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Smooth Local-Path Planning for Autonomous Vehicles[1]

Abstract

We propose a new method for generating smooth paths for vehicle path planning. The problem of finding the "smoothest path" joining two given configurations is solved (a configuration is a position and a direction). Generally, we solve the problem using two "simple" path segments. The most important issue is how to define the smoothness cost of a path. We propose two distinct definitions: the first is the path curvature, and the second is the derivative of the path curvature. We use circular arcs with the first definition, and we use "cubic spirals" with the second for "simple curves" respectively. The set of cubic spirals has several advantages when used in smooth-path planning, one of which is curvature continuity. This theory also reveals other important concepts in robotics, such as "symmetric configurations" and "symmetric means." This algorithm has been successfully implemented on the autonomous mobile robot Yamabico-11 at the University of California at Santa Barbara and at the Naval Postgraduate School.

1. Introduction

An autonomous mobile robot has the obvious advantage of freedom in motion. While the robot's motion enables intelligent sensing, material handling, and other tasks, it can also result in positional uncertainty. We must make every effort to maintain its positional precision. If smooth paths are used, path-tracking tasks become easier and better localization is attained. The use of a smooth path decreases unwelcome curvature discontinuity and unnecessary stops, and thus lowers the possibility of slippage. Furthermore, smoother motion plans make faster navigation possible. This is why the smooth-path planning problem is important.

Komoriya and Hongo independently adopted path-description methods using a sequence of straight lines and circular arcs (Hongo et al. 1985; Komoriya, Tachi, and Tanie 1986). Cox also uses a similar method for local path planning (Cox 1988). The use of clothoid curves, or Cornu spirals, as proposed in Kanayama and Miyake (1986), are considered to be smooth in the sense that the path curvature is continuous. Further fundamental analysis in this article using cost functions reveals that the set of "cubic spirals" is theoretically more meaningful than the set of clothoids.

The goal of this research is to find the "smoothest" path joining a given pair (q_1, q_2) of configurations. Our general method is to find the best intermediate configuration q and to join two configuration pairs (q_1, q) and (q, q_2) by "symmetric curves." We propose two distinct definitions of "smoothness," and we define a "smoothness cost" in two ways. One definition is (the square of) the curvature $\kappa(s) = \dot{\theta}(s)$ of a path. Another is (the square of) the derivative of the curvature $\dot{\kappa}(s) = \ddot{\theta}(s)$ with respect to s, the path length. With the first definition, we simply obtain circular arcs as the solution set of symmetric (and simple) curves. With the second, we obtain a new class of curves, which we call *cubic spirals*. This naming comes from the fact that the tangent direction θ of a cubic spiral is a cubic function of path length s. This concept was introduced in the robotics field for the first time in Kanayama and Hartman (1989).

One by-product of this algorithm is that we have developed a method for describing a path with a minimal amount of information. This property is valuable in designing a high-level language for motion control of vehicles (Kanayama and Onishi 1991).

Horn solved the problem of finding the minimum energy path joining two configurations $(-1, 0, \pi/2)$ and $(1, 0, \pi/2)$ only for this special input case, where the

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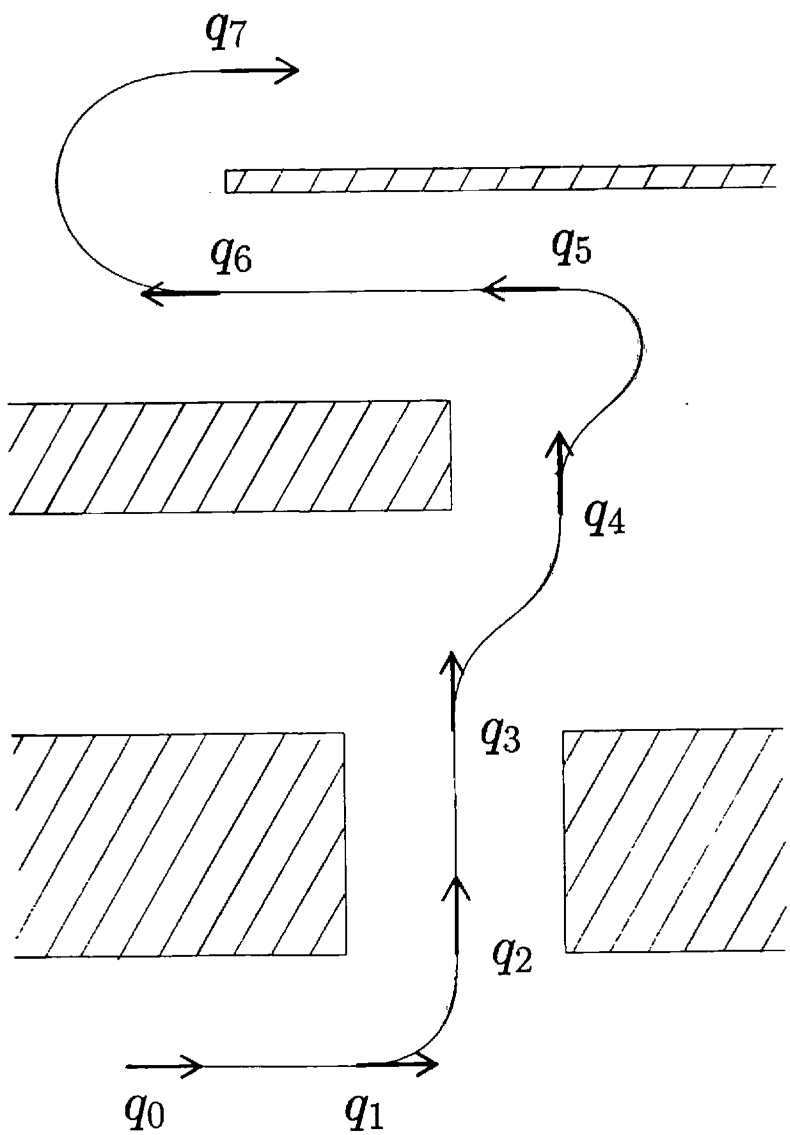

Fig. 1. A path described by configurations.

"cost of a path" is (the square of) the path curvature (Horn 1981). It is interesting that the resultant minimum cost path looks close to the cubic spiral solution given in Figure 12 in this article, although the general form of the solution, $\pm c\kappa = \sqrt{\cos(\psi - \phi)}$, is distinct from the solution this article proposes. In both analyses, calculus of variation is used, but the difference might have come from the distinct treatment of path length.

2. Problem Statements

2.1. Path Specification by Configurations

Suppose there is a 2D rigid-body mobile vehicle located on a 2D plane where a global Cartesian coordinate system is defined. We put a reference point and reference direction on this vehicle, which define the vehicle's position (x, y) and direction θ. We assume that the vehicle is able to move only in this reference direction, i.e., the vehicle is nonholonomic. A triple

$$q = (x, y, \theta) \tag{1}$$

is said to be the *configuration* of the vehicle, which represents the three degrees of freedom that a rigid-body vehicle possesses in the 2D plane (Lozano-Pérez 1983). For an arbitrary configuration q, $[q]$ denotes its position (x, y) and (q) its direction θ.

A path-description method using a sequence of configurations was proposed in Kanayama and Onishi (1991). For instance, a path shown in Figure 1 is described by a sequence $(q_0, q_1, \ldots, q_7)$ of eight configurations. Each path segment is constrained by an adjacent configuration pair (q_i, q_{i+1}).

2.2. Problem Definition

The problem is to find the "smoothest" directed path segment joining a given ordered pair of configurations $\left((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)\right)$, where $(x_1, y_1) \neq (x_2, y_2)$. [A directed path segment is said to *join* (x_1, y_1, θ_1) and (x_2, y_2, θ_2) if their end points are (x_1, y_1) and (x_2, y_2), and the tangential orientations at the end points are θ_1 and θ_2, respectively.] Therefore, one of the objectives of this article is to define the "smoothness" of paths.

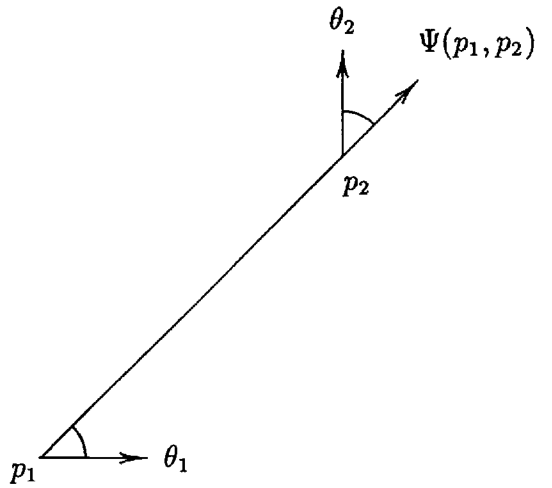

Fig. 2. Symmetric relation.

3. A Symmetric Configuration Pair

The notion of a "symmetric" configuration pair is essential in this theory. For two distinct points $p_1 \equiv (x_1, y_1)$ and $p_2 \equiv (x_2, y_2)$, let $\Psi(p_1, p_2)$ denote the direction

$$\Psi(p_1, p_2) = \text{atan2}(y_2 - y_1, x_2 - x_1) \tag{2}$$

of the directed segment $\overline{p_1, p_2}$ in the four quadrants. For instance, $\Psi\left((2,1),(3,2)\right) = \pi/4$ and $\Psi\left((3,2),(2,1)\right) = -3\pi/4$.

A configuration pair (q_1, q_2) is said to be *symmetric* if the mean value of (q_1) and (q_2) is equal to $\Psi([q_1],[q_2])$ or $\Psi([q_2],[q_1])$ (see Figure 2). For instance, in Figure 1, the pairs (q_0, q_1), (q_1, q_2), (q_2, q_3), (q_5, q_6), and (q_6, q_7) are symmetric; the others are not. The following configuration pairs are also symmetric, where a is an arbitrary positive real number.

$$\left((0,0,0),(\pm a,0,0)\right)$$
$$\left((0,0,0),(a,\pm a,\pm\pi/2)\right)$$
$$\left((0,0,0),(-a,\pm a,\mp\pi/2)\right)$$
$$\left((0,0,0),(0,\pm a,\pi)\right)$$

An angle-normalizing function Φ is defined as

$$\Phi(\theta) \equiv \theta - 2\pi \left\lfloor \frac{\theta + \pi}{2\pi} \right\rfloor. \tag{3}$$

For instance, $\Phi(5\pi/2) = \Phi(-3\pi/2) = \Phi(\pi/2) = \pi/2$.

LEMMA 1. If (q_1, q_2) are symmetric,

$$\tan\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{if } x_1 \neq x_2, \tag{4}$$

and

$$\Phi\left(\frac{\theta_1 + \theta_2}{2}\right) = \pm\frac{\pi}{2} \qquad \text{if } x_1 = x_2. \tag{5}$$

For a symmetric configuration pair (q_1, q_2), the *size* is the distance between the two points $[q_1]$ and $[q_2]$, and the *angle* is the difference between the two directions (q_1) and (q_2).

$$size(q_1, q_2) \equiv d([q_1],[q_2]) \tag{6}$$
$$angle(q_1, q_2) \equiv \Phi((q_2) - (q_1)) \tag{7}$$

where the function d denotes the Euclidean distance between two points. These two degrees of freedom completely characterize the relation of a symmetric configuration pair. Figure 3 shows a set of symmetric configuration pairs with the same sizes and distinct angles.

A configuration pair (q_1, q_2) is said to be *singular* if $(q_1) = (q_2) = \Psi([q_2],[q_1])$. For instance, a pair $\left((0,0,0),(-a,0,0)\right)$ is singular for any positive number a. If a configuration pair is singular, it is symmetric. The following proposition demonstrates one of the reasons why the symmetric property is so essential in the "configuration calculus." (In this article, a directed line segment is considered to be a special directed circular arc.)

PROPOSITION 1. Let (q_1, q_2) be a nonsingular configuration pair. Pair (q_1, q_2) is symmetric if and only if (q_1, q_2) is joined by some directed circular arc.

Therefore, from now on, we only deal with configuration pairs that are not singular.

