Automatica 89 (2018) 28–36

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Efficient path planning algorithms in reach-avoid problems☆

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ARTICLE INFO

Article history:
Received 10 January 2014
Received in revised form 7 September 2017
Accepted 10 October 2017
Available online 16 December 2017

Keywords:
Reach-avoid games
Fast marching methods
Path planning

ABSTRACT

We consider a multi-player differential game, referred to as a reach-avoid game, in which one set of attacking players attempts to reach a target while avoiding both obstacles and capture by a set of defending players. Unlike pursuit–evasion games, in this reach-avoid game one set of players must not only consider the other set of players, but also the target. This complexity makes finding solutions to such games computationally challenging, especially as the number of players grows. We propose an approach to solving such games in an open-loop sense, where the players commit to their control actions prior to the beginning of the game. This reduces the dimensionality of the required computations, thus enabling efficient computation of feasible solutions in real time for domains with arbitrary obstacle topologies. We describe two such formulations, each of which is conservative towards one side, and derive numerical algorithms based upon modified fast-marching methods (FMM) for computing their solutions.

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1. Introduction

In a reach-avoid game, one set of players (attackers) attempts to arrive at a target set in the state-space, while avoiding a set of unsafe states, as well as interceptions by an opposing set of players (defenders). Such problems encompass a large number of robotics and control applications. For example, many safe motion-planning problems (see Karaman and Frazzoli, 2011b and the references therein) may be formulated as reach-avoid games, in which the objective is to control one or more agents into a desired target region, while avoiding a set of obstacles or possibly adversarial agents. Reach-avoid games considered here belong to the general class of differential games (Friedman, 1971; Isaacs, 1967), which include a variety of interesting problems (e.g. pursuit–evasion games Flynn, 1974, Lewin, 1986, network consensus problems under adversarial attacks Khanafer, Touri, & Basar, 2012, 2013, motion planning Chen, Zhou, & Tomlin, 2014, 2017; Karaman & Frazzoli, 2011a) and have been a subject of significant past research. For reach-avoid related games, the approaches that have been proposed often feature trade-offs between optimality of the solution (with respect to the time to achieve a player's objective), and the complexity of the computation.

Our approach in this paper to reach-avoid games is to formulate them as an open-loop game, and more specifically, as a framework of open-loop games that includes the open-loop upper value game and open-loop lower values game. The open-loop games for the reach-avoid problem formulated here are an instantiation of a general Stackelberg game (Başar & Olsder, 1999), an important class of games for modeling strategic behavior in dynamic games. In an open-loop game, the granularity of a strategy at which a player chooses is a control function that maps the entire time horizon to a trajectory. Depending on which player(s) choose(s) first, and which one(s) choose(s) in response, they are leader(s) and follower(s) respectively.

The open-loop games are conservative towards one side of players: the side that chooses first. Consequently, this level of conservatism offers performance guarantees of the solution. Namely, if a solution exists, the player is guaranteed to achieve the desired objective, irrespective of the actions of the opponent, without needing to incorporate any future state information. This is particularly well-suited for certain safety-critical applications, where state update is hard or costly to obtain, for example in GPS-denied environments. Another safety-critical application is robotic surgery, where the robotic system needs to navigate inside human body to

☆ This work has been supported in part by NSF under CPS:ActionWebs (CNS-931843), by ONR under the HUNT (N0014-08-0696) and SMARTS (N00014-09-1-1051) MURIs and by grant N00014-12-1-0609, by AFOSR under the CHASE MURI (FA9550-10-1-0567). The material in this paper was presented at Proceedings of the IEEE International Conference on Robotics and Automation, May 14–18, 2012, St. Paul, Minnesota, USA; Proceedings of the 51st IEEE Conference on Decision and Control, December 10–13, 2012, Maui, Hawaii, USA. This paper was recommended for publication in revised form by Associate Editor Hideaki Ishii under the direction of Editor Ian R. Petersen.

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https://doi.org/10.1016/j.automatica.2017.11.035
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eliminate certain tissues while ensuring that no invasive damage is induced. Those safety-critical applications tend to demand the solution to satisfy strong performance guarantees (e.g. achieving the goal regardless of what the disturbance does, adversarial or non-adversarial alike), a property guaranteed by the open-loop framework.

In a static Stackelberg game (only one-round of interaction between players), an exhaust tree search can be used to find the optimal action (Başar & Olsder, 1999). However, at least in the context of reach-avoid open-loop dynamic games, no efficient algorithms have been devised to produce the optimal controls. Consequently, this paper is particularly focused upon the study of efficient and scalable computational algorithms for solving open-loop reach-avoid games. We emphasize that the proposed approach does not involve solving *any* Hamilton–Jacobi–Isaacs (HJI) equations[1] in the high dimensional space of all player states. Instead, as discussed in detail later, we reduce the problem to solving Hamilton–Jacobi–Bellman (HJB) equations in the low dimensional space of individual player states.

1.1. Related work

For certain games, it is possible to construct strategies (sometimes even optimal) for the players analytically or geometrically. Defending a line segment on a plane without obstacles has been considered in Kawecki, Kraska, Majcherek, and Zola (2009) and Rzymowski (2009), where the motion has been restricted to fixed maximum speed and moving along line segments, respectively. In both cases, optimal strategies have been found for defending the target set (a line segment). A class of methods has been proposed for safe motion-planning in the presence of moving obstacles by computing the future set of states an obstacle may occupy, given the dynamics of the controlled agent and the obstacles. This set of states are then treated as obstacles in the joint state-time space, and paths are planned which avoid these states (Fiorini & Shiller, 1998; Fraichard & Asama, 2004; Van Den Berg & Overmars, 2008). They are well suited for scenarios in which the obstacle motion can be unpredictable or even adversarial, so that in the presence of hard constraints on safety, one needs to account for the worst-case possibility that the disturbances may actively attempt to collide with the agent. However, these approaches tend to be limited to simple game configurations without complex static obstacle configurations or inhomogeneous speed constraints from varying terrain, issues that our formulation (Section 2.1) captures and that our computational framework (Section 4) addresses.

For general cases, the classical approach is to formulate the game as a minimax problem, in which a value function is defined representing the time-to-reach of the attackers at the target, subject to the constraint that it is not captured by the defender. This value is then computed via a related Hamilton–Jacobi–Isaacs (HJI) partial differential equation (PDE), with appropriate boundary conditions (Başar & Olsder, 1999; Evans & Souganidis, 1984; Isaacs, 1967). Solutions are typically found either using the method of characteristics (Başar & Olsder, 1999; Isaacs, 1967), where optimal trajectories are computed by integrating backward from a known terminal condition, or via numerical approximation of the HJI equation on grids (Falcone & Ferretti, 2002). For the HJI method, the number of grid points needed to approximate the value function typically grow exponentially in the number of continuous states. As such, finding solution strategies for such games can be computationally expensive, even for games involving a single attacker and a single defender. For differential games with multiple players, the computational burden also scale exponentially in the number of players. On a related note, in particular differential game scenarios, approximate dynamic programming (ADP) techniques have been developed to efficiently compute game solutions. For example, clever policy iterations have been designed to solve certain two-player zero-sum games (Johnson, Bhasin, & Dixon, 2011; Vamvoudakis & Lewis, 2012). However, the differential game scenario considered in this paper has a non-smooth value function. The application of ADP methods to such scenarios has been found to be challenging from a numerical convergence standpoint (Munos, Baird, & Moore, 1999). In closing, we mention that another related research thread concerns reach-avoid games under imperfect or incomplete information (Doyen & Raskin, 2010; Sebbane, 2014), where the players do not necessarily know the locations of the other players at all times.

As a final note, closely related to reach-avoid problems is the class of pursuit–evasion problems (Flynn, 1974; Lewin, 1986; Liu, Zhou, Tomlin, & Hedrick, 2013; Zhou et al., 2016). A reach-avoid game shares some similar aspects of a pursuit–evasion game: the attacker (similar to an evader) needs to avoid the capture of the defender (similar to a pursuer) because capture would result in the attackers losing the game instantly. However, there is a key distinction between the two: the utility of an agent in a reach-avoid game is fundamentally different from that of a pursuit–evasion game. In the latter, an evader's utility is solely based on whether it will ever be captured, whereas in the former, the attacker's utility depends on how long it takes to reach the target. Consequently, a reach-avoid problem is much more challenging because the attacker not only needs to avoid capture, but also needs to reach a per-determined target set. The recent work (Zhou et al., 2016) studies a multi-pursuer–single-evader pursuit–evasion problem, where an analytical cooperative pursuit strategy for the pursuers is derived using Voronoi partitions. This geometric approach (and the analytical results therein) are feasible because all agents in the pursuit–evasion problem are assumed to have constant speed. Conversely, in our current reach-avoid setting, in addition to the complexity just mentioned, we also allow for arbitrary speed profiles for all agents. As a result, these two levels of generality dictate that a completely different and non-analytical approach be taken.

