I. Bag of words (BoW) and distances

1. A: [1,1,1,1,0,1,0]
   B: [1,1,2,1,1,2,0]
   C: [1,0,1,0,0,1,1,]
2. L1 distance between A and B: |1-1|+|1-1|+|1-2|+|1-1|+|0-1|+|1-2|+|0-0|=3
   L1 distance between A and C: |1-1|+|1-0|+|1-1|+|1-0|+|0-0|+|1-1|+|0-1|=3
3. L1 normalization of A, B and C:
   Sum of A: |1|+|1|+|1|+|1|+|0|+|1|+|0|=5
   A = [1/5, 1/5, 1/5, 1/5, 0, 1/5, 0]
   Sum of B: |1|+|1|+|2|+|1|+|1|+|2|+|0| = 8
   B = [1/8, 1/8, 1/4, 1/8, 1/8, 1/4, 0]
   Sum of C: |1|+|0|+|1|+|0|+|0|+|1|+|1|= 4
   C=[1/4, 0, 1/4, 0, 0, 1/4, 1/4]

L1 distance between A and B: |1/5-1/8|+|1/5-1/8|+|1/5-1/4|+|1/5-1/8|+|0-1/8|+|1/5-1/4|+|0-0|=9/20

L2 distance between A and C: |1/5-1/4|+|1/5-0|+|1/5-1/4|+|1/5-0|+|0-0|+|1/5-1/4|+|0-1/4|=4/5

II. Histogram and Parzen window

1. First histogram
   A = [0,4,0,4,0]
   L1-normalized histogram = [0,1/2,0,1/2,0]
   B = [2,0,4,0,2]
   L1-normalized histogram= [1/4, 0, 1/2, 0, 1/4]
   L1 distance between A and B: |0-1/4|+|1/2-0|+|0-1/2|+|1/2-0|+|0-1/4|=2
2. Offset histogram:
   A = [2,2,2,2]
   L1-normalized histogram = [1/4, 1/4, 1/4, 1/4]
   B = [2,2,2,2]
   L1-normalized histogram = [1/4, 1/4, 1/4, 1/4]
   L1 distance between A and B: |1/4-1/4|+|1/4-1/4|+|1/4-1/4|+|1/4-1/4|=0
3. Narrow bin histogram:
   A = [0,2,2,0,0,2,2,0]
   L1-normalized histogram = [0, 1/4, 1/4, 0, 0, 1/4, 1/4, 0]
   B = [2,0,0,2,2,0,0,2]
   L1-normalized histogram = [1/4, 0, 0, 1/4, 1/4, 0, 0, 1/4]
   L1 distance between A and B: |0-1/4|+|1/4-0|+|1/4-0|+|0-1/4|+|0-1/4|+|1/4-0|+|1/4-0|+|0-1/4|=2
4. Kernel Density Estimation
   p(u=1.5) = 1/8*(k(1.2-1.5)+k(1.4-1.5)+ k(1.6-1.5)+ k(1.8-1.5)+ k(3.2-1.5)+ k(3.4-1.5)+ k(3.6-1.5)+ k(3.8-1.5))=1/8*(0.8+1.6+1.6+0.8)=0.6

p(u=2.5) = 1/8*(k(1.2-2.5)+k(1.4-2.5)+k(1.6-2.5)+k(1.8-2.5)+k(3.2-2.5)+k(3.4-2.5)+k(3.6-2.5)+k(3.8-2.5))=1/8*(0+0+0+0+0+0+0+0)=0

III. Covariance, z-score, whitening, and PCA

1. Covariance

$$x_1 = \begin{bmatrix} 20 \\ 5 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 8 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -6 \\ -3 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\mu = \left(\begin{bmatrix} 20 \\ 5 \end{bmatrix} + \begin{bmatrix} 8 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right) \div 4 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$(x_1 - \mu) \cdot (x_1' - \mu') = \left(\begin{bmatrix} 20 \\ 5 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right) \times \left(\begin{bmatrix} 20 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 13 \\ 4 \end{bmatrix} \times \begin{bmatrix} 13 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 169 & 52 \\ 52 & 16 \end{bmatrix}$$

$$(x_2 - \mu) \times (x_2' - \mu') = \left(\begin{bmatrix} 8 \\ -2 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right) \times \left(\begin{bmatrix} 8 & -2 \end{bmatrix} - \begin{bmatrix} 7 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

$$(x_3 - \mu) \times (x_3' - \mu') = \left(\begin{bmatrix} -6 \\ -3 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right) \times \left(\begin{bmatrix} -6 & -3 \end{bmatrix} - \begin{bmatrix} 7 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -13 \\ -4 \end{bmatrix} \times \begin{bmatrix} -13 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 169 & 52 \\ 52 & 16 \end{bmatrix}$$

$$(x_4 - \mu) \times (x_4' - \mu') = \left(\begin{bmatrix} 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right) \times \left(\begin{bmatrix} 6 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 & 3 \end{bmatrix}$$

