be a one-dimensional signals. If we are working with a multi-dimensional signals, we may first vectorize it into a long vector. When compressing digital signals, we are usually interested in *lossy compression*.

Any vector in $\mathbb{R}^M$ can be expressed as a linear combination of M *basis vectors* $\boldsymbol{\psi}_j \in \mathbb{R}^M$:

$$\boldsymbol{v} = \sum_{j=1}^{M} w_j \boldsymbol{\psi}_j \tag{2.1}$$

where w_j is the coefficient (or weight) associated with $\boldsymbol{\psi}_j$.

By forming the *basis matrix* $\boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \cdots \boldsymbol{\psi}_M]$, we can express equation (2.1) in matrix form

$$\boldsymbol{v} = \boldsymbol{\Psi}\boldsymbol{w}$$

where $\boldsymbol{w} = (w_1, \cdots, w_M)^T$. For simplicity, we assume that the basis $\boldsymbol{\Psi}$ is orthonormal, so that $\boldsymbol{\Psi}\boldsymbol{\Psi}^T = \boldsymbol{I}_M$ and $\boldsymbol{\psi}_i^T \boldsymbol{\psi}_j$ is 1 if $i = j$ and 0 otherwise. Thus, the coefficient w_j is given by $w_j = \boldsymbol{v}^T \boldsymbol{\psi}_j$.

We now have two equivalent representations of the same signal, $\boldsymbol{v}$ in the original basis and $\boldsymbol{w}$ in the $\boldsymbol{\Psi}$ basis. Since $\boldsymbol{\Psi}$ is orthogonal, $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same ℓ_2-norm, $||\boldsymbol{v}||_2 = ||\boldsymbol{\Psi}\boldsymbol{w}||_2 = ||\boldsymbol{w}||_2$. However, in the original signal, $\boldsymbol{v}$, the energy is typically spread over many of its components. On the other, it is possible to find a basis $\boldsymbol{\Psi}$ such that the energy of the transformed signal, $\boldsymbol{w}$, is concentrated in only a few large components w_j and a large fraction of its entries are very close to zero.

Suppose that we delete the entries w_j that are very small and replace them with zero to obtain $\hat{\boldsymbol{w}}$. Let $\hat{\boldsymbol{v}} = \boldsymbol{\Psi}\hat{\boldsymbol{w}}$ the approximate signal in the original domain. Since $\hat{\boldsymbol{w}}$ is very close to $\boldsymbol{w}$, it follows that

$$||\hat{\boldsymbol{v}} - \boldsymbol{v}||_2 = ||\Psi\hat{\boldsymbol{w}} - \Psi\boldsymbol{w}||_2 = ||\Psi(\hat{\boldsymbol{w}} - \boldsymbol{w})||_2 = ||\hat{\boldsymbol{w}} - \boldsymbol{w}||_2$$

is very small.

Thus, a viable method for lossy compression of the signal $\boldsymbol{v} \in \mathbb{R}^M$ would be the following:

(a) Uncompressed Image (b) Compressed Image via DCT

Fig. 2.2 The uncompressed image has a resolution of 512×512, i.e. 262144 pixels. We compress the image by performing a Discrete Cosine Transform and storing only the largest 27832 coefficients. The compression ratio is 9.42.

1. Compute the full set of transform coefficients $\{w_j\}_{j=1}^{M}$ via $\boldsymbol{w} = \Psi^T \boldsymbol{v}$.
2. Locate all the coefficients w_j whose absolute value is above a certain threshold (suppose there are K of them).
3. Discard all the $(M - K)$ small coefficients
4. Store the values and locations of the K large coefficients

In order to view the compressed signal in the original domain, we reconstruct it via the transform: $\boldsymbol{\Psi}\hat{\boldsymbol{w}} = \hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{w}}$ is $\boldsymbol{w}$ with the $(M - K)$ smallest coefficients replaced by zero.

It is possible to find basis matrices $\boldsymbol{\Psi}$ that result in very high compression ratios for a wide range of natural signals without any noticable reduction in the signal quality. Furthermore, many of the commonly used basis transforms can be computed very efficiently.

Audio signals and a wide class of communication signals are highly compressible in the localized Fourier basis. Images and video signals, on the other hand, can often be compressed via the *Discrete Cosine Transform* (DCT) or the *Discrete Wavelet Transform* (DWT). For instance, the JPEG standard for image compression is based on the DCT, while the more modern JPEG2000 format uses the CDF 9/7 wavelet transform or the CDF 5/3 wavelet transform [6].

In Figure 2.2, we compress the standard test image "Lenna" via a DCT. We are only storing about 10% of the transform coefficients. Yet, the difference between the original image and the compressed image is hardly noticable.

We will discuss the DCT and the DWT in more detail in Chapter 3.

2.2 Compressive Sensing

The conventional approach to data acquisition and compression is very effective and has been highly influential. However, it is also extremely wasteful. We acquire a huge amount of data at the signal sampling stage and then proceed to discard a large part of it at the compression stage.

Compressive Sensing is a more general approach that lets us *acquire signals directly in a compressed format*. This is clearly more efficient as it allows us to skip the intermediate stage of taking N samples.

In this section we will formulate the Compressive Sensing problem. For simplicity, the focus will be on discrete signals such as digital images or videos.

2.2.1 The Compressive Sensing Problem

Let $\boldsymbol{v} \in \mathbb{R}^M$ be the signal of interest. As an example, $\boldsymbol{v}$ could be a digital photograph, such as Figure 2.2a, that has been unrolled into a long vector of length M, where M is the number of pixels.

Suppose $\boldsymbol{v}$ is currently unknown and we want to acquire a compressed representation of it without measuring $\boldsymbol{v}$ directly. To do so, consider the following linear sensing scheme: We measure inner products between the signal $\boldsymbol{v}$ and a collection of M-dimensional vectors $\{\boldsymbol{\theta}_i\}_{i=1}^N$ to obtain the measurements $y_j = \boldsymbol{v}^T\boldsymbol{\theta}_i$ for $i = 1, \cdots, N$. This relation can be expressed more succinctly as

$$\boldsymbol{y} = \boldsymbol{\Theta}\boldsymbol{v} \tag{2.2}$$

where $\boldsymbol{y} = (y_1, \cdots, y_N)^T$ and $\boldsymbol{\Theta}$ is the $N \times M$ *sensing matrix* given by

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\theta}_1^T \\ \vdots \\ \boldsymbol{\theta}_N^T \end{bmatrix}. \tag{2.3}$$

We are interested in the *undersampled* situation where $N < M$. In particular, we would like to acquire a compressed representation $\boldsymbol{y}$ of $\boldsymbol{v}$ using as few measurements as possible, while still being able to recover the original signal.

There are two problems that stand out at this point.

1. Given that $N << M$, how can we ensure that, during the measurement process, we do not lose any information contained in $\boldsymbol{v}$ (i.e. $\boldsymbol{y}$ captures all the information in $\boldsymbol{v}$)?

2. Given our measurements $\boldsymbol{y}$ and knowledge of the sensing matrix $\boldsymbol{\Theta}$, how do we recover the signal of interest $\boldsymbol{v}$?

At an intuitive level, we would expect at least some information to be destroyed during the measurement process, especially if the number of measurements, N, is substantially lower than the length of the desired signal, M. Moreover, even if $\boldsymbol{y}$ did contain the same information as $\boldsymbol{v}$, we are still faced with solving the linear system in equation (2.2). This system is underdetermined and hence there are infinitely many vectors $\boldsymbol{v}$ that satisfy $\boldsymbol{\Theta v} = \boldsymbol{y}$.

2.2.2 Sparse Signals

In general, we cannot resolve these issues. However, if we restrict ourselves to a certain class of signals, it is possible to make progress.

In particular, we will focus on signals that have *sparse representations*. A signal $\boldsymbol{v} \in \mathbb{R}^M$ has a sparse representation if there exists a $M \times M$ basis matrix $\boldsymbol{\Psi}$ so that the transformed signal $\boldsymbol{w}$, where $\boldsymbol{v} = \boldsymbol{\Psi w}$, is *sparse*.

We say that $\boldsymbol{w}$ is K*-sparse* if $\boldsymbol{w}$ has K non-zero components. Equivalently, from (1.2)[1], $\boldsymbol{w}$ is K-sparse if $||\boldsymbol{w}||_0 = K$.