4. The Symmetric Mean

Let (q_1, q_2) be an arbitrary pair of configurations with $[q_1] \neq [q_2]$. A configuration q is said to be a *symmetric mean* of q_1 and q_2 if both pairs (q_1, q) and (q, q_2) are symmetric. For instance, one of the symmetric means of

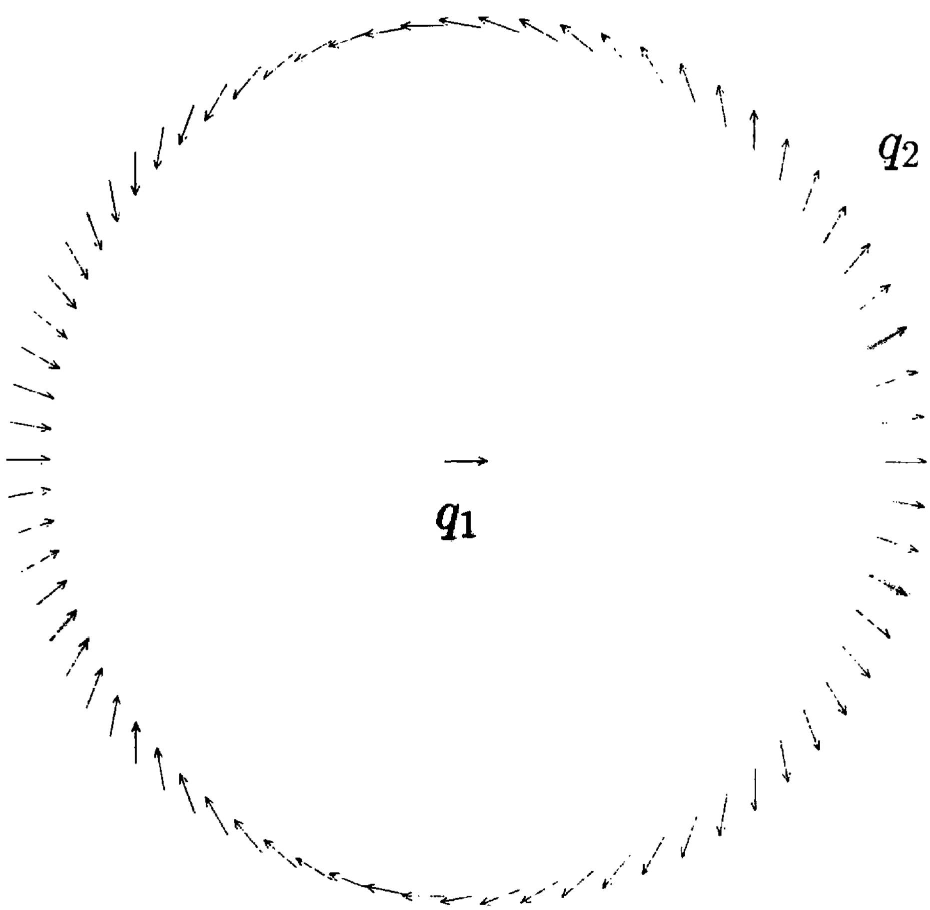

Fig. 3. Symmetric configurations with a unit distance.

a configuration pair $\big((0,0,0),(1,0,\pi/2)\big)$ is $(1/2,(1-\sqrt{2})/2,-\pi/4)$, because both pairs $\big((0,0,0),(1/2,(1-\sqrt{2})/2,-\pi/4)\big)$ and $\big((1/2,(1-\sqrt{2})/2,-\pi/4),(1,0,\pi/2)\big)$ are symmetric.

Now we consider the general problem of finding a symmetric mean of a given pair of configurations. Is there a symmetric mean for any pair? If there is, is it unique? The following proposition gives the answers to these questions. A pair (q_1, q_2) of configurations are said to be *parallel* if $(q_1) = (q_2)$.

PROPOSITION 2. Let $(q_1, q_2) \equiv \big((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)\big)$ be a pair of configurations with $(x_1, y_1) \neq (x_2, y_2)$. A configuration $q \equiv (x, y, \theta)$ is a symmetric mean of (q_1, q_2) if and only if q satisfies the following equations:

$$\big((x-x_1)(x-x_2)+(y-y_1)(y-y_2)\big)\tan\left(\frac{\theta_2-\theta_1}{2}\right) = (x-x_1)(y-y_2)-(x-x_2)(y-y_1), \tag{8}$$

$$\theta = \Psi(p_1, p) + \Phi\big(\Psi(p_1, p) - \theta_1\big). \tag{9}$$

Note that the points $[q_1]$ and $[q_2]$ themselves also satisfy eq. (8).

Proof: Let $q \equiv (x, y, \theta)$ be a symmetric mean of q_1 and q_2. By Lemma 1,

$$\frac{\theta+\theta_2}{2} = \arctan\left(\frac{y_2-y}{x_2-x}\right),$$

and

$$\frac{\theta_1+\theta}{2} = \arctan\left(\frac{y-y_1}{x-x_1}\right).$$

By taking the differences of both sides to cancel θ,

$$\frac{\theta_2-\theta_1}{2} = \arctan\left(\frac{y-y_2}{x-x_2}\right) - \arctan\left(\frac{y-y_1}{x-x_1}\right).$$

Applying the tangent function to both sides,

$$\tan\left(\frac{\theta_2-\theta_1}{2}\right) = \tan\left(\arctan\left(\frac{y-y_2}{x-x_2}\right) - \arctan\left(\frac{y-y_1}{x-x_1}\right)\right)$$

$$= \left(\frac{y-y_2}{x-x_2} - \frac{y-y_1}{x-x_1}\right) / \left(1 + \left(\frac{y-y_2}{x-x_2}\right)\left(\frac{y-y_1}{x-x_1}\right)\right)$$

$$= \frac{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)}{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}.$$

This is equal to eq. (8). Thus, eq. (8) is a necessary condition for q to be a symmetric mean. Eq. (9) is also a necessary condition, because of the definition of the symmetric property.

We describe a rough sketch of the sufficiency proof. If (q_1, q_2) is not parallel and p_c satisfies eq. (11), we can prove that

$$\Phi\Big(\Psi(p_c, p_2) - \Psi(p_c, p_1)\Big) = \Phi(\theta_2 - \theta_1). \qquad (10)$$

The symmetric property follows this relation.

If (q_1, q_2) is parallel, by eq. (9),

$$\frac{\theta_1+\theta}{2} = \frac{\theta_2+\theta}{2}$$
$$= \frac{\theta_1 + \Psi(p_1,p) + \Phi\Big(\Psi(p_1,p) - \theta_1\Big)}{2}$$
$$= \frac{\theta_1 + \Psi(p_1,p_2) + \Phi\Big(\Psi(p_1,p_2) - \theta_1\Big)}{2}$$
$$= \frac{\Psi(p_1,p_2) + \Psi(p_1,p_2)}{2}$$
$$= \Psi(p_1,p_2).$$

Therefore, q is a symmetric mean of q_1 and q_2. □

PROPOSITION 3. If a configuration $q \equiv (x, y, \theta)$ is a symmetric mean of

$$(q_1, q_2) = \Big((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)\Big),$$

then (I) If (q_1, q_2) is not parallel, the locus (x, y) of the symmetric means is a circle with its center at

$$p_c = (x_c, y_c) = \left(\frac{x_1+x_2-D(y_2-y_1)}{2}, \frac{y_1+y_2+D(x_2-x_1)}{2}\right), \qquad (11)$$

where

$$D \equiv \cot\left(\frac{\theta_2-\theta_1}{2}\right). \qquad (12)$$

The radius of the circle is the distance $d(p_c, [q_1])$ between the center and $[q_1]$. (II) If (q_1, q_2) is parallel, the locus (x, y) of the symmetric mean becomes a line,

$$(x-x_1)(y-y_2)-(x-x_2)(y-y_1) = 0. \qquad (13)$$

The proof of this proposition is straightforward from eq (8). For instance, if $(q_1, q_2) \equiv \Big((0,0,0), (1,0,\pi/2)\Big)$, eq (8) becomes $x^2 + y^2 - x - y = 0$. And hence,

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}. \qquad \textbf{(14)}$$

This represents a circle with a center of $(1/2, 1/2)$ and radius of $1/\sqrt{2}$ (Figure 4). The symmetric mean $(1/2, (1-\sqrt{2})/2, -\pi/4)$ given earlier as an example is the bottom one among the symmetric means in this figure. As another example, if $(q_1, q_2) \equiv \Big((0,0,0), (2,0,0)\Big)$ (which is parallel), the locus of symmetric means is

$$x - 2y = 0, \qquad (15)$$

which is shown in Figure 5. As shown in Figures 4 and 5, the locus of symmetric means is a circle or a line.

However, not all of these configurations are used in solving the smooth-path planning problem. Only "proper" symmetric mean configurations will be used. (If nonproper symmetric mean configurations are used, the resultant paths are extremely unnatural.) In parallel cases, a symmetric mean q is said to be proper if $[q]$ lies between $[q_1]$ and $[q_2]$. In nonparallel cases, let $\gamma_1 \equiv \Psi(p_c, [q_1])$, $\gamma_2 \equiv \Psi(p_c, [q_2])$, and $\gamma \equiv \Psi(p_c, [q])$. A symmetric mean q is said to be proper if

$$\Phi^+(\gamma - \gamma_1) < \Phi^+(\gamma_2 - \gamma_1), \qquad (16)$$

where the function Φ^+ (read as "positively normalize") is an angle-normalization function that maps an angle α to an angle in the range of $[0, 2/\pi]$ with modulo $2/\pi$. Its actual definition is

$$\Phi^+(\alpha) \equiv \alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor. \qquad (17)$$

5. Simple Curves

We deal with directed curves for vehicle-motion planning. A curve in this article is supposed to have a unique tangent direction θ at every point (x, y) on it. Therefore, a configuration (x, y, θ) is defined at every point in a natural manner. A curve with a finite arc length is said to be *simple* if its end-configuration pair (q_s, q_g) is symmetric. Recollect that every circular arc is simple.

By the result of the previous section, it is known that we can join any configuration pair using one or two simple curves (for instance, using circular arcs). However, there are two remaining questions: (1) Which class of simple curves is appropriate to use? and (2) If a given configuration pair is not symmetric, how do we choose the best specific symmetric mean among the infinitely many possibilities given by Proposition 2? Both questions will be answered through definitions of smoothness costs for curves.

