Cheat Sheet: Comp Sc BSc, LinAlg

Complex Numbers

Basics

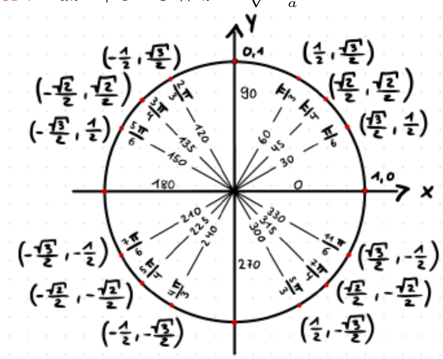
Def : $z = a + bi \Leftrightarrow \Re(z) = a, Im(z) = b$
Def : $z = a + bi \Leftrightarrow \bar{z} = a - bi \Leftrightarrow r \cdot e^{2\pi - \phi}$
Def : $z = r \cdot cos(\phi) + i \cdot sin(\phi)$
Def : $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$
Def : $\phi = \begin{cases} arctan\frac{y}{x} & \text{1. Q} \\ arctan\frac{y}{x} + \pi & \text{2./3. Q} \\ arctan\frac{y}{x} + 2pi & \text{4. Q} \end{cases}$

Operations

Def : $z_1 \pm z_2 : (x_1 + x_2) \pm i(y_1 + y_2)$
Def :
$z_1 \cdot z_2 : (x_1 + i \cdot y_1) + (x_2 + i \cdot y_2) = r_1 \cdot r_2 e^{i(\phi_1 + \phi_2)}$
Def : $\frac{z_1}{z_2} : \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)} = \frac{z_1 \cdot \bar{z_2}}{|z_2|^2}$
Def : $\sqrt[n]{a} \Leftrightarrow a = z^n \Leftrightarrow |a| \cdot e^{i\phi} = r^n \cdot e^{i\omega n} \Leftrightarrow r = \sqrt[n]{|a|}, \omega = \frac{\phi + 2k\pi}{n}$

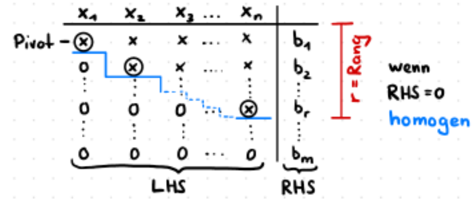
Polynomials

The roots of a complex polynomial are pairwise conjugated. **Def :** $z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
Def : $az^n + c = 0 \Leftrightarrow z = \sqrt[n]{-\frac{c}{a}}$



SLE

Gauss Algorithm



Compability Conditions: $b_{r+1} = ... = bm = 0$
S 1.1: $Ax = b$ hat min eine Lösung $\Leftrightarrow r = m$ oder $r < m + VB$ dann: $r = n \Leftrightarrow 1$ Lösung , $r < n \Leftrightarrow \infty$ Lösungen
Cor 1.7: For a quadratic SLE with n equations and n variables we have the following set of equivalence, of wich ONLY one of them can be true; So, EITHER

- i $Rank(A) = n$ (A is regular)
- ii for every b there exist at least one solution
- iii for every b there exists exactly one solution
- iv the corresponding homogeneous system has only the trivial solution

OR the following equivalences hold

- v $Rank(A) < n$ (A is singular)
- vi for some b there exits no solution
- vii for no b a unique solution exists
- viii for some b infinity many solution exists
- ix the corresponding homogeneous system has non-trivial solutions

Matrices and Vectors

Definitions

A $m \times n$ matrix hat m row (Zeilen)↓ and n columns (Spalten)→ , in which the i,j element gets noted by $a_{i,j}$ or $(A)_{i,j}$
Def nullmatrix: Has in every entry 0
Def diagonalmatrix: Has in every entry 0 except for the diagonal: $(D)_{ij} = 0$ for $i \neq j$ one can write $Diag(d_{11}, \cdots, d_{nn})$
Def identity: The identity is written as $I_n = Diag(1, \cdots, 1)$ It holds that $AI = IA = A$
Def upper triangular matrix: We have $(R)_{ij} = 0$ for $i > j$ (Rechtsdreiecksmatrix)
Def lower triangular matrix: We have $(R)_{ij} = 0$ for $i < j$ (Linksdreiecksmatrix)
Def Matrix-set: The set of $m \times n$-matrices is written as: $\mathbb{E}^{m \times n}$ For vectors we have: $\mathbb{E}^n$, where $\mathbb{E}$ is $\mathbb{R}$ or $\mathbb{C}$
Def matrix multiplication: If $C = AB$ then one can write $C_{ij} = (AB)_{ij} = \sum_{k=1}^n (A)_{ik}(B)_{kj} = \sum_{k=1}^n a_{ik}b_{kj}$
S 2.1:

- $(\alpha\beta)A = \alpha(\beta A)$
- $(A + B) + C = A + (B + C)$
- $(\alpha A)B = \alpha(AB)$
- $(AB) \cdot C = A \cdot (BC)$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $(A + B) \cdot C = AC + BC$
- $A \cdot (B + C) = AB + AC$
- $\alpha(A + B) = \alpha A + \alpha B$
- $A + B = B + A$

S 2.20: Let A and B be unitary(orthogonal). It holds:

- A is regular and $A^{-1} = A^H(A^T)$
- $AA^H(AA^T) = I$
- A^{-1} is unitary (orthogonal)
- AB is unitary (orthogonal)

Def Zerodiviser: If $AB = 0 \Leftrightarrow$ A,B Zerodiviser, Nullteiler
Def transposes: $(A^T)_{ij} = A_{ji}$
Def conjugate transposed: $A^H = (\overline{A})^T = \overline{A^T}$
Def symmetric: $A^T = A \Leftrightarrow$ A symmetric
Def skew-symmetric: $A^T = -A \Leftrightarrow$ A skew-symmetric
Def hermitian: $A^H = A \Leftrightarrow$ A hermitian
S 2.6: Also accounts for A^T instead of A^H. $\overline{\alpha}$ simplifies to α

- $(A^H)^H = A$
- $(\alpha A)^H = \overline{\alpha} A^H$
- $(A + B)^H = A^H + B^H$
- $(AB)^H = B^H A^H$

S 2.7: For symmetric matrices A and B it holds that: $AB = BA \Leftrightarrow AB$ is symmetric It holds for arbitrary matrix C that: C^TC and CC^T are symmetric. The same holds for the hermitian case

Scalarproduct and Norm

Def euclidian scalarproduct:
$\langle x, y \rangle = x^T y = \sum_{k=1}^n \overline{x_k} \cdot y_k \xrightarrow[\mathbb{R}]{in} \sum_{k=1}^n x_k \cdot y_k$
S 2.9:

- **S1** $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$(linear in 2nd factor)
- **S1** $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ (linear in 2nd factor)
- **S2** for $\mathbb{E} = \mathbb{R}$: $\langle x, y \rangle = \langle y, x \rangle$ (symmetric)
- **S2'** for $\mathbb{E} = \mathbb{C}$: $\langle x, y \rangle = \overline{\langle y, x \rangle}$(hermitian)
- **S3** $\langle x, x \rangle > 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0$(positiv definite)

Cor 2.10:

- **S4** for $\mathbb{E} = \mathbb{R}$: linear in 1st factor $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$ $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- **S4'** for $\mathbb{E} = \mathbb{C}$: conjugate-linear in 1st factor $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$ $\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$

Def norm: $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^T x} = \sqrt{\sum_{k=1}^n (|x_k|)^2} \xrightarrow[\mathbb{R}]{in} \sqrt{\sum_{k=1}^n x_k^2}$
S 2.11: $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$(Cauchy-Schwarz inequality, "=" holds when y is a multiple of x or vice verca)
Def CBS: CBS is a property of the scalar product: CBS squared yields: $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$
S 2.12: For the euclidian norm holds:

- **N1** $\|x\| > 0, \|x\| = 0 \Leftrightarrow x = 0$(positiv definit)
- **N2** $\|\alpha x\| = \alpha \|x\|$ (homogeneous)
- **N3** $\|x \pm y\| \leq \|x\| + \|y\|$ (Triangle-inequality)

Def : Angle ϕ between x, y: $, \phi = arccos\frac{Re(\langle x, y \rangle)}{\|x\| \cdot \|y\|} \xrightarrow[\mathbb{R}]{in} arccos\frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$
Def : x, y are orthogonal: $\langle x, y \rangle = 0 \Leftrightarrow x \perp y$
S 2.13: $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow x \perp y$ (Pythagoras) **Def p-norm:** $\|x\|_p = (|x_1|^p \cdots |x_n|^p)^{\frac{1}{p}}$

