Chapter 2

Compressive Sensing

2.1 The Compressive Sensing Problem

- *Intro to CS. Outline main ideas*
- *Conventional Sampling, Conversion*
- *Problem Statement for discrete. Img, Vids as unrolled vec. Sparsity -> Compression (adaptive). Why wasteful.*
- *CS is NON-ADAPTIVE. Sensing Matrix. Much fewer measurements.*
- *Against Fund Thm of LA -> resolve by sparsity (show a graphic).*
- *What Sensing matrix? We've only done masks, so maybe no point in talking about RIP etc -> just reference.*
- *Signal Recovery. L2 vs L0 vs L1 + Geometry?.*
- *We focus on Reconstruction. Side note to single-px cam (in sensing). Perhaps this should go in Ch1:Background*

This section is based on [3]. The problem to be solved can be formulated as follows: Let $\boldsymbol{x} \in \mathbb{R}^N$ be a signal of interest. We do not measure $\boldsymbol{x}$ directly and it is thus unknown. Instead, we have a measurement $\boldsymbol{y} \in \mathbb{R}^M$, with $M << N$, from which we want to reconstruct $\boldsymbol{x}$. The signals $\boldsymbol{x}$ and $\boldsymbol{y}$ are related as follows:

$$\boldsymbol{\Omega x} = \boldsymbol{y} \tag{2.1}$$

Fig. 2.1 Example of a signal pair $\boldsymbol{x}$ (left) and $\boldsymbol{y}$ (right). We wish to reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$.

where $\boldsymbol{\Omega}$ is a known $M \times N$ matrix referred to as the *sensing matrix*.

For example, in [3], the signal of interest $\boldsymbol{x}$ is an image, so that N is equal to the total number of pixels in the image and x_i is equal to the intensity of the corresponding pixel. However, we imagine that we have only access to a corrupted version of $\boldsymbol{x}$ in which random pixel values have been deleted. This is our measurement $\boldsymbol{y}$. See Figure 2.1 for an example. The sensing matrix $\boldsymbol{\Omega}$ corresponding to this scenario is obtained by taking the $N \times N$ identity matrix and deleting the rows that correspond to the missing entries in $\boldsymbol{x}$.

Compressive Sensing (CS) is a collection of signal processing techniques that allow for efficient *reconstruction* (and indeed *aquisition*) of such signals by solving the underdetermined system (2.1).

Of course, there are infinitely many solutions to an underdetermined system. In the CS framework, we seek to find a solution $\hat{\boldsymbol{x}}$ that is *sparsest in some domain*. By that, we mean that we want to find $\hat{\boldsymbol{x}}$ that satisfies (2.1), such that there exists a basis transformation of $\hat{\boldsymbol{x}}$ in which it has the smallest number of nonzero entries.

More concretely, we assume there exists a domain in which the desired signal $\boldsymbol{x}$ is sparse. I.e. there exists a $N \times N$ basis matrix Ψ such that $\boldsymbol{x} = \boldsymbol{\Psi w}$ and $\boldsymbol{w}$ is sparse.

The CS problem can then be expressed as follows:

$$\min ||\boldsymbol{w}||_0 \qquad \text{subject to} \qquad \boldsymbol{\Omega \Psi w} = \boldsymbol{y} \tag{2.2}$$

where $||.||$ denotes the l_0 norm, i.e. the number of nonzero components.

2.2 The Solution

2.2.1 Stable Measurement Matrix

Matrices

2.2.2 Reconstruction Algorithms

L0, L1, L2, Deterministic, Geometry

For a more detailed review of the CS framework, see [1].

References

[1] Candès, E. J. and Wakin, M. B. (2008). An introduction to compressive sampling. *Signal Processing Magazine, IEEE*, 25(2):21–30.

[2] DeVore, R. A., Jawerth, B., and Lucier, B. J. (1992). Image compression through wavelet transform coding. *Information Theory, IEEE Transactions on*, 38(2):719–746.

[3] Pilikos, G. (2014). Signal reconstruction using compressive sensing. MPhil thesis, University of Cambridge.

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