In Section 2.1.2 we noted that many manmade and natural signals are compressible. They have an almost-sparse representation, meaning that, when expressed in the basis $\boldsymbol{\Psi}$, almost all of their energy is contained in only K components and the remaining components are very close to zero. Such signals are well-approximated by K-sparse representations.

[1]Definition of ℓ_0-norm.

2.3 Solving the Compressive Sensing Problem

So let us suppose then, that the desired signal $\boldsymbol{v}$ is K-sparse when expressed in the $\boldsymbol{\Psi}$ basis, where K is small ($K \leq M$). We can substitute $\boldsymbol{v} = \boldsymbol{\Psi w}$ into (2.2) to get

$$\boldsymbol{y} = \boldsymbol{\Theta v} = \boldsymbol{\Theta \Psi w}$$

Let us define $\boldsymbol{\Phi} = \boldsymbol{\Theta \Psi}$, so that

$$\boldsymbol{y} = \boldsymbol{\Phi w}. \tag{2.4}$$

We have arrived at another underdetermined linear system. The CS measurements $\boldsymbol{y}$ are still the same as in (2.2), but the sensing matrix $\boldsymbol{\Theta}$ has been replaced by the new sensing matrix $\boldsymbol{\Phi}$.

However, we now want to recover the signal $\boldsymbol{w}$ and we can abuse the fact that $\boldsymbol{w}$ is K-sparse for some K. This allows us to address to problems above.

The information in $\boldsymbol{w}$ is highly localized. It is fully contained in only $K << M$ of its entries. Thus, as long as $N \geq K$, it should, at least in principle, be possible for a measurement vector $\boldsymbol{y}$ of length $N << M$ to completely capture the information within $\boldsymbol{w}$.

Furthermore, since $(M - K)$ entries of $\boldsymbol{w}$ are zero, it follows that in (2.4), $\boldsymbol{y}$ is a actually linear combination of only K columns of Φ. So, if $N \geq K$, we might expect to find suitable constraints that allow us to recover $\boldsymbol{w}$. Once $\boldsymbol{w}$ is recovered, we can compute $\boldsymbol{v}$ via $\boldsymbol{v} = \boldsymbol{\Psi w}$.

2.3.1 Constructing the Sensing Mechanism

In practice, we do not know the locations of the K nonzero entries in $\boldsymbol{w}$. So the first challenge in a compressive sensing system is to design a $N \times M$ sensing matrix $\boldsymbol{\Theta}$ that ensures we measure all the information *for any* signal $\boldsymbol{v}$ that has a K-sparse representation in the basis $\boldsymbol{\Psi}$.

We will not go into the theoretical details of designing an optimal CS measurement process, Instead we will refer the reader to [2] and simply state three particular sensing matrices that have been used successfully:

1. Form $\boldsymbol{\Theta}$ by sampling its entries θ_{ij} independently from the Gaussian distribution with mean zero and variance $1/M$: $\theta_{ij} \overset{iid}{\sim} \mathcal{N}(0, 1/M)$.

2. Sample the entries θ_{ij} independently from a symmetric Bernoulli distribution, $P(\theta_{ij} = \pm 1/\sqrt{M}) = 1/2$.

3. Form $\boldsymbol{\Theta}$ by starting with the $M \times M$ identity matrix I_M and deleting $M - N$ of its rows at random. This sensing matrix corresponds to measuring a down-sampled version of the signal $\boldsymbol{v}$ directly. If the signal of interest $\boldsymbol{v}$ is a digital image, then $\boldsymbol{\Theta}$ measures $\boldsymbol{v}$ after an *image mask* has been applied to it. This sensing mechanism has been succesfully used in [4].

2.3.2 Signal Recovery

Using the sensing mechanisms outlined above, we are able to measure a signal $\boldsymbol{v}$ directly in a compressed format $\boldsymbol{y}$. The final part of a Compressive Sensing system is a signal reconstruction algorithm that decompresses $\boldsymbol{y}$ and recovers $\boldsymbol{v}$ or, equivalently, its sparse representation $\boldsymbol{w}$.

This can be achieved by searching for the sparsest solution $\boldsymbol{w}$ that satisfies equation (2.4) [1]. That is, the desired signal is the solution to the optimization problem:

$$\min_{\boldsymbol{w}} ||\boldsymbol{w}||_0 \qquad \text{subject to} \qquad \boldsymbol{\Phi w} = \boldsymbol{y} \tag{2.5}$$

Unfortunately, (2.5) is an NP-complete problem that can only be solved by an exhaustive search through all possible combinations for the locations of the non-zero entries in $\boldsymbol{w}$.

Luckily, and somewhat surprisingly, it can be shown that, if $N = O(K \log(M/K))$, it is possible to reconstruct K-sparse signals exactly by solving the ℓ_1-optimization problem [1, 2]

$$\min_{\boldsymbol{w}} ||\boldsymbol{w}||_1 \qquad \text{subject to} \qquad \boldsymbol{\Phi w} = \boldsymbol{y} \tag{2.6}$$

The solution $\hat{\boldsymbol{w}}$ can be expressed in the original domain via $\hat{\boldsymbol{v}} = \Psi \hat{\boldsymbol{w}}$.

In practice, signals $\boldsymbol{v}$ usually only have an almost-sparse representation. Thus, the reconstruction is not completely exact and and the solution $\hat{\boldsymbol{v}}$ will be a close approximation to $\boldsymbol{v}$.

A large part of the CS literature is focused on developing algorithms that recover a signal $\boldsymbol{v}$ from a set CS measurement $\boldsymbol{y}$, either by solving (2.6) or through some alternative route.

For a discussion and comparison of some of the most widely used algorithms, see [4]. In Chapter 5 we will discuss a particular framework for CS signal reconstruction known as *Bayesian Compressive Sensing*.

Chapter 3

Basis Functions

In Section 2.1.2, we introduced transform coding. We said that any discrete signal $\boldsymbol{v} \in \mathbb{R}^M$ can be expressed in a different basis via a basis transform:

$$\boldsymbol{v} = \boldsymbol{\Psi}\boldsymbol{w}$$

where $\boldsymbol{\Psi}$ is the $M \times M$ basis matrix and $\boldsymbol{w}\mathbb{R}^M$ is the representation of $\boldsymbol{v}$ in the $\boldsymbol{\Psi}$ basis.

The particular classes of signals $\boldsymbol{v}$ that we are interested in are digital images and digital video. The aim of this chapter is to construct a basis matrix $\boldsymbol{\Psi}$ that gives us a (near-) sparse representation of a wide range of such signals $\boldsymbol{v}$. Finding a set of basis functions $\boldsymbol{\Psi}$ that achieve such a transformation lies at the heart of many lossy compression techniques.

It is important to note here that the choice of basis functions $\boldsymbol{\Psi}$ typically has a significant effect on the performance of the reconstruction algorithms.

3.1 Discrete Cosine Transform

The first basis transform that we will use is the Discrete Cosine Transform (DCT), one of the most widely used transforms in signal processing. It underlies JPEG image compression and is used in various video compression algorithms such as MJPEG, MPEG, H.261 and H.263 [10]. [1]

A DCT decomposes a signal in terms of cosine functions with different frequencies. Its extensive use in lossy compression algorithms is due to the DCT's *energy*

[1] A related transform, known as the *Modified DCT* is used in many lossy audio compression formats such as MP3, AAC and Vorbis.

(a) Original signal $\boldsymbol{v}$ (b) DCT of $\boldsymbol{v}$

Fig. 3.1 Panel (a) shows the original signal $\boldsymbol{v}$, a 256×256 grayscale image known as "cameraman". Panel (b) illustrates the 2-D DCT of $\boldsymbol{v}$. The brightness of a an element increases with the absolute value of the corresponding DCT coefficient. (The high-frequency coefficients have been enhanced to show more detail).

compaction properties. The majority of a signal's energy is contained within relatively few coefficients - typically those corresponding to the lower frequency basis functions.

On a side note, the DCT comes in a various versions that have minor differences between them. In the following, we will describe the most widely used version, known as the *DCT-II*, as well as its inverse transform, the *DCT-III*. We will refer to them simply as "the DCT" and "the Inverse DCT (IDCT)", respectively.

