On-line Trajectory Generation for Safe and Optimal Vehicle Motion Planning

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Abstract This paper presents a framework for motion planning of autonomous vehicles, it is characterized by its efficient computation and its safety guarantees. An *optimal control* based approach generates comfortable and physically feasible maneuvers of the vehicle. Therefore, a combined optimization of the lateral and longitudinal movements in street-relative coordinates with carefully chosen cost functionals and terminal state sets is performed. The collision checking of the trajectories during the planning horizon is also performed in street-relative coordinates. It provides continuous collision checking, which covers nearly all situations based on an algebraic solution and has a constant response time. Finally, the problem of safety assessment for partial trajectories beyond the planning horizon is addressed. Therefore, the *Inevitable Collision States* (ICS) are used, extending the safety assessment to an infinite time horizon. To solve the ICS computation nonlinear programming is applied. An example implementation of the proposed framework is applied to simulation scenarios that demonstrates its efficiency and safety capabilities.

1 Introduction

A number of projects in the field on autonomous driving have been initiated in the last decades. They range from driving in unstructured off-road environments to structured urban driving. Each environment poses special requirements to be fulfilled. Whereas

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the main challenges in off-road driving are to find drivable routes while obtaining a good localization of the vehicle, the main challenges in structured urban driving are to cope with other traffic participants and to guarantee a maximum level of safety.

In this paper we focus on high-speed driving in structured environments like country roads as well as highways. The special demands for such kind environments are: Smooth maneuvers, high level of safety, hard real-time constraints and a long foresight. Recently, the project *HAVEit* compared two approaches for such kind of environments in [7]. As a result, the quintic polynomial based approach outperformed the other method based on searching the optimal solution in the discretized command space.

Our approach also makes use of quintic polynomials, but instead of generating trajectories in world coordinates, they are represented in lane coordinates of the road. This allows to generate comfortable minimum jerk maneuvers at a wide speed range. Therefore, an optimal control approach is used excluding other vehicles, which is presented in Sect. 2. Former limitations of this approach which is presented in detail in [9] regarding safety guarantees of the trajectories are addressed in Sect. 3. This safety assessment is performed by continuous collision checking during the planning horizon as well as a safety evaluation of the last state, allowing to guarantee safety assuming a reliable motion prediction of the other road users. Since no on-line optimization is included, the presented approach has a bounded response time and is on-line capable.

2 Optimal Control Formulation

Street relative coordinates and cost decomposition In most traffic situations the human driver plans the vehicle's lateral movement relative to the lanes rather than to the absolute ground. Imitating this approach, the trajectory generation problem is formulated in the so-called Frenet frame $[\mathbf{n}_c, \mathbf{t}_c]$ of the street, shown in Fig. 1. Here, the offset to the lane center is denoted by $d(t)$ and $s(t)$ describes the covered arc length of the frame's root point $\mathbf{r}(s)$ along the center line. This allows the fast computable closed-form reparameterization $\widetilde{\mathbf{u}}(s(t), d(t)) = \mathbf{r}(s(t)) + d(t)\mathbf{n}_c(s(t))$ of the planned trajectory $\widetilde{\mathbf{u}}(t)$. Next, we assume that the trajectory costs J may be separated into a lateral and a longitudinal component, J_d and J_s, according to the weighted sum $J[d, s] = J_d[d] + k_s J_s[s], k_s > 0$. Focusing in a first step on its component $d(t)$, we now define the lateral cost functional to be

$$J_d[d] := \frac{1}{2} \int_0^{\tau} \dddot{d}^{\,2}(t)\mathrm{d}t + (h(d(t), t))_{\tau} \quad (1)$$