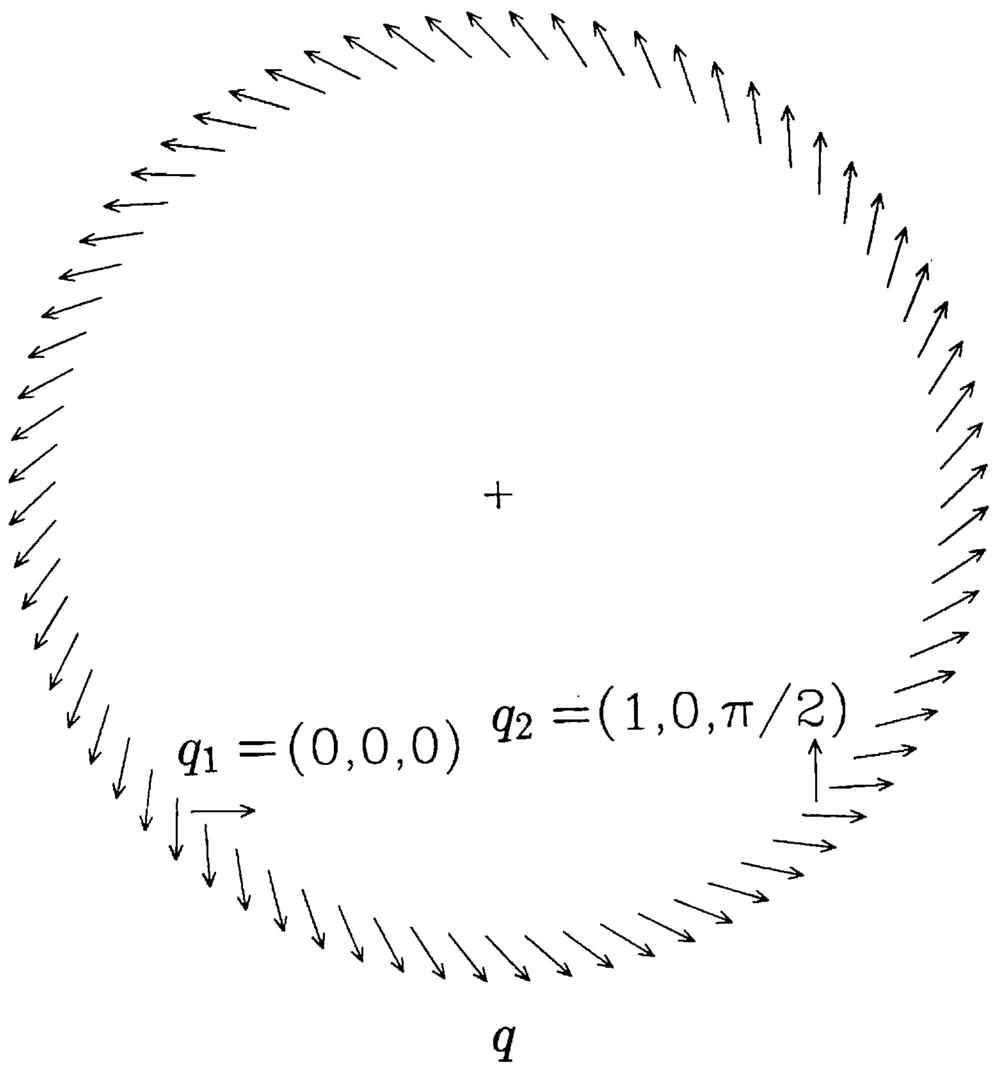

Fig. 4. Symmetric means: circular case.

5.1. Representation of Curves

In this article, we define a (directed) curve Π with a finite arc length $\ell(> 0)$ by a triple

$$\Pi \equiv (\ell, \kappa, q_0), \tag{18}$$

where $\kappa : [0, \ell] \rightarrow \Re$ is its curvature and $q_0 \equiv (x_0, y_0, \theta_0)$ is its initial configuration (Lipshutz 1969). This representation is called the *natural equation* of a curve. Its direction θ and position (x, y) at an arc length s are evaluated by

$$\theta(s) = \theta_0 + \int_0^s \kappa(t)dt, \tag{19}$$

$$x(s) = x_0 + \int_0^s \cos\theta(t)dt, \text{ and} \tag{20}$$

$$y(s) = y_0 + \int_0^s \sin\theta(t)dt. \tag{21}$$

At the initial point (x_0, y_0), s is defined as 0. A configuration $q(s) = \left(x(s), y(s), \theta(s)\right)$ is naturally defined by this set of simultaneous equations. The first configuration, $q(0)$, and the last, $q(\ell)$, are said to be the *end configurations* of Π.

5.2. Symmetric Properties in Configuration and Curvature

A curvature function κ of a path Π is said to be *symmetric* if

$$\kappa(s) = \kappa(\ell - s), \text{ for all } s \in [0, \ell]. \tag{22}$$

PROPOSITION 4. Let a path $\Pi \equiv (\ell, \kappa, q_0)$. If κ is symmetric, Π is simple.

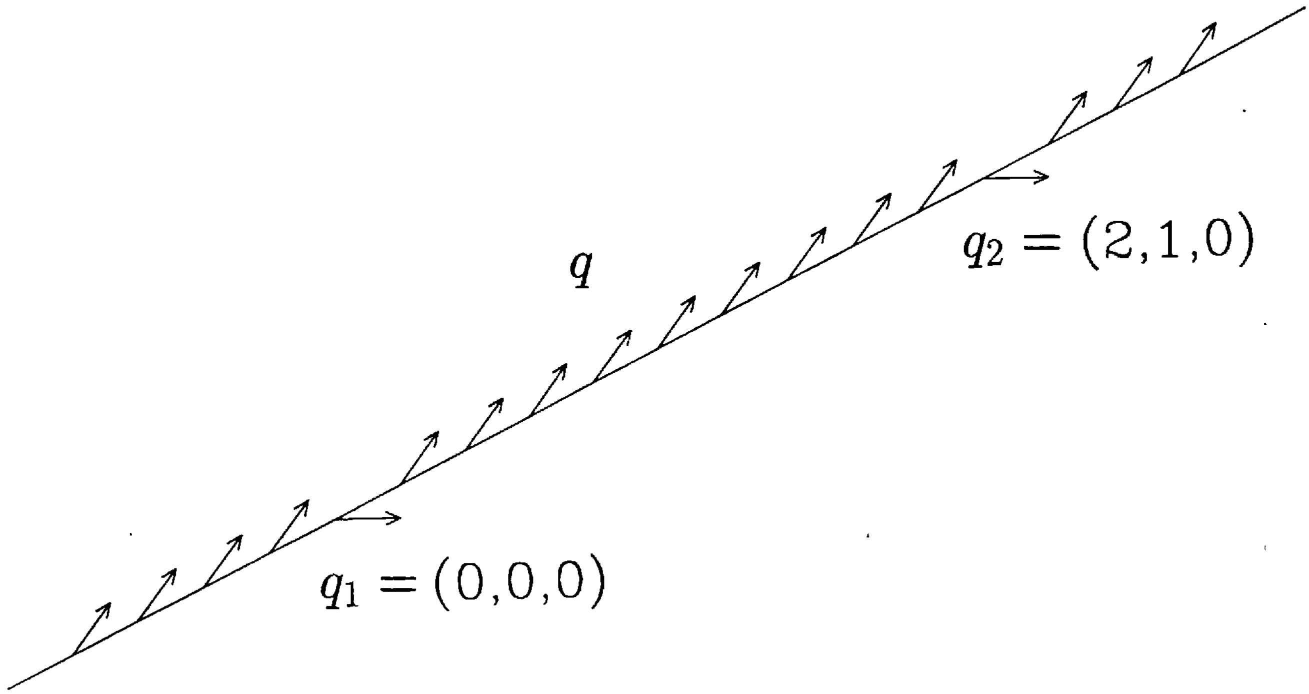

Fig. 5. Symmetric means: linear case.

Proof: Let us define a new coordinate system (x', y', θ') with the origin at $(x(\ell/2), y(\ell/2))$ and with the new x-axis as the direction $\theta(\ell/2)$ in the original coordinate system. Furthermore, a new path length s' is defined as

$$s' \equiv s - \ell/2. \tag{23}$$

Then

$$x'(0) = y'(0) = \theta'(0) = 0 \tag{24}$$

and

$$\kappa(s') = \kappa(-s') \text{ for all } s' \in [-\ell/2, \ell/2]. \tag{25}$$

First, we prove that $\theta'(s') = -\theta'(-s')$ for any $s' \in [-\ell/2, \ell/2]$.

$$\theta'(s') = \int_0^{s'} \kappa(s')ds' = \int_{\ell/2}^{\ell/2+s'} \kappa(s)ds = \int_{\ell/2-s'}^{\ell/2} \kappa(s)ds$$
$$= -\int_{\ell/2}^{\ell/2-s'} \kappa(s)ds = -\int_0^{-s'} \kappa(s')ds' = -\theta'(-s').$$

Therefore

$$y'(\ell/2) = \int_0^{\ell/2} \sin\theta'(s')ds' = \int_0^{\ell/2} \sin(-\theta'(-s'))ds'$$
$$= \int_0^{-\ell/2} \sin\theta'(t)dt = y'(-\ell/2),$$

where $t \equiv -s'$. Therefore,

$$\frac{\theta'(-\ell/2) + \theta'(\ell/2)}{2} = 0 = \arctan\left(\frac{y'(\ell/2) - y'(-\ell/2)}{x'(\ell/2) - x'(-\ell/2)}\right),$$

and the end-configuration pair is symmetric by eq. (4) in this new coordinate system. Since the symmetric property is invariant with coordinate transformations, this proposition is proved. □

Therefore, any symmetric curvature produces a simple curve. Obviously, a constant curvature

$$\kappa(s) = A \text{ for all } s \in [0, \ell] \tag{26}$$

is symmetric, where A is a constant. This is, in fact, a directed circular arc. Also, another "triangle" curvature function

$$\kappa(s) = \begin{cases} As & \text{if } 0 \le s \le \ell/2 \\ A(\ell - s) & \text{if } \ell/2 < s \le \ell \end{cases} \tag{27}$$

is symmetric (A is a constant). This piecewise linear curvature generates a clothoid pair (Kanayama and Miyake 1986). Thus, the set of all the directed circular arcs and the set of all the clothoid pairs are candidates of simple curves, and may be used in solving this smooth-path planning problem.

6. The Smoothest Curves

6.1. Smoothness Cost

In this section, we define the smoothness cost of a curve as a function of κ. One candidate is the curvature itself, and the other is the derivative $d\kappa/ds$ of curvature (actually, the square of each). A rationale for taking the curvature is as follows. Assuming a constant speed for a vehicle, the centripetal (lateral) acceleration

$$\frac{mv^2}{r} = mv^2\kappa \tag{28}$$

of the vehicle is proportional to its curvature κ, where m is its mass, r is the radius of the circular arc representing the current short portion of the curve, and v is its constant speed. A larger centripetal force may cause slippage and also may make its passenger uncomfortable. From this standpoint, the cost of smoothness of a path Π can be defined as

$$cost_0(\Pi) \equiv \int_0^\ell F_0 ds = \int_0^\ell \kappa^2 ds. \tag{29}$$

On the other hand, there might be another mathematical model whereby a constant lateral centripetal force is tolerable, but its change is not. The variation of the lateral acceleration may cause energy loss and discomfort to passengers on the vehicle. Assuming a constant-speed motion, the time derivative of the acceleration given by eq. (28) is

$$\frac{d}{dt}(mv^2\kappa) = mv^2\frac{d\kappa}{dt} = mv^2\frac{d\kappa}{ds}\frac{ds}{dt} = mv^3\frac{d\kappa}{ds}. \tag{30}$$

From this observation, another definition for smoothness cost becomes

$$cost_1(\Pi) \equiv \int_0^\ell F_1 ds = \int_0^\ell \left(\frac{d\kappa}{ds}\right)^2 ds. \tag{31}$$

Using these cost functions, let us investigate a general curvature function that minimizes the cost for each smoothness cost definition.

6.2. The Circular Arc Solution

First, we will find curves that minimize the curvature cost $cost_0(\Pi)$.

PROPOSITION 5. If the smoothness cost of a path $\Pi \equiv (\ell, \kappa, q_0)$ is defined by eq. (29), and if its path length ℓ is fixed, a curve with a constant curvature function

$$\kappa(s) = c_0 = constant \tag{32}$$

minimizes the path cost.

Proof: Since $F_0 = \kappa^2 = \dot{\theta}^2$, by the theorem of calculus of variations (Weinstock 1974), we have

$$\frac{\partial F_0}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial F_0}{\partial \dot{\theta}}\right) = 0.$$

Hence,

$$-2\frac{d\dot{\theta}}{ds} = -2\frac{d^2\theta}{ds^2} = 0.$$

By simply integrating this, eq. (32) is obtained. □

Then it is shown that if the curvature is taken as a smoothness cost, circular arcs are optimal solutions. Since each circular arc is simple, we are able to use it as the solution of a given symmetric configuration.

Let us obtain the characteristics of a circular arc as functions of the symmetric configuration's size d and angle α.