1.2. Our contributions

In this paper, we provide a computationally tractable framework for solving open-loop reach-avoid games. Our major contribution is the algorithmic framework for computing open-loop values. Specifically, we develop (Section 4) efficient and novel numerical algorithms (a set of modified fast-marching methods) that allow us to quickly compute solutions to these open-loop games, in the form of a set of open-loop player trajectories with provable properties. To the best of our knowledge, this is the first set of efficient algorithms for computing open-loop values. We note that the two open-loop values, in addition to being interesting for study in their own right, also provide bounds on the closed-loop value (in general the solution to an HJI equation, which is typically intractable to solve) of the reach-avoid game. In this sense, our open-loop framework can be interpreted as a computationally efficient approximation of the closed-loop value for reach-avoid games, through the trade-off of a certain degree of optimality for a reduction in computational complexity. In particular, there exist non-trivial cases where solving for open-loop values also yield closed-loop values. We also emphasize the independent modeling value of the open-loop formulations: since they are conservative towards one set of players, the resulting solutions provide worst-case guarantees that will be useful in safety-critical applications. Some of the theoretical results were presented in two previous publications (Takei, Huang, Ding, & Tomlin, 2012; Zhou, Takei, Huang, & Tomlin, 2012). This work unifies the presentation of the

[1] This corresponds to a closed-loop formulation. See Section 1.1 for a discussion and related work.

open-loop games and their relationship to each other. Specifically, we establish a theoretical framework (Section 3) that puts on rigorous footing the procedure for relating various optimal time-to-reach functions. We also expand the discussion on the novel aspects of the developed algorithmic framework, all of which were previously omitted in the conference papers due to space limitation.

2. The reach-avoid problem

2.1. Payoff function

Suppose there are two players P_A and P_D, whose states are confined in a bounded, open domain $\Omega \subset \mathbb{R}^n$. The domain Ω can be partitioned as: $\Omega = \Omega_{free} \cup \Omega_{obs}$, where Ω_{free} is a compact set representing the free space in which the two players can move, while $\Omega_{obs} = \Omega \setminus \Omega_{free}$ corresponds to obstacles in the domain. Let $x_A, x_D \in \mathbb{R}^n$ denote the state of players P_A and P_D, respectively and $x_A^0, x_D^0 \in \Omega$ be their initial conditions. We have the following dynamics for $t \geq 0$:

$$\begin{aligned} \dot{x}_A(t) &= f_A(x_A(t))a(t), \quad x_A(0) = x_A^0, \\ \dot{x}_D(t) &= f_D(x_D(t))d(t), \quad x_D(0) = x_D^0, \end{aligned} \tag{1}$$

where f_A, f_D represent the spatially-varying maximum speeds for P_A and P_D, and a, d represent the directions in which P_A and P_D are moving. It is assumed that $f_A, f_D : \Omega \to [0, \infty)$ are bounded and Lipschitz continuous, and that a, d are drawn from the set $\Sigma = \{\sigma : [0, \infty) \to \overline{B}_n \mid \sigma \text{ is continuous}\}$, where $\overline{B}_n$ denotes the closed unit ball in $\mathbb{R}^n$. Per the rules of the reach-avoid game, we constrain the controls of both players to be those which allow the player states to remain within the confines of the free space. In particular, for initial conditions $x_A^0, x_D^0 \in \Omega_{free}$, we define the admissible control set $\Sigma_A(x_A^0, x_D^0)$ for P_A to be those controls in Σ such that $x_A(t) \in \Omega_{free}$, $\forall t \geq 0$. The admissible control set $\Sigma_D(x_A^0, x_D^0)$ for P_D is defined in a similar fashion. For convenience, we will drop the dependence of these sets on the initial condition when the context is clear.

We now define the payoff for the *reach-avoid game*. We first define two sets: $\mathcal{T} \subset \Omega_{free}$, the *target set*, and $\mathcal{A} \subset \Omega^2$, the *avoid set*, both assumed to be compact. The avoid set describes the capture condition in the reach-avoid game, namely the set of joint player states (x_A, x_D) at which P_A is captured by P_D. A typical example of an avoid set is given by $\mathcal{A} = \{(x_A, x_D) \in \Omega^2 \mid \|x_A - x_D\| \leq r\}$, for some $r \geq 0$, corresponding to a scenario in which P_A is captured if it comes within a capture radius r of P_D. In a reach-avoid game, the goal for P_A is to reach the target set $\mathcal{T}$ as quickly as possible, while steering clear of the avoid set $\mathcal{A}$. On the other hand, the goal for P_D is to either capture P_A before P_A reaches the target set, or to delay P_A from entering $\mathcal{T}$ for as long as possible. Denoting the joint state by $\mathbf{x} = (x_A, x_D)$ and the joint initial condition by $\mathbf{x^0} = (x_A^0, x_D^0)$, we define the payoff function $\mathcal{J} : \mathbb{R}^n \times \Sigma \times \Sigma \to \mathbb{R}$ as that of a minimum time differential game problem with state constraints (see for example Appendix A of Bardi & Capuzzo-Dolcetta, 1997):

$$\mathcal{J}(\mathbf{x^0}, a, d) = \inf\{t \geq 0 \mid x_A(t) \in \mathcal{T}, \mathbf{x}(s) \notin \mathcal{A}, \forall s \in [0, t]\}, \tag{2}$$

where the infimum of the empty set is by convention $+\infty$. Given that this is a minimum time problem, the payoff function (2) is in general discontinuous. In particular, it is finite on the set of initial conditions $\mathbf{x^0} \in \mathbb{R}^n$ for which P_A can reach the target set $\mathcal{T}$ without being captured by P_D. It is equal to $+\infty$ elsewhere.

2.2. Closed-loop game formulation

Given the payoff function (2), the value function of the reach-avoid game depends on the information pattern employed by each player. In this section, we will introduce the value functions for a closed-loop game, defined using a *non-anticipative* information pattern, as commonly adopted in the differential game literature (see for example Elliott & Kalton, 1972, Varaiya, 1967). Under this information pattern, a strategy for P_A defines the responses of P_A to possible control selections by P_D, and vice versa for player P_D. In order to preclude strategies which allow foreknowledge of the control selections by the opposing player, we only consider strategies which select controls in a causal fashion. In particular, the set $\mathcal{F}_A$ of admissible strategies for P_A is given by

$$\begin{aligned} \mathcal{F}_A := \{\gamma : \Sigma_D \to \Sigma_A \mid \forall t, \sigma_1(\tau) = \sigma_2(\tau), \text{ for a.e. } \tau \in [0, t] \\ \Rightarrow \gamma[\sigma_1](\tau) = \gamma[\sigma_2](\tau), \text{ for a.e. } \tau \in [0, t]\}. \end{aligned}$$

The set $\mathcal{F}_D$ of admissible strategies for P_D is defined similarly.

For $\mathbf{x^0} \in \Omega_{free}^2$, the upper value and lower value of the closed-loop reach-avoid game are respectively given by:

$$\overline{V}(\mathbf{x^0}) = \overline{V}(x_A^0, x_D^0) = \sup_{\gamma_D \in \mathcal{F}_D} \inf_{a \in \Sigma_A} \mathcal{J}(\mathbf{x^0}, a, \gamma_D[a]); \tag{3}$$

$$\underline{V}(\mathbf{x^0}) = \underline{V}(x_A^0, x_D^0) = \inf_{\gamma_A \in \mathcal{F}_A} \sup_{d \in \Sigma_D} \mathcal{J}(\mathbf{x^0}, \gamma_A[d], d). \tag{4}$$

2.3. Open-loop game formulation

In principle, one can compute $\overline{V}(\mathbf{x^0})$ (or $\underline{V}(\mathbf{x^0})$) by solving a corresponding HJI PDE (Isaacs, 1967): they are shown to be viscosity solutions to HJI equations (Evans & Souganidis, 1984). See Zhou et al. (2012) for more details. Although numerical methods for solving an HJI equation on a Cartesian grid exist, they suffer from the curse of dimensionality. Specifically, with a uniform grid, the number of grid nodes grows exponentially with the dimension of the joint state space, while the dimension scales linearly with the number of players. This results in an exponential growth in complexity with the number of players, thus making the problem computationally intractable even for a modest number of players. Moreover, for cases in which computational solutions are possible (e.g. with two players and two-dimensional domains), the closed-loop value function often cannot be computed in real time (Huang, Ding, Zhang, & Tomlin, 2011). Motivated by the difficulty of computing the closed-loop value function, we now consider the open-loop formulations of the reach-avoid game, where each player selects *controls* from the control spaces Σ_A or Σ_D, rather than *strategies* from the non-anticipative strategy space $\mathcal{F}$.

