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Path Planning and Integrated Collision Avoidance for Autonomous Vehicles

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Abstract— **This paper discusses some of the current state-of-the-art and remaining challenges in research on path planning and vehicle control of autonomous vehicles. Reliable path planning is fundamental for the proper operation of an autonomous vehicle. Typically, the path planner relies on an incomplete model of the surroundings to generate a reference trajectory, used as input to a vehicle controller that tracks this reference trajectory. Depending on how much complexity is put into the path-planning block, the path planning and vehicle-control blocks can be viewed as independent of each other, connected to each other, or merged into one block. There are several types of path-planning techniques developed over the last decades, each with its own set of benefits and drawbacks. We review different techniques for path planning and trajectory tracking, and give examples of its use in relation to autonomous vehicles. We report on our own recent findings and give an outlook on potential research directions.**

I. INTRODUCTION

One of several key problems to solve before autonomous vehicles become a reality is how to determine a collision-free trajectory of the autonomous vehicle, when the sensor information provides an incomplete, and occasionally even wrong, picture of the surroundings. The sensing and mapping module use various sensor information, such as Radar, Lidar, camera, and global positioning system (GPS) information, together with prior map information, to estimate the parts of the surroundings relevant to the driving scenario. The outputs from the sensing and mapping block provides the basis for the path planner, which produces a desired trajectory that the vehicle should follow. Autonomous vehicle control is commonly divided into trajectory generation (path planning) and trajectory tracking (vehicle control). However, the two modules should not be seen as isolated parts of the autonomous vehicle, but should ideally work together.

Given a system that is subject to a set of differential constraints (the dynamics), an initial state, a final state, a set of obstacles, a set of environmental constraints, and a goal region, the path-planning problem is to find a trajectory that drives the system from its initial state to the goal region. For general environments the path-planning problem is known to be computationally difficult [1], and there exist many different approaches for solving the path-planning problem. There has been extensive developments in graph-search methods, for example, A* [2], D* [3], [4], and D*-lite [5], with several applications to vehicle autonomy [6], [7]. The use of probabilistic planners based on rapidly-exploring random trees (RRTs) has been an active area of research over the last two decades [8]–[12], and has also been used successfully in autonomous vehicles [13]–[15]. RRT relies on random exploration of the state space. Randomized planners are widely employed because they provide a path whenever one exists, and optimal variants of the suboptimal RRT have been developed recently. Oftentimes structure and prior knowledge of the particular problem instance can be imposed. In such cases, deterministic trajectory generation based on optimal control can be used [16]–[18], but nonconvexity of the underlying planning problem is often an issue.

For proper overall system performance, the path-planning and vehicle-control blocks should exchange information with each other. For example, the vehicle control should be able to closely track the path computed by the path planner. Hence, limitations on the paths that can be tracked should be transferred to the path planner. Similarly, depending on the type of path (e.g., lane change or overtaking) the specific type of vehicle control that is being used may change and should be propagated to the vehicle control. Both path planning and vehicle control should consider prediction of the environment.

Autonomous vehicles must react to emergency situations, for example, emergency turning to avoid suddenly appearing pedestrians and/or emergency braking to avoid collision with rapidly approaching vehicles. This is traditionally handled in the vehicle control module. Methods based on constrained control such as model-predictive control (MPC) [19]–[21] have been proven well suited for this problem, because of their ability to efficiently handle constraints and that the trajectory tracking problem is often convex, or at least well approximated by a convex problem.

In this paper, we review techniques related to integrating path planning and collision avoidance for autonomous vehicles, and talk about some of the main research challenges associated with control of autonomous vehicles. We give examples from our own research and end with a discussion.

II. VEHICLE MODELS

The detail level of the employed models vary with the intended purpose of the model. It is important that the path-planning module at some point considers the constraints imposed by the kinematics of the vehicle, which implies that the models need to consider dynamic feasibility. The vehicle control should also cover the interaction between road and vehicle, and in case of emergency maneuvers the models need to be complex enough to cover, or at least be able to predict, vehicle behavior in the nonlinear operating regime

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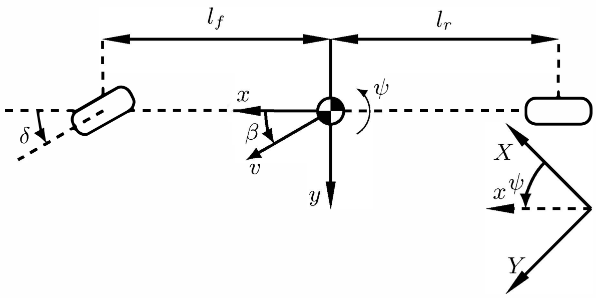


Fig. 1. Schematics of the single-track model. The capital letters denote the inertial frame.

