Chapter 2

Compressive Sensing

This section is based on [3]. The problem to be solved can be formulated as follows: Let $\boldsymbol{x} \in \mathbb{R}^N$ be a signal of interest. We do not measure $\boldsymbol{x}$ directly and it is thus unknown. Instead, we have a measurement $\boldsymbol{y} \in \mathbb{R}^M$, with $M << N$, from which we want to reconstruct $\boldsymbol{x}$. The signals $\boldsymbol{x}$ and $\boldsymbol{y}$ are related as follows:

$$\boldsymbol{\Omega x} = \boldsymbol{y} \tag{2.1}$$

where $\boldsymbol{\Omega}$ is a known $M \times N$ matrix referred to as the *sensing matrix*.

For example, in [3], the signal of interest $\boldsymbol{x}$ is an image, so that N is equal to the total number of pixels in the image and x_i is equal to the intensity of the corresponding pixel. However, we imagine that we have only access to a corrupted version of $\boldsymbol{x}$ in which random pixel values have been deleted. This is our measurement $\boldsymbol{y}$. See Figure **??** for an example. The sensing matrix $\boldsymbol{\Omega}$ corresponding to this scenario is obtained

Fig. 2.1 Example of a signal pair $\boldsymbol{x}$ (left) and $\boldsymbol{y}$ (right). We wish to reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$.

by taking the $N \times N$ identity matrix and deleting the rows that correspond to the missing entries in $\boldsymbol{x}$.

Compressive Sensing (CS) is a collection of signal processing techniques that allow for efficient *reconstruction* (and indeed *aquisition*) of such signals by solving the underdetermined system (2.1).

Of course, there are infinitely many solutions to an underdetermined system. In the CS framework, we seek to find a solution $\hat{\boldsymbol{x}}$ that is *sparsest in some domain.* By that, we mean that we want to find $\hat{\boldsymbol{x}}$ that satisfies (2.1), such that there exists a basis transformation of $\hat{\boldsymbol{x}}$ in which it has the smallest number of nonzero entries.

More concretely, we assume there exists a domain in which the desired signal $\boldsymbol{x}$ is sparse. I.e. there exists a $N \times N$ basis matrix Ψ such that $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{w}$ and $\boldsymbol{w}$ is sparse.

The CS problem can then be expressed as follows:

$$\min ||\boldsymbol{w}||_0 \qquad \text{subject to} \qquad \boldsymbol{\Omega}\boldsymbol{\Psi}\boldsymbol{w} = \boldsymbol{y} \tag{2.2}$$

where $||.||$ denotes the l_0 norm, i.e. the number of nonzero components.

For a more detailed review of the CS framework, see [1].

References

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