Outer Product and Projections

Def outer product: m-vector x and n-vector y: xy^T
S 2.14: A $m \times n$-matrix has rank 1 if it is the outer product of an m-vector $\neq 0$ and n-vector $\neq 0$
S 2.15: The orthogonal projection $P_y x$ of the n-vector x onto y is defined as: $P_y x = \frac{1}{\|y\|^2} \cdot yy^H x = uu^H = P_u$ where $u = \frac{y}{\|y\|}$
Def projections matrix: $P_y = \frac{1}{\|y\|^2} \cdot yy^H$ It has the properties: $P_y^H = P_y$ (hermitian/symmetric) and $P_y^2 = P_y$ (idempotent)

Inverse

Def invertible: $\exists A^{-1} \Leftrightarrow A^{-1} \cdot A = A \cdot A^{-1} = I$
S 2.17: A is invertible$\Leftrightarrow \exists X : AX = I \Leftrightarrow X$ is unique $\Leftrightarrow A$ is regular
S 2.18: If A, B are regular:

- A^{-1} is regular and $A^{-1^{-1}} = A$
- AB is regular and $(AB)^{-1} = B^{-1}A^{-1}$
- A^H is regular and $(A^H)^{-1} = (A^{-1})^H$

S 2.19: If A is regular das LGS $Ax = b$ has the unique solution $x = A^{-1}b$
Finding an inverse $[A|I] \xrightarrow[op]{row} [I|A^{-1}]$ if

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $det(A) \neq 0 \Leftrightarrow$ A is invertible
$\Leftrightarrow A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
$A = \left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right] \Leftrightarrow A^{-1} \left[\begin{array}{c|c} a_{11}^{-1} & a_{12}^{-1} \\ \hline a_{21}^{-1} & a_{22}^{-1} \end{array}\right]$

Orthogonal and unitary matrices

Def unitary/orthogonal: $AA^H = I, AA^T = I \Leftrightarrow$ A is unitary/orthogonal $\Leftrightarrow det(A) = \pm 1$
S 2.20: A,B are unitary/orthonormal:

- A is regular and $A^{-1} = A^H$
- $AA^H = I_n$
- A^{-1} is unitary/orthogonal
- AB is unitary/orthogonal
- columns are orthonormal

S 2.21: Images from unitary/orthonormal matrices are conformal (längen-winkeltreu)
Def 2d rotation: $R(\phi) = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}$
Def 3d rotation:
$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{pmatrix}, R_y(\phi) = \begin{pmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{pmatrix}, R_z(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

LU-Decomposition

The LU-decomposition is useful when multiple SLE have the same A

- Find $PA = LR$
- solve $Lc = Pb$
- solve $Rx = c$

$$\left(\begin{array}{ccc|ccc|ccc} 1&0&0&1&2&-4&1&0&0\\ 0&1&0&-4&-8&13&0&1&0\\ 0&0&1&2&-5&-3&0&0&1 \end{array}\right) \begin{array}{l} \\ +4I \\ -2I \end{array}$$

negak!

$$\left(\begin{array}{ccc|ccc|ccc} 1&0&0&1&2&-4&1&0&0\\ 0&1&0&0&0&-3&-4&1&0\\ 0&0&1&0&-9&5&2&0&1 \end{array}\right) \begin{array}{l} \\ III \\ II \end{array}$$

$$\left(\begin{array}{ccc|ccc|ccc} 1&0&0&1&2&-4&1&0&0\\ 0&0&1&0&-9&5&2&1&0\\ 0&1&0&0&0&-3&-4&0&1 \end{array}\right)$$

P U (R) L (L)

Vectorspaces

Def : A vectorspace V over $\mathbb{K}$ is a non-empty set, on which vectoraddition and scalarmultiplication is defined

Def Axioms:

- **V1** : $x + y = y + x$
- **V2** : $(x + y) + z = x + (y + z)$
- **V3** : $\exists 0 \in V : x + 0 = x$
- **V4** : $\forall x \exists - x : x + (-x) = 0$
- **V5** : $\alpha(x + y) = \alpha \cdot x + \alpha \cdot y$
- **V6** : $(\alpha + \beta)x = \alpha x + \beta x$
- **V7** : $(\alpha\beta)x = \alpha(\beta x)$
- **V8** : $1 \cdot x = x$

S 4.1:

- **i** : $0 \cdot x = 0$
- **ii** : $\cdot 0 = 0$
- **iii** : $\alpha \cdot x = 0 \to x = 0 \lor \alpha = 0$
- **iv** : $(-\alpha)x = \alpha(-x) = -(\alpha \cdot x)$

Def polynomial space: $\mathcal{P}_n$ is defined as all polynomials of degree n. Further: $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$

S 4.1: I a vectorspace the following holds for a scalar α and $x \in V$:

- $0x = 0$
- $0\alpha = 0$
- $\alpha x = 0 \Rightarrow \alpha = 0$ or $x = 0$
- $(-\alpha)x = \alpha(-x) = -(\alpha x)$

S 4.12: $\{b_1, \cdots, b_n\} \subset V$ is a basis of V $\Leftrightarrow$ every vector $x \in V$ can be uniquely represented as: $x = \sum_{k=1}^{n} \xi_k b_k$

S 4.2: $\forall x \in V, \forall y \in V \exists z \in V : x + z = y$ where z is unique and $z = y + (-x)$

Subspace

Def : A subspace (unterraum) U is a non-empty subset of V. It is closed under vector addition and scalar multiplication. U contains the zero-vector

S 4.3: Every subspace is a vectorspace

Def spanning set: The vectors $v_1, \cdots, v_n$ are a spanning set (erzeugendes System) of V, if $\forall w \in span\{v_1, \cdots, v_n\}$

Linear dependency, basis, dimensions

Def linear dependency: Vectors $v_1, \cdots, v_n$ are linearly dependent $\Leftrightarrow \sum_{k=1}^{n} \alpha_k \cdot v_k = 0 \to \alpha_1 = \cdots = \alpha_n = 0$

Def dimension: the dimension of V is $dimV = |spanV|$ $(dim\{0\} = 0)$

Lem 4.8: Every set $\{v_1, \cdots, v_m\} \subset V$ with $|\mathbb{B}_v| < m$ is linear dependant

Cor 4.10: in an finite vectorspace, a set with n independant vectors is basis of V if $dim(V) = n$

Def : The coefficients ξ_k are coordinates of x with respect to a basis $\mathbb{B}$ $\xi = (\xi_1, \cdots, \xi_n)^T$ is a coordinate vector

Def : Two subspaces $U, U' \subset V$ are complementary if every $v \in V$ has a unique representation in $U and U'$. Namely, $v = u \in U + u' \in U'. \to V = U \bigoplus U'$

Linear Maps

Definitions

Def linearity: $F : V \to W$ is linear:

- $F(v + w) = F(v) + F(w)$
- $\alpha F(v) = F(\alpha v)$

Def injective: $\forall x, x' \subset X : f(x) = f(x') \Leftrightarrow x = x'$

Def surjective: $\forall y \subset Y, \exists x \subset X, f(x) = y$

Def bijective: surjective and injective$\Leftrightarrow f^{-1}$ exists

Matrix representation

Let F be a linear map $X \to Y$. One can write $F(b_i) \in Y$ as a linear combination of the basis of Y: $F(b_i) = \sum_{k=1}^{m} a_{k,l} \cdot c_k$ **Def :** The matrix $A^{m \times n}$ witht the elements $a_{k,l}$ is a matrix (Abbildungsmatrix) with respect X, Y $F(x) = y \Leftrightarrow \underset{\smile}{A}\xi = \eta$ [H]

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, shift right, "k_x"'] & Y \arrow[d, shift right, "k_y"'] \\
\mathbb{E}^n \arrow[r, "A"] \arrow[u, shift right, "k_x^{-1}"'] & \mathbb{E}^m \arrow[u, shift right, "k_y^{-1}"']
\end{tikzcd}$$

Def isomorphism: F is bijective $\Leftrightarrow$ F is an isomorphism

Def automorphism: F is isomorphism and $X = Y \Leftrightarrow$ F is an automorphism

S 5.1: F is isomorphism $\Leftrightarrow F^{-1}$ exists and is an isomorphism and linear