Definition 1. The *open-loop upper value* and the *open-loop lower value* of the reach-avoid game are defined as follows:

$$\begin{aligned} \overline{v}(\mathbf{x^0}) &= \inf_{a \in \Sigma_A} \sup_{d \in \Sigma_D} \mathcal{J}(\mathbf{x^0}, a, d), \\ \underline{v}(\mathbf{x^0}) &= \sup_{d \in \Sigma_D} \inf_{a \in \Sigma_A} \mathcal{J}(\mathbf{x^0}, a, d). \end{aligned} \tag{5}$$

These two open-loop values can be viewed as conservative calculations of the payoff, under the assumption that the opposing player has complete knowledge of one's choice of control, whether in an anticipative or non-anticipative fashion. Specifically, for the upper value, P_A first chooses its control over the whole time horizon $[0, \infty)$ and makes this control known to P_D, which then chooses its control in response. Thus the upper value is conservative towards P_A, who must consider all possible responses by P_D when choosing its control a. Similarly, the lower value is conservative towards P_D, who chooses first in this case. It is known that:

$$\underline{v}(\mathbf{x^0}) \leq \underline{V}(\mathbf{x^0}) \leq \overline{V}(\mathbf{x^0}) \leq \overline{v}(\mathbf{x^0}), \ \forall \mathbf{x^0} \in \Omega_{free}^2. \tag{6}$$

Consequently, for initial states at which the open-loop upper and lower values coincide, the closed-loop value is simply given by

$V(\mathbf{x^0}) = \underline{v}(\mathbf{x^0}) = \overline{v}(\mathbf{x^0})$. In such cases, the value of the closed-loop game can be obtained for the initial state $\mathbf{x^0}$ without explicitly solving the HJI equation. As a very important note, the open-loop values $\overline{v}(\mathbf{x^0})$ and $\underline{v}(\mathbf{x^0})$ are not solutions to HJI equations. Instead, they can be computed by solving a certain constrained Hamilton–Jacobi–Bellman equations discussed in Section 4, as opposed to HJI equations.

The optimal controls $\bar{a} \in \Sigma_A$, $\bar{d} \in \Sigma_D$, $\underline{a} \in \Sigma_A$ and $\underline{d} \in \Sigma_D$ for the open-loop upper and lower value games, are respectively:

$$\begin{aligned} &\bar{a} \in \arg\min_{a\in\Sigma_A}\max_{d\in\Sigma_D} \mathcal{J}(\mathbf{x^0}, a, d), \quad \bar{d} \in \arg\max_{d\in\Sigma_D} \mathcal{J}(\mathbf{x^0}, \bar{a}, d),\\ &\underline{d} \in \arg\max_{d\in\Sigma_D}\inf_{a\in\Sigma_A} \mathcal{J}(\mathbf{x^0}, a, d), \quad \underline{a} \in \arg\min_{a\in\Sigma_A} \mathcal{J}(\mathbf{x^0}, a, \underline{d}). \end{aligned} \tag{7}$$

Note that $(\bar{a}, \bar{d})$ can be interpreted as the Stackelberg equilibrium of a two-player zero-sum game (Başar & Olsder, 1999); similarly for $(\underline{a}, \underline{d})$. We emphasize again that the controls here (e.g. $\bar{a}$, $\bar{d}$) are control functions, rather than instantaneous control values.

3. The open-loop upper value and lower value

This section provides the theoretical foundations for solving the open-loop reach-avoid games.

3.1. Upper value

We first examine the upper value game, which is conservative towards P_A. The value function for this game is defined in (5). The open-loop value $\overline{v}$ can be found by characterizing the set of points in the state space that P_A can reach no matter what P_D does and selecting a path within this set. Given a joint initial condition $\mathbf{x}^0$, a point $y \in \Omega_{free}$ is *safe-reachable* if there exists a player P_A control such that for every player P_D control, P_A can reach y in finite time, while avoiding capture by player P_D. More precisely, we define a *safe-reachable set* for P_A:

$$\begin{aligned} \mathcal{S} :=& \{y \in \Omega_{free} \mid \exists a \in \Sigma_A, \forall d \in \Sigma_D, \exists t \geq 0, x_A(t) = y,\\ &\mathbf{x}(s) \notin \mathcal{A}, \forall s \in [0, t]\}, \end{aligned} \tag{8}$$

where $\mathbf{x}(\cdot) = (x_A(\cdot), x_D(\cdot))$ is the solution to (1). Note that the set $\mathcal{S}$ implicitly depends on the initial condition $\mathbf{x}^0$ of the reach-avoid game. The safe reachable set provides us with a necessary and sufficient condition for $\overline{v}$ to be finite. Clearly, $\overline{v} < \infty$ if and only if $\mathcal{S} \cap \mathcal{T} \neq \emptyset$. Fig. 1 illustrates $\mathcal{S}$ for an example where P_A is twice as fast as P_D. In this case, $\mathcal{A} = \{(x_A, x_D) \in \Omega^2 \mid x_A = x_D\}$. As shown here, the safe-reachable set is in general not the set of equal time-to-reach points for the two agents, as there may be points that are reachable by P_A, but the choice of controls to achieve this condition could result in capture by P_D. For comparison, the Apollonius circle showing the equal time-to-reach points of P_A and P_D is overlaid. In the following, we will provide a characterization of the set $\mathcal{S}$ in terms of two minimum-time functions, defined respectively from the perspectives of P_A and P_D.

Definition 2. Given $x_D^0 \in \Omega_{free}$ and $y \in \Omega_{free}$, the minimum time-to-capture for P_D is

$$\bar{t}(y; x_D^0) := \inf_{d\in\Sigma_D} \inf\{t \geq 0 \mid (y, x_D(t)) \in \mathcal{A}, x_D(0) = x_D^0\}. \tag{9}$$

That is, for a stationary P_A at y, $\bar{t}(y; x_D^0)$ is the shortest time for P_D, starting at x_D^0, to capture P_A. For compactness of notation, we shall also write $\bar{t}(y)$ when the context is clear.

Definition 3. Given an initial state $x_A^0 \in \Omega_{free}$, a set $R \subseteq \Omega_{free}$, and a location $y \in \Omega_{free}$, the minimum time-to-reach for P_A with constraint R is

$$\begin{aligned} \overline{w_R}(y; x_A^0) :=& \inf_{a\in\Sigma_A} \inf\{t \geq 0 \mid x_A(t) = y, x_A(0) = x_A^0,\\ &x_A(s) \in R, \forall s \in [0, t]\}. \end{aligned} \tag{10}$$

Intuitively, $\overline{w_R}(y; x_A^0)$ quantifies how quickly P_A can reach a point y from an initial condition x_A^0, while remaining within the set R. We can now relate the two minimum-time functions to the safe-reachable set $\mathcal{S}$.

Lemma 1. *Given a joint initial state $\mathbf{x^0} = (x_A^0, x_D^0) \in \Omega_{free}^2 \setminus \mathcal{A}$, let $\mathcal{M} \subseteq \Omega_{free}$ be the maximal set such that $\overline{w_{\mathcal{M}}}(y; x_A^0) < \bar{t}(y; x_D^0)$, $\forall y \in \mathcal{M}$. Then $\mathcal{M} = \mathcal{S}$.*

Proof. Let $\mathcal{E}$ be the collection of all sets $E \subseteq \Omega_{free}$ satisfying the inequality in the proposition, namely $\overline{w_E}(y; x_A^0) < \bar{t}(y; x_D^0)$, $\forall y \in E$. Note that $\mathcal{E}$ is non-empty, as $E = \{x_A^0\}$ is an element of $\mathcal{E}$. Now consider the set $E^* := \bigcup_{E\in\mathcal{E}} E$. We claim that $E^* \in \mathcal{E}$, and hence E^* is the maximal element of $\mathcal{E}$. Indeed, for any sets $R, R' \subseteq \Omega_{free}$ such that $R \subseteq R'$, we have by the definition for the minimum time-to-reach function in (10) that $\overline{w_{R'}}(y; x_A^0) \leq \overline{w_R}(y; x_A^0)$, for all $y \in \Omega_{free}$. In particular, for any set $E \in \mathcal{E}$, we have $\overline{w_{E^*}}(y; x_A^0) \leq \overline{w_E}(y; x_A^0)$, for all $y \in \Omega_{free}$. Let $y \in E^*$, then given the definition of the set E^*, $y \in E$ for some $E \in \mathcal{E}$. Thus, $\overline{w_{E^*}}(y; x_A^0) \leq \overline{w_E}(y; x_A^0) < \bar{t}(y; x_D^0)$. This proves the claim, which in turn implies $\mathcal{M} = E^*$.