of the vehicle. The models can broadly be characterized into kinematic and dynamic models, respectively. The kinematic single-track model is a model that mimics the vehicle dynamics well under mild driving conditions where wheel slip can be neglected. Fig. 1 provides a schematic of the model. The dynamic equations are:

$$\dot{x} = \begin{bmatrix} \dot{p}_X \\ \dot{p}_Y \\ \dot{\psi} \\ \dot{v}_x \end{bmatrix} = f(x, u) = \begin{bmatrix} v_x \cos(\psi + \beta)/\cos(\beta) \\ v_x \sin(\psi + \beta)/\cos(\beta) \\ v_x \tan(\delta)/L \\ a_x \end{bmatrix},$$

where $\dot{\psi}$ is the yaw rate; v_x is the longitudinal velocity at the center of mass; l_f, l_r are the distances from the center of mass to the front and rear wheel base $L = l_f + l_r$; the inputs u are given by the steering angle δ and the acceleration a_x; and β is the body slip angle, defined as $\beta = \arctan\left(l_r \tan(\delta)/L\right)$. The dynamic single-track model has the same kinematics as in Fig. 1 but the equations are instead given by [17]

$$\begin{aligned} \dot{v}_x - v_y\dot{\psi} &= \frac{1}{m}(F_{x,f}\cos(\delta) + F_{x,r} - F_{y,f}\sin(\delta)), \\ \dot{v}_y + v_x\dot{\psi} &= \frac{1}{m}(F_{y,f}\cos(\delta) + F_{y,r} + F_{x,f}\sin(\delta)), \\ I_{zz}\ddot{\psi} &= l_f F_{y,f}\cos(\delta) - l_r F_{y,r} + l_f F_{x,f}\sin(\delta), \end{aligned} \tag{1}$$

where m is the vehicle mass; I_{zz} is the vehicle inertia about the z-axis; and $\{F_{x,i}, F_{y,i}\}_{i=f,r}$ are the longitudinal and lateral tire forces acting at the front and rear wheels, respectively. The forces are in general modeled as nonlinear functions of the wheel slip λ and wheel slip angle α, where the slip angles α_f, α_r are described by [22]

$$\alpha_f = \delta - \arctan\left(\frac{v_y + l_f\dot{\psi}}{v_x}\right), \quad \alpha_r = -\arctan\left(\frac{v_y - l_r\dot{\psi}}{v_x}\right). \tag{2}$$

For moderate driving, small-angle approximations and linear tire-force expressions can be used, resulting in a linear dynamic model of the vehicle.

For the path planning problem, the models provided here are sufficiently accurate, but the vehicle control may, depending on how aggressive maneuvering should be captured, need to consider even more complex models. Considering even simpler models can be possible depending on how low-level the path plan should be. For instance, curvature-bounded point-mass models may be sufficient in certain scenarios. However, ideally path planning and vehicle control should at least share common characteristics, to be able to provide guarantees on performance of the overall system.

III. PATH PLANNING

Let $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{Y} \subseteq \mathbb{R}^p$, and $\mathcal{U} \subseteq \mathbb{R}^m$, and assume that the vehicle dynamics can be expressed by a nonlinear differential equation on the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), x(0) = x_0 \\ y(t) &= h(x(t), u(t)), \end{aligned} \tag{3}$$

where the state $x(t) \in \mathcal{X}$, the ouput $y(t) \in \mathcal{Y}$, and control input $u(t) \in \mathcal{U}$. Further, let $\mathcal{X}_{\text{obs}}$ denote the, potentially dynamic, obstacle space, $\mathcal{X}_{\text{goal}}$ the goal region, and let $\mathcal{X}_{\text{free}} = \mathcal{X} \setminus \mathcal{X}_{\text{obs}}$ denote the drivable space. A basic formulation of the path-planning problem we consider is to solve the following problem,

$$\begin{aligned} \underset{x(t),u(t)}{\text{minimize}} \quad & \int_0^T g(x(t), u(t))\,\mathrm{d}t \\ \text{s.t.} \quad & x(t) \in \mathcal{X}_{\text{free}}, \\ & \dot{x}(t) = f(x(t), u(t)), x(0) = x_0, \\ & u(t) \in \mathcal{U}, \\ & x(T) \in \mathcal{X}_{\text{goal}}, \end{aligned} \tag{4}$$

with possibly free final time T. In autonomous-driving applications where the environment changes with time and the differential constraints of the vehicle need to be enforced, exact solutions to (4) are unrealistic. Instead, numerical solutions are sought that provide feasible solutions in short enough time. Because of the dynamic environment, (4) needs to be solved sequentially, either at a fixed update rate or whenever the sensing module provides an environment that does not match what was used in previous time steps.

Path-planning techniques can broadly be divided into variational methods, graph-search methods, and sampling-based methods. Variational, or optimal-control based, methods can be efficient when the problem size is sufficiently small such that computations of new solutions can be done fast, and/or when the environment is nearly static, as shown in the DARPA Urban Challenge [6]. However, oftentimes the nonconvexity of the underlying nonlinear optimization problem can cause convergence to local minima, slow convergence, or even lack of convergence.