3.1.1 One-Dimensional DCT

Formally, the DCT $\boldsymbol{w}$ of a one-dimensional signal $\boldsymbol{v}$ of length M is given by

$$w_k = c(k) \sum_{m=1}^{M} v_m \cos\left(\frac{\pi(2i-1)(k-1)}{2M}\right) \qquad k = 1, \cdots, M \tag{3.1}$$

where

$$c(k) = \begin{cases} \sqrt{\frac{1}{M}} & \text{if } k = 1 \\ \sqrt{\frac{2}{M}} & \text{otherwise} \end{cases}$$

This transforms a signal $\boldsymbol{v}$ in the original domain (time or space) into its representation $\boldsymbol{w}$ in the DCT domain.

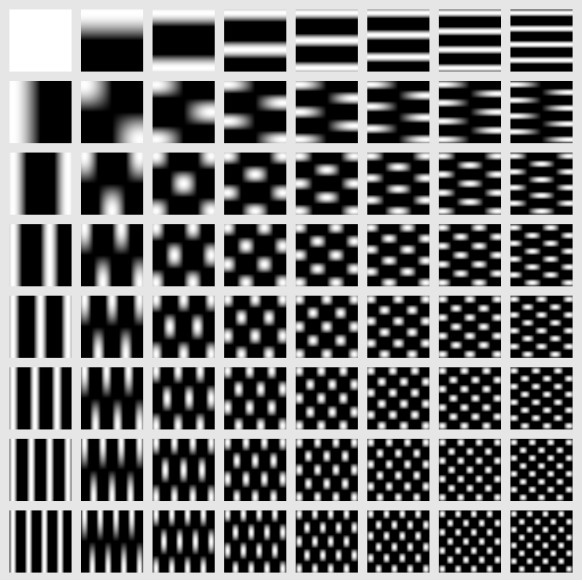


Fig. 3.2 The 2-D DCT basis functions that are used by the DCT to decompose a 8×8 image. The spatial frequency increases towards the bottom right corner.

Conversely, given a signal $\boldsymbol{w}$ in the DCT domain, we can transform it back to the orignal (time or space) domain via the IDCT defined by

$$v_n = \sum_{k=1}^{M} c(k) w_k \cos\left(\frac{\pi(2i-1)(k-1)}{2M}\right) \qquad n = 1, \cdots, M \tag{3.2}$$

We can express equation (3.2) in the desired form $\boldsymbol{v} = \boldsymbol{P}\boldsymbol{w}$. The entries of the basis matrix $\boldsymbol{P}$ are given by

$$P_{n,k} = c(k) \cos\left(\frac{\pi(2i-1)(k-1)}{2M}\right). \tag{3.3}$$

Note that the basis matrix $\boldsymbol{P}$ is orthogonal, $\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}_M$.

3.1.2 Multi-Dimensional DCT

Once we know how to perform the DCT on a one-dimensional signal, we can easily extend the transform to multi-dimensional signals (images, video, etc). To do so, we simply perform successive 1-D transforms along each dimension of the signal. This property is known as *seperability*.

Suppose the signal of interest is a digital image. That means that $\boldsymbol{v}$ is a $M_1 \times M_2$ matrix where $M_1 \times M_2$ is the resolution of the image. To transform the signal, we first

perform the DCT on every row of the matrix. Following that, we perform the DCT on every column of the resulting matrix to get the final transformed signal.

Figure 3.1 shows an example of a 2-D signal $\boldsymbol{v}$ and its transform $\boldsymbol{w}$. Note that the majority of the energy of the transformed signal is concentrated in the top left corner. Most of the DCT coefficients are zero or very close to zero.

In Figure 3.2, we show the 2-D basis functions that would used by the DCT to decompose a signal of size 8×8. Each basis function is characterised by a horizontal and vertical spatial frequency. Typically, natural images are mostly made up of low-frequency components and the corresponding coefficients are therefore relatively large. The highest-frequency components are usually only needed to describe very fine details.

The DCT can be used to decompose video signals with 3-D basis functions. Besides the spatial frequencies, the 3-D basis functions have an additional temporal frequency component. To perform the DCT on a video, we could first perform the 2-D DCT on every frame of the video followed by a 1-D DCT across the temporal axis for each pixel.

For a discussion on the properties of the DCT, see [3]

3.2 Discrete Wavelet Transform

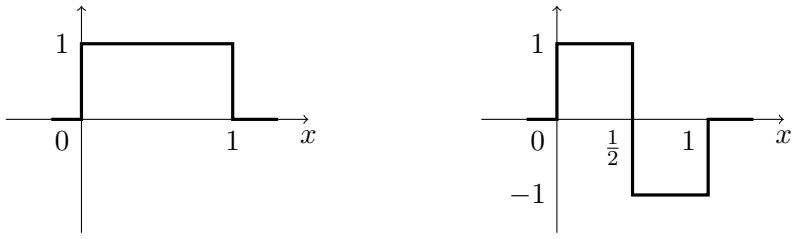
Wavelets have become a very popular tool in signal processing. Their energy compaction properties are on par and often superior to those of the DCT for a wide range of signal classes. In 2000, the JPEG committee released a new image coding standard, JPEG2000, that is gradually replacing the original JPEG standard. The new format moved away from the DCT and uses a Discrete Wavelet Transform (DWT) instead.

[CAVEAT and INTRO]

3.2.1 Introduction to Wavelets

To motivate wavelets, consider again the one-dimensional signal $\boldsymbol{v}$ of length M. Suppose, for simplicity, that M is a power of 2, $M = 2^q$ say. We can view the $\boldsymbol{v}$ as a piecewise-constant function $v(x)$ on the half-open interval $[0, 1)$, where $v(x) = v_i$ if $x \in [\frac{i-1}{M}, \frac{i}{M})$.

Let V^j denote the vector space containing all piecewise-constant functions f defined on the interval $[0, 1)$ that consist of 2^j pieces, each of which is constant across a sub-interval of size 2^{-j}. Thus, V^0 consists of all functions that are constant on $[0, 1)$, while V^1 consists of all functions that have two constant pieces, one over $[0, 1/2)$ and one over $[1/2, 1)$. In particular, our signal $v(x)$ resides in the space V^q.



(a) Haar scaling function $\phi(x)$ (b) Haar mother wavelet $\psi(x)$

Fig. 3.3 The scaling function and wavelet function for the Haar wavelets.

Note that if $f \in V^j$, then $f \in V^{j+1}$. Thus, the vector spaces V^j are nested: $V^0 \subset V^1 \subset V^2 \subset \cdots$.

Next, we need to choose a basis for each vector space V^j. To do so, we introduce a *scaling function* (also known as *scalet*, or *father wavelet*) that is usually denoted $\phi(x)$. The form of the scaling function depends on the particular choice wavelet decomposition.

For example, for the *Haar wavelets* the scaling function is given by

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

See Figure 3.3a for a plot of $\phi(x)$.

Given the scaling function $\phi(x)$, we can define the following basis for V^j:

$$\phi_k^j(x) := 2^{j/2}\phi(2^j x - k) \qquad k = 0, \cdots, 2^j - 1$$

Using this basis, we can decompose our signal $v(x) \in V^q$ as

$$v(x) = \sum_{k=0}^{2^q-1} c_k^q \phi_k^q(x)$$

For the scaling function defined in equation (3.4), we have that $c_k^q = v_{k+1}$.

To obtain *wavelets*, consider the *orthogonal complement* of V^j in V^{j+1} and denote it W^j. That is, $W^j = \{f \in V^{j+1} : \quad \langle f, g\rangle = 0 \quad \forall g \in V^j\}$ where the inner product $\langle f, g\rangle$ is given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

By forming a basis for W^j, we obtain a set of *wavelet functions* $\{\psi_k^j,\ k = 0, \cdots, 2^j - 1\}$. Wavelet functions can be constructed by scaling and shifting a so-called *mother wavelet*

$\psi(x)$ as follows:

$$\psi_k^j(x) = 2^{j/2}\psi(2^j x - k) \qquad k = 0, \cdots, 2^j - 1$$

For the Haar wavelets, the mother wavelet is given by:

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The Haar mother wavelet is shown in Figure 3.3b.