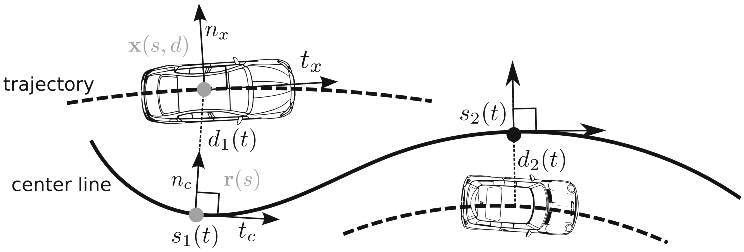

Fig. 1 Vehicles represented in the Frenet frame

with yet both unspecified end costs $(h(\xi(t),t))_\tau$ and end time τ. It can be shown (see e.g. [9]) that the unconstrained[1] movement of $d(t)$ that transfers the vehicle from the initial state $[d(0), \dot{d}(0), \ddot{d}(0)]$ to a given end state $[d(\tau), \dot{d}(\tau), \ddot{d}(\tau)]$ while minimizing (1), is a fifth-order polynomial. This gives us the general shape of the lateral trajectory, so that only the end state and the end time is left for optimization. In doing so, the target application narrows a priori the set of reasonable solutions. Also on a partially blocked road, the vehicle should generally progress along it and not crosswise. Thus, we constrain $d(t)$'s first and second derivative at time τ to be zero.

As for the choice of the terminal time, reaching the end state early might lead to uncomfortable, energetically wasteful actions, whereas a late arrival implies lagging movements. Since these issues are also strongly coupled with the end state, we seek to find the best trade-off by defining the terminal costs in (1) to be

$$(h(d(t),t))_\tau := (k_\tau t + \tfrac{1}{2} k_d [d(t) - d_{\mathrm{ref}}]^2)_\tau \qquad (2)$$

with $k_\tau, k_d > 0$, which penalize both slow convergence and final deviations from the reference trajectory with e.g. $d_{\mathrm{ref}} = 0$.

Generating trajectory sets In order to reduce the number of end states, we now only allow the lateral trajectory to arrive at certain points in absolute time as well as with certain discrete distances to the reference trajectory d_{ref}. Consequently, all admissible polynomials form an entire fan-shaped trajectory set evenly covering the maneuver space as shown on the left of Fig. 2. The associated costs to each trajectory can be quickly evaluated with (1) and (2) in closed form. Analog considerations lead to a set of polynomial movements for the longitudinal velocity $\dot{s}(t)$, which can be seen on the right of Fig. 2. Before we crosswise superimpose the respective longitudinal with the lateral set and back-transform to global coordinates the generated trajectories are checked for collision in the Frenet frame. After the best, collision-free trajectory is obtained, the Frenet coordinates are projected back in the world coordinates and handed over to a low-level controller.

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[1] no restrictions such as obstacles.

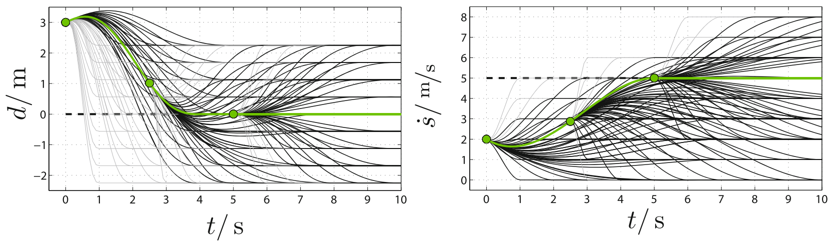

Fig. 2 Simulation of an optimal transfer to the *dashed* reference by replanning. In each step green is the optimal trajectory, *black* are the valid and *gray* the invalid alternatives

3 Safety Assessment

The presented trajectory generation approach can be characterized as a Partial Motion Planning (PMP) [6] method, since the trajectories have a limited time horizon and do not reach the global navigation goal. Thereby, the problem arises, that the final state of the vehicle may have a non-zero velocity. As pointed out in [6] it is indispensable to assess the safety of the partial trajectory by using the Inevitable Collision States (ICS) [4] for evaluating the final state of the trajectory. The idea of ICS is to identify states of the vehicle which will eventually lead to a collision at some point in the future, meaning to assess the safety beyond the planning horizon of the trajectories.