A. Determining the drivable space

An important part of a trajectory-generation system is to determine the drivable space $\mathcal{X}_{\text{free}}$, or equivalently to determine the obstacle space $\mathcal{X}_{\text{obs}}$. Determining the obstacle space essentially boils down to predicting the motion of the moving obstacles. Without motion predictions, the path planner has to assume a static, or perfectly known, environment, which is unrealistic because the environment is typically dynamic and highly uncertain. If precise prediction of the environment is available, path planners can compute safer, less conservative, and more robust trajectories.

Motion prediction (potentially including driver-intention recognition) can be approached in several ways. For example, deterministic methods predict a single future trajectory, while stochastic methods represent the future trajectories with probability density functions (PDFs), which are estimated using statistical methods such as Monte-Carlo sampling [23]. Another common approach is to base the prediction on Markov chains [21], [24] for reachable set computations [25]. In [26], the reachable set computation was done by means of Monte Carlo and the driver-intention recognition was implicitly performed by a biased driver-preference distribution, while we have investigated how to predict individual vehicles by the notion of similarity with other vehicles [27].

B. Graph-Search Methods

Graph-search methods approach the path-planning problem by representing the state space $\mathcal{X}_{\text{free}}$ as a discretized graph $\mathcal{G}$ of vertices (states) $\mathcal{V}$ and edges (paths/trajectories) $\mathcal{E}$ connecting the vertices. The initial state x_0 is a vertex of the graph and there is at least one vertex associated with the goal region $\mathcal{X}_{\text{goal}}$. Graph-search methods have been rather commonly used in autonomous driving [6], [28], [29], especially as a high-level path planning method on road networks. Different methods to construct the graph are surveyed in [30], such as geometric methods and sampling-based methods. Looking at the particular application of autonomous driving on structured road networks, it might also be feasible to discretize the preferred locations in each lane, for example, the middle of the lane lane. By modeling the path-planning problem in the road-aligned frame, as is commonly done in vehicle control and estimation [26], [31], [32] the dimensionality of the graph can be heavily reduced, which is one reason why graph-based methods may be a preferred choice for path planning. Given a static graph, Dijkstra's algorithm [33] or A* [2] can be used for computing the shortest path. As in any path-planning method, however, the possibility of replanning with the path plan using the previous plan is important. Thus, when the environment changes, or perhaps more importantly, when the environment changes in a way that is inconsistent with the predictions of the drivable space, a new plan has to be computed. To this end, D* and its related methods are efficient replanning methods [4], [34].

C. Sampling-Based Methods

Sampling-based methods are focused toward incrementally building a feasible path, or a sequence of feasible paths that converge to an optimal path, given enough computation time. RRT is an important instance of sampling-based methods, which has found various applications [9]. RRT-type algorithms incrementally build a tree by selecting a random sample and expanding the tree towards that sample. Checking if the sample and the corresponding edge is in $\mathcal{X}_{\text{free}}$ amounts to pointwise comparison. RRT does therefore naturally integrate with some of the methods for determing the drivable space, since it does not require a geometric expression for $\mathcal{X}_{\text{free}}$, as opposed to some methods for constructing the graph in graph-search methods. RRT provides a path whenever one exists, but is suboptimal [10]. The quality of the path in RRT can vary heavily between any two planning instants, which is highly undesirable for autonomous driving. There exist various heuristic techniques to provide good performance [9], one of which was developed for the DARPA Urban challenge [13]. Optimal variants of RRT have recently been provided, such as RRT* [10], [35] and RRT$^{\#}$ [11], [36], under the assumption of known obstacles. RRT^{X} [37] is an extension that allows for optimality in uncertain environments. A difficulty in RRT-type algorithms is to allow for differential constraints, which are important to consider in vehicle applications, where the vehicle dynamics limits the drivable region. Work towards alleviating this problem can be found in [13], [14], [38]–[40]. A common underlying assumption is the availability of a steering function that connects two vertices. However, this amounts to solving a two-point boundary value problem, which is computationally difficult to solve in general. Local linearization [41] or focusing on linear dynamics [38] simplifies the solution to the boundary-value problem, and differential flatness has been utilized for autonomous high-speed driving [14].