PROPOSITION 6. Let (q_1, q_2) be a symmetric configuration pair with $d = size(q_1, q_2)$ and $\alpha = angle(q_1, q_2)$. The length ℓ, the curvature κ, and the cost $cost_0(\Pi)$ of the circular arc solution Π for the configuration pair are given by the following:

$$\ell = \begin{cases} d & \text{if } \alpha = 0 \\ d\alpha/(2\sin(\alpha/2)) & \text{if } \alpha \neq 0, \end{cases} \tag{33}$$

$$\kappa = \frac{2\sin(\alpha/2)}{d}, \text{ and} \tag{34}$$

$$cost_0(\Pi) = \frac{2\alpha\sin(\alpha/2)}{d}. \tag{35}$$

Proof: When $\alpha = 0$, the proposition is obvious. Otherwise, by a geometrical analysis, the (signed) radius r of the solution-directed circle is

$$r = \frac{d/2}{\sin(\alpha/2)}.$$

Therefore, $\kappa = 1/r = 2\sin(\alpha/2)/d$. The length ℓ is obtained as $|\alpha r| = \alpha r$, which is equal to eq. (33). Its smoothness cost is

$$cost_0(\Pi) = \int_0^\ell \kappa^2 ds = \ell\kappa^2 = \alpha r\kappa^2 = \alpha\kappa = \frac{2\alpha\sin(\alpha/2)}{d}.$$

□

6.3. The Cubic Spiral Solution

If we define the derivative of curvature as the path-smoothness cost, we obtain the following result.

PROPOSITION 7. If the smoothness cost of a path $\Pi \equiv (\ell, \kappa, q_0)$ is defined as eq. (31), and if a path length ℓ is fixed, then a curve with a quadratic curvature function of s

$$\kappa(s) = c_0 s^2 + c_1 s + c_2 \tag{36}$$

minimizes the path cost.

Proof: Since $F_1 = \dot{\kappa}^2 = \ddot{\theta}^2$, we have

$$\frac{\partial F_1}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial F_1}{\partial \dot{\theta}}\right) + \frac{d^2}{ds^2}\left(\frac{\partial F_1}{\partial \ddot{\theta}}\right) = 0.$$

Therefore,

$$2\frac{d^2\ddot{\theta}}{ds^2} = 2\frac{d^4\theta}{ds^4} = 0.$$

Eq. (36) is obtained by simply integrating this three times. □

Then, if the derivative of curvature is taken, curves with quadratic curvature functions are optimal. Since their direction functions θ are cubic in this case, we name the class of curves *cubic spirals*. An example of a cubic spiral and its curvature and direction functions are shown in Figures 6 and 7.

If there exist s_1 and s_2 such that $\kappa(s_1) = \kappa(s_2) = 0$ with $s_1 < s_2$, the points $p(s_1)$ and $p(s_2)$ on the cubic spiral are inflection points where the sign of the curvature changes. Since an entire cubic spiral has an infinite length, we need to cut it to obtain a finite portion whose end configurations are symmetric. The best way to do this is to cut it at the inflection points, s_1 and s_2. If we take $s_1 \equiv 0$ and its length $\ell \equiv s_2 - s_1$, the curvature function becomes

$$\kappa(s) = As(\ell - s) \text{ for all } s \in [0, \ell] \tag{37}$$

where A is a constant. Obviously, this curvature function is symmetric. We will call this finite portion a cubic spiral also. Since the curvature at each end point is 0, the path curvature becomes continuous if a cubic spirals are used to construct a complex path. This is the most significant advantage of the cubic spirals. Figure 8 shows cubic spirals with the same length ℓ and distinct angles α.

Let us evaluate the constant A in eq. (37).

LEMMA 2. The curvature function of a cubic spiral with a length of ℓ and an angle of α is

$$\kappa(s) = \frac{6\alpha}{\ell^3} s(\ell - s). \tag{38}$$

Proof: Since the integration of curvature equals the difference in the directions of the end configurations, by eq. (37),

$$\alpha = \int_0^\ell \kappa ds = \int_0^\ell As(\ell - s)ds = A\int_0^\ell (\ell s - s^2)ds$$
$$= A\left[\frac{\ell}{2}s^2 - \frac{s^3}{3}\right]_0^\ell = \frac{A\ell^3}{6}.$$

Therefore, $A = 6\alpha/\ell^3$. □

The difficult part of using the cubic spiral is that even if a cubic spiral's size and angle are given, there is no closed form for evaluating its length—a task that was straightforward in the circle case (eq. (33). The following equation is the best we have.

LEMMA 3. If the length of a cubic spiral is 1, its size is given by

$$D(\alpha) \equiv 2\int_0^{1/2} \cos\left(\alpha\left(\frac{3}{2} - 2t^2\right)t\right) dt, \tag{39}$$

where α is its angle.

Proof: By eq. (38) and by the fact that $\ell = 1$,

$$\kappa(s) = 6\alpha s(1 - s).$$

Let us use the same transformation adopted in the proof of Proposition 4. Since $s' \equiv s - 1/2$, the curvature is

$$\kappa(s') = 6\alpha\left(s' + \frac{1}{2}\right)\left(\frac{1}{2} - s'\right) = 6\alpha\left(\frac{1}{4} - s'^2\right).$$

Therefore, the tangent direction $\theta'(s')$ is

$$\theta'(s') = \int_0^{s'} 6\alpha\left(\frac{1}{4} - t^2\right) dt = 6\alpha\left[\frac{t}{4} - \frac{t^3}{3}\right]_0^{s'}$$
$$= 6\alpha\left[\frac{s'}{4} - \frac{s'^3}{3}\right] = \alpha\left(\frac{3}{2} - 2s'^2\right)s'.$$

Therefore, the x-coordinate of one of the end points is

$$x'\left(\frac{1}{2}\right) = \int_0^{1/2} \cos\left(\alpha\left(\frac{3}{2} - 2t^2\right)t\right) dt,$$

since the distance of both end points is equal to $x'(1/2) - x'(-1/2) = 2x'(1/2)$. □

Figure 9 shows the function $D(\alpha)$. The nonexistence of a closed form for D is a similar situation to the case of clothoid curves or Cornu spirals (Kanayama and Miyake 1986). However, if we precalculate this well-defined D function, we can solve all the problems related to the cubic spirals (as shown later).

PROPOSITION 8. Let (q_1, q_2) be a symmetric configuration pair with $d = size(q_1, q_2)$ and $\alpha = angle(q_1, q_2)$. The length ℓ, the curvature κ, and the cost $cost_1(\Pi)$ of the cubic spiral solution Π for the configuration pair are given by the following:

$$\ell = \frac{d}{D(\alpha)}, \tag{40}$$

$$\kappa = \frac{6\alpha D(\alpha)^3}{d^3} s\left(\frac{d}{D(\alpha)} - s\right), \text{ and} \tag{41}$$

$$cost_1(\Pi) = \frac{12\alpha^2}{\ell^3} = \frac{12\alpha^2 D(\alpha)^3}{d^3}. \tag{42}$$

Proof: Since all cubic spirals are similar,

$$\frac{\ell}{d} = \frac{1}{D(\alpha)}.$$

Therefore, eq. (40) for ℓ is obtained. Substituting this ℓ into eq. (38) gives eq. (41). To evaluate the cost, we need to derive an equation for $\dot{\kappa}$:

$$\frac{d\kappa}{ds} = \frac{6\alpha}{l^3}\frac{d}{ds}(s(l - s)) = \frac{6\alpha}{l^3}(\ell - 2s).$$

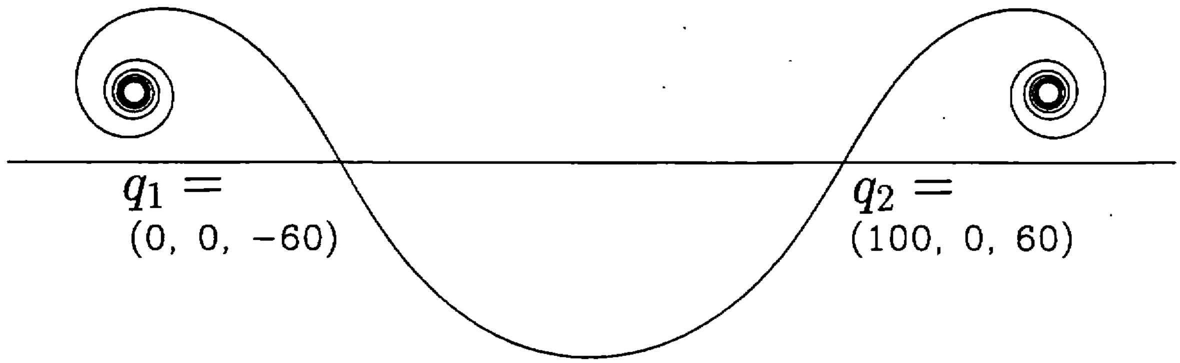

Fig. 6. Entire cubic spiral.

Therefore, its cost is

$$
\begin{aligned}
cost_1(\Pi) &= \int_0^\ell \left(\frac{d\kappa}{ds}\right)^2 ds = \frac{36\alpha^2}{\ell^6}\int_0^\ell (\ell^2 - 4\ell s + 4s^2)ds \\
&= \frac{36\alpha^2}{\ell^6}\left[\ell s^2 - 2\ell s^2 + \frac{4}{3}s^3\right]_0^\ell \\
&= \frac{36\alpha^2}{\ell^6}\left(\ell^3 - 2\ell^3 + \frac{4}{3}\ell^3\right) \\
&= \frac{12\alpha^2}{\ell^3} = \frac{12\alpha^2 D(\alpha)^3}{d^3}.
\end{aligned}
$$

□

Figure 10 shows how three distinct simple curves (a circular arc, a clothoid pair, and a cubic spiral) can be used to make a 90° turn. Figure 11 compares the curvature functions of the curves shown in the previous figure. All the curvature functions are symmetric. Note that the area between each curvature function and the s-axis is equal to $\pi/2$, because the integration of κ is equal to α (the difference in θ). Maximum curvature of the cubic spiral is less than that of the clothoid curve. This is another advantage of using cubic spirals. Figure 12 demonstrates the differences of using the three distinct simple curves to make a U-turn.

Actually, we propose the use of the second definition of smoothness cost, $i = 1$. The implementation of this algorithm on the autonomous robot Yamabico-11 is done with the curvature-derivative cost.

7. Solution

7.1. A Global Description of the Algorithm

Now we are ready to solve the original problem stated in Section 2.2. A pair of configurations (q_1, q_2) (with $[q_1] \neq [q_2]$) and a cost function $cost_i$ are given ($i = 1, 2$). Our general algorithm is:

1. If (q_1, q_2) is symmetric, use one simple curve.
2. If (q_1, q_2) is not symmetric, find a proper symmetric mean q which minimizes the "total smoothness cost" of the two simple curves.

Since the first case is trivial, we will discuss the second case. The most important step is determining how to find the best symmetric mean. There are two distinct definitions for smoothness cost: $i = 0$ means that the curvature cost of equation (29) is adopted, and that the set of directed circular arcs is used; $i = 1$ means that the curvature-derivative cost of eq. (31) is adopted, and that the set of cubic spirals is used.