Note that, since the scaling functions ϕ_k^j form a basis of V^j and the wavelet functions ψ_k^j form a basis of W^k, and since W^j is the orthogonal complement to V^j in V^{j+1}, it follows that the set $\{\phi_k^j, \psi_k^j : k = 0, \cdots, 2^j - 1\}$ forms a basis of the vector space V^{j+1}.

This allows us to express our signal $v \in V^q$ as

$$v(x) = \sum_{k=0}^{2^{q-1}-1} d_k^{q-1}\psi_k^{q-1}(x) + \sum_{k=0}^{2^{q-1}-1} c_k^{q-1}\phi_k^{q-1}(x)$$

This gives us the first level of the discrete wavelet transform of v. The coefficients c_k and d_k are sometimes referred to as "approximation" coefficients and "detail" coefficients, respectively.

We can continue the decomposition by splitting the basis for V^{q-1} into the bases for V^{q-2} and W^{q-2} to get the next level of the transform:

$$v(x) = \sum_{k=0}^{2^{q-1}-1} d_k^{q-1}\psi_k^{q-1}(x) + \sum_{k=0}^{2^{q-2}-1} d_k^{q-2}\psi_k^{q-2}(x) + \sum_{k=0}^{2^{q-2}-1} c_k^{q-2}\phi_k^{q-2}(x)$$

To get the full decomposition, we continue in this fashion up to the qth level:

$$v(x) = \sum_{j=0}^{q-1}\sum_{k=0}^{2^j-1} d_k^j\psi_k^j(x) + c_0^0\phi(x)$$

The full DWT of $\boldsymbol{v}$ consists of the coefficients $\{c_0^0, d_k^j : j = 0, \cdots, q-1,\ k = 0, \cdots, 2^j - 1\}$.

3.2.2 Computing the DWT

In practice, we can compute one level of the DWT coefficients by passing the signal $\boldsymbol{v}$ through a *low-pass filter* $\boldsymbol{h}$ and a high-pass filter $\boldsymbol{g}$, respectively, and then downsampling the results by a factor of two. Overall, these computations can be done by multiplying

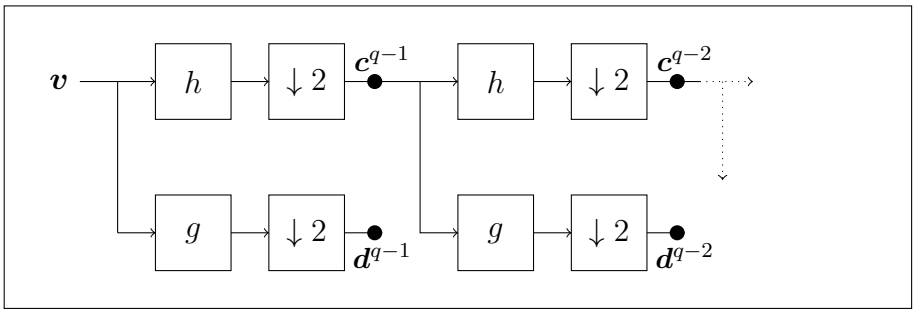


Fig. 3.4 The first two levels of the DWT of the signal $\boldsymbol{v}$ of length 2^q via a filter bank.

the vector $\boldsymbol{v}$ by a matrix $\boldsymbol{H}$ and a matrix $\boldsymbol{G}$ to get the approximation and detail coefficients, repectively.

To compute the next level, we take the approximation coefficients of the current stage and pass them again through the filter bank. The procedure is depicted in Figure 3.4.

Chapter 4

Sparse Bayesian Learning

Sparse Bayesian Learning [8] is a general Bayesian framework within supervised Machine Learning. It can be applied to both regression and classification tasks. The *Relevance Vector Machine*, or *RVM*, is a particular specialisation of the Sparse Bayesian Learning model which has identical functional form to the Support Vector Machine (SVM). However, the RVM comes with a number of key advantages over the SVM. The solution produced by a RVM is typically much sparser than the solution by a comparable SVM. Furthermore, the RVM is a probabilistic model and as such, allows us to estimate error bounds in its predictions.

In this chapter, we will derive the Sparse Bayesian Learning model for regression. We will summarise both the original inference algorithm [8] and also the faster "Sequential Sparse Bayesian Learning Algorithm" [9].

4.1 Model Specification

We are given a data set of N input vectors $\{\boldsymbol{x}^{(i)}\}_{i=1}^{N}$ and their associated *targets* $\{y^{(i)}\}_{i=1}^{N}$. The input vectors live in D-dimensional space, $\boldsymbol{x} \in \mathbb{R}^D$. The targets are real values, $y \in \mathbb{R}$. [1]

We model the data using a linearly-weighted sum of M fixed basis functions $\{\phi_j(\cdot)\}_{j=1}^{M}$ and base our predictions on the function $f(\cdot)$ defined as

$$f(\boldsymbol{x}; \boldsymbol{w}) = \sum_{j=1}^{M} w_j \phi_j(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}) \tag{4.1}$$

[1] When using the Sparse Bayesian model for regression, we assume the targets are real-valued. It is also possible to use the model for classification in which cased the targets are assumed to be discrete class labels.

where $\boldsymbol{w} = [w_1, \ldots, w_M]^T$ and $\boldsymbol{\phi}(\cdot) = [\phi_1(\cdot), \ldots, \phi_M(\cdot)]^T$. Using a large number M of non-linear basis functions $\phi_j : \mathbb{R}^D \to \mathbb{R}$ allows for a highly flexible model.

The *Relevance Vector Machine*, or RVM, is a specialisation of the Sparse Bayesian Learning model in which the basis functions take the form of *kernel functions*

$$\phi_j(\cdot) \equiv K\left(\cdot\,,\, \boldsymbol{x}^{(j)}\right).$$

This defines a basis function for each training data point $\boldsymbol{x}^{(i)}$. Typically, we also include an additional *bias* term $\phi_0(\cdot) \equiv 1$, so that $M = N + 1$. The RVM has identical functional form to the popular Support Vector Machine (SVM), but superior properties. It typically gives sparser solutions than the SVM and has the additional advantage of providing confidence measures for its predictions.

However, in the following derivation, we will stick to the case of general basis functions $\phi_j : \mathbb{R}^D \to \mathbb{R}$. Thus M need not equal $N+1$ and may, in fact, be a lot larger.

To train the model (4.1), i.e. find values for $\boldsymbol{w}$ that are optimal in some sense, we make the standard assumption that our training data are samples from the model with additive noise:

$$\begin{aligned} y^{(i)} &= f(\boldsymbol{x}^{(i)}; \boldsymbol{w}) + \epsilon^{(i)} \\ &= \boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}^{(i)}) + \epsilon^{(i)} \qquad i = 1, \ldots, N. \end{aligned} \tag{4.2}$$

The errors $\{\epsilon^{(i)}\}_{i=1}^N$ are assumed to be independent samples from a zero-mean Gaussian distribution with variance σ^2

$$p(\epsilon^{(i)}) = \mathcal{N}(\epsilon^{(i)} \,|\, 0, \sigma^2) \qquad i = 1, \ldots, N. \tag{4.3}$$

Combining equation (4.2) with equation (4.1), we may express the model for the complete data using matrix notation:

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{w} + \boldsymbol{\epsilon} \tag{4.4}$$

where $\boldsymbol{\epsilon} = \left[\epsilon^{(1)}, \cdots, \epsilon^{(N)}\right]^T$. The $N \times M$ matrix $\boldsymbol{\Phi}$ is known as the design matrix. The ith row of $\boldsymbol{\Phi}$ is given by $\boldsymbol{\phi}(\boldsymbol{x}^{(i)})^T$. The jth column of $\boldsymbol{\Phi}$ is given by $\boldsymbol{\phi}_j = \left[\phi_j\left(\boldsymbol{x}^{(1)}\right), \cdots, \phi_j\left(\boldsymbol{x}^{(N)}\right)\right]^T$, which is also referred to as the jth *basis vector*. Thus

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}(\boldsymbol{x}^{(1)})^T \\ \vdots \\ \boldsymbol{\phi}(\boldsymbol{x}^{(N)})^T \end{bmatrix}$$

Combining equation (4.4) and equation (4.3), we find that the complete data likelihood function is given by

$$\begin{aligned} p\left(\boldsymbol{y} \,|\, \boldsymbol{w}, \sigma^2\right) &= \mathcal{N}\left(\boldsymbol{y} \,|\, \boldsymbol{w}, \sigma^2 \boldsymbol{I}_M\right) \\ &= (2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{1}{2\sigma^2}||\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{w}||^2\right\} \end{aligned} \tag{4.5}$$

where $\boldsymbol{I}_M$ is the $M \times M$ identity matrix.