$$(x_4 - \mu)\times(x_4' - \mu') = (\begin{bmatrix}6\\4\end{bmatrix} - \begin{bmatrix}7\\1\end{bmatrix}) \times (\begin{bmatrix}6 & 4\end{bmatrix} - \begin{bmatrix}7 & 1\end{bmatrix})$$

$$= \begin{bmatrix}-1\\3\end{bmatrix} \times \begin{bmatrix}-1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix}$$

$$C[d,d'] = \frac{1}{4}(\begin{bmatrix}169 & 52\\52 & 16\end{bmatrix} + \begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix} + \begin{bmatrix}169 & 52\\52 & 16\end{bmatrix} + \begin{bmatrix}1 & -3\\-3 & 9\end{bmatrix})$$

$$= \frac{1}{4}\begin{bmatrix}340 & 98\\98 & 50\end{bmatrix}$$

$$= \begin{bmatrix}85 & 24.5\\24.5 & 12.5\end{bmatrix}$$

2. Z-score normalization

$x_1 = \begin{bmatrix} 20 \\ 5 \end{bmatrix}, \; x_2 = \begin{bmatrix} 8 \\ -2 \end{bmatrix}, \; x_3 = \begin{bmatrix} -6 \\ -3 \end{bmatrix}, \; x_4 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$\mu[d] = [7 \; 1]^T$

$$\sigma_x = \sqrt{\frac{(20-7)^2 + (8-7)^2 + (-6-7)^2 + (6-7)^2}{4}}$$

$$= \sqrt{\frac{169 + 1 + 169 + 1}{4}}$$

$$= \sqrt{85}$$

$$= 9.2195$$

$$\approx 9.22$$

$$\sigma_y = \sqrt{\frac{(5-1)^2 + (-2-1)^2 + (-3-1)^2 + (4-1)^2}{4}}$$

$$= \sqrt{\frac{16 + 9 + 16 + 9}{4}}$$

$$= \sqrt{12.5}$$

$$= 3.5355$$

$$\approx 3.54$$

$z_{x_1}[d] = (20-7)/9.22 = 1.40997 \approx 1.41$

$z_{y_1}[d] = (5-1)/3.54 \approx 1.12994 \approx 1.13$

$z_{x_2}[d] = (8-7)/9.22 \approx 0.10845 \approx 0.11$

$z_{y_2}[d] = (-2-1)/3.54 \approx -0.84745 \approx -0.85$

$z_{x_3}[d] = (-6-7)/9.22 \approx -1.40997 \approx -1.41$

$z_{y_3}[d] = (-3-1)/3.54 \approx -1.12994 \approx -1.13$

$z_{x_4}[d] = (6-7)/9.22 \approx -0.10845 \approx -0.11$

$z_{y_4}[d] = (4-1)/3.54 \approx 0.84745 \approx 0.85$

z score for X: $\{[1.41, 1.13]^T, [0.11, -0.85]^T, [-1.41, -1.13]^T, [-0.11, 0.85]^T\}$

3. Unbiased and uniformly scaled

z score for X: $\{[1.41, 1.13]^T, [0.11, -0.85]^T, [-1.41, -1.13]^T, [-0.11, 0.85]^T\}$

$$\mu[d] = \left[\frac{1.41 + 0.11 + (-1.41) + (-0.11)}{4}, \frac{1.13 + (-0.85) + (-1.13) + (0.85)}{4}\right]$$

$$= [0, 0]$$

$$\sigma_x[d] = \sqrt{\frac{(1.41-0)^2 + (0.11-0)^2 + (-1.41-0)^2 + (-0.11-0)^2}{4}}$$

$$= \sqrt{\frac{4}{4}}$$

$$= 1$$

$$\sigma_y[d] = \sqrt{\frac{(1.13-0)^2 + (-0.85-0)^2 + (-1.13-0)^2 + (0.85-0)^2}{4}}$$

$$= \sqrt{\frac{4}{4}}$$

$$= 1$$

$$\sigma[d] = [1, 1]$$

4. Whitening

$x_1 = [20\ -5]^T, \quad x_2 = [8\ 2]^T, \quad x_3 = [-6\ 3]^T, \quad x_4 = [6\ -4]^T$

$$z_1 = C^{-0.5}(x_1 - \mu)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \left(\begin{bmatrix} 20 \\ -5 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.35 \\ -0.42 \end{bmatrix}$$

$$z_2 = C^{-0.5}(x_2 - \mu)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \left(\begin{bmatrix} 8 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.42 \\ 1.35 \end{bmatrix}$$

$$z_3 = C^{-0.5}(x_3 - \mu)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \left(\begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \begin{bmatrix} -13 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1.35 \\ 0.42 \end{bmatrix}$$

$$z_4 = C^{-0.5}(x_4 - \mu)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.133 & 0.096 \\ 0.096 & 0.418 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -0.42 \\ -1.35 \end{bmatrix}$$

$$Z = \{[1.35, -0.42]^T, [0.42\ 1.35]^T, [-1.35, 0.42]^T, [-0.42\ -1.35]^T\}$$