IV. VEHICLE CONTROL AND COLLISION AVOIDANCE

MPC [19], [20] has recently evolved as an important approach in the research literature for automotive control in general and vehicle-dynamics control in particular. In its general form, MPC admits nonlinear dynamics as in (3). MPC solves at each time step a finite-horizon optimal-control problem and applies the first of the inputs to the system. The MPC formulation depends on the nature of the models, constraints, computational resources, and performance guarantees. A basic discrete-time formulation is

$$\begin{aligned} \underset{u(k)}{\text{minimize}} \quad & F(x(N_p)) + \sum_{k=0}^{N_p-1} L(x(k), y(k), u(k)) \\ \text{s.t.} \quad & x(k+1) = f(x(k), u(k)), \\ & x_{\min} \leq x(k) \leq x_{\max}, \quad k \in \{1, \ldots, N_p\}, \\ & u_{\min} \leq u(k) \leq u_{\max}, \quad k \in \{0, \ldots, N_c\}, \\ & x(0) = x_0, \end{aligned} \tag{5}$$

where N_p is the prediction horizon and N_c the control horizon. Today there exist efficient software and implementations for real-time solution of MPC, especially for convex MPC formulations. Furthermore, various MPC approaches have been proposed in relation to autonomous-driving applications. For example, [42] suggests MPC for trajectory generation of an active front steering vehicle. MPC can also be used in a hierarchical framework, where a high-level MPC provides trajectory generation to a low-level trajectory-tracking MPC [43], [44]. Here, robust control invariant (RCI) sets can be exploited to provide the desired guarantess. For instance, [45] uses robust positive invariant sets to bound the maximum deviation of the trajectory from a nominal trajectory and provide collision avoidance, and [46] uses RCIs for vehicle stabilization.

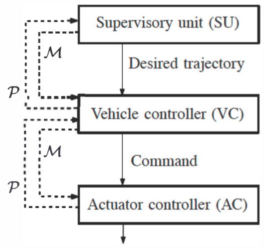


Fig. 2. Example of planning and control stack for driver assistance and automated driving features.

V. Example: Integrating MPC with Path Planning

Fig. 2 gives an example of the planning and control stack for autonomous driving, where the vehicle control (VC) is interconnected with the actuator control (AC) and path planner (SU), respectively. For instance, the AC needs to guarantee that it can achieve the commands selected by the VC. The VC must guarantee a tracking error bound along the desired trajectory, so that the SU can plan a robust trajectory accounting for such potential tracking error. In general the tracking error will depend on the reference trajectory, and hence the error bounds will depend on the classes of desired trajectories, according to "agreements" such as: "The VC will ensure performance measure $\mathcal{M}$ as long as the SU generates reference trajectories satisfying property $\mathcal{P}$". In this example we investigate MPC for tracking control of a motion plan generated by a path planner that incorporates obstacle prediction. Hence, the explicit collision avoidance is done at the planning level with given robustness margins, whereas an MPC ensures that the generated reference trajectory is tracked with a preassigned tolerance.

When considering vehicle control through steering, the shape of the reference trajectory is related to the road shape. Road segments are often similar to clothoids [31], in which the curvature changes at a constant rate.The road is therefore well represented by a piecewise-clothoidal (PWCL) curve, and here we consider PWCL trajectories with bounded curvature *and* curvature rate of change. As a consequence, the trajectories are subject to state-dependent constraints.

In [32] we designed a steering controller for PWCL trajectories ensuring a preassigned bound on the tracking error. For PWCL trajectories, algorithms for maximal RCI sets generally results in nonconvex sets, which makes it hard to do real-time control for PWCL trajectories. Based on a recently developed method [47] for constructing convex RCI sets, in [32] we provided a guarantee of the vehicle controller performance measure ($\mathcal{M}$), when the trajectories generated by the path planner are in the class of trajectories satisfying the PWCL curvature and curvature rate bounds ($\mathcal{P}$). The method can be used to determine which curvature rate bounds are acceptable, and hence determines at design time the property $\mathcal{P}$ and the control algorithm that enforces the performance measure $\mathcal{M}$ in the "agreement" between the path planner and vehicle controller. Actuator constraints are accounted for, to avoid negative interactions between the vehicle and actuator control modules.

The reference trajectory from the path planner is modeled as generated by a particle moving at constant speed v_x along a curve with curvature $\kappa = 1/R_T$, R_T being the turn radius. This results in the model for the desired vehicle yaw, $\psi_{\rm des}$, and yaw rate, $\dot{\psi}_{\rm des}$, $\ddot{\psi}_{\rm des} = v_x\dot{\kappa}$. Sampling with sampling period T_s yields

$$\dot{\psi}_{\rm des}(t+1) = \dot{\psi}_{\rm des}(t) + \gamma(t), \tag{6}$$

where $\gamma(t) = v_x T_s \dot{\kappa}(t)$ is an exogenous variable that describes the change of desired yaw rate. Eq. (6) models PWCL trajectories where the change of curvature is constant in time periods of length (at least) T_s. The model we use for vehicle control is described by (1)–(2), but with respect to a trajectory with yaw rate $\dot{\psi}_{\rm des}$. The state $x_e = [e_y\ \dot{e}_y\ e_\psi\ \dot{e}_\psi]'$, e_y, e_ψ is the lateral and yaw rate tracking errors, respectively, the input is steering angle $v_e = \delta$, the disturbance is $d = \dot{\psi}_{\rm des}$. To ensure that the difference between the reference trajectory and the vehicle motion is bounded in a preassigned range, which allows the SU to plan with appropriate safety margins, we enforce $e_{y_{\rm min}} \leq e_y \leq e_{y_{\rm max}}$. Often, additional constraints have to be imposed on the vehicle system. In particular, the steering angle and angular rate are bounded, for both physical limitations and for safety reasons, as

$$\delta_{\rm min} \leq v \leq \delta_{\rm max}, \quad \Delta\delta_{\rm min} \leq u \leq \Delta\delta_{\rm max}. \tag{7}$$