To comply with these safety criteria the presented safety assessment approach is separated in two parts: assessment of the trajectories during the planning horizon and an ICS check of the last state of the trajectory beyond the planning horizon. First, safety assessment during the planning horizon is addressed by an algebraic solution for trajectories in the Frenet frame.

3.1 Trajectory Collision Checking

The generated trajectories are checked for collision in a fixed horizon with a time-continuous approach, i.e. without the need to sample the trajectories in time space, which is the common approach. Instead, the presented approach relates the collision assessment problem to a root finding problem. The derived algorithm is fast, reliable and conservative.

The Frenet frame assures in high-speed freeway scenarios that the angular offset from the axis s can be considered zero i.e. the vehicles' orientation is always equal and aligned with the course of the road. For the algorithm, we assume the vehicles to have a rectangular shape with length l_i and width w_i for the ith vehicle. In the remainder, the vehicles are checked for collisions pairwise.

The relative position function of the two vehicles i and j is defined as $s_{ij}(t) := s_j(t) - s_i(t)$, $d_{ij}(t) := d_j(t) - d_i(t)$ again polynomials. The Minkowski sum of the two vehicles has the size $l_{ij} = l_i + l_j$ times $w_{ij} = w_i + w_j$. A collision state C of the two considered vehicles in the horizon $[0, T)$ is defined as

$$C \Leftrightarrow \exists t \in [0, T):\ s_{ij}(t) \in (-\tfrac{l_{ij}}{2}, \tfrac{l_{ij}}{2})\ \wedge\ d_{ij}(t) \in (-\tfrac{w_{ij}}{2}, \tfrac{w_{ij}}{2}).$$

Its complementary event that no collision will occur is depicted as $\overline{C}$. A trajectory will collide if the hulls intersect in the considered horizon,

$$\exists t \in (0, T):\ s_{ij}(t) = s_c\ \wedge\ d_{ij}(t) = d_c \quad \Rightarrow C$$

with d_c and s_c the point of collision on one of the edges of the Minkowski sum

$$\Big[(d_c = -\tfrac{w_{ij}}{2}\ \vee\ d_c = \tfrac{w_{ij}}{2})\ \wedge\ s_c \in (-\tfrac{l_{ij}}{2}, \tfrac{l_{ij}}{2})\Big] \vee \Big[(s_c = -\tfrac{l_{ij}}{2}\ \vee\ s_c = \tfrac{l_{ij}}{2})\ \wedge\ d_c \in (-\tfrac{w_{ij}}{2}, \tfrac{w_{ij}}{2})\Big].$$

Hence the collision assessment problem is a root finding problem of the four polynomials, $s_{ij}(t) \pm \frac{l_{ij}}{2} = 0$ and $d_{ij}(t) \pm \frac{w_{ij}}{2} = 0$. If one of the polynomials has a degree lower than five, the problem can be solved directly by solving the polynomial equation and evaluating the other polynomial equation at these roots.

But, in case $s_{ij}(t) \pm \frac{l_{ij}}{2}$ and $d_{ij}(t) \pm \frac{w_{ij}}{2}$ are square-free quintic polynomials this is not possible as, according to the Abel-Ruffini theorem (see [8]), no general algebraic solution for the polynomials can found. To deal with this case an algorithm has been developed which is presented in detail in (Lawitzky, A., Buss, D.M., 2012, Maneuver-based risk assessment for high-speed automotive scenarios, unpublished.) and sketched here. This algorithm is divided into consecutive, algebraic tests.