Kernel, Image and Rank

Def Kern: $kerF = \{x \in X | F(x) = 0\}$

S 5.6: F injective $\Leftrightarrow kerF = \{0\}$

Def Image: $ImF = \{F(x)| x \in X\}$

S 5.6: F surjective $\Leftrightarrow imF = Y$

ker A

is the solution set of $Ax = 0$. $Im(A)$ set of all b, such that $Ax = b$ is solvable **S 5.7:** $dimX - dim(kerF) = dim(imF) = Rank(F)$

Def : The rank F is equal to $dim(im(F))$

Cor 5.8:

- $F : X \mapsto Y$ injective $\Leftrightarrow$ Rank F = dim X
- $F : X \mapsto Y$ surjective $\Leftrightarrow$ Rank F = dim Y
- $F : X \mapsto Y$ bijective (isomorphism) $\Leftrightarrow$ Rank F = dim X = dim Y
- $F : X \mapsto Y$ bijective (automorphism) $\Leftrightarrow$ Rank F = dim X, ker F = $\{0\}$

Cor 5.10:

- $Rank(G \circ F) \leq min(RankF, RankG)$
- G is injective $Rank(G \circ F) = RankF$
- G is surjective $Rank(G \circ F) = RankG$

Matrices as linear mapping

Def columnspace: The columnspace (Spaltenraum) of A is the subspace $\Re(A) = im(A) = span\{a_1, \cdots, a_n\}$

Def nullspace: The nullspace (Nullraum) of A is a subspace $\mathcal{N}(A) = kerA = L_0(Ax = 0)$

Def : # free variables = $dimN(A)$

S 5.12: Rank A =r: and L_0 Solution of $Ax = 0 \Rightarrow dimL_0 = dimN(A) = dim(KerA) = n - r$

S 5.13: Rank $A \in M^{m \times n}$:

- pivots in Row-echelon-form
- $dim(im(A))$ of $A : \mathbb{E}^n \mapsto \mathbb{E}^m$
- dimension of the linear independent columns/rows

Cor 5.14: $RankA^T = RankA^H = RankA$

S 5.16: for $A \in \mathbb{E}^{m \times n}$ and $Bin\mathbb{E}^{p \times m}$:

- $RankBA \leq min(RankA, RankB)$
- $RankB = m \leq p \Rightarrow RankBA = RankA$
- $RankA = m \leq n \Rightarrow RankBA = RankB$

Cor 5.17: From S.5.16 it follows for quadratic matrices. $A \in \mathbb{E}^{m \times m}$ and $Bin\mathbb{E}^{m \times m}$

- $RankBA \leq min(RankA, RankB)$
- $RankB = m \Rightarrow RankBA = RankA$
- $RankA = m \Rightarrow RankBA = RankB$

S 5.18: For quadratic matrix $\mathbb{E}^{n \times n}$ the following statements are equivalent:

- A is regular
- $RankA = n$
- Columns are linearly independent
- Rows are linearly independent
- $kerA = N(A) = \{0\}$
- A is invertible
- $ImA = \Re(A) = \mathbb{E}^n$

S 5.19: For $Ax = b, b \neq 0$ with the solution x_0 and L_0 the solutionset is defined by $L_b = x_0 + L_0$ and is called affine subspace (not a real subspace since $0 \notin L_b$) $dim(Im(A)) = n - dim(ker(A)) = n - (n - r) = r$

RC Find Basis of Im A= $\mathcal{R}(A)$:

1. bring into row echelon form
2. mark rows with pivots
3. marked columns in the normal form are a Basis

ex

$$\begin{matrix} -1 & -4 & 7 & 3 \\ 3 & 0 & -6 & 0 \\ -3 & 4 & 1 & -3 \\ 1 & -4 & 3 & 3 \end{matrix} \xrightarrow[(1)]{} \begin{matrix} -1 & -4 & 7 & 3 \\ 0 & -12 & 15 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \xrightarrow[(2)]{}$$

$$\begin{matrix} -1 & -4 & 7 & 3 \\ \uparrow & -12 & 15 & 9 \\ 0 & \uparrow & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \xrightarrow[(3)]{} span\left\{\begin{matrix} -1 & -4 \\ 3 & 0 \\ -3 & 4 \\ 1 & -4 \end{matrix}\right\}$$

RC Find Basis of ker A $= \mathcal{N}(A), A \in E^{m \times n}$:

1. bring into row echelon form
2. create a vector for every row, which does not have a pivot. The dimensions of the vectors are $E^{1 \times n}$ [3]Solve SLE $Ax = 0$ with the yielded vectors.
4. Write the solution as vector

ex

$$\begin{matrix} -1 & -4 & 7 & 3 \\ 3 & 0 & -6 & 0 \\ -3 & 4 & 1 & -3 \\ 1 & -4 & 3 & 3 \end{matrix} \xrightarrow[(1)]{} \begin{matrix} -1 & -4 & 7 & 3 \\ 0 & -12 & 15 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \xrightarrow[(2)]{}$$

$$\begin{matrix} -1 & -4 & 7 & 3 \\ 0 & -12 & 15 & 9 \\ 0 & 0 & \uparrow & \uparrow \\ 0 & 0 & 0 & 0 \end{matrix} \xrightarrow[(3)]{} \begin{matrix} x_4 = \alpha \\ x_3 = \beta \\ x_2 = \frac{5\beta + 3\alpha}{4} \\ x_1 = 2\alpha \end{matrix} \xrightarrow[(4)]{}$$

$$\textbf{span} \left\{\begin{matrix} 2 & 0 \\ \frac{5}{4} & \frac{3}{4} \\ 1 & 0 \\ 0 & 1 \end{matrix}\right\}$$

Def Maps: Let X, Y be vector spaces with $dimX = n, dimY = m$

- $F : X \mapsto Y$ a linear map
- $A : \mathbb{E}^n \mapsto \mathbb{E}^m, \xi \mapsto \eta$ *
- $B : \mathbb{E}^n \mapsto \mathbb{E}^m, \xi' \mapsto \eta'$ *
- $T : \mathbb{E}^n \mapsto \mathbb{E}^{n\prime}, \xi \mapsto \xi'$ **
- $S : \mathbb{E}^m \mapsto \mathbb{E}^{m\prime}, \eta \mapsto \eta'$**

*Abbildungsmatrix
**Transformationsmatrix

$$\begin{tikzcd}
x \in X \arrow[r, "F"] \arrow[d, shift right, "k_x"'] & Y \arrow[d, shift right, "k_y"'] \\
\xi \in \mathbb{E}^n \arrow[r, "A"] \arrow[u, shift right, "k_x^{-1}"'] \arrow[d, shift right, "T^{-1}"'] & \eta \in \mathbb{E}^m \arrow[u, shift right, "k_y^{-1}"'] \arrow[d, shift right, "S^{-1}"'] \\
\xi' \in \mathbb{E}^n \arrow[r, "B"] \arrow[u, shift right, "T"'] & \eta \in \mathbb{E}^m \arrow[u, shift right, "S"']
\end{tikzcd}$$

$$\boxed{\begin{matrix} A = SBT^{-1} \\ B = S^{-1}AT \end{matrix}}$$

Wenn $F: X \to X$,
$A = TBT^{-1}$
$B = T^{-1}AT$

S 5.20: $RankF = r$ has the mappingmatrix* $A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

Vector spaces with scalar products

Definitions

Def Norm: A norm is a function $\| \cdot \| : V \to \mathbb{R}, x \to \|x\|$ in a vector space which satisfies:

- **N1** $\|x\| > 0, \|x\| = 0 \Leftrightarrow x = 0$ (positiv definit)
- **N2** $\|\alpha x\| = \alpha \|x\|$ (homogenous)
- **N3** $\|x \pm y\| \leq \|x\| + \|y\|$ (Triangle-inequality)

A normed vector space has a norm **Def scalar product:** is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{E}, x, y \mapsto \langle x, y \rangle$, which satisfies:

- **S1** $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$(linear in 2nd factor)
- **S1** $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ (linear in 2nd factor)
- **S2** $\langle x, y \rangle = \overline{\langle x, y \rangle}$(symmetric, hermitian)
- **S3** $\langle x, x \rangle > 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0$(positiv definite))

Def unitsphere: the set $\{x \in V | \, \|x\| = 1\}$

Def induced norm: The length of a vector is defined as: $\|\cdot\| : V \mapsto \mathbb{R}, \|x\| \mapsto \sqrt{\langle x, x \rangle}$

Def angle ϕ: $\phi = \sphericalangle(x, y), 0 \leq \phi \leq \pi$ is defined by: $\phi = \frac{\langle x, x \rangle}{\|x\| \cdot \|y\|} = \frac{\Re\langle x, y \rangle}{\|x\| \cdot \|y\|}$