We stipulate that $\Pi_0(q, q')$ and $\Pi_1(q, q')$ mean a circular arc and a cubic spiral defined by the end configurations q and q', respectively, if (q, q') is a symmetric configuration pair. Furthermore, if configuration pairs (q, q') and (q', q'') are symmetric, $\Pi_i(q, q', q'')$ is a path that is a concatenated path of the simple paths $\Pi_i(q, q')$ and $\Pi_i(q', q'')$. A symmetric mean q satisfies eqs. (8) and (9). Using eqs. (35) and (42), the total cost for both cases ($i = 0$ and 1) becomes

$$
\begin{aligned}
&cost_0\Big(\Pi_0(q_1, q, q_2)\Big) \\
&\quad = cost_0\Big(\Pi_0(q_1, q)\Big) + cost_0\Big(\Pi_0(q, q_2)\Big) \\
&\quad = 2\left(\frac{\alpha_1 \sin(\alpha_1/2)}{d_1} + \frac{\alpha_2 \sin(\alpha_2/2)}{d_2}\right), \qquad (43)
\end{aligned}
$$

and

$$
\begin{aligned}
&cost_1\Big(\Pi_1(q_1, q, q_2)\Big) \\
&\quad = cost_1\Big(\Pi_1(q_1, q)\Big) + cost_1\Big(\Pi_1(q, q_2)\Big) \qquad (44) \\
&\quad = 12\left(\frac{{\alpha_1}^2}{{\ell_1}^3} + \frac{{\alpha_2}^2}{{\ell_2}^3}\right) \qquad (45) \\
&\quad = 12\left(\frac{{\alpha_1}^2 D(\alpha_1)^3}{{d_1}^3} + \frac{{\alpha_2}^2 D(\alpha_2)^3}{{d_2}^3}\right), \qquad (46)
\end{aligned}
$$

where

$$\alpha_1 \equiv \Phi(\theta - \theta_1), \quad \alpha_2 \equiv \Phi(\theta_2 - \theta),$$

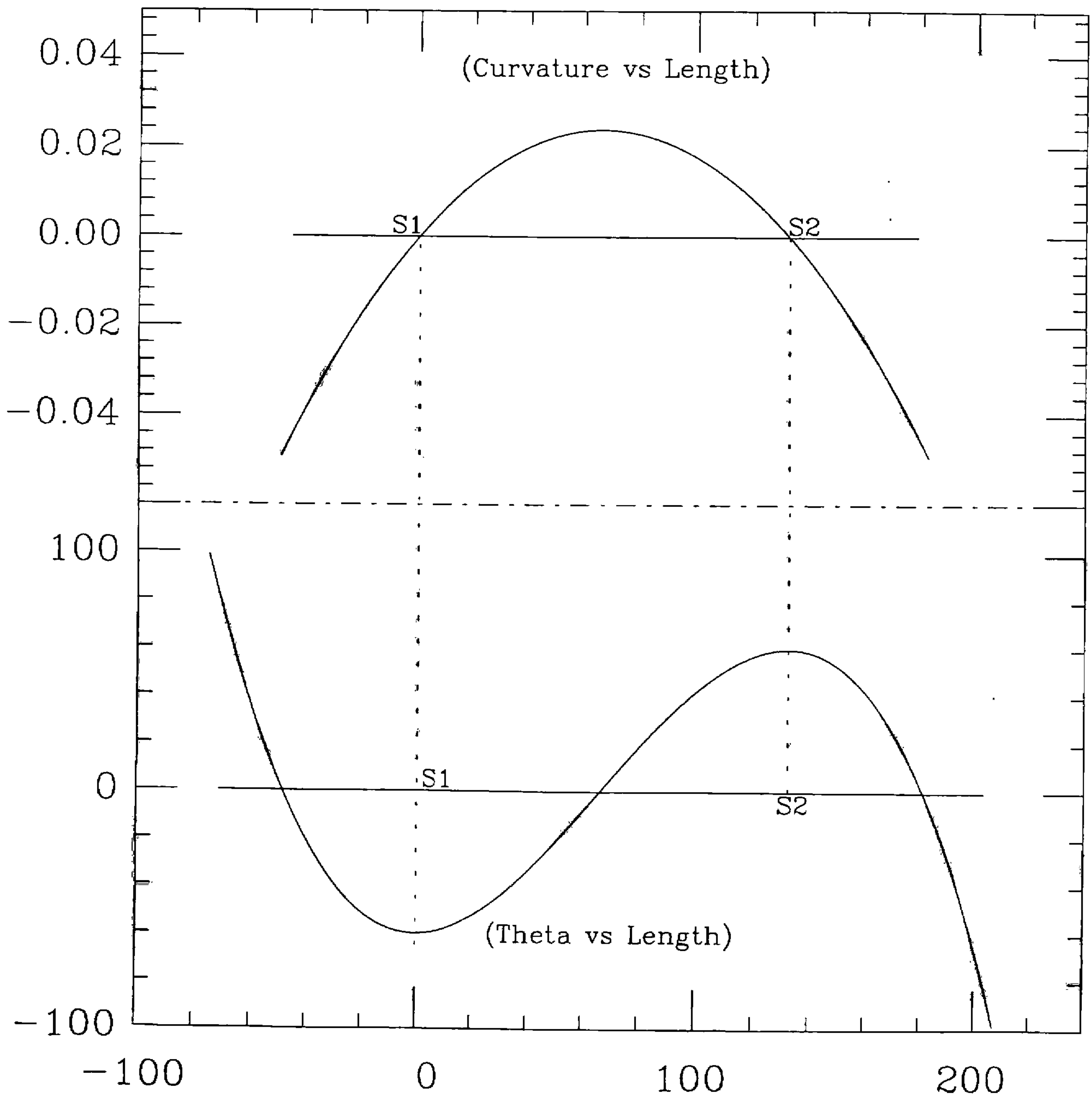

Fig. 7. Curvature and tangent direction of a cubic spiral.

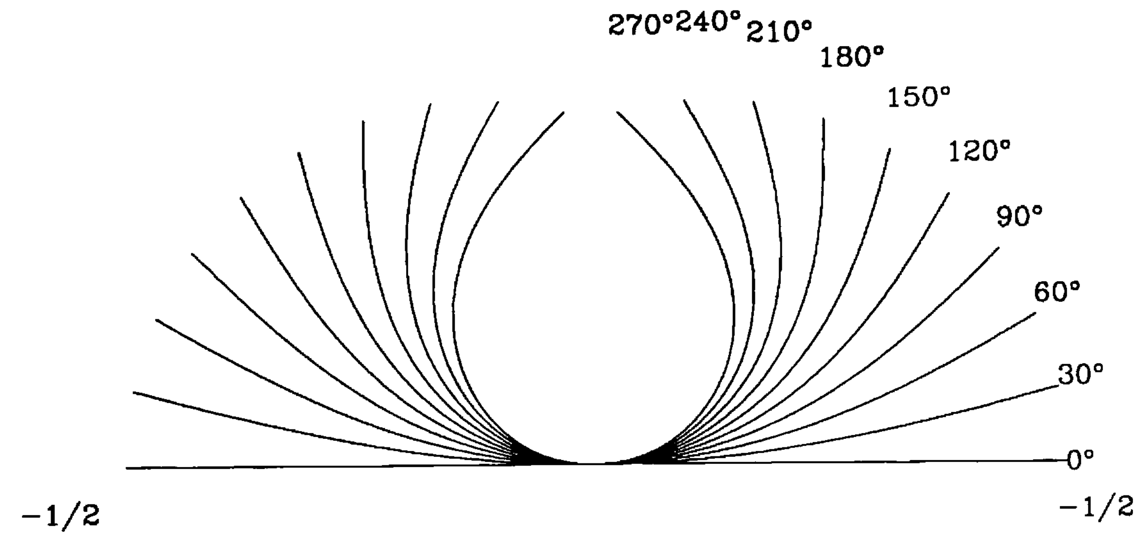

Fig. 8. Cubic spirals with distinct angles.

$$d_1 \equiv d([q_1],[q]), \text{ and } d_2 \equiv d([q],[q_2]). \quad (47)$$

The total cost of $cost_i\Big(\Pi_i(q_1, q, q_2)\Big)$ becomes infinity for each i, because d_1 and d_2 appear in the denominators of either of the two terms in the total cost equations. It is obvious that the cost is always positive. Therefore, there exists at least one minimum value of the total cost.

7.2. The Parallel Case

In a parallel case, the optimal solution is obtained in a closed form for each i (as shown in this section).

LEMMA 4. If (q_1, q_2) is a parallel configuration pair and q is one of its symmetric means, the direction of q is a constant,

$$(q) = \Psi([q_1],[q_2]) + \Phi(\Psi([q_1],[q_2]) - (q_1)), \quad (48)$$

and the angles of the two symmetric configuration pairs are given by

$$angle(q_1, q) = 2\Phi(\Psi([q_1],[q_2]) - (q_1)), \text{ and} \quad (49)$$

$$angle(q, q_2) = -angle(q_1, q). \quad (50)$$

Proof: Use eq. (9). □

As shown in the following proposition, the middle point of $[q_1]$ and $[q_2]$ gives the optimal solution in parallel cases with both smoothness cost definitions:

PROPOSITION 9. Let $(q_1, q_2) \equiv \Big((x_1, y_1, \theta_1), (x_2, y_2, \theta_2)\Big)$ be a pair of parallel configurations with $\theta_1 = \theta_2$. A symmetric mean

$$q_m \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \Psi([q_1],[q_2]) + \Phi(\Psi([q_1],[q_2]) - \theta_1)\right) \quad (51)$$

using the middle point of q_1 and q_2 minimizes the total cost $cost_i\Big(\Pi_i(q_1, q, q_2)\Big)$ for each i.

Proof: Let q denote a proper symmetric mean configuration. By Lemma 4, $angle(q_1, q)$ is a constant. Let us name this α. Then $angle(q, q_2) = -\alpha$. Furthermore, let $w_0 \equiv d([q_1],[q_2])$ and $w \equiv d([q_1],[q])$. Note that if q is proper, $0 < w < w_0$ and $d([q],[q_2]) = w_0 - w$. Now, evaluate the total cost for each case using eqs. (43) and (46).

1. If $i = 0$,

$$cost_0\Big(\Pi_0(q_1, q, q_2)\Big) = \frac{2\alpha\sin(\alpha/2)}{w} + \frac{2\alpha\sin(\alpha/2)}{w_0 - w} = 2\alpha\sin\left(\frac{\alpha}{2}\right)\left(\frac{1}{w} + \frac{1}{w_0 - w}\right). \quad (52)$$

2. If $i = 1$,

$$cost_1\Big(\Pi_1(q_1, q, q_2)\Big) = \frac{12\alpha^2 D(\alpha)^3}{w^3} + \frac{12\alpha^2 D(\alpha)^3}{(w_0 - w)^3} = 12\alpha^2 D(\alpha)^3\left(\frac{1}{w^3} + \frac{1}{(w_0 - w)^3}\right). \quad (53)$$

In both cases, by taking the total cost derivatives with respect to w, we prove that each total cost is minimized at $w = w_0/2$. □

A solution using the center point is a path of two simple curves that are mirror images of each other. Figure 13 shows several resultant paths for parallel cases (this and the following figures show the results for $i = 1$, i.e., the curvature-derivative-cost solutions).

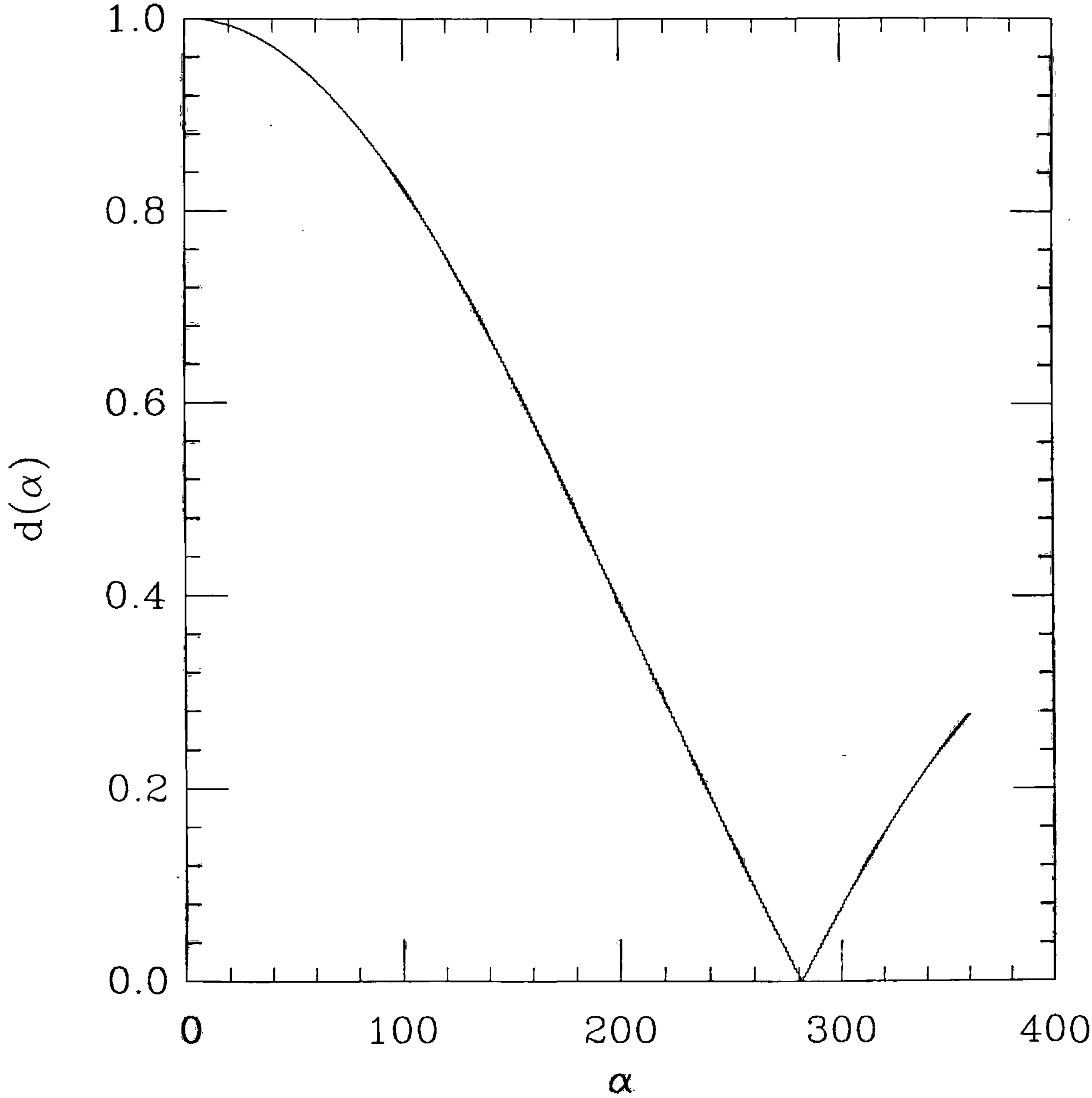

Fig. 9. Distance function of cubic spirals.