First, two sufficient conditions can be formulated for a collision-free trajectory by separating the dimensions s and d, saying that no collision will occur if the vehicles do not come close enough for a collision in either dimension

$$s_0 \notin (-\tfrac{l_{ij}}{2}, \tfrac{l_{ij}}{2}) \wedge \forall t \in [0, T):\ s_{ij}(t) \pm \tfrac{l_{ij}}{2} \neq 0 \quad \Rightarrow \overline{C}$$

where both conditions of $\pm$ need to evaluate true. A similar condition can be obtained for d_{ij}. Note that it is not necessary to determine the roots of $s_{ij}(t)$ and $d_{ij}(t)$, but only whether or not there are any in the horizon.

An implicit formula $F(s_{ij}, d_{ij}) = 0$ can be obtained using the *resultant* [3], $F(s_c, d_c) = \mathrm{Res}(s_{ij}(t) - s_c, d_{ij}(t) - d_c)$ a quintic polynomial in s_c and d_c, for fixed other dimension.

If $F(s_c, d_c)$ has no roots along the borders $\pm w_{ij}$ and $\pm l_{ij}$ no collision will occur. As the resultant $F(s_c, d_c)$ looses the time information completely, it is impossible to determine whether the combined trajectory intersects with the hull during the considered timespan or outside of it. The number of roots of $F(s_c, d_c)$ along the four borders $i = 1 \ldots 4$ is depicted as r_i.

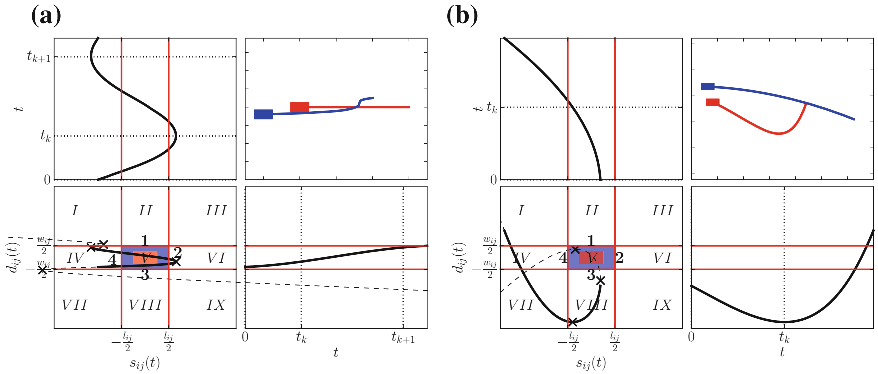

Fig. 3 The Minkowski sum's rectangle and the trajectories are shown in s and d coordinates (*left bottom*) as well as their temporal trajectories $s_{ij}(t)$ and $d_{ij}(t)$ (*left top and right bottom*). Additionally, an example scene for two vehicles is sketched (*right top*). The fields around the Minkowski sum's rectangle are depicted with roman numerals *I–IX*. The lines are depicted with numbers 1–4. The *black line* depicts the trajectory in the time horizon (*solid*) and outside of it (*dashed*). **a** In this example scenario the trajectory starts in field *IV*. The trajectory is next evaluated at the next extremum at $t = t_k$ which belongs to field *VI*. Note that by design of the regarded trajectory points, the trajectory has to cross *lines* 4 and 2 of the rectangle—other ways are not possible as otherwise an extremum would have to be in between. Consequently, a collision occurs. **b** In this scenario the trajectory starts in field *VIII* and heads to field *I*. All previous tests could not make a decision whether or not the trajectory pair collides. Note that $r_3 = r_4 = 1$ and the number of obvious hits with *line* 3, $n_3 = 1$ can be determined with the discussed technique. With this $p_3 = 1 - 1 = 0$, so there is no way that the trajectory collides (as the rectangle is a connected space). Hence, the trajectory is classified as safe

With the extrema values of $s_{ij}(t)$ and $d_{ij}(t)$ another test is performed based on the fact that the Minkowski sum is a connected space. This is illustrated in Fig. 3a where an example scene sketches a trajectory. For this test we separate the trajectory in pieces between points in time of extrema of either $s_{ij}(t)$ or $d_{ij}(t)$. With this, we argue that e.g. all trajectories coming from field II in Fig. 3a going to field $VIII$ had to intersect with lines 1 and 3. Similar conditions can be obtained for the other fields. Let $n_{i,(a,b)}$ be the number of the obtained obvious intersections with line i in timespan (a, b). If there occured such an intersection with one of the lines the trajectory is considered unsafe.