Def orthogonal vectors: two vectors x, y are orthogonal $\Leftrightarrow \langle x, y \rangle = 0$

Def orthogonal sets : two sets X, Y are orthogonal $\Leftrightarrow \forall x \in X, \forall y \in Y \langle x, y \rangle = 0$

S 6.1: $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle = \|x\|^2 \cdot \|y\|^2$(**Cauchy Schwarz Inequality**)
S 6.2: $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow x \perp y$(**Pythagoras**)
Def orthogonal basis: $\Leftrightarrow \forall i, \forall j, i \neq j : \langle b_i, b_j\rangle = 0$
Def orthonormal basis: $\Leftrightarrow$ **orthogonal basis with vectors of length 1**
S 6.3: **A set** M **of pairwise orthogonal vectors are linearly independant if** $0 \notin M$
S 6.4: **Let** $\{b_1, \cdots, n\}$ **a orthonormal basis,** $x \in V$: $x = \sum_{k=1}^{n} \langle b_k, x\rangle\, b_k \to \xi_k = \langle b_l, x\rangle$
S 6.5: **from** $\xi_k = \langle b_l, x\rangle_v, \eta_k = \langle b_l, x\rangle_v$ **follows** $\langle x, y\rangle_v = \sum_{k=1}^{n} \overline{\xi_n}\eta_k = \xi^H\eta = \langle \xi, \eta\rangle_{\mathbb{E}^n}$ **Which implies that if a basis in** V **is orthonormal the scalar product is valid in** V
From that follows:
$\|x\|_v = \|\xi\|_{\mathbb{E}^n}, \sphericalangle(x,y)_v = \sphericalangle(\xi,\eta)_{\mathbb{E}^n}, x \perp y \Leftrightarrow \xi \perp \eta$

RC Gram-Schmidt:

- $b_1 = \frac{a_1}{\|a_1\|_v}$
- $\widetilde{b_k} = a_k - \sum_{j=1}^{k-1} \langle b_j, a_k\rangle_v \cdot b_j$
- $b_k = \frac{\widetilde{b_k}}{\|\widetilde{b_k}\|_v}$

ex

$A = \begin{matrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{matrix} \underset{1}{\to} a_1 = \begin{matrix} 2 \\ 2 \\ 1 \end{matrix} / \left\| \begin{matrix} 2 \\ 2 \\ 1 \end{matrix} \right\| = \begin{matrix} 2/3 \\ 2/3 \\ 1/3 \end{matrix}, \widetilde{a_2} = \begin{matrix} 3 \\ 4 \\ 1 \end{matrix} - \left\langle \begin{matrix} 2/3 \\ 2/3 \\ 1/3 \end{matrix}, \begin{matrix} 3 \\ 4 \\ 1 \end{matrix} \right\rangle = \begin{matrix} -1/3 \\ 2/3 \\ -2/3 \end{matrix}, a_2 = \begin{matrix} -1/3 \\ 2/3 \\ -2/3 \end{matrix} / \left\| \begin{matrix} -1/3 \\ 2/3 \\ -2/3 \end{matrix} \right\| = \begin{matrix} -1/3 \\ 2/3 \\ -2/3 \end{matrix} \mapsto$
$A = \begin{matrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 2/3 & -2/3 \end{matrix}$

S 6.6: **After k-steps the set** $\{b_1, \cdots, b_k\}$ **is pairwise orthonormal.**$\{b_1, \cdots, b_k\}$ **is a basis** $\Leftrightarrow$ $\{a_1, \cdots, a_k\}$ **is a basis Every vectorspace** ($\neq \infty$) **has a orthonormal basis**
Cor 6.7: **To a vector space with scalar product with finite or countably infinite many dimensions a orthonormal basis exists.**
Def orthogonal complement: $U^\perp$ **is the orthogonal complement of a subspace** U.
$U \bigoplus U^\perp = V$
S 6.9: **For a complex matrix with** $rank A = r$ **it holds:**

- $N(A) = \Re(A^H)^\perp \subset \mathbb{E}^n$
- $N(A^H) = \Re(A)^\perp \subset \mathbb{E}^m$
- $N(A) \bigoplus \Re(A^H) = \mathbb{E}^n$
- $N(A^H) \bigoplus \Re(A) = \mathbb{E}^m$
- $dim\Re(A) = r$
- $dim\Re(A^H) = r$
- $dimN(A) = n - r$
- $dimN(A^H) = m - r$

Those are the fundamental subspaces

Change of Basis

$\mathcal{B}, \mathcal{B}'$ **are orthonormalbasis. Hence:**
$b_k' = \sum_{n}^{j=1} \mathcal{T}_{jk} b_j$ **Matrix for change of basis** $\mathcal{T}$:
$\mathcal{T}^{-1} = \mathcal{T}^H$ **since both basis are orthonormal.**
Therefore it holds that:

- $\xi = \mathcal{T}\xi'$
- $\xi' = \mathcal{T}^{-1}\xi$
- $\mathcal{B} =, \mathcal{B}'\mathcal{T}$
- $\mathcal{B}' =, \mathcal{B}\mathcal{T}^H$

S 4.13: **Let** $\xi = (\xi_1 \cdots \xi_n)^T$ **be a coordinate vector of an arbitrary vector** $v \in V$ **with respect to the old basis**
. Let $\xi' = (\xi_1' \cdots \xi_n')^T$ **be the new representation of a vector** x **with respect to the new basis.**
$x = \sum_{i=1}^{n} \xi_i b_i = \sum_{k=1}^{n} \xi_k' b_k'$

All matrices are unitary/orthogonal Cor 6.12:
$\langle x, y\rangle_v = \xi^H\eta = \langle \xi, \eta\rangle_v = \langle \xi', \eta'\rangle_v = \xi'^H\eta' \Rightarrow T$ **is conformal (längen-winkeltreu)**
Note Convention: **A representation of a vector with respect ot the basis** $\mathcal{B}_1$ **is written as** $[v]_{\mathcal{B}_1}$
Therefore: $[v]_{\mathcal{B}_2} = Mat(\mathcal{B}_1)_{\mathcal{B}_2}[v]_{\mathcal{B}_1}$ **and** $[v]_{\mathcal{B}_1} = Mat(\mathcal{B}_2)_{\mathcal{B}_1}[v]_{\mathcal{B}_2}$ **where** $Mat(\mathcal{B}_2)_{\mathcal{B}_1}$ **is the matrix of change ob basis from** $\mathcal{B}_1$ **to** $\mathcal{B}_1$ **Hence:** $Mat(\mathcal{B}_1)_{\mathcal{B}_2} = \left([b_1]_{\mathcal{B}_2} \quad | \cdots | \quad [b_n]_{\mathcal{B}_2}\right)$

RC Calculating the matrix of F **with respect to Basis** $\mathcal{B}$**: We have a function** $F : X \mapsto X$ **and the basis of X** $:= \mathcal{B}$**:**

1. **calculate for all** $\forall a \in \mathcal{B}$: $F(b_a)$
2. **solve for all** $F(b_a) = \alpha_a b_1 + \beta_a b_2 \cdots \gamma_a b_n$
3. **write coordinate vectors as:** $\xi_a = (\alpha_a \beta_a \cdots \gamma_a)^T$
4. **write matrix as** $F_{[\mathcal{B}]} = \xi_1 | \cdots | \xi_n$

RC Compute Basistransformationsmatrix from $\mathcal{B}$ **to** $\mathcal{S}$**:**
Since $\mathcal{S}$ **is a standardbasis we have:**

1. $\mathcal{S} \to \mathcal{B}$ **is given by** $B = (s_1|s_2|\cdots|s_n)$ **(Columns of** B **are the basis vectors of** $\mathcal{B}$**)**
2. **Compute inverse of** B **to get** $\mathcal{B} \to \mathcal{S}$

RC Basistransformationsmatrix from 2×2 **matrices with respect to the standard basis:**
The standard basis of 2×2 **matrices is given by:** $\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
and the new Basis is defined by: $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ c & e \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \begin{pmatrix} i & j \\ k & l \end{pmatrix}, \begin{pmatrix} m & n \\ o & p \end{pmatrix} \right\}$ **Then the matrix (Abbildungsmatrix) is defined by**
$F = \begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{pmatrix}$

RC Prove that $\mathcal{B}$ **is Basis:**
$\mathcal{S}$ **denotes the standard basis.**