To see that $E^* = \mathcal{S}$, first we note that both E^* and $\mathcal{S}$ are non-empty. In particular, given an initial condition $\mathbf{x^0} = (x_A^0, x_D^0) \in \Omega_{free}^2 \setminus \mathcal{A}$, x_A^0 is an element of both E^* and $\mathcal{S}$. Now take any point $y \in \mathcal{S}$. By definition, there exists $a^* \in \Sigma_A$, such that for every $d \in \Sigma_D$, there exists $t \geq 0$ so that $x_A(t) = y$ and $\mathbf{x}(s) \notin \mathcal{A}$, $\forall s \in [0, t]$. Let $t^* := \min\{t \geq 0 \mid x_A(t) = y\}$, where $x_A(\cdot)$ is the trajectory of P_A under (1), with the choice of control a^*. Then it can be seen that $x_A(\cdot)$ satisfies $x_A(s) \in \mathcal{S}$, $\forall s \in [0, t^*]$.

This implies that $\overline{w_{\mathcal{S}}}(y; x_A^0) \leq t^*$. Furthermore, by the properties of a^*, we have that for any $d \in \Sigma_D$, the joint path satisfies $\mathbf{x}(t^*) \notin \mathcal{A}$, or equivalently $(y, x_D(t^*)) \notin \mathcal{A}$. By the compactness of $\mathcal{A}$ and the definition of the minimum time-to-capture function in (9), this in turn implies that $t^* < \bar{t}(y; x_D^0)$. Thus, $\overline{w_{\mathcal{S}}}(y; x_A^0) < \bar{t}(y; x_D^0)$, $\forall y \in \mathcal{S}$, implying $\mathcal{S} \in \mathcal{E}$, and so $\mathcal{S} \subseteq E^*$.

We will now proceed to show that $\mathcal{S}$ is a superset of every element in $\mathcal{E}$. Let $E \in \mathcal{E}$ and $z \in E$. For simplicity of argument, we assume that there exists a minimum-time control with respect to $\overline{w_E}(z; x_A^0)$. Namely, there exists $a^* \in \Sigma_A$ such that the corresponding trajectory x_A^* satisfies $t^* := \min\{t \geq 0 \mid x_A^*(t) = z\} = \overline{w_E}(z; x_A^0)$ and $x_A^*(s) \in E, \forall s \in [0, t^*]$. Since $E \in \mathcal{E}$, we have $\overline{w_E}(x_A^*(s); x_A^0) < \bar{t}(x_A^*(s); x_D^0)$, $\forall s \in [0, t^*]$. Moreover, by the fact that x_A^* is a time-optimal path, we have $\overline{w_E}(x_A^*(s); x_A^0) = s$, $\forall s \in [0, t^*]$. By Definition 2, one can then infer that for every $d \in \Sigma_D$, the joint path under a^* and d satisfies $(x_A^*(s), x_D(s)) \notin \mathcal{A}$, $\forall s \in [0, t^*]$. This implies that $z \in \mathcal{S}$. In the case that the infimum with respect to $\overline{w_E}(z; x_A^0)$ is not achieved, one can apply a similar line of reasoning to ϵ-optimal controls. From this result, it then follows that $E^* = \bigcup_{E\in\mathcal{E}} E \subseteq \mathcal{S}$. Combining the two set inclusions, we have $\mathcal{S} = E^* = \mathcal{M}$. □

Intuitively, the above result states that $\mathcal{S}$ is the largest set in the free space Ω_{free} such that player P_A can reach all points in $\mathcal{S}$ safely by traveling along a path contained in $\mathcal{S}$, regardless of the controls of player P_D. Such a path will be referred to as a *safe-reachable path*. The upper value $\overline{v}$ can now be found using the minimum time-to-reach function $\overline{w_{\mathcal{S}}}$.

Theorem 1. *Given compact sets $\mathcal{T}, \mathcal{A} \subset \Omega_{free}$, and initial condition $\mathbf{x^0} = (x_A^0, x_D^0) \in \Omega_{free}^2$, let $\mathcal{S}$ be as defined in (8), and $\overline{w_{\mathcal{S}}}(y; x_A^0)$ be as defined in (10). Let the upper value $\overline{v}$ be as defined in (5). Then*

$$\overline{v}(\mathbf{x^0}) = \inf\{\overline{w_{\mathcal{S}}}(y; x_A^0) \mid y \in \mathcal{T}\}.$$

See Zhou, Ding, Huang, Takei, and Tomlin (2017) for proof.

Given this result, the problem of evaluating the upper value $\overline{v}(\mathbf{x}^0)$ becomes a problem of computing the minimum time-to-reach function $\overline{w_{\mathcal{S}}}$. However, the function $\overline{w_{\mathcal{S}}}$ also depends on the

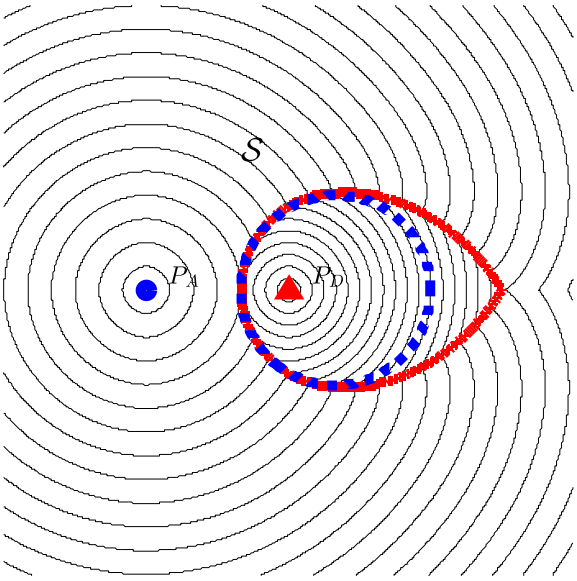

Fig. 1. $\mathcal{S}$ is the subset partitioned by the solid red curve containing x_A^0. Level sets of $\overline{w_{\mathcal{S}}}$ on $\mathcal{S}$ and $\bar{t}_c$ on $\Omega \setminus \mathcal{S}$ are plotted. The dotted blue circle is the set of equal time-to-reach points of P_A and P_D when capture is not considered. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

safe reachable set $\mathcal{S}$, thus $\overline{w_{\mathcal{S}}}$ and $\mathcal{S}$ must be computed simultaneously. We will demonstrate how to compute $\overline{w_{\mathcal{S}}}$ and $\mathcal{S}$ simultaneously, along with the minimum time-to-capture function $\bar{t}$, in Section 4. For now, it is important to observe that, as a consequence of Theorem 1, the problem of computing $\overline{v}(\mathbf{x}^0)$ in the high dimensional *joint state space* of the two agents reduces to computing $\overline{w_{\mathcal{S}}}$ in the lower dimensional state space of each individual player. This result not only speeds up the computation for the game with two agents, but also has important consequences for finding solutions when additional defending agents are introduced into the game.

3.2. Lower value

This section studies the second value function defined in Eqs. (5). In general, computing $\underline{v}$ is not a trivial task. This is due to an asymmetry inherent in the games: unlike P_A, P_D does not have a target set which solely depends on the state of P_D. Instead, the goal of P_D is to prevent P_A from entering $\mathcal{T}$ through the possibility of capture via the set $\mathcal{A}$, which depends on the joint states of P_A and P_D. This interdependence makes the exact computation of the open-loop lower value difficult, since P_D must consider P_A's actions with respect to P_D's entire trajectory. Therefore, we seek to find a computable lower bound to $\underline{v}$ using a particular choice of strategy. To simplify the task for P_D, we consider a strategy in which P_D moves to a particular location and then remains stationary at that location for the remainder of the game. Thus, instead of evaluating options over entire trajectories, P_D essentially considers locations where it may place itself as an obstacle. The resulting value of this strategy will provide a lower bound on the value of the game. Furthermore, we show that, in some cases, the lower bound is equal to $\underline{v}$.