$n_i = n_{i,(-\infty,\infty)}$ is defined as the number of all obvious intersections with line i. As noted before, r_i is the number of roots on the four lines of the resultant. Let $p_i = r_i - n_i, i = 1 \ldots 4$ the number of possible hits with line i. There will be no collision, if there is no possible connection when you go through each of the sections between 0 and T. E.g. a trajectory section starting at field II going to field VI can only collide if both, $p_1 \neq 0$ and $p_2 \neq 0$ are fulfilled. Similar rules can be given for all trajectory sections, see Fig. 3b.

The presented approach offers an fast, algebraic solution for the problem of collision assessment. The theoretical case that none of the tests returned a clear statement is interpreted as unsafe. The fact that it is possible to construct situations where this conservative algorithm gives a wrong answer is irrelevant in practice.

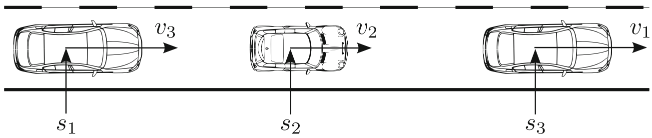

Fig. 4 Example lane scenario

3.2 Inevitable Collision States

As mentioned in Sect. 3 it is necessary to evaluate the safety of the final state of the trajectories. The presented ICS checker provides continuous results and considers the collision avoidance behavior of the other vehicles. Thus, the ICS based safety assessment is performed for the complete road scene instead of only considering the ego vehicle. First the definition of ICS is recalled from literature [4].

The state x is an Inevitable Collision State (ICS) if all possible control inputs lead at one time to a collision with at least one object.

In the remainder the problem of reasoning about the safety of road scenes is addressed for an infinite time horizon (T, ∞). To make use of the structured environment of roads, we assess the safety of each lane separately. Therefore, we use the nonlinear one-dimensional vehicle model presented in [1], since it shows a good trade-off between accuracy and complexity. The complete state $\mathbf{x}$ of a lane containing n vehicles is described by the initial velocities $\mathbf{v}(T) = [v_1(T), \ldots, v_n(T)]$ and the initial distances between the vehicles $\mathbf{s}(T) = [s_{12}(T), \ldots, s_{n-1,n}(T)]$. An example road scene is illustrated in Fig. 4. In order to classify a whole lane as safe we extend the common ICS definition. This new definition is based on the assumption that all workspace objects move independently from each other. We consider environments populated by human controlled vehicles which react on each other in order to prevent collisions. Thus, the ICS definition is extended to a cooperative ICS definition. This entails that instead of evaluating the safety of a single object, the safety of a group of objects is evaluated. A system state $\mathbf{x}$ is an Cooperative Inevitable Collision State (ICS^c) if

$$\text{ICS}^c \Leftrightarrow \forall \tilde{\mathbf{u}} \in \tilde{\mathcal{U}} \; \exists \, (i, j | i \neq j) \; \exists t \in (T, \infty) \colon \; |s_{ij}(t)| < \frac{l_{ij}}{2}.$$

where $\tilde{\mathbf{u}} = [\tilde{u}_1, \ldots, \tilde{u}_n]$ are the future trajectories of all objects including the ego vehicle and $\tilde{\mathcal{U}}$ is the set of input trajectories. Loosely speaking, we need to find one trajectory $\tilde{u}$ for each vehicle which will be collision-free regarding an infinite time horizon. In this paper we assume constant control inputs for all vehicles $\tilde{u} \mathrel{\widehat{=}} u$. Thus, the goal is to find the set of control inputs

$$\mathbf{u} = [u_1, \ldots, u_n] \text{ subject to } \;\; \forall (i, j | i \neq j) \;\; \forall t \in (T, \infty) \;\;\; |s_{ij}(t)| - \frac{l_{ij}}{2} \geq 0. \quad (3)$$