1. **compute for all** $\forall a \in \mathcal{B}$: $b_a = \alpha s_1 + \beta s_2 \cdots \gamma s_n$
2. **Since** $span(\mathcal{B}) = span(\mathcal{S})$**,** $\mathcal{B}$ **has to be basis.**

RC Prove that F **is a bijective mapping** $F : X \mapsto Y$ **with the basis** $\mathcal{X}$ **for** X **and** $\mathcal{Y}$ **for** Y **:**

1. **calculate for all** $\forall a \in \mathcal{X}$: $F(x_a)$
2. **solve for all** $F(x_a) = \alpha_a y_1 + \beta_a y_2 \cdots \gamma_a y_n$
3. **write coordinate vectors as:** $\xi_a = (\alpha_a \beta_a \cdots \gamma_a)^T$
4. **write matrix as** $F_{[\mathcal{X}]} = \xi_1 | \cdots | \xi_n$
5. **As** $F_{[\mathcal{X}]}$ **is quadratic and has full rank we have that** $dim(\mathcal{X}) = dim(\mathcal{Y})$ **and thus by Cor.5.8 that F is bijective**

unitary/orthogonal mapping

Def unitary: **A linear mapping**$F : X \mapsto Y$ **is unitary/orthogonal if** $\langle F(v), F(w)\rangle_y = \langle v, w\rangle_x$
S 6.13:

1. F **is isometric (längentreu):** $\|F(v)\|_y = \|v\|_X$
2. F **is conformal (winkeltreu):** $v \perp w \Leftrightarrow F(v) \perp F(w)$
3. $kerF = \{0\}$**, F is injective**

- **if** $n = dimX = dimY < \infty$

4. F **is isomorphism**
5. $\{b_1, \cdots, b_n\}$ **is orthonormal basis of X** $\Leftrightarrow \{F(b_1), \cdots, F(b_n)\}$ **is a orthonormal basis of Y**
6. F^{-1} **is unitary/orthogonal**
7. **The mapping matrix (Abbildungsmatrix) A is unitary/orthogonal**

Least Squares

Let $Ax = b$ **a overdetermined SLE (Equations ¿ Variables). No exact solution exists.**
$\to x^* = argmin_{x \in \mathbb{E}^n} \|Ax - b\|_2^2 \Rightarrow (Ax - b) \perp \Re(A)$

Def Pseudoinverse: **If Rank A = n:**
$A^+ = (A^HA)^{-1}A^H \Rightarrow A^+A = I$
~~**Def normalequations:**~~ $\cancel{(A^TA)x = A^Ty}$

RC Least Square Method for functions:
We assume that $ker(A) = \{0\}$ **and** A^HA **is regular**

0. **bring problem in a form where everything is numerically determined except the coefficients**
1. **calculate** A^Ty
2. **calculate** A^TA
3. **solve the equation** $(A^TA)x = A^Ty$
4. **calculate error** $r = y - Ax$

ex
The equation is given $y(t) = x_1t + x_2t^2$ **We have** $t_n(1,2,3,4)$ **and** $y(t)_n = (13.0, 35.5, 68.0, 110.5) \underset{0}{\to}$
$A = \begin{matrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{matrix}, y = \begin{matrix} 13.0 \\ 35.5 \\ 68.0 \\ 110.5 \end{matrix} \underset{1}{\to} A^Ty = \begin{matrix} 730.0 \\ 2535.0 \end{matrix} \underset{2}{\to} A^TA = \begin{matrix} 30 & 100 \\ 100 & 354 \end{matrix} \underset{3}{\to} \left.\begin{matrix} 30 & 100 \\ 100 & 354 \end{matrix}\right| \begin{matrix} 730 \\ 2535.0 \end{matrix} \mapsto x_1 = 7.9355, x_2 = 4.919$

RC Least Square Method for 2D-points:

0. **Write X- / Y-coordinate alternately in the form** $(x, 0)$**,** $(0, y)$ **in** A **for every point. Write X- / Y-coordinate alternately in y.**
1. **Rest as usually**

ex
The points $P = \{(-1,1), (1,1), (1,-1), (-1,-1)\}$ **should be transformed with respect to the squared distance to the points** $P' = \{(0,2), (1,3), (0,-2), (-1,-3)\}$**. The transformation is defined as** $T(P) = T\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} s_x \cdot p_x \\ s_y \cdot p_y \end{pmatrix}$

$$Ax = y \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \\ 0 \\ -2 \\ -1 \\ -3 \end{pmatrix}$$

RC Least Squares with QR-decomposition:
We have the normal equations $A^HAx = A^Tb \Rightarrow (QR)^H(QR)x = (QR)^Hb \Rightarrow R^HQ^HQRx = R^HQ^H \Rightarrow R^HRx = R^TQ^Tb \Rightarrow Rx = Q^Hb$ **Therefore we have:**

1. **compute QR-decomposition of A**
2. **solve** $Rx = Q^Tb$

RC Least Squares with SVD:
$\|Ax - b\|_2^2 = \left\| \Sigma \underbrace{V_x^H}_{y} - \underbrace{U^Hb}_{c} \right\|_2^2 = \|\Sigma y - c\|_2^2$;$x^* = V\Sigma^+U^Hb \Rightarrow \infty$ **solutions, here: smallest 2-norm** ($y^* = \Sigma^+U^Hb$) **Where** Σ^+ **is the pseudoinverse of** Σ**, hence it holds**

Determinants

Def :
$det\,(a_{11}) = a_{11}, det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$,
$det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{matrix} a_{11}a_{22}a_{33} & +a_{21}a_{32}a_{13} \\ +a_{31}a_{12}a_{23} & -a_{13}a_{22}a_{31} \\ -a_{12}a_{21}a_{33} & -a_{11}a_{23}a_{32} \end{matrix}$
S 8.12: $det(A) = \sum_{i=1}^{n} a_{ki}\mathcal{K}_{ki} = \sum_{i=1}^{n} a_{il}\mathcal{K}_{il}$ **for a fixed** k **and** l**.**
Def cofactor $\mathcal{K}_{ki}$**:** $\mathcal{K}_{ki} = (-1)^{k+i}det(A_{[k,i]})$
Def$\mathbf{A}_{[k,i]}$**:** **Is defined as the matrix** A **without the** $k - th$ **row and** $i - th$ **column**
Def : $det A = 0 \Leftrightarrow A$ **is singular**
Def : $det A \neq 0 \Leftrightarrow A$ **is regular**
S 8.3:

i. $det(A)$ **is linear in every row**
ii. **swapping two rows changes the sign of** $det(A)$
iii. $det(I) = 1$

S 8.4:

iv. **if A has a row with 0** $\Rightarrow det(A) = 0$
v. $det(\gamma A) = \gamma^n det(A)$
vi. **if A has to equal rows** $\Rightarrow det(A) = 0$
vii. **adding a multiple of a row to another row doesnt affect the det**
viii. **is A a diagonalmatrix:** $det(A) = \prod_{i=1}^{n} a_{ii}$
iv. **is A a triangularmatrix:** $det(A) = \prod_{i=1}^{n} a_{ii}$

Cor 8.10: **Every statement of Satz 8.3 and 8.4 also holds for columns instead of rows**
S 8.5: **Using Gauss on A results in:**
$det(A) = (-1)^v \prod_{k=1}^{n} r_{kk}$
where v **is # swappings of rows and** r_{kk} **are the diagonal elements of the row echelon form S 8.7:** $det(AB) = det(A) \cdot det(B)$
Cor 8.8: **if A is regular** $\Rightarrow det(A^{-1}) = \frac{1}{det(A)}$
S 8.9: $det(A^T) = det(A)$ **and** $det(A^H) = \overline{det(A)}$
Def det of block matrices:
$det\left[\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right] = det(A) \cdot det(B)$
Def det of unitary matrices: **Let** A **be unitary/orthogonal** $|det(A)| = \pm 1$ **proof:**
$det(U^TU) = det(I) = (det(U))^2 = 1 \Rightarrow det(U) = \pm 1$

Eigenvalues and -vectors

Def eigenvector: **A number** $\lambda \in \mathbb{E}^n$ **is called eigenwert of a linear mapping:** $F : X$ **if** $\exists v \in V, v \neq 0$ **such that** $F(v) = \lambda v$**. v is an eigenvector. The set of all eingenvectors, which corespond to** λ **form a subspace**
$E_\lambda = \{v \in V | F(v) = \lambda v\}$
Def spectrum: **The set of all eigenvalues of F is called spectrum**
Def : $\xi \in \mathbb{E}^n$ **is a eigenvector of** $\lambda \Leftrightarrow A\xi = \lambda\xi$
Lem 9.1: **A linear map** F **and its matrix representation have the same eigenvalues and the eigenvectors are connected by the coordinaterepresentation** k_v
Lem 9.2: λ **is eigenvalue** $\Leftrightarrow ker(A - \lambda I)$ **is singular** ($E_\lambda = ker(A - \lambda I)$)
Def multiplicity: **The geometric multiplicity of** $\lambda = dim(E_\lambda)$