For this strategy, we must evaluate possible positions for P_D to place itself and the resulting payoff of the game for each position. For any candidate position, we must compute the arrival time of P_A to the target if P_D were to place itself at that location, serving as a static obstacle. Computing this for all points in the state space is computationally expensive, but the set of candidate points can be substantially reduced via the following observations. First, a candidate location is only viable if P_D can reach that position before P_A has a chance to end the game by arriving at the target. Second, a location for P_D can only affect P_A if P_D can arrive there before P_A, otherwise P_A can effectively treat that location as being empty. Once this set of candidate locations is found, the defender can choose as its target location the position which results in the longest arrival time for the attacker.

Definition 4. Given an initial state $x_A^0 \in \Omega_{free}$ for P_A and a location $y \in \Omega_{free}$ for P_D, the minimum time-to-reach for P_A with respect to a stationary P_D is

$$\underline{t}(y; x_A^0) := \inf_{a \in \Sigma_A} \inf\{t \geq 0 \mid x_A(t) \in \mathcal{T}, x_A(0) = x_A^0, \\ (x_A(s), y) \notin \mathcal{A}, \forall s \in [0, t]\}. \tag{11}$$

That is, suppose P_D remains at the point x throughout, then $\underline{t}(x; x_A^0)$ is the minimum time for P_A, starting at x_A^0, to reach the target without being captured. Note that x_D^0 obviously fills the requirements of a candidate point that P_D can reach first, before P_A could end the game by entering the target set $\mathcal{T}$. For any other candidate point y, we must ensure that the game does not terminate for any point along the path between x_D^0 and y (i.e. there is no way for the attacker to end the game while the defender is enroute to its destination). We will call such paths *non-terminating*, since the game cannot be terminated while the defender is on such a path. In a similar manner to the safe-reachable set of the upper value game, we will find such paths by putting appropriate state constraints on the motion of P_D.

Definition 5. Given an initial state $x_D^0 \in \Omega_{free}$, a set $R \subset \Omega_{free}$, and a location $y \in \Omega_{free}$, the minimum time-to-reach for P_D with constraint R is

$$\underline{w_R}(y; x_D^0) := \inf_{d \in \Sigma_D} \inf\{t \geq 0 \mid x_D(t) = y, x_D(0) = x_D^0, \\ x_D(t) \in R, \forall s \in [0, t]\}. \tag{12}$$

Intuitively, $\underline{w_R}(y; x_D^0)$ is the minimum time for P_D to reach y from x_D^0 by traveling along a path that is contained in R. Given that our objective is to find controls for P_D such that P_D can reach its desired destination without P_A first terminating the game, we will proceed to define the largest set of states in Ω_{free} for which this is possible. As a first step, for any initial state $\mathbf{x^0} = (x_A^0, x_D^0) \in \Omega_{free}^2$ such that $x_A^0 \notin \mathcal{T}$, let $\mathcal{E}$ be the collection of all sets $E \subseteq \Omega_{free}$ which satisfy $\underline{w_E}(y; x_D^0) < \underline{t}(y; x_A^0)$, $\forall y \in E$. Note that $\mathcal{E}$ is non-empty, as $\{x_D^0\}$ is an element of $\mathcal{E}$. We define a set $\mathcal{Z} \subseteq \Omega_{free}$ as the union of all sets in $\mathcal{E}$, namely $\mathcal{Z} := \bigcup_{E \in \mathcal{E}} E$. Then by a similar argument as in the proof of Lemma 1, one can show that $\mathcal{Z}$ belongs to $\mathcal{E}$ and hence is the maximal set in Ω_{free} for which the following inequality holds: $\underline{w_{\mathcal{Z}}}(y; x_D^0) < \underline{t}(y; x_A^0)$, $\forall y \in \mathcal{Z}$. Now to find our desired candidate set, we must find the points in $\mathcal{Z}$ that the defender can reach first, since the avoid set $\mathcal{A}_y := \{x \in \Omega \mid (x, y) \in \mathcal{A}\}$ for a fixed location y of P_D can only serve as a meaningful obstruction to P_A if P_D can arrive at y before the attacker. In order to do this, we will need to introduce the notion of a t-reachable set $\mathcal{R}_1(t)$ for P_A, defined as follows.

$$\mathcal{R}_1(t) := \{x \in \Omega_{free} \mid \exists a \in \Sigma_A, \exists s \in [0, t], x_A(s) = x, \\ x_A(0) = x_A^0\}. \tag{13}$$

In other words, this is the set of all states in Ω_{free} that P_A can reach within t time units. We can now use this definition, along with that of $\mathcal{Z}$, to produce the desired lower bound.

Definition 6. Given a joint initial state $\mathbf{x^0} \in \Omega_{free}^2$, define:

$$\underline{v}(\mathbf{x^0}) := \begin{cases} \sup_{y \in \mathcal{Z}^*} \underline{t}(y; x_A^0), & x_A^0 \notin \mathcal{T} \wedge \mathbf{x^0} \notin \mathcal{A} \\ 0, & x_A^0 \in \mathcal{T} \wedge \mathbf{x^0} \notin \mathcal{A} \\ \infty, & \mathbf{x^0} \in \mathcal{A} \end{cases} \tag{14}$$

where $\mathcal{Z}^* := \{y \in \mathcal{Z} \mid \mathcal{A}_y \cap \mathcal{R}_1(T(y)) = \emptyset\}$,

$$T(y) := \underline{w_{\mathcal{Z}}}(y; x_D^0). \tag{15}$$

$\mathcal{Z}^*$ contains all points y from $\mathcal{Z}$ at which P_D can reach before P_A, and along a path that ensures P_A cannot end the game before P_D reaches y. Thus, if P_D reaches a point in $\mathcal{Z}^*$ via a shortest path contained in $\mathcal{Z}^*$, then by the time P_D reaches that point, it is guaranteed that P_D still has not entered the target yet. $\underline{\underline{v}}(\mathbf{x^0})$ encodes the time it takes for the game to end if the defender picks the best point in $\mathcal{Z}^*$ (points in the pink region in this picture), reaches that point via a time-optimal path and thereafter stays at that point. This leads to the following result.

Theorem 2. *Given initial condition $\mathbf{x^0} = (x_A^0, x_D^0) \in \Omega^2_{free}$, let function $\underline{\underline{v}}$ be as defined in (14), and lower value $\underline{v}$ be as defined in (5). Then $\underline{\underline{v}}(\mathbf{x^0}) \leq \underline{v}(\mathbf{x^0})$.*

See Zhou et al. (2017) for proof.

With this understanding of $\mathcal{Z}^*$, we may select the control for P_D as moving to a point $x_D^* \in \arg\max_{x \in \mathcal{Z}^*} \underline{t}(x; x_A^0)$, whenever the supremum in (14) is achieved, and remaining stationary at x_D^* thereafter. Since P_D is guaranteed to arrive at x_D^* before P_A, $\mathcal{A}_{x_D^*}$ appears to P_A as though a permanent obstacle, forcing P_A to take at least $\underline{t}(x_D^*, x_A^0)$ time units to reach the target.

4. The modified fast marching method for computing open-loop values

Computing solutions to the open-loop upper and lower value games as described in the previous section requires finding $\overline{w_{\mathcal{S}}}$ and $\mathcal{S}$ for the upper value game and $\underline{w_{\mathcal{Z}}}$ and $\mathcal{Z}$ for the lower value game. In each case, the desired value is a minimum time-to-reach function constrained within some set. If the set is known *a priori*, the computation of the function values can be straightforwardly performed by obtaining the solution to a constrained optimal control problem. However, in both cases the set is also unknown and actually depends upon the time-to-reach function, so both must be computed together. To do this in an efficient manner, we utilize modified versions of the fast marching method (FMM) to compute both the desired function values and the corresponding sets simultaneously on a grid. This section briefly summarizes the FMM algorithm, and then presents the computations of $\overline{v}$ and $\underline{\underline{v}}$ via modified FMM. We refer the readers to Sethian (1999) for a discussion on vanilla FMM.

4.1. Upper value

We present the modified FMM in detail to compute the safe-reachable set $\mathcal{S}$ and the corresponding minimum time-to-reach function $\overline{w_{\mathcal{S}}}$. Our algorithm computes the solution to the following HJB equation in $\mathcal{S}$:

$$-\inf_{a \in \bar{B}_n} \{\nabla \overline{w_{\mathcal{S}}}(y; x_A^0) \cdot f_A(x_A) a\} = 1 \tag{16}$$

along with the set $\mathcal{S}$ itself, using the boundary conditions

$$\overline{w_{\mathcal{S}}}(x_A^0; x_A^0) = 0; \ \overline{w_{\mathcal{S}}}(y; x_A^0) = \infty, \ y \in \Omega \setminus \mathcal{S}. \tag{17}$$

We note that if the speed function f_A is identified with $v(\cdot)$ in the Eikonal equation, then the previous HJB equation is equivalent to the Eikonal equation, provided f_A is isotropic, which is true for the assumptions in this paper.