So far, we have specified the general linear regression model. To get to the sparse Bayesian formulation, we define a zero-mean Gaussian prior distribution over the parameters $\boldsymbol{w}$

$$p(\boldsymbol{w} \,|\, \boldsymbol{\alpha}) = \prod_{j=1}^{M} \mathcal{N}\left(w_j \,|\, \alpha_j^{-1}\right)$$

where $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_M]^T$ is a vector of M hyperparameters. It is important to note that each hyperparameter α_j is solely responsible for controlling the strength of the prior of its associated weight w_j. If α_j is large, the prior over w_j is very strongly peaked at zero. This form of the prior distribution is, more than anything, responsible for the dramatic sparsity in the final model.

To complete the specification, we must define a prior over the noise parameter σ^2 and the a hyperprior over the hyperparameters $\boldsymbol{\alpha}$. Following the derivation in [8], we use the following Gamma [2] priors

$$p(\boldsymbol{\alpha} \,|\, a, b) = \prod_{j=1}^{M} \text{Gamma}(\alpha_j \,|\, a, b)$$

$$p(\beta \,|\, c, d) = \text{Gamma}(\beta \,|\, c, d)$$

where $\beta \equiv \sigma^{-2}$.

As a side note, consider the prior of $\boldsymbol{w}$ after marginalising out the dependence on the hyperpriors $\boldsymbol{\alpha}$. Since each w_j is normally distributed with an unknown precision parameter α_j and since the (hyper)prior over α_j is the Gamma distribution and

[2] The Gamma distribution is defined by

$$\text{Gamma}(z \,|\, a, b) = \Gamma(a)^{-1} b^a z^{a-1} \exp(-bz) \qquad z, a, b > 0$$

where $\Gamma(.)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt.$$

therefore conjugate to $p(w_j \,|\, \alpha_j)$, it follows that the resulting integral can be evaluated analytically

$$\begin{aligned} p(\boldsymbol{w} \,|\, a, b) &= \int p(\boldsymbol{w} \,|\, \boldsymbol{\alpha}) p(\boldsymbol{\alpha} \,|\, a, b) d\boldsymbol{\alpha} \\ &= \prod_{j=1}^{M} \int \mathcal{N}(w_j \,|\, 0, \alpha_j^{-1}) \,\mathrm{Gamma}(\alpha_j | \, a, b) d\alpha_j \\ &= \prod_{j=1}^{M} \frac{b^a \Gamma(a + \frac{1}{2})}{(2\pi)^{\frac{1}{2}} \Gamma(a)} \left(b + \frac{w_j^2}{2} \right)^{-(a+\frac{1}{2})} . \end{aligned}$$

This corresponds to a product of independent Student-t density functions over the weights w_j. The choice $a = b = 0$ implies that $p(\boldsymbol{w} \,|\, a, b) \propto \prod_{j=1}^{M} 1/|w_j|$. As discussed in [8], it is this hierarchical formulation of the weight prior that is ultimately responsible for encouraging sparse solutions.

4.2 Model Inference

We have specified the likelihood model for the data and a prior distribution over the model parameters. The next step in Bayesian inference is to compute the posterior distribution of the parameters. We begin by setting up Bayes' Rule

$$p(\boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2 \,|\, \boldsymbol{y}) = \frac{p(\boldsymbol{y} \,|\, \boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2) p(\boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2)}{\int p(\boldsymbol{y} \,|\, \boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2) p(\boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2) d\boldsymbol{w} d\boldsymbol{\alpha} d\sigma^2} \tag{4.6}$$

The integral in the denominator of (4.6) is computationally intractable and we must resort to an alternative strategy. First, we decompose the left-hand-side of equation (4.6) as

$$p(\boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2 \,|\, \boldsymbol{y}) = p(\boldsymbol{w} \,|\, \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2) p(\boldsymbol{\alpha}, \sigma^2 \,|\, \boldsymbol{y}).$$

Next, we use Bayes' Rule to compute the posterior distribution of the weights given $\boldsymbol{\alpha}$ and σ^2

$$p(\boldsymbol{w} \,|\, \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2) = \frac{p(\boldsymbol{y} \,|\, \boldsymbol{w}, \sigma^2) p(\boldsymbol{w} \,|\, \boldsymbol{\alpha})}{p(\boldsymbol{y} \,|\, \boldsymbol{\alpha}, \sigma^2)} \tag{4.7}$$

The denominator of the right-hand-side is known as the *marginal likelihood* and given by

$$p(\boldsymbol{y} \,|\, \boldsymbol{\alpha}, \sigma^2) = \int p(\boldsymbol{y} \,|\, \boldsymbol{w}, \sigma^2) p(\boldsymbol{w} \,|\, \boldsymbol{\alpha}) d\boldsymbol{w} \tag{4.8}$$

Since $\boldsymbol{\alpha}$ and σ^2 are treated as fixed quantities in equation (4.7), the Gaussian density $p(\boldsymbol{w} \,|\, \boldsymbol{\alpha})$ is the conjugate prior to the Gaussian likelihood function $p(\boldsymbol{y} \,|\, \boldsymbol{w}, \sigma^2)$. Thus,

the integral in equation (4.8) is a convolution of two Gaussians and therefore equal to another Gaussian:

$$\begin{aligned} p(\boldsymbol{y} \mid \boldsymbol{\alpha}, \sigma^2) &= \mathcal{N}(\boldsymbol{y} \mid \mathbf{0}, \boldsymbol{C}) \\ &= (2\pi)^{-N/2} |\boldsymbol{C}|^{-1/2} \exp\left\{-\frac{1}{2}\boldsymbol{y}^T \boldsymbol{C}^{-1} \boldsymbol{y}\right\} \end{aligned}$$

where

$$\boldsymbol{C} = \sigma^2 \boldsymbol{I}_N + \boldsymbol{\Phi} \boldsymbol{A}^{-1} \boldsymbol{\Phi}^T. \tag{4.9}$$

The posterior distribution for $\boldsymbol{w}$ is a also a Gaussian:

$$p(\boldsymbol{w} \mid \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2) = \mathcal{N}(\boldsymbol{w} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{4.10}$$

Its mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ are given by

$$\boldsymbol{\Sigma} = \left(\sigma^{-2} \boldsymbol{\Phi}^T \boldsymbol{\Phi} + \boldsymbol{A}\right)^{-1} \tag{4.11}$$

$$\boldsymbol{\mu} = \sigma^{-2} \boldsymbol{\Sigma} \boldsymbol{\Phi}^T \boldsymbol{y} \tag{4.12}$$

with $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{\alpha})$.

Finally, we need to find the posterior of the hyperparameters, $p(\boldsymbol{\alpha}, \sigma^2 \mid \boldsymbol{y})$. This part is computationally intractable, so instead we approximate the posterior by a delta-function at its mode. Hence, the problem reduces to finding the values of $\boldsymbol{\alpha}$ and σ^2 that maximise $p(\boldsymbol{\alpha}, \sigma^2 \mid \boldsymbol{y}) \propto p(\boldsymbol{y} \mid \boldsymbol{\alpha}, \sigma^2) p(\boldsymbol{\alpha}) p(\sigma^2)$.