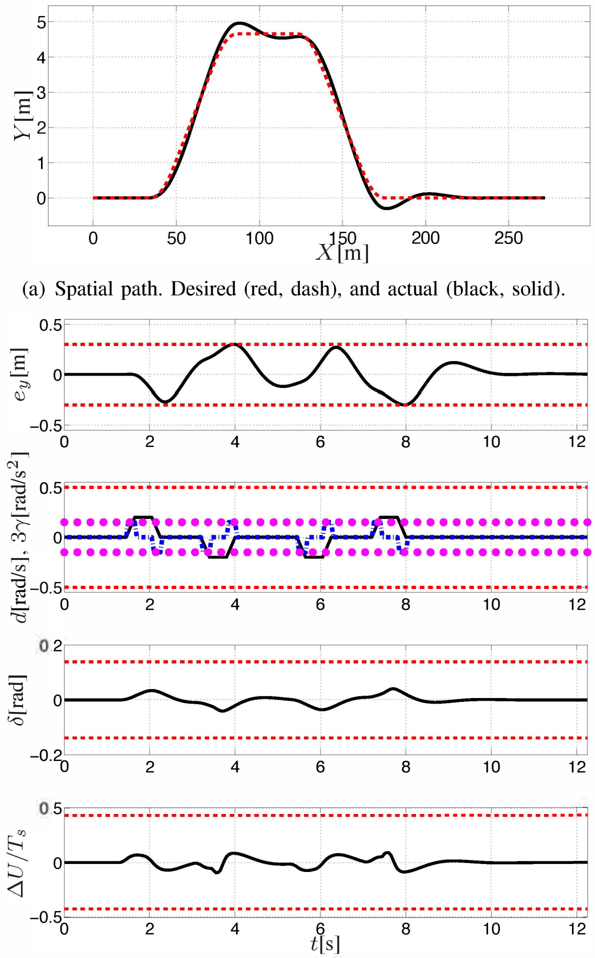
Further constraints can be imposed on the state variables, resulting in the state bounds

$$x_{e\,{\rm min}} \leq x_e \leq x_{e\,{\rm max}}. \tag{8}$$

Thus, (7), (8) describe the performance measure $\mathcal{M}$ that the vehicle control has to guarantee in the "agreement" with respect to the path planner.

Path Planning: The path planner can be of any type, as long as it can generate constrained PWCL reference trajectories according to the parameters determined determined together with the steering control law. We have previously designed [15], [48] an RRT-type path planner that instead of sampling the state-space (as in RRT) we sample the input space. We formulate the tree expansion in the RRT as a nonlinear, possibly multimodal, estimation problem, which makes the method computationally efficient. Enforcing constraints on the PWCL trajectories amounts to constraining $\dot{\psi}_{\rm des}$ in the path planner, which is straightforward.

Results: We use as simulation vehicle a mid-size SUV on a dry asphalt road. The vehicle speed is 80 km/h and the single-track model is discretized with sampling-time $T_s = 0.05$ s. The bounds on steering and maximum tolerable lateral deviation are $\delta_{\rm min} = -\delta_{\rm min} = 0.165$ [rad], $\Delta\delta_{\rm max} = -\Delta\delta_{\rm min} = 0.420$ [rad/s], and $e_{y_{\rm max}} = -e_{y_{\rm min}} n = 0.3$ m. For the values used in simulations, from $d_{\rm max}$ we obtain that the minimum turn radius at 80 km/h is 44.44 m. The simulation reported in Fig. 3 shows the results for a double lane change with maximum rate of change $\gamma_{\rm max}$, $-\gamma_{\rm max}$ and where $N = 4$. Indeed, Fig. 3 shows that the tracking constraints, as well as the steering and steering rate constraints are satisfied, despite the short horizon.



(a) Spatial path. Desired (red, dash), and actual (black, solid).

(b) Trajectories of lateral error, desired yaw rate, steering, input (black, solid), and constraints (red, dash). Second plot from top: yaw rate change (blue, dash-dot) and constraints (magenta, dot).

Fig. 3. Double lane change with maximum curvature rate of change.