To compute $\bar{t}_{(i,j)}$, we define $\mathcal{A}^x := \{y \in \Omega \mid (x, y) \in \mathcal{A}\}$ as a slice of the avoid set at a fixed P_A location $x \in \Omega$. Intuitively, this corresponds to the set of points which allows P_D to capture P_A at x. Then $\bar{t}_{(i,j)}$ for any node $y_{i,j}$ can be found as $\inf_{y \in \mathcal{A}^x} \phi_D(y)$, where $\phi_D(y)$ is the unconstrained minimum time-to-reach function representing the shortest time to reach y starting from x_D^0.

In the following, we provide a schematic description of the modified FMM, with the numerical approximation of $\overline{w_{\mathcal{S}}}(y; x_A^0)$ at a grid node (i, j) denoted as $\overline{w_{\mathcal{S}}}^{(i,j)}$.

1. Initialize $\overline{w_{\mathcal{S}}}^{(i,j)} = \infty$, $\forall$ node $(i, j) \in \mathcal{G}$.
2. Compute $\bar{t}_{(i,j)}$ for every $(i, j) \in \mathcal{G}$ from $\inf_{y_{i,j} \in \mathcal{A}^x} \phi_D(y_{i,j})$ by using standard FMM to compute the defender time-to-reach ϕ_D for every node (i, j).
3. Set $\overline{w_{\mathcal{S}}}^{(i,j)} = 0$, if node (i, j) is the initial position of P_A, set it to be in Accepted.
4. For all nodes (i, j) adjacent to a node in Accepted, set $\overline{w_{\mathcal{S}}}^{(i,j)} = \overline{w_{\mathcal{S}}}^{*(i,j)}$ via the Eikonal update and label them to be in NarrowBand.
5. Choose a node (i, j) in Narrow-Band with the smallest $\overline{w_{\mathcal{S}}}^{(i,j)}$ value. If there is no node remaining or $\overline{w_{\mathcal{S}}}^{(i,j)} = \infty$, continue to Step 6. Otherwise, if $\overline{w_{\mathcal{S}}}^{(i,j)} \geq \bar{t}_{(i,j)}$, then set $\overline{w_{\mathcal{S}}}^{(i,j)} = \infty$. Place node (i, j) in Accepted, return to Step 4.
6. Return the two arrays containing the values of $\overline{w_{\mathcal{S}}}^{(i,j)}$ and $\bar{t}_{(i,j)}$. Compute $\mathcal{S}$ by taking all nodes (i, j) with finite $\overline{w_{\mathcal{S}}}^{(i,j)}$ values. Compute $\overline{v}$ by first intersecting $\mathcal{S}$ with $\mathcal{T}$ and pick the smallest $\overline{w_{\mathcal{S}}}^{(i,j)}$ in the intersection and designate the corresponding node (i, j) to be the final point in $\mathcal{T}$.

The algorithm terminates in a finite number of iterations, since the total number of nodes is finite. On a grid with M nodes, the complexity is $O(M \log M)$ and the algorithm naturally extends to three or higher dimensions (Sethian, 1999).

Remark 1. Here we highlight the novel aspects of the modified FMM as given above. First, at a high level, we note that FMM is a numerical method for computing shortest paths (i.e. as solutions to certain minimum-to-reach type PDE as in HJB). However, note that here it is far from sufficient to just compute the shortest path from the attacker to the target: such a path must be computed subject to a crucial constraint that at each point along this path, the attacker must be safe (i.e. arriving there before the defender can intercept it).

Next, we give a detailed description of the modified aspects employed here. The algorithm presented above differs from the standard FMM in the addition of Step 5, which rejects points that are reachable by P_D in less time than P_A. To see why this modification is sufficient, suppose that, at the start of Step 4 of the current iteration, all Accepted nodes have the correct $\overline{w_{\mathcal{S}}}^{(i,j)}$ values. Suppose also that node (i, j) is the smallest element in NarrowBand ordered by $\overline{w_{\mathcal{S}}}$ value and $\overline{w_{\mathcal{S}}}^{(i,j)} \geq \bar{t}_{(i,j)}$. Since $\overline{w_{\mathcal{S}}}^{(i,j)}$ is computed using neighboring Accepted nodes (which are assumed to have the correct values), this implies that an optimal path in $\mathcal{S}$ would take longer than $\bar{t}_{(i,j)}$ to reach (i, j). This in turn implies that P_D will capture P_A should P_A attempt to reach (i, j). Therefore, $\overline{w_{\mathcal{S}}}^{(i,j)} = \infty$, since (i, j) cannot be safe-reachable. On the other hand, if $\overline{w_{\mathcal{S}}}^{(i,j)} < \bar{t}_{(i,j)}$, there is a safe-reachable path in $\mathcal{S}$ that takes less than $\bar{t}_{(i,j)}$ time to reach (i, j). Thus, right before Step 6, $\overline{w_{\mathcal{S}}}^{(i,j)} < \infty$ if and only if $(i, j) \in \mathcal{S}$. Since all Accepted nodes are initially correct (Step 3), the above argument holds inductively until all Accepted nodes are computed correctly. The result of the computation above is a grid $\mathcal{G}$ with nodes where $\overline{w_{\mathcal{S}}}^{(i,j)}$ approximates the value of $\overline{w_{\mathcal{S}}}$ for each node (i, j). The safe-reachable set $\mathcal{S}$ can then be approximated as $\mathcal{S} \approx \{(i, j) \mid \overline{w_{\mathcal{S}}}^{(i,j)} < \infty\}$.

4.2. Lower value

We use a different modified FMM algorithm to compute $\underline{v}$, the bound on the open-loop lower value. Computing $\underline{v}$ has some conceptual similarity to the computation of $\overline{v}$, but requires some extra

steps. Here, instead of computing $\overline{w_{\mathcal{S}}}$, $\bar{t}$, and $\mathcal{S}$, we are interested in computing $\underline{t}(x; x_A^0)$, $\underline{w_{\mathcal{Z}}}(x; x_D^0)$, $\mathcal{Z}$ and $\mathcal{Z}^*$. The computation of $\underline{t}$ requires more computation than that of $\overline{w_{\mathcal{S}}}$ and $\bar{t}$, as for each point in $\mathcal{Z}^*$ a separate FMM computation must be performed to find P_A's arrival time at the target. However, by using a modified FMM method, we are able to compute the $\underline{t}$, $\underline{w_{\mathcal{Z}}}$, and $\mathcal{Z}$ simultaneously and avoid computing $\underline{t}$ unnecessarily, reducing computation time.

In the following, we denote the numerical approximation of the functions $\underline{t}(y; x_A^0)$ and $\underline{w_{\mathcal{Z}}}(y; x_D^0)$ as $\underline{t}_{(i,j)}$ and $\underline{w_{\mathcal{Z}}}^{(i,j)}$, respectively. The notation $\mathcal{A}_{(i,j)}$ is used to denote the slice $\mathcal{A}_{y_{i,j}} = \{x \in \Omega \mid (x, y_{i,j}) \in \mathcal{A}\}$ of the avoid set at a fixed P_D location $y_{i,j}$. The sets Accepted and NarrowBand have the same meaning as before: Accepted represents the set of nodes whose corresponding $\underline{w_{\mathcal{Z}}}^{(i,j)}$ values have been computed. NarrowBand represents the set of nodes that are about to be added to Accepted. $W_{(i,j)}$ and $T_{(i,j)}$ are two arrays which are used to store values for $\underline{w_{\mathcal{Z}}}^{(i,j)}$ and $\underline{t}_{(i,j)}$, respectively. The algorithm then proceeds as follows:

1. Initialize $W_{(i,j)} = \infty$, $\forall$ node $(i, j) \in \mathcal{G}$.
2. $W_{(i,j)} = 0$, if (i, j) is the initial position of P_D, and set (i, j) to be in Accepted.
3. For each node (i, j) adjacent to a node in Accepted, run the Eikonal update to obtain the $\underline{w_{\mathcal{Z}}}^{(i,j)}$ value for (i, j). Set $W_{(i,j)} = \underline{w_{\mathcal{Z}}}^{(i,j)}$ for all (i, j) adjacent to a node in Accepted and place these nodes in NarrowBand.
4. Choose a node (i, j) in NarrowBand with the smallest $W_{(i,j)}$ value. If there is no node remaining or if it is equal to ∞ return W and T and continue to Step 6. Otherwise compute $\underline{t}_{(i,j)}$ by doing the following: treat $\mathcal{A}_{(i,j)}$ as an obstacle, compute $\underline{t}_{(i,j)}$ using standard FMM. Set $T_{(i,j)}$ to be $\underline{t}_{(i,j)}$ and put (i, j) into Accepted. If $W_{(i,j)} < \underline{t}_{(i,j)}$, set $W_{(i,j)}$ to $\underline{t}_{(i,j)}$; otherwise set $W_{(i,j)}$ to be ∞. Now return to Step 3.
5. Find a node (i, j) with the largest $T_{(i,j)}$ value, record this value in a variable M and set $T_{(i,j)}$ to be $-\infty$. Compute the reachable set $\mathcal{R}_1(M)$ using FMM and test if it has nonempty intersection with $\mathcal{A}_{(i,j)}$. If so, return to Step 5. Otherwise, set $\underline{v} = M$ and return $\underline{v}$.