Here, we make the simplifying assumption that $a = b = c = d = 0$, giving us uniform (but improper) hyperpriors (see [8] for the general case). Maximising $p(\boldsymbol{\alpha}, \sigma^2 \mid \boldsymbol{y})$ is then equivalent to maximising the marginal likelihood, or equivalently, its logarithm

$$\begin{aligned} \mathcal{L}(\boldsymbol{\alpha}, \sigma^2) &= \log p(\boldsymbol{y} \mid \boldsymbol{\alpha}, \sigma^2) = \log \mathcal{N}(\boldsymbol{y} \mid \mathbf{0}, \boldsymbol{C}) \\ &= -\frac{1}{2}\left[N \log 2\pi + \log |\boldsymbol{C}| + \boldsymbol{y}^T \boldsymbol{C}^{-1} \boldsymbol{y}\right] \end{aligned} \tag{4.13}$$

The procedure of finding $\boldsymbol{\alpha}$ and σ^2 that maximise the (log) marginal likelihood (4.13) is also known as *type-II Maximum likelihood* and *evidence approximation.*

Algorithm 1 Sparse Bayesian Learning: Original Training Algorithm

Choose some initial positive values for σ^2 and α_j for $j = 1, \cdots, M$
repeat
  $\boldsymbol{A} = \text{diag}(\boldsymbol{\alpha})$
  $\boldsymbol{\Sigma} = \left(\sigma^{-2}\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{A}\right)^{-1}$
  $\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^T\boldsymbol{y}$

  for $j = 1, \cdots, M$ **do**
    $\gamma_j = 1 - \alpha_j\Sigma_{jj}$
    $\alpha_j = \gamma_j/\mu_j^2$
  end for
  $\sigma^2 = ||\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\mu}||^2/(N - \sum_j \gamma_j)$
until Convergence

4.2.1 Original Training Algorithm

The original training algorithm in [8] is derived by setting the derivatives of (4.13) to zero. We obtain the following update equations for $\boldsymbol{\alpha}$ and σ^2:

$$\alpha_j^{\text{new}} = \frac{\gamma_j}{\mu_j^2} \tag{4.14}$$

$$(\sigma^2)^{\text{new}} = \frac{||\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\mu}||^2}{N - \sum_j \gamma_j} \tag{4.15}$$

where μ_j is the jth component of the posterior mean $\boldsymbol{\mu}$ (4.12). The quantities γ_j are defined by

$$\gamma_j = 1 - \alpha_j\Sigma_{jj}$$

where Σ_{jj} is the jth diagonal element of the posterior covariance $\boldsymbol{\Sigma}$ (4.11).

To train the model, we can start by giving $\boldsymbol{\alpha}$ and σ^2 some initial values and evaluate the mean and covariance of the weights posterior using equations (4.12) and (4.11), respectively. Next we alternate between re-estimating the hyperparamaters $\boldsymbol{\alpha}$ and σ^2 using (4.14) and (4.15) and updating the posterior mean and covariance parameters using (4.12) and (4.11). We continue until a relevant convergence criterion is met. For example, we may choose to stop if the change in the marginal likelihood - or, alternatively, the change in the parameter values - between two iterations is below a certain pre-defined threshold.

This procedure is summarised in Algorithm 1.

During training, it is typically observed that many of the hyperparameters α_j tend to infinity. Equations (4.12) and (4.11) imply that the weights w_j corresponding to these hyperparameters have a posterior distribution where the mean and the variance are both zero, meaning their posterior is infinitely peaked at zero. As a consequence, the corresponding basis functions $\phi_j(\cdot)$ are effectively removed from the model and we achieve sparsity.

In the case of the RVM, where $\phi_j(\cdot) \equiv K(\cdot, \boldsymbol{x}^{(j)})$, the input vectors $\boldsymbol{x}^{(j)}$ corresponding to the remaining non-zero weights are known as the *relevance vectors* of the model.

4.2.2 Sequential Sparse Bayesian Learning Algorithm

A central drawback of the training algorithm discussed in the previous section is its speed. The computational complexity scales with the cube of the number of basis functions. During training, as basis functions are pruned from the model, the algorithm accelerates. Nevertheless, if M is very large, the procedure can be very expensive to run.

An alternative strategy of maximising the marginal likelihood (4.13) was developed by [9], resulting in a highly accelerated training algorithm: the *Sequential Sparse Bayesian Learning Algorithm*. It starts with a single basis function and maximises the marginal likelihood by sequentially adding and deleting candidate basis functions. This significantly reduces the computational complexity of the algorithm.

To derive the algorithm, we follow the analysis in [7] and consider the dependence of the marginal likelihood $\mathcal{L}(\boldsymbol{\alpha}, \sigma^2)$ on a single hyperparameter α_j. First, we decompose the matrix $\boldsymbol{C}$, defined in (4.9), as follows:

$$\begin{aligned}\boldsymbol{C} &= \sigma^2 \boldsymbol{I}_N + \sum_{m \neq j} \alpha_m^{-1} \boldsymbol{\phi}_m \boldsymbol{\phi}_m^T + \alpha_j^{-1} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^T \\ &= \boldsymbol{C}_{-j} + \alpha_j^{-1} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^T\end{aligned}$$

where $\boldsymbol{C}_{-j} \equiv \sigma^2 \boldsymbol{I}_N + \sum_{m \neq j} \alpha_m^{-1} \boldsymbol{\phi}_m \boldsymbol{\phi}_m^T$ is $\boldsymbol{C}$ without the contribution of the jth basis vector $\boldsymbol{\phi}_j$. Making use of standard identities [WHICH ONES?] for matrix inverses and determinants, we can express $|\boldsymbol{C}|$ and $\boldsymbol{C}^{-1}$ as

$$\boldsymbol{C}^{-1} = \boldsymbol{C}_{-j}^{-1} - \frac{\boldsymbol{C}_{-j}^{-1} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^T \boldsymbol{C}_{-j}^{-1}}{\alpha_j + \boldsymbol{\phi}_j^T \boldsymbol{C}_{-j}^{-1} \boldsymbol{\phi}_j}$$

$$|\boldsymbol{C}| = |\boldsymbol{C}_{-j}| \left| 1 + \alpha_j^{-1} \boldsymbol{\phi}_j^T \boldsymbol{C}_{-j}^{-1} \boldsymbol{\phi}_j \right|$$

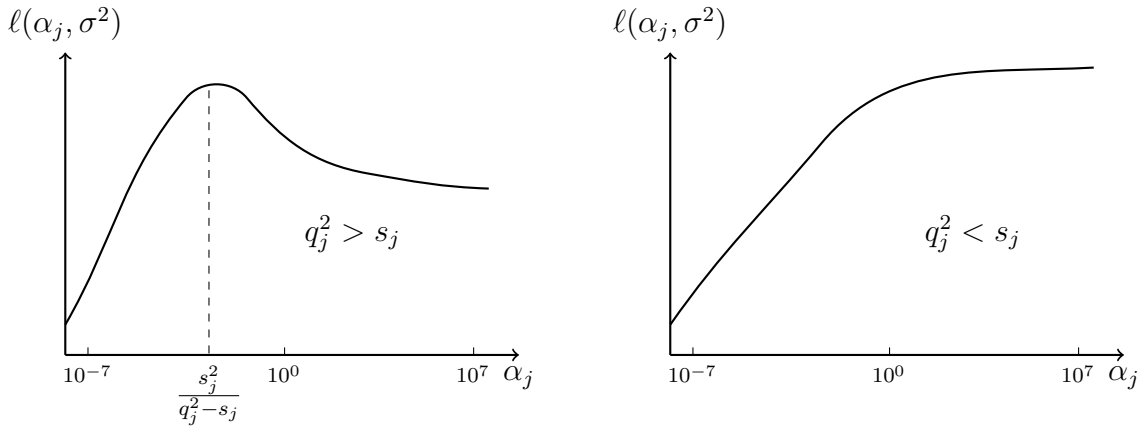


Fig. 4.1 Example plots of $\ell(\alpha_j, \sigma^2)$ against α_j illustrating the stationary points when $q_j^2 > s_j$ (left) and $q_j^2 < s_j$ (based on [7]).

This allows us to decompose the marginal likelihood:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\alpha}, \sigma^2) &= \mathcal{L}(\boldsymbol{\alpha}_{-j}, \sigma^2) + \frac{1}{2}\left[\log \alpha_j - \log(\alpha_j + s_j) + \frac{q_j^2}{\alpha_j + s_j}\right] \\ &\equiv \mathcal{L}(\boldsymbol{\alpha}_{-j}, \sigma^2) + \ell(\alpha_j, \sigma^2) \end{aligned} \tag{4.16}$$

This conveniently seperates terms in α_j in $\ell(\alpha_j, \sigma^2)$ from the remaining terms in $\mathcal{L}(\boldsymbol{\alpha}_{-j}, \sigma^2)$, which is the (log) marginal likelihood with the basis vector $\boldsymbol{\phi}_j$ excluded.