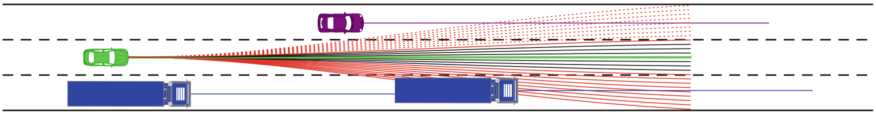

Fig. 5 Trajectories leading to collision during the planning horizon are illustrated by *solid red lines*, *dashed red lines* represent trajectories leading to an ICS^c.The supposedly best trajectory is shown by the *green line*

The problem described in (3) is solved by pairwise nonlinear programming as presented in [2]. In order to reduce computational time and to guarantee deterministic response times, the solution of the nonlinear programming problem is stored in a 4D lookup table (LUT). The input of the LUT are both initial velocities $v_r(T)$, $v_f(T)$, the initial free space $|s_{fr}(T)| - \frac{l_{fr}}{2}$ and the constant control input u_f. The indices f and r refer to the front or rear vehicle respectively. The output of the LUT is the maximum possible control input $u_r^{\max}$ of the rear vehicle.

ICS Checker The basic idea of the ICS^c checker is to perform an ICS check for each rear vehicle regarding the front vehicle. By applying the ICS check pairwise to all vehicles the lane is classified as safe if a valid control input is found for each vehicle. The control input of the front car is the maximum possible control input u_r of the previously evaluated pair of vehicles. Except for the first pair, whose control input needs to be determined a priori.

4 Simulation Results

In this section, simulations are provided to evaluate the presented approach. We first show the individual steps of the complete approach through the use of an example scenario representing a common highway situation. The ego vehicle (green) is driving in the middle lane and generates trajectories for all three lanes. To avoid illustration difficulties, only trajectories for the current velocity are drawn. The snapshot of the scenario as well as the generated trajectories and the results of the safety assessment are illustrated in Fig. 5. The motion of the other vehicles are predicted with constant velocity in their current lane. As can be seen, all trajectories leading to collision during or beyond the planning horizon are correctly identified: The ego car will collide with the blue trucks on the left inside the planning horizon, checked with the algorithm of Sect. 3.1. Furthermore it will collide with the purple vehicle beyond the planning horizon according to Sect. 3.2, since the velocity of the ego vehicle is higher than the velocity of the purple one. Ignoring constraints by traffic regulations and aiming to keep the current velocity, the trajectory with the lowest cost is the constant velocity trajectory.

To evaluate the computational performance of the presented algorithm we used a 6:23 min simulation highway drive. The current C++ implementation was tested on a Intel core I5-2500 using only a single core. The average response time was 0.010 s with an average of 469.26 generated trajectories for the ego vehicle and 9.25 surrounding vehicles. At the worst instant of our scenario, the algorithm had to check 677 trajectories with 17 vehicles for collision resulting in a worst case response time of 0.121s.

The performance of the presented algorithm has been exhaustively tested for hours of collision-free driving under various conditions.

5 Conclusion and Future Work

In this paper the optimal control based motion planner [9] was extended to address former limitations. By introducing an algebraic based continuous collision checker in combination with a novel ICS checker, this approach fulfills the three safety criteria proposed by [5]. The presented safety assessment guarantees motion safety during and beyond the planning horizon. This novel framework for autonomous driving is especially suitable for high-speed navigation on freeways. The derived optimal-control-based solution allows lane-changes, distance-keeping, velocity-keeping, merging, etc. amidst moving and stationary obstacles. Simulation results showed the online capabilities of the complete framework. It is planned to implement this approach on an experimental platform to verify the results in real-world experiments.

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