Def characteristic polynomial: It is defined by $\mathcal{X}_A(\lambda) = det(A - \lambda I) = 0$
Def Trace: $tr(A) = \sum_{k=1}^{n} a_{kk}$
S 9.5: $\lambda \in \mathbb{E}$ is eigenvalue of A $\Leftrightarrow \mathcal{X}_A(\lambda) = 0$
Lem 9.4:
$\mathcal{X}_A(\lambda) = (-1)^n \cdot \lambda^n (-1)^{n-1} \cdot tr(A) \cdot \lambda^{n-1} + \cdots det(A)\lambda^0 = a_n \cdot^n + a_{n-1} \cdot \lambda^{n-1} + \cdots det(A)$
Lem 9.6: A (quadratic) matrix is singular if and only if it has 0 as an eigenvalue
Def algebraic multiplicity: is the multiplicity of an eigenvalue in the char. polynomial.
S 9.13: geometric multiplicity $\leq$ algebraic multiplicity

RC Find Eigenvalues and -vectors:
1. find char. polynomial $\mathcal{X}_A(\lambda) = det(A - \lambda I)$
2. find roots of $\mathcal{X}_A$
3. for every λ_k find the solution for $(A - \lambda_k I)x = 0$

S 9.7: for similar matrices $C = T^{-1}AT$ (C and A are similar) holds that $tr(A) = tr(C), det(A) = det(C), \mathcal{X}_A = \mathcal{X}_C$ and they have the same eigenvalues
S 9.11: Eigenvectors for different Eigenvalues are linearly independant $\Rightarrow$ max $dimV$ different Eigenvalues
Def : λ is eigenvalue for $A \Rightarrow \lambda^q$ is eigenvalue of A^q
Note Trace: $trace(A)$ is equal to the sum of all eigenvalues of A:$trace(A) = \lambda_1 + \cdots + \lambda_n$
Note Trace: $det(A)$ is equal to the producct of all eigenvalues of A:$det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n$

Decompositions

Spectral-/Eigenvaluedecomposition

Def : $A = V\Lambda V^{-1}$ (A and Λ are similar) Precondition for diagonalisation
: **S 9.14:** $A \in \mathbb{C}^{n\times n}$ is diagonalisible $\Leftrightarrow \forall$ Eigenvalues (geom. mult. = alg. mult.)
S 9.15: If $A \in \mathbb{C}^{n\times n}$ is unitary ($A^H = A$)it holds that:
- i all eigenvalues are real
- ii the eigenvectors are pairwise orthogonal
- iii an orthonormal basis U exists, which consists of all the eigenvectors
- iv for the unitary matrix U holds that $U^HAU = \Lambda$

Cor 9.16: The previous statements is also valid for real-symmetric matrices

RC Eigenvaluedecomposition:
1. find the eigenvalues of A λ_k and write $\Lambda = diag(\lambda_1 \cdots \lambda_n)$
2. find the according eigenvectors v_k of λ_k write them as $V = (v_1|\cdots|v_n)$ (sorted according to Λ)
3. find inverse V^{-1}

Cor 9.10: If A is diagonisable it can be composed as a sum of 1-rank-matrices
: $A = \sum_{k=1}^{n} V_k\lambda_k w_k^T$ with $V = (V_1 \cdots V_n)$ and
$V^{-1} = \begin{pmatrix} w_1^T \\ \vdots \\ w_n^T \end{pmatrix}$ from that follows: $Av_k = v_k\lambda_k$
and $w_k^TA = \lambda_k w_k^T$: w^T is a left eigenvector

RC Eigenvaluedecomposition with SVD:
The SVD is given by $A = U\Sigma V^H$
1. Expand $U\Sigma V^H \Rightarrow UI\Sigma V^H \Rightarrow UI_1I_2\Sigma V^H$ where $UI_1 = V$ and $I_1I_2 = I$
2. Calculate $(UI_1)(I_2\Sigma)V^H$

RC Composition of 1-rank- matrices:
1. write $A = V\Lambda V^{-1}$
2. rewrite $A = \sum_{k=1}^{n} V_k\lambda_k w_k^T$ V_k = *row* ($\downarrow$) *of* V, w_k^T = *column* ($\rightarrow$) *of* V^{-1}

RC Powers of A:
1. write $A = V\Lambda V^{-1}$
2. calculate $A^m = V\Lambda^m V^{-1}$

note: $\Lambda^m = diag(a_{11}^m, \cdots, a_{nn}^m)$

Singularvaluedecomposition

Def SVD for A^HA: Spectral-decomposition exists for every matrix A^HA. Since A^HA is hermetian (hermetisch) and positive semidefinite.$\rightarrow$ A^HA has real, non-negative eigenvalues $\lambda \in \mathbb{R}$ and $\lambda \geq 0$ Therefore one can rewrite:$A^HAV = V\Lambda \xrightarrow[\lambda=\sigma^2]{} A^HAV_r = V_r\Sigma_r^2 \Rightarrow$
$V_r^HA^HAV_r = \Sigma_r^2 \Rightarrow \underbrace{(\Sigma_r^{-1}V_r^HA^H)}_{=U_r^{-1}}\underbrace{(AV_r\Sigma_r^{-1})}_{=U_r} = I$
Def SVD: SVD exists for every matrix, such that U, V are unitary and Σ is diagonal and positive. $A = U\Sigma V^H$ it follows $AA^H = U\Sigma_m^2U^H$,$A^HA = V\Sigma_n^2V^H$, $A^H = V\Sigma^TU^H$
Def A invertible: If A is invertible $A^{-1} = V\Sigma^{-1}U^H$
S 11.11f: For Rank r it holds that:
- $\{u_1, \ldots, u_1\}$: Basis of $Im(A) = \mathcal{R}(A)$
- $\{u_{r+1}, \ldots, u_m\}$: Basis of $Ker(A^H) = \mathcal{N}(A^H)$
- $\{v_1, \ldots, v_1\}$: Basis of $Im(A^H) = \mathcal{R}(A^H)$
- $\{v_{r+1}, \ldots, v_m\}$: Basis of $Ker(A) = \mathcal{N}(A)$

Def Automorphism: In a self-image (Selbstabbildung) it holds that
$A = U\Sigma V^H = V\underbrace{V^HU}_{R}\Sigma V^H = \underbrace{V}_{1}\underbrace{R}_{2}\underbrace{\Sigma}_{3}\underbrace{V^H}_{4}$
- 1,4 Change to orthonormal basis
- 2 rotation, mirroring
- 3 scaling of unit-axes

Def singular values: singular values: $\sigma_i = \sqrt{\lambda_i}$ sorted in descending order $\sigma_a \leq \sigma_b \cdots \sigma_r \leq 0 \cdots$
Def Eigenbasis: V is the orthonormal eigenbasis of A^HA such that: $AA^H = U\Sigma_m^2U^H$.
Similar for U as eigenbasis of AA^H:
$A^HA = V\Sigma_n^2V^H$

RC SVD of $A \in \mathbb{E}^{m\times n}$with A^HA:
1. Calculate $(A^HA) \in \mathbb{E}^{n\times n}$
2. find eigenvalue of A^HA
3. write $\Sigma_r = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}) \in \mathbb{E}^{n\times n}$
4. rewrite: $\Sigma \in \mathbb{E}^{m\times n}$: $\Sigma = \begin{matrix}\Sigma_r & 0 \\ 0 & 0\end{matrix}$
5. find eigenvectors of $A^HA \Rightarrow v_1, \cdots, v_r$
6. norm eigenvectors and compute $V = (\frac{v_1}{\|v_1\|}|\cdots|\frac{v_n}{\|v_n\|}) \in \mathbb{R}^{n\times n}$
7. solve for $U = AV\Sigma^{-1}$
8. if $U_r \neq U$: $U_r \xrightarrow[gram]{schmidt} U \in \mathbb{E}^{m\times m}$
9. write $A = U\Sigma V^H$