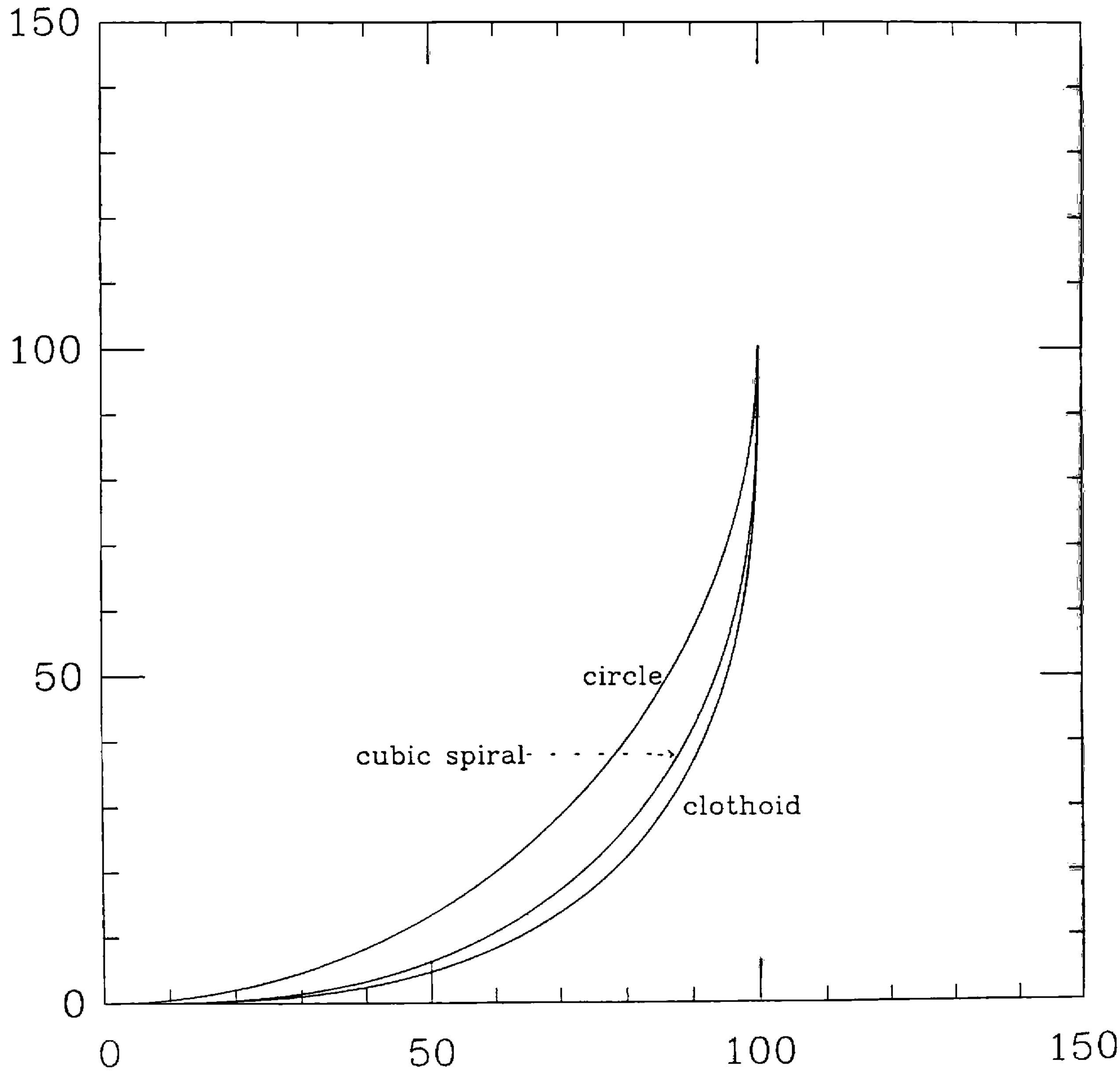

Fig. 10. 90° turns.

7.3. The Nonparallel Case

No closed-form equation for the minimum-cost point has been found for nonparallel cases. However, we do have the following conjecture.

CONJECTURE 1. There is not more than one local minimum-cost point among the set of symmetric means.

No exception to this conjecture has yet been found. Thus, a trisection numerical search is effectively being used, even in real time, to find the minimum-cost or "smoothest" solution.

7.4. A Final Algorithm and Examples

Thus, the final algorithm for this problem follows ($i = 0$ or 1).

Smooth-Path Planning(i, q_1, q_2)
begin
  if $symmetric(q_1, q_2)$ **then** return$\Big(simple_i(q_1, q_2) \Big)$;
  if $parallel(q_1, q_2)$
    then $q = \Big((x_1 + x_2)/2, (y_1 + y_2)/2, \Psi([q_1], [q_2]) + \Phi(\Psi([q_1], [q_2]) - \theta_1)\Big)$;
    else q = proper-symmetric-mean-that-minimizes

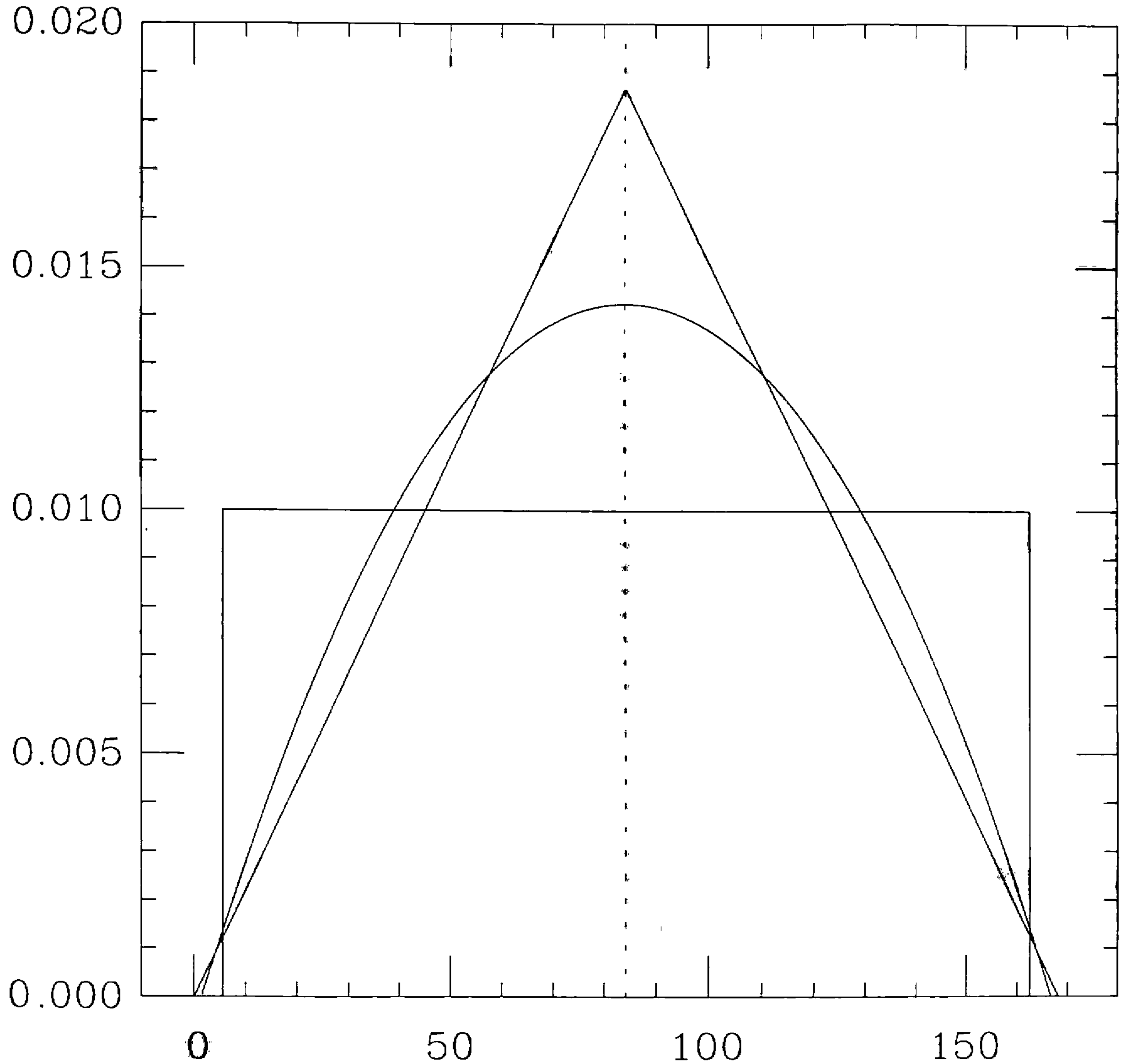

Fig. 11. Curvature for 90° turns.

$$cost_i\Big(\Pi_i(q_1, q, q_2)\Big);$$
$$\text{return}\Big(simple_i(q_1, q), simple_i(q, q_2)\Big);$$
end

In Figure 14, the input configuration pair is $\big((0,\ 0,\ 0), (100, 100, -\pi/4)\big)$. Since both directions are not equal, this pair is not parallel. The circular locus of symmetric means is depicted by the broken line whose center is (136.603, −36.603) and is out of bounds of this figure. The cost at each symmetric mean is shown in Figure 15 as a function of the direction γ (in degrees). The total cost is minimized at $\gamma = 134.517°$. The minimum symmetric mean point is (41.1614, 61.0433). Figure 16 shows another example with an input pair that is almost symmetric. Hence, one of the two symmetric cubic spirals is short and almost straight. Figure 17 shows a series of solution paths for nonparallel cases. The path at the bottom in this figure is the solution to the input (q_1 and q_2) for Figure 4.