VI. Discussion

There have been many advancements over the last decades in relation to autonomous vehicles. Over the next decade, the capabilities of production vehicles, eventually resulting in fully autonomous vehicles, are expected to increase as a result of advancements in algorithms, computing, and sensing. Autonomous vehicles are complex decision-making systems and include a number of intricate subsystems, where the output of one system depends on the output of the previous. Thus, a research challenge is to integrate the different modules to ensure performance and safety of the vehicles. The performance of the complete system is limited by the weakest link in the chain, and a sophisticated vehicle controller alone cannot guarantee performance of the vehicle. Integrating MPC with path planning is one challenge. We presented one possible approach to this based on agreements between the MPC and path planner, but there are still things to be improved on. For instance, our approach considers the steering wheel as main actuator and mainly uses motion prediction at the path-planning stage. There are also methods for unifying path planning and vehicle control/collision avoidance into one module, for example, by utilizing the structured environment of one-way roads [49]. Unifying the planning and control modules has potential, but typically also makes for a more complex and difficult problem to solve [50], [51]. Completely integrated path planning and vehicle control may therefore be out of reach in the general autonomous-driving scenario, but certainly has potential for several situations (e.g., lane-change assist).

Achieving trajectory generation in the path planner that is robust to sensing and prediction errors is far from trivial, for example, how to seamlessly integrate the motion prediction with the path planner and vehicle control. There are several methods for determining the safe space, some based on statistical methods and other on geometric methods, as surveyed in this paper. Still, how to efficiently incorporate this at the path-planning stage is not a fully solved problem.

When using graph-search methods for trajectory planning (i.e., path with time information), it becomes important to consider the differential constraints imposed by the vehicle dynamics/kinematics. Some work on this can be found in [9]. Traditional methods have been based on investigating approximate solutions (e.g., [52]), because in general exact solutions are infeasible to obtain. Similarly, avoiding the need for exact steering procedures in sampling-based planners is clearly desired for autonomous-driving applications. One approach toward this is the Stable sparse tree (SST) [53], which is based on generating random controls and propagating them through the dynamic model, whereas we in [39] developed an extension to the closed-loop prediction in [13], thus providing certain optimality without imposing a steering function. In this context, feedback motion planning [12], [40] can be useful because the edges are computed by feedback control, and are therefore dynamically feasible by construction. Feedback-based path planning may also have the potential to better integrate with vehicle control, because then the two modules share common theory. In addition, certain guarantees on robustness and invariance can possibly be derived. Feedback-based sampling methods seem like a fruitful approach for increasing robustness and computational efficiency. Using feedback-based planning bridges the gap between the path-planning and vehicle-control modules, and can potentially offer a more integrated approach to path planning and collision avoidance in autonomous driving.

References

[1] J. H. Reif, "Complexity of the movers problem and generalizations," in *IEEE Conf. Foundations of Computer Science*, 1979, pp. 421–427.
[2] N. J. Nilsson, *Principles of Artificial Intelligence*. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 1980.
[3] A. Stentz, "Optimal and efficient path planning for partially-known environments," in *IEEE Int. Conf. Robotics and Automation*, San Diego, CA, 1994.
[4] ——, "The focussed D* algorithm for real-time replanning," in *Int. Joint Conf. on Artificial Intelligence*, Montreal, Quebec, Canada, 1995.
[5] S. Koenig and M. Likhachev, "Fast replanning for navigation in unknown terrain," *IEEE Trans. Robot.*, vol. 21, no. 3, pp. 354–363, 2005.
[6] C. Urmson, J. Anhalt, D. Bagnell, C. Baker, R. Bittner, M. Clark, J. Dolan, D. Duggins, T. Galatali, C. Geyer, *et al.*, "Autonomous driving in urban environments: Boss and the Urban challenge," *J. Field Robotics*, vol. 25, no. 8, pp. 425–466, 2008.

[7] M. Montemerlo, J. Becker, S. Bhat, H. Dahlkamp, D. Dolgov, S. Ettinger, D. Haehnel, T. Hilden, G. Hoffmann, B. Huhnke, *et al.*, "Junior: The Stanford entry in the Urban challenge," *J. Field Robotics*, vol. 25, no. 9, pp. 569–597, 2008.

[8] L. E. Kavraki and J.-C. Latombe, "Randomized preprocessing of configuration space for fast path planning," Stanford University, Stanford, CA, Tech. Rep. STAN-CS-93-1490, 1993.

[9] S. M. LaValle, *Planning Algorithms*. Cambridge, UK: Cambridge University Press, 2006.

[10] S. Karaman and E. Frazzoli, "Sampling-based algorithms for optimal motion planning," *Int. J. R. Res.*, vol. 30, no. 7, pp. 846–894, 2011.

[11] O. Arslan and P. Tsiotras, "Use of relaxation methods in sampling-based algorithms for optimal motion planning," in *IEEE Int. Conf. Robotics and Automation*, Karlsrühe, Germany, May 2013.