The quantity s_j is the *sparsity factor*, defined as

$$s_j = \boldsymbol{\phi}_j^T \boldsymbol{C}_{-j}^{-1} \boldsymbol{\phi}_j.$$

It serves as a measure of how much the marginal likelihood would decrease if we added $\boldsymbol{\phi}_j$ to the model. The quantity q_j, on the other hand, is known as the *quality factor*. It is defined as

$$q_j = \boldsymbol{\phi}_j^T \boldsymbol{C}_{-j}^{-1} \boldsymbol{y}$$

and measures the extent to which ϕ_j increases $\mathcal{L}(\boldsymbol{\alpha}, \sigma^2)$ by helping to explain the data $\boldsymbol{y}$. Thus, a particular basis vector $\boldsymbol{\phi}_j$ should not be included in the model if its sparsity factor s_j is large, unless it is offset by a large quality factor q_j.

We can see this more explicitly if we consider the first derivative of $\ell(\alpha_j, \sigma^2)$ with respect to α_j [7]

$$\frac{\partial \ell(\alpha_j, \sigma^2)}{\partial \alpha_j} = \frac{\alpha_j^{-1} s_j^2 - (q_j^2 - s_j)}{2(\alpha_j + s_j)^2}$$

Equating it to zero (and noting that α_j is an inverse-variance and therefore positive), we obtain the following solution for α_j:

$$\alpha_j = \begin{cases} s_j^2/(q_j^2 - s_j) & \text{if } q_j^2 > s_j \\ +\infty & \text{otherwise} \end{cases} . \tag{4.17}$$

The solution (4.17) is illustrated in Figure 4.1.

It follows that, if, during training, a candidate basis vector $\boldsymbol{\phi}_j$ is currently included in the model (meaning $\alpha_j < \infty$) even though $q_j^2 \leq s_j$, then α_j should be set to ∞ and $\boldsymbol{\phi}_j$ should be pruned from the model. On the other hand, if $\boldsymbol{\phi}_j$ is currently excluded from the model (i.e. $\alpha_j = \infty$), but $q_j^2 > s_j$, then α_j should be set to $s_j^2/(q_j^2 - s_j)$ and $\boldsymbol{\phi}_j$ should be added to the model. Furthermore, if $\boldsymbol{\phi}_j$ is included and $q_j^2 > s_j$, then we may also re-estimate α_j. Each step in the algorithm (weakly) increases the marginal likelihood. Thus we are guaranteed to find a maximum.

During the algorithm, we must maintain and update values of the quality factors and sparsity factors for all basis functions, as well as the posterior mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ of the weights $\boldsymbol{w}$. In practice, it easier to keep track of the quantities $Q_m = \boldsymbol{\phi}_m^T \boldsymbol{C}^{-1} \boldsymbol{\phi}_m$ and $S_m = \boldsymbol{\phi}_m^T \boldsymbol{C}^{-1} \boldsymbol{y}$ which can also be written as (using the Woodbury Identity)

$$S_m = \sigma^{-2} \boldsymbol{\phi}_m^T \boldsymbol{\phi}_m - \sigma^{-4} \boldsymbol{\phi}_m^T \boldsymbol{\Phi} \boldsymbol{\Sigma} \boldsymbol{\Phi}^T \boldsymbol{\phi}_m \tag{4.18}$$

$$Q_m = \sigma^{-2} \boldsymbol{\phi}_m^T \boldsymbol{y} - \sigma^{-4} \boldsymbol{\phi}_m^T \boldsymbol{\Phi} \boldsymbol{\Sigma} \boldsymbol{\Phi}^T \boldsymbol{y} \tag{4.19}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Phi}$ contain only the basis functions that are currently included in the model.

The factors s_m and q_m can by obtained from S_m and Q_m as follows:

$$s_m = \frac{\alpha_m S_m}{\alpha_m - S_m} \tag{4.20}$$

$$q_m = \frac{\alpha_m Q_m}{\alpha_m - S_m} \tag{4.21}$$

Note that if $\alpha_m = \infty$, then $q_m = Q_m$ and $s_m = S_m$.

We have summarized the procedure in Algorithm 2. After initializing the standard deviation σ^2 in step 1, we add the first basis function $\boldsymbol{\phi}_j$ to the model. We could initialize with any basis vector, but in step 2, we pick the one with the largest normalized projection on the target vector $\boldsymbol{y}$, i.e. we choose $j = \arg\max_m \left\{ ||\boldsymbol{\phi}_m^T \boldsymbol{y}||^2 / ||\boldsymbol{\phi}_m||^2 \right\}$. In

Algorithm 2 Sequential Sparse Bayesian Learning Algorithm [9]

1: Initialise σ^2.
2: Add basis function $\boldsymbol{\phi}_j$ to the model, where $j = \arg\max_m \left\{||\boldsymbol{\phi}_m^T \boldsymbol{y}||^2/||\boldsymbol{\phi}_m||^2\right\}$. Set $\alpha_j = \frac{||\boldsymbol{\phi}_j||^2}{||\boldsymbol{\phi}_j^T \boldsymbol{y}||^2/||\boldsymbol{\phi}_j||^2 - \sigma^2}$. Set $\alpha_m = \infty$ for $m \neq j$.
3: Compute $\boldsymbol{\Sigma} = \left(\sigma^{-2}\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{A}\right)^{-1}$ and $\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^T\boldsymbol{y}$ which are scalars initially. Compute S_m, Q_m, s_m and q_m for $m = 1, \cdots, M$ using (4.18) − (4.21).
4: **repeat**
5: Select some candidate basis vector $\boldsymbol{\phi}_j$.
6: **if** $q_j^2 > s_j$ and $\alpha_j = \infty$ **then add** $\boldsymbol{\phi}_j$ to the model and update α_j.
7: **if** $q_j^2 > s_j$ and $\alpha_j < \infty$ **then re-estimate** α_j.
8: **if** $q_j^2 < s_j$ and $\alpha_j < \infty$ **then delete** $\boldsymbol{\phi}_j$ from the model and set $\alpha_j = \infty$.
9: Update $\sigma^2 = ||\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{w}||/(N - M + \sum_m \alpha_m \boldsymbol{\Sigma}_{mm})$[8].
10: Update $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}$ and S_m, Q_m, s_m, q_m for $m = 1, \cdots, M$.
11: **until** Convergence

step 3 we compute the model statistics and in step 4 we begin the large loop of the algorithm. There are two things to note here. First, in step 5, we need to select a candidate basis vector $\boldsymbol{\phi}_j$. We are free to pick one at random. Alternatively, it is possible to compute the change in the marginal likelihood for each ccandidate basis vector and choose the one that would give us the largest increase. Second, we would usually like to estimate the noise variance σ^2 from the data, as is done in step 9. However, in practice, we may decide to set σ^2 in advance in step 1 and keep it fixed throughout the algorithm. If we decide to do so, then we can perform the updates in step 10 using very efficient update formulae that do not require matrix inversions. The formulae can be found in the appendix of [9]. If we do decide to update σ^2 in step 9, then we must use the full equations (4.11), (4.12) and (4.18)-(4.21).