RC SVD of $A \in \mathbb{E}^{m\times n}$with AA^H:
1. Calculate $(AA^H) \in \mathbb{E}^{m\times m}$
2. find eigenvalue of AA^H
3. write $\Sigma_r = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}) \in \mathbb{E}^{m\times m}$
4. rewrite: $\Sigma \in \mathbb{E}^{m\times n}$: $\Sigma = \begin{matrix}\Sigma_r & 0 \\ 0 & 0\end{matrix}$
5. find eigenvectors of $AA^H \Rightarrow v_1, \cdots, v_r$
6. norm eigenvectors and compute $V = (\frac{v_1}{\|v_1\|}|\cdots|\frac{v_n}{\|v_n\|}) \in \mathbb{R}^{n\times n}$
7. solve for $U = AV\Sigma^{-1}$
8. if $U_r \neq U$: $U_r \xrightarrow[gram]{schmidt} U \in \mathbb{E}^{m\times m}$
9. write $A = U\Sigma V^H$

RC SVD with Spectral decomposition $A = V\Lambda V^{-1}$:
One has to sort the singular values of Λ according to their value. Then one can do:
1. We have $U = V, \Sigma = \sqrt{\Lambda}$
2. Rewrite: $A = V\sqrt{\Lambda}V^{-1}$

Cor 11.4: If $A \in \mathbb{E}^{m\times n}$ and $rank(A) = r$ then: eiganvalues of $A^HA \in \mathbb{E}^{m\times m}$ and $AA^H \in \mathbb{E}^{n\times n}$ are the same but the mulitplicity of the eigenvalue 0 is $n - r$ or $m - r$
Def spectral norm: It is defined as: $\|A\|_2 = \sigma_1$

QR-Decomposition

Def QR-Decomposition: A matrix A can be composed as $A = QR$ where Q is ortohogonal and R is an upper triangular matrix. The decomposition is unique if $m \leq n$ and $Rank(A) = n$

RC QR-Decomposition:
1. Gram Schmidt on the rows ($\downarrow$) of $A \rightarrow Q$
2. solve $R = Q^TA \rightarrow R$

ex
$A = \begin{matrix}2 & 3 \\ 2 & 4 \\ 1 & 1\end{matrix} \xrightarrow[1]{} q_1 = \begin{matrix}2/3 \\ 2/3 \\ 1/3\end{matrix}, q_2 = \begin{matrix}-1/3 \\ 2/3 \\ -2/3\end{matrix} \mapsto Q = \begin{matrix}2/3 & -1/3 \\ 2/3 & 2/3 \\ 2/3 & -2/3\end{matrix} \xrightarrow[2]{} R = Q^TA = \begin{matrix}3 & 5 \\ 0 & 1\end{matrix}$

Definitions

Def nullmatrix: Has in every entry 0
- $\forall A(A + 0 = 0 + A = A)$ (S.2.2)

Def diagonalmatrix: Has in every entry 0 except for the diagonal: $(D)_{ij} = 0$ for $i \neq j$ one can write $Diag(d_{11}, \cdots, d_{nn})$
- is symmetric
- $det(A) = A_{11} \cdot \cdots \cdot A_{nn}$(S.8.4)
- $A^{-1} = diag(\frac{1}{A_{11}} \cdots \frac{1}{A_{nn}})$
- $A^m = diag(a_{11}^m, \cdots a_{nn}^m)$

Def identity: The identity is written as $I_n = Diag(1, \cdots, 1)$
- $AI = IA = A$
- $A^{-1} = I$

Def upper triangular matrix: $(R)_{ij} = 0$ for $i > j$
- is nilpotent

Def lower triangular matrix: $(R)_{ij} = 0$ for $i < j$
- is nilpotent

Def Zerodiviser: If $AB = 0 \Leftrightarrow$ A,B Zerodiviser, Nullteiler
Def symmetric: $A^T = A \Leftrightarrow$A symmetric
- A and B symmetric $\Rightarrow AB = BA \Leftrightarrow AB$ is symmetric (S.2.7)
- if positiv definit $\Rightarrow$ regular (L3.7)

Def skew-symmetric: $A^T = -A \Leftrightarrow$A skew-symmetric
- $tra(A) = 0$
- if A has odd order
  - $det(A)$=0
  - do inverse exists
  - A is singular

if A has even order
- inverse is skew-symmetric if it exists

Def hermitian: $A^H = A \Leftrightarrow$A hermitian
- if positiv definit $\Rightarrow$ regular (L.3.7)

Def unitary/orthogonal: $A^HA = I \Leftrightarrow$A is unitary/orthogonal
- A is regular (S.2.20)
- $A^{-1} = A^H$(S.2.20)
- A^{-1} is unitary (S.2.20)
- A and B is unitary $\Rightarrow AB$ is unitary (S.2.20)
- $det(A) = \pm 1$

Multiple Choice

General

- C If the solutions of an SLE are $x_1 = 0, x_2 = 0, x_3 = 1$ the system has infinite many solutions
- C Let A be a real 2×4 matrix with rank 2. Then the SLE $Ax = b$ has a non-trivial solution
- W If A is invertible it holds: $ABA^{-1} = B$
- C $\mathbb{E}^{n\times n} \rightarrow \mathbb{E}, A \mapsto trace(A)$ is linear
- W $\mathbb{E}^{n\times n} \rightarrow \mathbb{E}, A \mapsto det(A)$ is linear
- C Let $D \in \mathbb{E}^{2\times 2}$, $dim(KerD) = 2$ only if $D = \begin{matrix}0 & 0 \\ 0 & 0\end{matrix}$
- C If A^2 is invertible, so is A^3
  $det(A^2) \neq= 0 \Rightarrow det(A) \neq 0 \Rightarrow det(A^3) \neq 0$
- C If A is regular and $A^2 = A$,then $A = I$
- W For linear dependant x, y, z it holds $x = \alpha y + \beta z$ Only x, y can be dependant
- C If A and $A^2 \in \mathbb{E}^{n\times n}$ and A^2 is regular, A^3 is invertible
- C $\forall x \in \mathbb{R}^n, \|Ax\|_2 \leq \|A\|_2\,\|x\|_2$
- W Let $f: \mathbb{R}^n \rightarrow \mathbb{R}, f(x) := \|Ax\|_2$ The function f is a norm in $\mathbb{R}^n$
- W $\|AB\|_2 \leq \|A\|_2$

Given are the orthogonal matrices A and B with the same dimension. Which of the following properties is true?
- W The matrix product AB is orthogonal, but BA is not orthogonal
- W The matrix product BA is orthogonal, but AB is not orthogonal.
- C The matrix product AB and the matrix product BA are orthogonal
- W The matrix product AB and the matrix product BA are not orthogonal It holds that $A^T = A^{-1}$ $BT = B^{-1}$ and further $(AB)^T = B^TA^T = B^{-1}A^{-1} = (AB)^{-1}$ and also vice versa $(BA)^T = (BA)^{-1}$

We have $\mathbb{R}^n$ with the standard scalar product $\langle\cdot,\cdot\rangle$ and 2-norm. Let A be a real $n \times n$ matrix. Which statements are correct
- C $\forall x, y \in \mathbb{R}^n$ it holds that $\left\langle x, A^Ty\right\rangle = \langle Ax, y\rangle$
- C $A^T = A^{-1} \Rightarrow \forall x, y \in \mathbb{R}^n \langle Ax, Ay\rangle = \langle x, y\rangle$
- C $A^T = A^{-1} \Rightarrow \forall x \in \mathbb{R}^n \|Ax\| = \|x\|$
- C Let B be another real $n \times n$ matrix. $A^T = A^{-1}$ and $B^T = B^{-1} \Rightarrow$ the inverse of AB exists and it is: $(AB)^{-1} = (AB)^T$

Given are orthogonal matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. Which of the following statements are correct?
- C The matrix A^T is orthogonal
- W The matrix $A + B$ is orthogonal
- W The matrix $A + A^T$ is orthogonal
- C The matrix AB^{-1} is orthogonal

Given is a lower triangular matrix $A \in \mathbb{R}^{3\times 3}$ whose entries are non-negative integers and whose entries either occur only once or are equal to zero. Which of the following options are possible for the value of the determinant $det(A)$?
- W 5
- W 7
- W -2
- C 35 A is triangular$\Rightarrow det(A) = a_{1,1}a_{2,2}a_{3,3} \Rightarrow$ so $det(A) = 0$ or $det(A) = a_{1,1}a_{2,2}a_{3,3}$

The dimensions of the subspace of all skew-symmetric real 3×3 matrices is:
- W 1
- C 3
- W 6
- W 9

Let $A \in \mathbb{R}^{2\times 3}$ and $b \in \mathbb{R}^2$. Assume a solution for $Ax = b$ exists
- C $Ax = b$ has always ∞ solutions min 1 free variable
- W The set of the solution (Lösungsmenge) of $Ax = b$ forms a line in 3D could also be a plance, 1 or 2 free variables
- C Geometrically $Ax = b$ coresponds to an intersection of two planes in 3D

Rank

W Let $B \in \mathbb{E}^{3\times 1}$ and $C \in \mathbb{E}^{1\times 3}$: BC can have rank 3.