8. Conclusion and Discussions

In this article, we solved the problem of finding the "smoothest" local path that joins two given configurations. In most cases, a path consists of two "simple" path

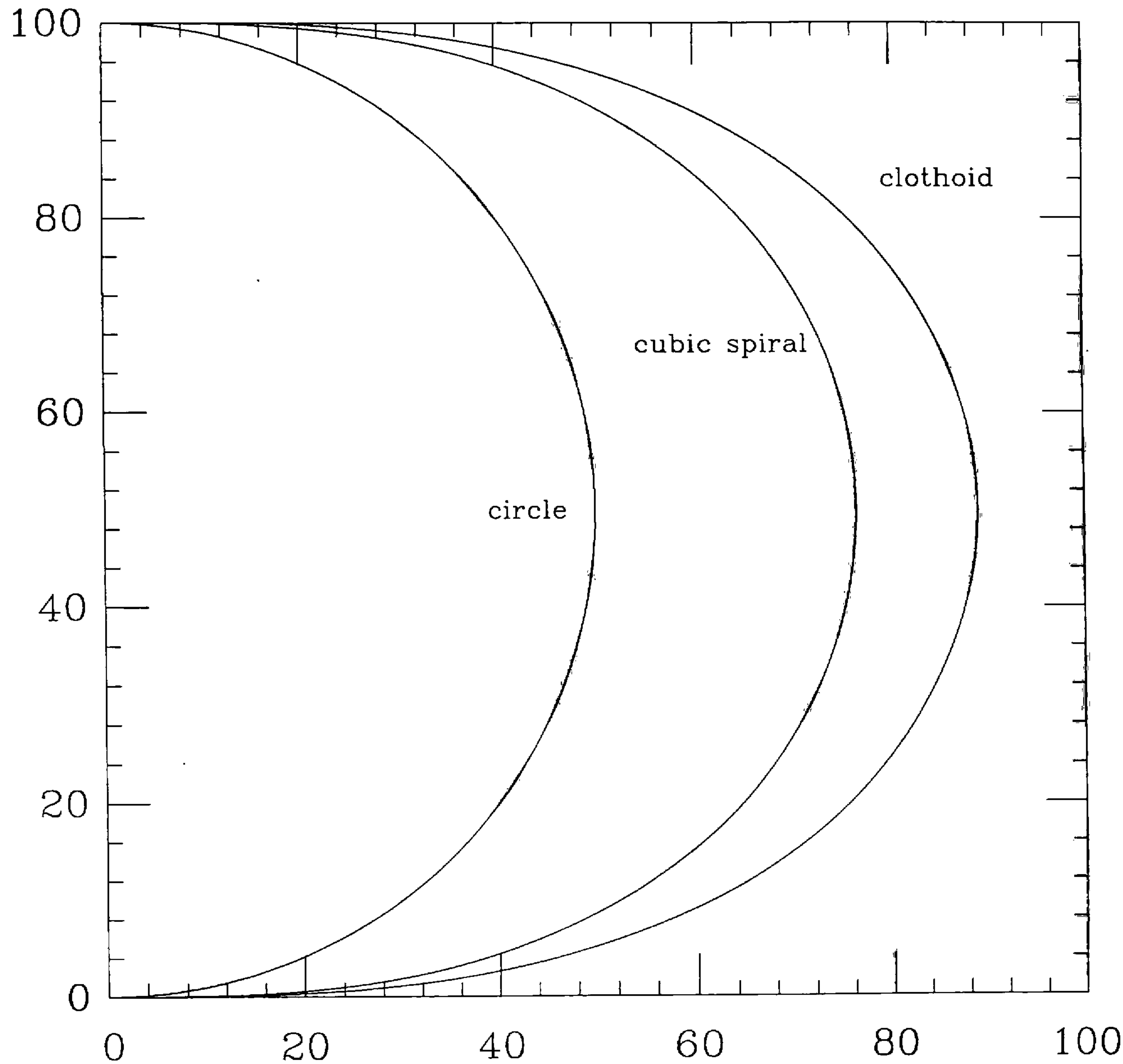

Fig. 12. 180° *turns.*

segments. How to define a path's smoothness cost is the most important issue. We propose two distinct definitions: the first is the path curvature, and the second is the derivative of the path curvature. With the first definition, we use circular arcs for "simple paths." With the second definition, we use cubic spirals, which are introduced in the robotics field for the first time. The set of cubic spirals have several advantages if used in the smooth-path planning, one of which is curvature continuity.

The authors have implemented this algorithm with cubic spirals on the Yamabico-11 autonomous mobile robot system, which has been developed at the University of California at Santa Barbara and the Naval Postgraduate School. The result was a success. The vehicle moves far more smoothly than when the clothoid method is used (Kanayama and Miyake 1986). As we expected, the curvature at the midpoint of each simple curve is smaller with cubic spirals than with clothoid pairs. This difference is especially important when a faster motion is needed.

In the MML language for the Yamabico-11 robot, we have the higher level functions "move(p)" and "stop(p)," where the argument p is a pointer to a configuration. These functions embody actual movement from the current configuration to the destination "p." The "stop" function makes the vehicle stop at the end configuration.

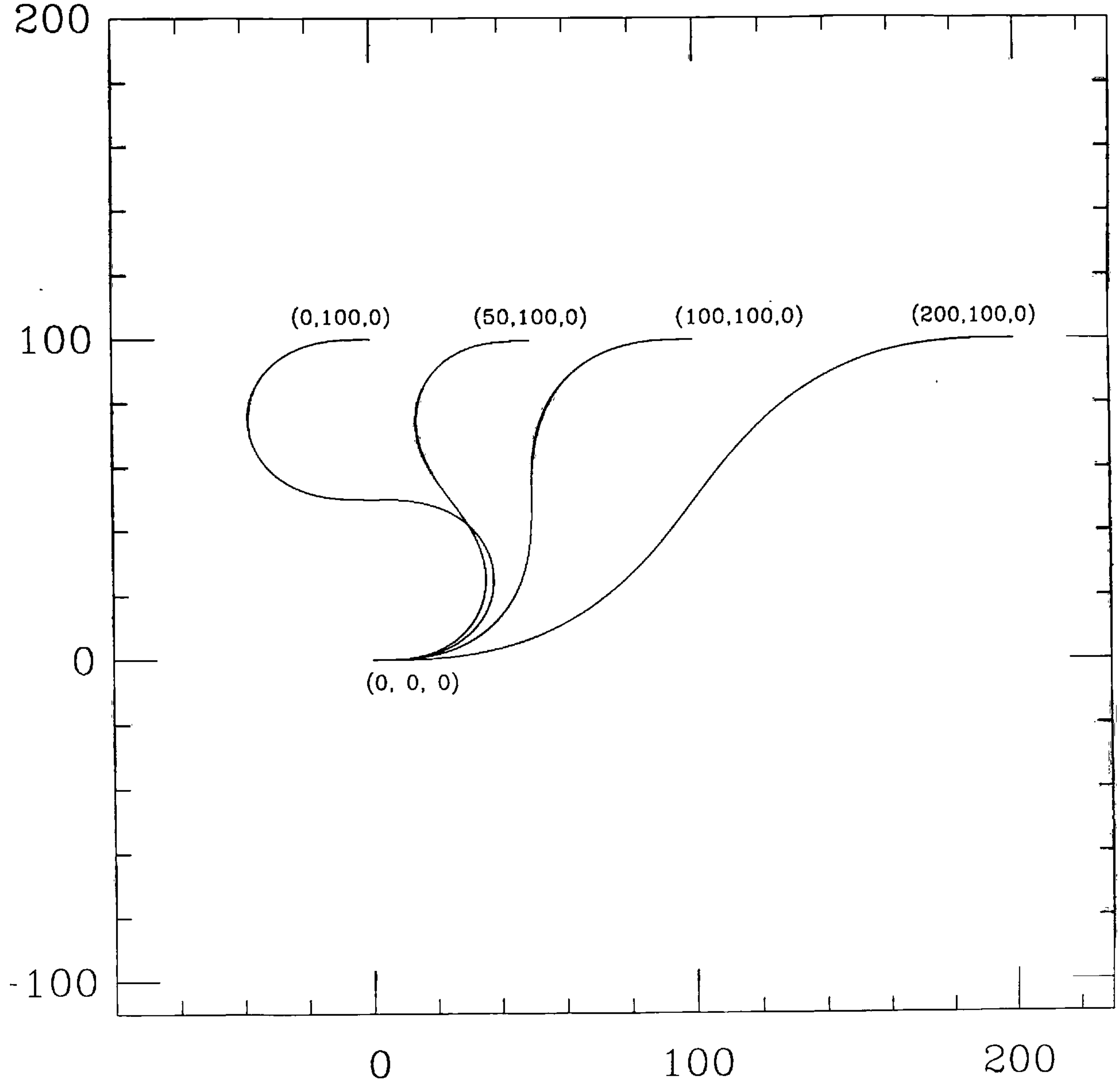

Fig. 13. Solution to parallel cases.

For example, the path shown in Figure 1 is executed by Yamabico-11 with six "move" and one "stop" functions. These "move" and "stop" functions are being used in sensor-based navigation in real time.

In this article, we solved the path-planning problem without curvature specification in the boundary conditions. More generally, the motion planning problem of finding a smooth path that joins a pair of "configurations" with curvature $q_1 = (x_1, y_1, \theta_1, \kappa_1)$ and $q_2 = (x_2, y_2, \theta_2, \kappa_2)$ is an open problem for the future. The output we expect is a path that satisfies the boundary conditions for curvature while minimizing a smoothness cost.

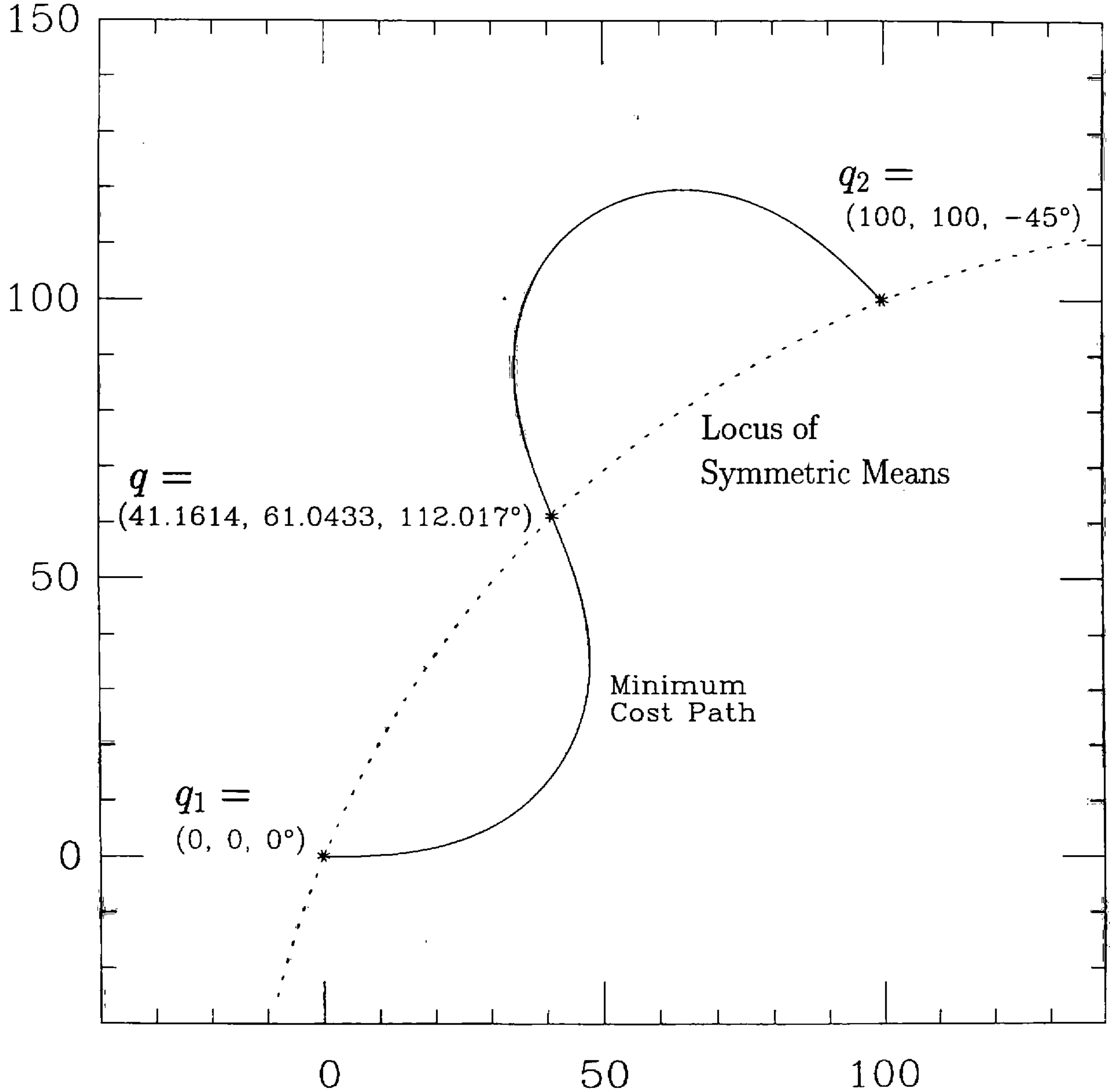

Fig. 14. Solution to a nonparallel case.