4.3 Making Predictions

Once we have trained the model, we may use it to predict the target y^* for a new input vector $\boldsymbol{x}^*$. To do so, we would like to compute the *predictive distribution*

$$p(y^* \mid \boldsymbol{y}) = \int p(y^* \mid \boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2) p(\boldsymbol{w}, \boldsymbol{\alpha}, \sigma^2 \mid \boldsymbol{y})\, d\boldsymbol{w}\, d\boldsymbol{\alpha}\, d\sigma^2.$$

We cannot compute this integral analytically, nor do we actually know the posterior of all the model parameters. Instead, we use the type-II maximum likelihood solutions for $\boldsymbol{\alpha}$ and σ^2 that we obtained during training and base our predictions on the posterior

distribution of the weights conditioned on $\boldsymbol{\alpha}$ and σ^2. The predictive distribution for $\boldsymbol{x}^*$ is then:

$$p(y^* \mid \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2) = \int p(y^* \mid \boldsymbol{w}, \sigma^2) p(\boldsymbol{w} \mid \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2)\, d\boldsymbol{w} \tag{4.22}$$

Both factors in the integrand are Gaussians, and we can therefore readily compute the integral to get

$$p(y^* \mid \boldsymbol{y}, \boldsymbol{\alpha}, \sigma^2) = \mathcal{N}(y^* \mid \mu^*, (\sigma^2)^*) \tag{4.23}$$

The predictive mean is given by

$$\mu^* = \boldsymbol{\mu}^T \boldsymbol{\phi}(\boldsymbol{x}^*) \tag{4.24}$$

and the predictive variance is given by

$$(\sigma^2)^* = \sigma^2 + \boldsymbol{\phi}(\boldsymbol{x}^*)^T \boldsymbol{\Sigma} \boldsymbol{\phi}(\boldsymbol{x}^*) \tag{4.25}$$

Equation (4.24) implies that, if we want to produce point predictions, we may simply set the weights $\boldsymbol{w}$ equal to posterion mean $\boldsymbol{\mu}$ which is typically very sparse. If we are also interested in error bars for our predictions, we can obtain them using Equation (4.25). The error bars consist of two parts, the noise in the data σ^2 and the uncertainty in the weights.

For more details and derivations on Sparse Bayesian Learning, see [7–9].

Chapter 5

Design of the Multi-Scale Cascade of Estimations Algorithm

Bringing all building blocks together. Description and explanation of the algorithm

1. *Bayesian CS*
2. *Basis Matrix ?*
3. *MSCE Algorithm*

So far, we have not addressed the central question: How do we solve the compressive sensing problem. Various deterministic approaches have been developed in recent years. See [4] for an overview.

In the MPhil project, we will employ a probabilistic technique based on Sparse Bayesian Learning. In particular, we will use the *Relevance Vector Machine (RVM)* [8, 9] to reconstruct $\boldsymbol{w}$ from the measurements $\boldsymbol{y}$. Following that, we obtain a reconstructed version of the desired signal $\boldsymbol{x}$ by pre-multiplying $\boldsymbol{w}$ by $\boldsymbol{\Psi}$ to obtain the desired signal.

5.1 Interpolator

We use a sensing matrix $\boldsymbol{\Omega}$ that acts as signal mask. That is, we obtain the $N \times M$ matrix $\boldsymbol{\Omega}$ by taking the $M \times M$ identity matrix $\boldsymbol{I}_M$ and deleting $(M - N)$ rows. This corresponds to a subsampled signal in which we only measured N pixel values. For this specific class of sensing matrices, the problem of reconstructing the original signal is also known as *interpolation*.

Fig. 5.1 Corrupted signal $\boldsymbol{y}$ (left) and reconstructed signal $\hat{\boldsymbol{x}}$ (right) using a cascade of 3 RVMs with Haar basis functions (see [4]).

In order to reconstruct the image, we use the estimated posterior mean to "predict" what a pixel value y^* should be at a location x^* in which information was missing:

$$y^* = \boldsymbol{w}^T \boldsymbol{\psi}(x^*) \tag{5.1}$$

Apart from achieving sparse solutions, one further desirable feature of the RVM is that the model provides error bars for its predictions. This is used in [4] to construct a multi-scale cascade of RVM estimations and achieve significant performance boosts.

An example of this can be seen in Figure 5.1.

Chapter 6

Implementation Details and Code optimization

1. Main Draw back: slow decoder -> parallel

This chapter gives a brief description of the current state of our implementation of the 3D signal reconstructer.

Comparing notation to the previous chapter, what we refer to as W here is the transpose of what was previously denoted as Ψ. And since $\boldsymbol{v} = W^T \hat{\boldsymbol{v}}$, we see that $\boldsymbol{v}$ corresponds to what was previously called $\boldsymbol{x}$.

6.1 Update formulae, details on RVM implementation

Chapter 7

Results

1. Metrics: PSNR, (Relative Error), (FSIM)
2. Full Res vs Pre-compressed: With full res, I need a lot of samples to get perfect reconstruction. With pre-comp, I'm not sure.
3. Curve: Performance vs Compression
4. Masks vs Gaussian vs Rademacher
5. Haar vs DCT (vs Daub)
6. What't the best performance I can achieve? How does it compare to the lit?
7. .
8. MSCE was done to solve the signal interpolation. In particular, to fix the problem with small support of wavelets while still using their power.
9. Problem: Slow. Solution: Parallelization.
10. Problem: Limited to signal interpolation. Solution: Use CS sensing matrices (but Cascade don't work yet).
11. Do we even need the cascade? What if we jump straight to the third scale.
12. Can we improve the performance of the interpolator?

We have obtained some results with our current implementation. The implementation uses the Haar wavelet transform at the first scale.

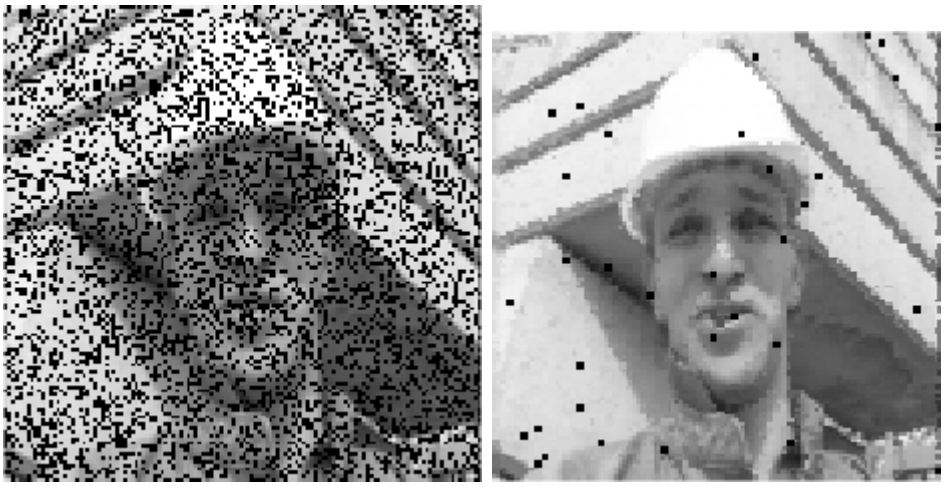


Fig. 7.1 Sample frame from corrupted video (left) and the reconstructed video (right)

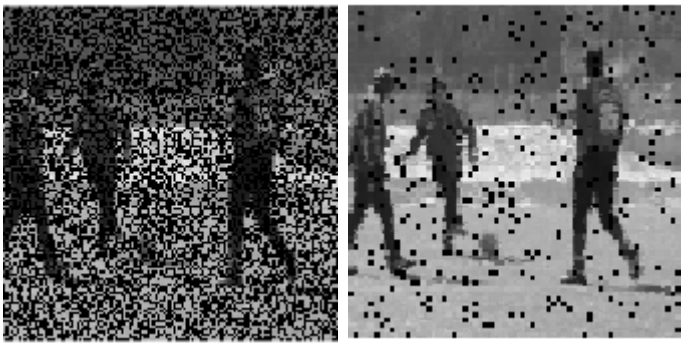


Fig. 7.2 Sample frame from corrupted video (left) and the reconstructed video (right)

Our example video has a resolution of 128 by 128 pixels and consists of a total of 64 frames. Thus, $r = 128$, $c = 128$ and $s = 64$. Note that even for such a relatively small sample, the size of the basis matrix Ψ is $(128 * 128 * 64) \times (128 * 128 * 64) = 1048576 \times 1048576$. Even in single precision, storing this matrix would require around 4 terrabytes.

For this reason, we have split the original input signal into $8 \times 8 \times 8$ blocks and perform the algorithm on the individual blocks.

In Figures 3.1 and 3.2, we have included a sample frame from the corrupted test video and the same frame after reconstruction.

In Figure 3.1, we corrupted the video by deleting 30% of the pixel values in the first frame and deleting the same pixel values in each subsequent frame (so the same pixels are missing in each frame). Figure 3.2 uses the same corruption scheme but we deleted 50% rather than 30% of pixel values.

These initial results are promising, though clearly there are still improvements to be made.

Chapter 8

Conclusion

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