[12] R. Tedrake, I. R. Manchester, M. Tobenkin, and J. W. Roberts, "Lqr-trees: Feedback motion planning via sums-of-squares verification," *Int. J. R. Res.*, vol. 29, no. 8, pp. 1038–1052, 2010.

[13] Y. Kuwata, S. Karaman, J. Teo, E. Frazzoli, J. How, and G. Fiore, "Real-time motion planning with applications to autonomous urban driving," *IEEE Trans. Control Syst. Technol.*, vol. 17, no. 5, pp. 1105–1118, 2009.

[14] J. Hwan Jeon, R. V. Cowlagi, S. C. Peters, S. Karaman, E. Frazzoli, P. Tsiotras, and K. Iagnemma, "Optimal motion planning with the half-car dynamical model for autonomous high-speed driving," in *Am. Control Conf.*, Washington, DC, June 2013.

[15] K. Berntorp and S. D. Cairano, "Joint decision making and motion planning for road vehicles using particle filtering," in *IFAC Symp. Advances in Automotive Control*, Kolmården, Sweden, June 2016.

[16] P. Dingle and L. Guzzella, "Optimal emergency maneuvers on highways for passenger vehicles with two- and four-wheel active steering," in *Amer. Control Conf.*, Baltimore, Maryland, June 2010.

[17] K. Berntorp, B. Olofsson, K. Lundahl, and L. Nielsen, "Models and methodology for optimal trajectory generation in safety-critical road–vehicle manoeuvres," *Veh. Syst. Dyn.*, vol. 52, no. 10, pp. 1304–1332, 2014.

[18] S. Anderson, S. Peters, T. Pilutti, and K. Iagnemma, "An optimal-control-based framework for trajectory planning, threat assessment, and semi-autonomous control of passenger vehicles in hazard avoidance scenarios," *Int. J. Vehicle Autonomous Systems*, vol. 8, no. 2/3/4, pp. 190–216, 2010.

[19] C. E. Garcia, D. M. Prett, and M. Morari, "Model predictive control: theory and practice–a survey," *Automatica*, vol. 25, no. 3, pp. 335–348, 1989.

[20] D. Hrovat, S. Di Cairano, H. E. Tseng, and I. V. Kolmanovsky, "The development of model predictive control in automotive industry: A survey," in *IEEE Int. Conf. Control Applications*, Dubrovnik, Croatia, 2012.

[21] A. Carvalho, S. Lefévre, G. Schildbach, J. Kong, and F. Borrelli, "Automated driving: The role of forecasts and uncertainty—A control perspective," *European J. Control*, vol. 24, pp. 14–32, 2015.

[22] H. B. Pacejka, *Tire and Vehicle Dynamics*, 2nd ed. Oxford, United Kingdom: Butterworth-Heinemann, 2006.

[23] A. Broadhurst, S. Baker, and T. Kanade, "Monte Carlo road safety reasoning," in *IEEE Intell. Veh. Symp.*, Las Vegas, NV, 2005.

[24] S. Lefèvre, D. Vasquez, and C. Laugier, "A survey on motion prediction and risk assessment for intelligent vehicles," *Robomech Journal*, vol. 1, no. 1, p. 1, 2014.

[25] M. Althoff, O. Stursberg, and M. Buss, "Model-based probabilistic collision detection in autonomous driving," *IEEE Trans. Intell. Transp. Syst.*, vol. 10, no. 2, pp. 299–310, 2009.

[26] A. Eidehall and L. Petersson, "Statistical threat assessment for general road scenes using Monte Carlo sampling," *IEEE Trans. Intell. Transp. Syst.*, vol. 9, no. 1, pp. 137–147, 2008.

[27] K. Okamoto, K. Berntorp, and S. Di Cairano, "Similarity-based vehicle-motion prediction," in *Amer. Control Conf.*, Seattle, WA, May 2017.

[28] A. Bacha, C. Bauman, R. Faruque, M. Fleming, C. Terwelp, C. Reinholtz, D. Hong, A. Wicks, T. Alberi, D. Anderson, *et al.*, "Odin: Team victorTango's entry in the DARPA Urban challenge," *J. Field Robotics*, vol. 25, no. 8, pp. 467–492, 2008.

[29] D. Dolgov, S. Thrun, M. Montemerlo, and J. Diebel, "Practical search techniques in path planning for autonomous driving," *Ann Arbor*, vol. 1001, p. 48105, 2008.

[30] B. Paden, M. p, S. Z. Yong, D. Yershov, and E. Frazzoli, "A survey of motion planning and control techniques for self-driving urban vehicles," *IEEE Trans. Intell. Veh.*, vol. 1, no. 1, pp. 33–55, 2016.

[31] R. Rajamani, *Vehicle Dynamics and Control*. Berlin Heidelberg: Springer-Verlag, 2006.

[32] S. D. Cairano, U. V. Kalabi, and K. Berntorp, "Vehicle tracking control on piecewise-clothoidal trajectories by MPC with guaranteed error bounds," in *IEEE Conf. Decision and Control*, Las Vegas, NV, Dec. 2016.