Note that instead of computing $\underline{t}$ at all points, we compute it "on the fly" in the sense that we stop immediately when the smallest $W_{(i,j)}$ in Narrowband is equal to ∞. This results in a significant amount of computational savings, and is justified by the fact that if $W_{(i,j)} \geq \underline{t}_{(i,j)}$, then (i, j) is not in $\mathcal{Z}$, which means $W_{(i,j)}$ should be ∞ by definition of the function $\underline{w_{\mathcal{Z}}}$. Later, if we want to extract $\mathcal{Z}$, we need only look at the nodes that have finite values.

Remark 2. Here we outline the novel aspects of the modified FMM in the open-loop lower value computation. Again, it is insufficient for the defender to just compute the shortest path because it must arrive there before the attacker can in order to serve effectively as a obstacle. Furthermore, here, as opposed to both the traditional FMM and the modified FMM for the open-loop upper value, the defender is interested in selecting the largest $\underline{t}_{(i,j)}$, value that encodes how long it can delay the attacker from reaching the target. The addition of Steps (4) and (5) serves precisely those two purposes. Finally we note that, for both the open-loop upper and lower computation, with more general dynamics that are not isotropic, the FMM is not directly applicable. However a similar causality-ordering procedure is possible via the Ordered Upwind Method (OUM) (Sethian & Vladimirsky, 2003), allowing the method to be eventually extended to anisotropic (Sethian & Vladimirsky, 2003) and non-holonomic (Takei, Tsai, Shen, & Landa, 2010) dynamics.

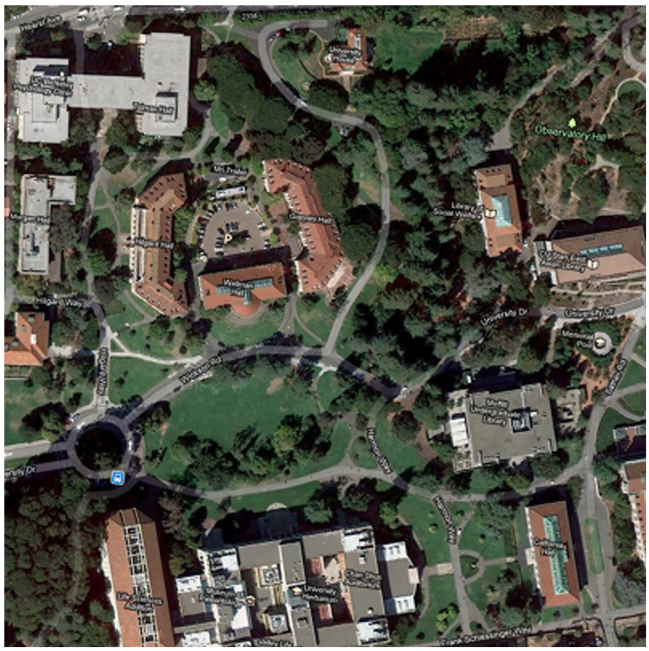

Fig. 2. A segment of the UC Berkeley campus used for the simulations (map image courtesy of maps.google.com).

4.3. Extracting control inputs

For the upper value game, if $\overline{v} < \infty$, the next step is to identify P_A's optimal control $\bar{a} \in \arg\min_{a \in \Sigma_A} \sup_{d \in \Sigma_D} \mathcal{J}(\mathbf{x^0}, a, d)$. Note that $\bar{a}$ is not necessarily unique, but all such inputs yield the same value $\overline{v}$. Given the result of Theorem 1, one can interpret $\bar{a}$ as a control input for P_A realizing a time-optimal safe-reachable path from x_A^0 to a final point $x_A^f \in \arg\inf_{y \in \mathcal{T} \cap \mathcal{S}} \overline{w_{\mathcal{S}}}(y; x_A^0)$. In fact, this final point is returned in Step 5 of our algorithm. Thus, $\overline{w_{\mathcal{S}}}$ can be used to extract the optimal control where it is smooth. In particular, given the dynamics in (1) and the assumption on the control set Σ_A, one can verify that the optimal control for P_A at a location $y \in \Omega_{free}$ where the value function $\overline{w_{\mathcal{S}}}$ is differentiable is given by $\mu(y) = -\frac{\nabla \overline{w_{\mathcal{S}}}(y; x_A^0)}{\|\nabla \overline{w_{\mathcal{S}}}(y; x_A^0)\|}$. In general, the value function $\overline{w_{\mathcal{S}}}$ may not be differentiable at every point $y \in \Omega_{free}$. Non-differentiable points typically correspond to locations at which the optimal input is not unique.

The optimal path $x_A^*(\cdot)$ of P_A can be then computed using $\mu(y)$ by solving the ordinary differential equation

$$\dot{x}_A^*(t) = f_A(x_A^*(t))\mu(x_A^*(t)) \tag{18}$$

from $t = \overline{w_{\mathcal{S}}}(x_A^f; x_A^0)$ to $t = 0$ backward in time, with the terminal condition $x_A^*(\overline{w_{\mathcal{S}}}(x_A^f; x_A^0)) = x_A^f$. Due to the construction of $\overline{w_{\mathcal{S}}}$, this results in $x_A^*(0) = x_A^0$. The realization of the optimal control is then given by $\bar{a}(t) = \mu(x_A^*(t))$.

For the open-loop lower value case, we can find the final point x_D^f for P_D by selecting the point within $\mathcal{Z}^*$ with the lowest $\underline{t}$ value. Then the optimal input map $\mu(\cdot)$ for P_D can be found using $\underline{w_{\mathcal{Z}}}(y; x_D^0)$ in a similar fashion $\overline{w_{\mathcal{S}}}(y; x_A^0)$, resulting in an optimal control $\underline{d}$ and an optimal path $x_D^*(\cdot)$.

4.4. Multilayer reach-avoid games and simulations

The modified fast marching methods can be extended to a multi-player reach-avoid game setting, where the attacking and/or

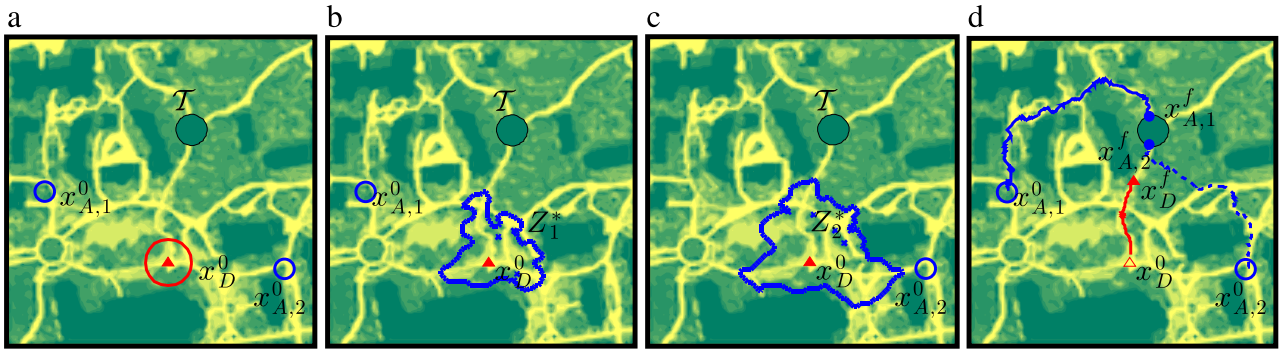

Fig. 3. Simulation of a game between two attackers and one defender for the lower value game, showing (a) the initial positions of the players and the target region, (b) $\mathcal{Z}^*$ with respect to the first attacker (solid blue line), (c) $\mathcal{Z}^*$ with respect to the second attacker (solid blue line), and (d) the paths taken by all of the players. The defender, by taking this route, can force the attackers to take the two routes respectively (of course they can take other suboptimal routes that may take them longer to reach the target), thereby guaranteeing that the attackers will not enter the target unless a certain amount of time has passed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

defending side can contain any number of players. Here we demonstrate the open-loop game formulation applied to more complex games and games with multiple players. The game scenarios are computed on a 2-D map of size 400^2 pixels, representing the UC Berkeley campus (see Fig. 2). To represent the varying terrain in the area, the map data $f_{i,j}$, $i, j \in 1, \ldots, 400$ represent the fraction of the players' maximum speeds that are allowed at each point on the map, with 1 being maximum speed and 0 representing impenetrable obstacles. They have values 0 in the buildings and 1 in the walkways; other regions have intermediate values in $(0, 1)$ selected in proportion to the estimated density of vegetation. Due to space limitation, we only show the set of simulations for the lower value game with two attackers and one defender (see Fig. 3). All computations were performed on a desktop computer with 3.33 GHz Intel Core-II duo processors.