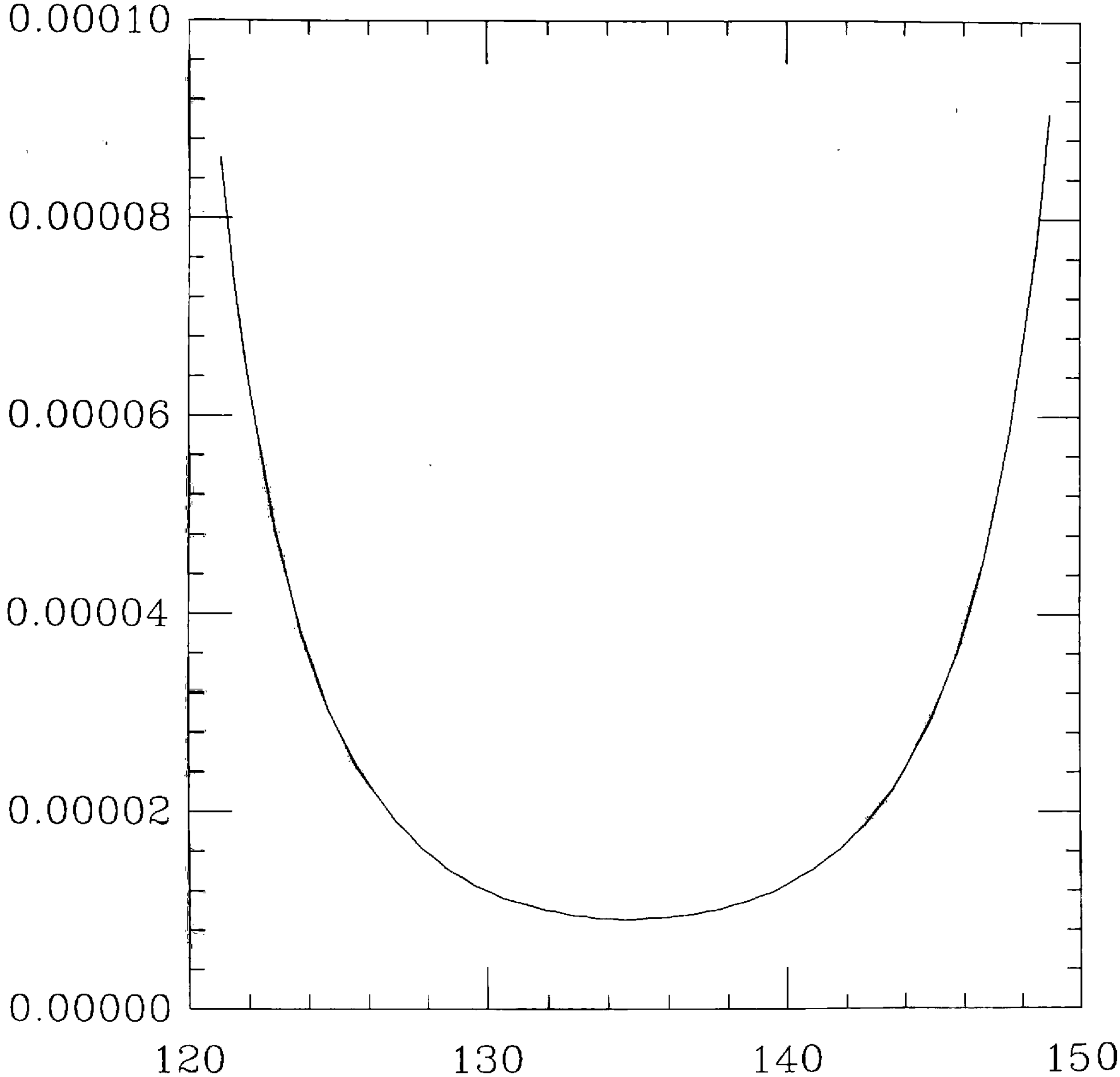

Fig. 15. Total cost.

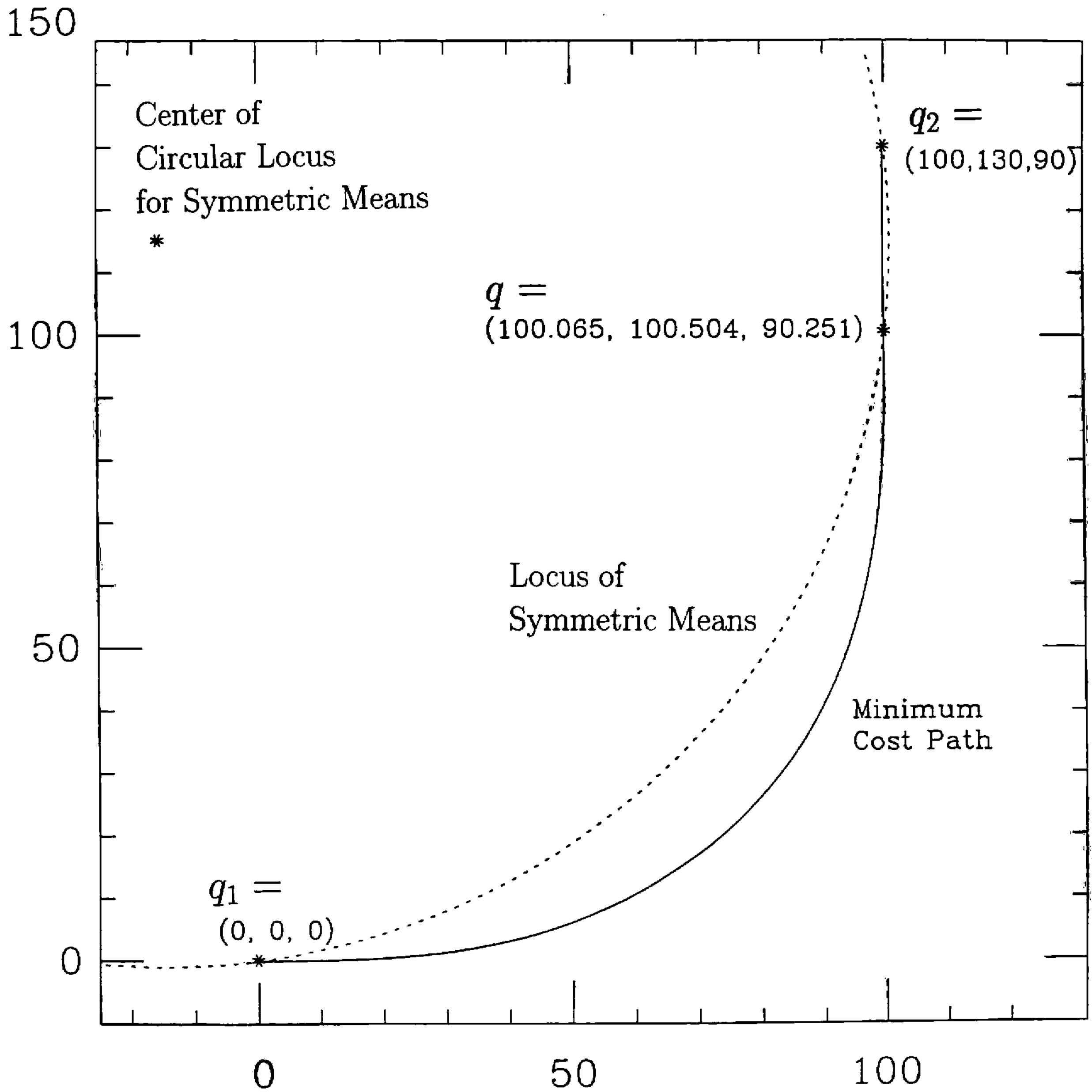

Fig. 16. Solution to a close-to-parallel case.

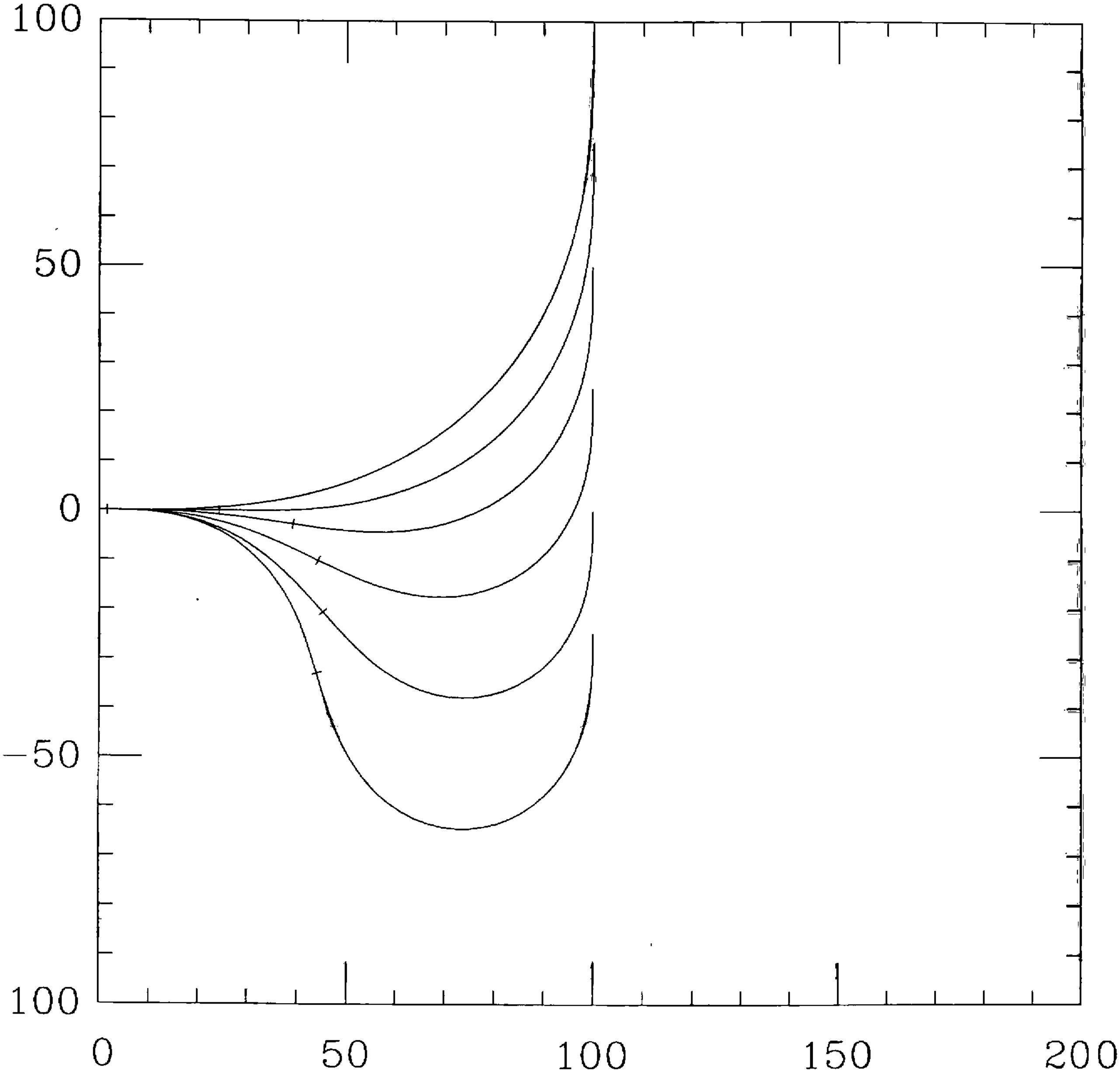

Fig. 17. Series of solutions to nonparallel cases.

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