[33] E. W. Dijkstra, "A note on two problems in connexion with graphs," *Numerische mathematik*, vol. 1, no. 1, pp. 269–271, 1959.

[34] S. Koenig and M. Likhachev, "Fast replanning for navigation in unknown terrain," *IEEE Trans. Robot.*, vol. 21, no. 3, pp. 354–363, 2005.

[35] S. Karaman, M. R. Walter, A. Perez, E. Frazzoli, and S. Teller, "Anytime motion planning using the RRT*," in *IEEE Int. Conf. Robotics and Automation*, Shanghai, China, May 2011.

[36] O. Arslan and P. Tsiotras, "Dynamic programming guided exploration for sampling-based motion planning algorithms," in *IEEE Int. Conf. Robotics and Automation*, Seattle, Washington, May 2015.

[37] M. Otte and E. Frazzoli, "RRT^X: Real-time motion planning/replanning for environments with unpredictable obstacles," in *Algorithmic Foundations of Robotics XI*, 2015, pp. 461–478.

[38] D. J. Webb and J. van den Berg, "Kinodynamic RRT*: Asymptotically optimal motion planning for robots with linear dynamics," in *IEEE Int. Conf. Robotics and Automation*, Karlsruhe, Germany, 2013.

[39] O. Arslan, K. Berntorp, and P. Tsiotras, "Sampling-based Algorithms for Optimal Motion Planning Using Closed-loop Prediction," *ArXiv e-print: 1601.06326*, Jan. 2016.

[40] C. Danielson, A. Weiss, K. Berntorp, and S. D. Cairano, "Path planning using positive invariant sets," in *IEEE Conf. Decision and Control*, Las Vegas, NV, Dec. 2016.

[41] A. Perez, R. Platt, G. Konidaris, L. Kaelbling, and T. Lozano-Perez, "LQR-RRT*: Optimal sampling-based motion planning with automatically derived extension heuristics," in *IEEE Int. Conf. Robotics and Automation*, St. Paul, MN, 2012.

[42] P. Falcone, F. Borrelli, J. Asgari, H. E. Tseng, and D. Hrovat, "Predictive active steering control for autonomous vehicle systems," *IEEE Trans. Control Syst. Technol.*, vol. 15, no. 3, pp. 566–580, 2007.

[43] P. Falcone, F. Borrelli, H. E. Tseng, J. Asgari, and D. Hrovat, "A hierarchical model predictive control framework for autonomous ground vehicles," in *Amer. Control Conf.*, Seattle, WA, 2008.

[44] K. Berntorp and F. Magnusson, "Hierarchical predictive control for ground-vehicle maneuvering," in *Amer. Control Conf.*, Chicago, Il, July 2015.

[45] Y. Gao, A. Gray, H. E. Tseng, and F. Borrelli, "A tube-based robust nonlinear predictive control approach to semiautonomous ground vehicles," *Veh. Syst. Dyn.*, vol. 52, no. 6, pp. 802–823, 2014.

[46] G. Palmieri, M. Barić, L. Glielmo, and F. Borrelli, "Robust vehicle lateral stabilisation via set-based methods for uncertain piecewise affine systems," *Veh. Syst. Dyn.*, vol. 50, no. 6, pp. 861–882, 2012.

[47] S. Di Cairano and F. Borrelli, "Reference tracking with guaranteed error bound for constrained linear systems," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2245–2250, 2016.

[48] K. Berntorp and S. D. Cairano, "Particle filtering for online motion planning with task specifications," in *Amer. Control Conf.*, Boston, MA, July 2016.

[49] J. Nilsson, P. Falcone, M. Ali, and J. Sjöberg, "Receding horizon maneuver generation for automated highway driving," *Control Eng. Pract.*, vol. 41, pp. 124–133, 2015.

[50] J. Funke, M. Brown, S. M. Erlien, and J. C. Gerdes, "Collision avoidance and stabilization for autonomous vehicles in emergency scenarios," *IEEE Trans. Control Syst. Technol.*, vol. PP, no. 99, pp. 1–13, 2016.

[51] V. Turri, A. Carvalho, H. E. Tseng, K. H. Johansson, and F. Borrelli, "Linear model predictive control for lane keeping and obstacle avoidance on low curvature roads," in *IEEE Int. Conf. Intell. Transp. Syst.*, The Hague, Netherlands, 2013.

[52] P. Jacobs and J. Canny, "Planning smooth paths for mobile robots," in *Nonholonomic Motion Planning*. Springer, 1993, pp. 271–342.

[53] Y. Li, Z. Littlefield, and K. E. Bekris, "Sparse methods for efficient asymptotically optimal kinodynamic planning," in *Algorithmic foundations of robotics XI*. Springer, 2015, pp. 263–282.