References

Bardi, M., & Capuzzo-Dolcetta, I. (1997). *Systems & control: Foundations & applications. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations* (p. xviii+570). Boston, MA: Birkhäuser, With appendices by Maurizio Falcone and Pierpaolo Soravia.

Başar, T., & Olsder, G. (1999). *Dynamic noncooperative game theory*. (2nd ed.). Philadelphia, PA: SIAM.

Chen, Mo, Zhou, Zhengyuan, & Tomlin, Claire J. (2014). Multiplayer reach-avoid games via low dimensional solutions and maximum matching. In *American control conference (ACC), 2014* (pp. 1444–1449). IEEE.

Chen, Mo, Zhou, Zhengyuan, & Tomlin, Claire J. (2017). Multiplayer reach-avoid games via pairwise outcomes. *IEEE Transactions on Automatic Control*, *62*(3), 1451–1457.

Doyen, Laurent, & Raskin, Jean-François (2010). *Lectures on games theory for the computer scientist*. *Games with imperfect information: Theory and algorithms* (pp. 185–212).

Elliott, R. J., & Kalton, N. J. (1972). The existence of value in differential games. In *Memoirs of the American mathematical society, Vol. 126*.

Evans, L. C., & Souganidis, P. E. (1984). Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana University Mathematics Journal*, *33*(5), 773–797.

Falcone, M., & Ferretti, R. (2002). Semi-Lagrangian schemes for Hamilton-Jacobi equations, discrete representation formulae and Godunov methods. *Journal of Computational Physics*, *175*(2), 559–575.

Fiorini, P., & Shiller, Z. (1998). Motion planning in dynamic environments using velocity obstacles. *International Journal of Robotics Research*, *17*(7), 760.

Flynn, James (1974). Lion and man: the general case. *SIAM Journal on Control*, *12*(4), 581–597.

Fraichard, Thierry, & Asama, Hajime (2004). Inevitable collision states - a step towards safer robots? *Advanced Robotics*, *18*(10), 1001–1024.

Friedman, A. (1971). *Pure and applied mathematics : a series of texts and monographs. Differential games*. Wiley-Interscience.

Huang, H., Ding, J., Zhang, W., & Tomlin, C. J. (2011). A differential game approach to planning in adversarial scenarios: a case study on capture-the-flag. In *Proceedings of the IEEE international conference on robotics and automation*, Shanghai, China.

Isaacs, R. (1967). *Differential games*. New York: Wiley.

Johnson, M., Bhasin, S., & Dixon, W. E. (2011). Nonlinear two-player zero-sum game approximate solution using a Policy Iteration algorithm. In *2011 50th IEEE conference on decision and control and European control conference, CDC-ECC*, (pp. 142–147). http://dx.doi.org/10.1109/CDC.2011.6160778, ISSN: 0743-1546.

Karaman, Sertac, & Frazzoli, Emilio (2011a). Incremental sampling-based algorithms for a class of pursuit-evasion games. In David Hsu, Volkan Isler, Jean-Claude Latombe, & Ming C. Lin (Eds.), *Springer tracts in advanced robotics*: *Vol. 68. Algorithmic foundations of robotics IX* (pp. 71–87). Springer Berlin Heidelberg.

Karaman, Sertac, & Frazzoli, Emilio (2011b). Sampling-based algorithms for optimal motion planning. *International Journal of Robotics Research*, *30*(7), 846–894.

Kawecki, P., Kraska, B., Majcherek, K., & Zola, M. (2009). Guarding a line segment. *Systems & Control Letters*, *58*(7), 540–545.

Khanafer, A., Touri, B., & Basar, T. (2012). Consensus in the presence of an adversary. In *3rd IFAC workshop on distributed estimation and control in networked systems, NecSys*.

Khanafer, A., Touri, B., & Basar, T. (2013). Robust distributed averaging on networks with adversarial intervention. In *2013 IEEE 52nd annual conference on decision and control, CDC*, (pp. 7131–7136). http://dx.doi.org/10.1109/CDC.2013.6761020, ISSN: 0743-1546.

Lewin, J. (1986). The lion and man problem revisited. *Journal of Optimization Theory and Applications*, *49*(3), 411–430.

Liu, Shih-Yuan, Zhou, Zhengyuan, Tomlin, Claire, & Hedrick, Karl (2013). Evasion as a team against a faster pursuer. In *American control conference (ACC), 2013* (pp. 5368–5373). IEEE.

Munos, Remi, Baird, Leemon C., & Moore, Andrew W. (1999). Gradient descent approaches to neural-net-based solutions of the Hamilton-Jacobi-Bellman equation. In *International joint conference on neural networks, 1999, Vol. 3.* IJCNN'99, (pp. 2152–2157). IEEE.

Rzymowski, W. (2009). A problem of guarding line segment. In *Proceedings of the 48th IEEE conference on decision and control, 2009 held jointly with the 2009 28th Chinese control conference. CDC/CCC 2009*, (pp. 6444–6447), http://dx.doi.org/10.1109/CDC.2009.5400251 ISSN: 0191-2216.

Sebbane, Y. B. (2014). *Intelligent systems, control and automation: Science and engineering*. *Planning and decision making for aerial robots*. Springer International Publishing.

Sethian, J. A. (1999). *Cambridge monographs on applied and computational mathematics, Vol. 3. Level set methods and fast marching methods*. (2nd ed.). (p. xx+378). Cambridge, UK: Cambridge University Press.

Sethian, J. A., & Vladimirsky, A. (2003). Ordered upwind methods for static Hamilton-Jacobi equations: theory and algorithms. *SIAM Journal of Numerical Analysis*, *41*(1), 325–363.

Takei, Ryo, Huang, Haomiao, Ding, Jerry, & Tomlin, C. J. (2012). Time-optimal multi-stage motion planning with guaranteed collision avoidance via an open-loop game formulation. In *Proceedings of the IEEE international conference on robotics and automation, Minneapolis, Minnesota*.

Takei, R., Tsai, R., Shen, H., & Landa, Y. (2010). A practical path-planning algorithm for a vehicle with a constrained turning radius: a Hamilton-Jacobi approach. In *Proceedings of the IEEE American control conference*, July.

Vamvoudakis, Kyriakos G., & Lewis, F. L. (2012). Online solution of nonlinear two-player zero-sum games using synchronous policy iteration. *International Journal of Robust and Nonlinear Control*, *22*(13), 1460–1483.

Van Den Berg, J., & Overmars, M. (2008). Planning time-minimal safe paths amidst unpredictably moving obstacles. *International Journal of Robotics Research*, *27*(11–12), 1274.

Varaiya, P. (1967). On the existence of solutions to a differential game. *SIAM Journal on Control*, *5*(1), 153–162.

Zhou, Zhengyuan, Ding, Jerry, Huang, Haomiao, Takei, Ryo, & Tomlin, Claire (2017). Efficient Algorithms for Open-Loop Reach-Avoid Games, UCLA CAM Report, ftp://ftp.math.ucla.edu/pub/camreport/cam17-49.pdf.

Zhou, Zhengyuan, Takei, Ryo, Huang, Haomiao, & Tomlin, C. J. (2012). A general, open-loop formulation for reach-avoid games. In *Proceedings of the IEEE international conference on decision and control*, Maui, Hawaii.

Zhou, Zhengyuan, Zhang, Wei, Ding, Jerry, Huang, Haomiao, Stipanović, Dušan M., & Tomlin, Claire J. (2016). Cooperative pursuit with Voronoi partitions. *Automatica*, *72*, 64–72.

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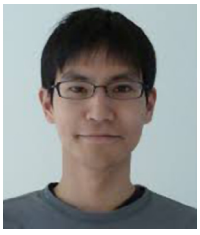

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