Vectorspaces

- W Let V be a vector space over $\mathbb{R}$ with scalar product $\langle\cdot,\cdot\rangle$ and let $F : V \mapsto V$ be a linear map. If it holds that $\forall v \in V, \langle v, F(v)\rangle = 0$, then F is necessarily the null map, i.e., $F(v) = 0$ for all $v \in V$.
- W Given a vector space V with a norm $\|\cdot\|$. For all $u, v \in V$, we have $\|v\| \le \|v + u\|$. counterexample: $u = -v = (1, 1)^T$
- C Let $S \subset V$ and W be a subspace of V: $S \subset W \Rightarrow span(S) \subset W$
- C In a vector space of finite dimension with scalar product, one can complete any set of orthonormal vectors to form an orthonormal basis.
- C A vector space of finite dimension with scalar product has an orthonormal basis. Corollary 6.7
- W Consider the vector space $\mathbb{R}^n$ with the Euclidean scalar product. The scalar product of two unit vectors can be arbitrarily large. Cauchy Schwarz, S 6.1: $\langle v, w\rangle^2 \le \langle v, v\rangle \langle w, w\rangle = 1 \cdot 1 = 1$
- W We again consider $\mathbb{R}^n$ with the Euclidean scalar product. Can we find any number of pairwise orthogonal unit vectors in this vector space?

Let $\mathcal{V}$ the standardvectorspace of all 2×2 matrices. Which of the following are subspaces of $\mathcal{V}$?($B = \begin{smallmatrix}1 & 2\\ 3 & 4\end{smallmatrix}$)

- W $\{A \in \mathcal{V} | A \text{ is invertible}\}$
- W $\left\{A \in \mathcal{V} | A^2 = 0\right\}$
- C $\left\{A \in \mathcal{V} | A^T = A\right\}$
- C $\left\{A \in \mathcal{V} | A^T B = BA\right\}$

Consider the vector space F of functions of $\mathbb{R} \mapsto \mathbb{R}$ with the operations addition$(f + g)(x) = f(x) + g(x)$ and scalar multiplication $(\lambda f)(x) = \lambda f(x)$. This includes the subspace $P2 = \{a_0 + a_1 x + a_2 x^2 | a_i \in \mathbb{R}\}$ of polynomials of degree ≤ 2. Which of the following statements are correct?

- C $Span\{x + 1, x1, x^2 + 1, x^2 1\}$ is equal to $\mathbb{P}_2$.
- W $x + 1, x1, x^2 + 1, x^2 1 \in \mathbb{P}_2$ are linearly independent
- W $x + 1, x1, x^2 + 1, x^2 1 \in \mathbb{P}_2$ form a generating set (spanning set) of F.
- C $x + 1, x1, x^2 + 1, x^2 1 \in \mathbb{P}_2$ form a generating set (spanning set) of $\mathbb{P}_2$.
- W The polynomials $x + 1, x1, x^2 + 1, x^2 1 \in \mathbb{P}_2$ form a basis of $\mathbb{P}_2$

Let V, W be finite dimensional vector spaces over a space K. Let $F : V \mapsto W$ be a linear map and $(v_1, \cdots, v_n)$ a basis of V . Then it holds that:

- W $F(v_1), \cdots, F(v_n)$ are linearly independent if F is surjective
- C $F(v_1), \cdots, F(v_n)$ are linearly independent if F is injective
- C $F(v_1), \cdots, F(v_n)$ form a generating end system if F is surjective
- W $F(v_1), \cdots, F(v_n)$ form a generating end system if F is injective
- C $F(v_1), \cdots, F(v_n)$ form a basis if and only if F is an isomorphism

Let V, W be two real vector spaces with scalar products, let $\mathcal{B}$ be an orthonormal basis of V and let $F : V \mapsto W$ be an orthogonal mapping. Which of the following statements is true?

- W F is an isomorphism. F is only an isomorphism if $dimV = dimW < \infty$
- C $\|F(v)\|_W = \|v\|_v$ for all $v \in V$. Follows directly from the definition of the scalar product induced norm and the orthogonality of F.
- W If $dimV, dimW < \infty$, it is possible that $dimV > dimW$ With $dimV > dimW$ there is no orthogonal mapping $F : V \mapsto W$.
- W F is not injective.
- C is angle-preserving, i.e., for all $v, w \in V$ it holds that $\sphericalangle(F(v), F(w)) = \sphericalangle(v, w)$.
- C F is an isomorphism to the image of F. F is injective as shown before and obviously surjective to its image. Since the inverses of linear mappings are also linear, F is therefore an isomorphism
- W The set of images $F(\mathcal{B})$ is an orthonormal basis of W. Since F is not necessarily an isomorphism, the image space can be smaller than W.
- C The set of images $F(\mathcal{B})$ is an orthonormal basis of $Im(F)$This follows from the isomorphism property of F from the question above
- C If it exists,$F^{-1} : Im(F) \mapsto V$ is orthogonal. It exists as seen above. The orthogonality of the inverse follows directly from the definition of orthogonal mappings.

Det

- W Let Q be unitary and $A \in \mathbb{E}^{n\times n}$, $det(QA) = det(A)$ $|detQ| = \pm 1$
- C if $A, B, P \in \mathbb{E}^{n\times n}$ and P is invertible with $A = PBP^{-1}$ then: $det(A) = det(B)$

Given is a matrix $A \in \mathbb{R}^{n\times n}$ with entries $a_{ij} = ij$ and $n > 1$. Which statement is correct?

- W $det(A) = 1$
- C $det(A) = 0$
- W $det(A) = (-1)^n$
- W $det(A) = (-2)^n$

Which of the following statements are not correct for arbitrary $n \times n$-matrices A and B?

- C $det(A + B) = det(A) + det(B)$
- W $det(AB) = det(BA)$
- W If A is singular then AB is also singular
- W $det(AA^T A) = (det(A))^3$

Let $A, B \in \mathbb{R}^{n\times n}$ with $AB = -BA$

- C $det(AB) = det(-BA)$
- W $det(A)det(B)) = -det(A)det(B)$ n has to be even $\Rightarrow$S8.4 v
- W Either A or B has a zero-determinant
- W A and B have to be singular
- W $ABx = 0$ has more than one solution (Lösungsschar)
- C $ABx = c$ can have no, one and ∞ many solutions, if $c \in \mathbb{R}^2, c \ne 0$
- W It has to be $A = 0$ or $B = 0$

Which of the following statements are correct for an arbitrary $n \times n$-matrix A and for arbitrary n?

- W $det(2A) = 2det(A)$
- W $det(-A) = det(A)$
- C $det(A^4) = det(A)^4$
- W Let A be a triangular matrix with the property $a_{i,j} = 0 fori + j > n + 1$ (so there are zeros at the bottom right). The determinant can be calculated using the formula $det(A) = a_{1,n} \cdot a_{2,n1} \cdots a_{n,1}$.

Let $A, P, Q \in \mathbb{R}^{n\times n}$ where P is permutation matrix and Q is a unitary matrix.

- W $det(PA) = det(A)$
- C $det(PAP) = det(A)$
- W $det(QA) = det(A)$

Eigenvalue/Eigenvectors

- C with $\mathcal{X}_A(\lambda) = (\lambda - 1)^3 + 3$ is $A \in \mathbb{E}^{3\times 3}$ invertible 0 isn't eigenvalue $\Rightarrow$ A is regular
- W Let v_1 and v_2 be eigenvectors of A, so is $v_1 + v_2$ an eigenvector
- C If $A \in \mathbb{E}^{n\times n}$ and $\mathcal{X}_A(\lambda) = (\lambda - 1)^n + 2$, A is invertible
- C Similar matrices have the same eigenvalues
- W Similar matrices have the same eigenvectors
- W Every $n \times n$ matrix has linear independant eigenvectors
- W Eigenvectors, which correspond to the same eigenvalue are always linear dependant
- C If a real matrix has a eigenvector, it follows that the matrix has infinity many eigenvectors
- C Every rotation in $\mathcal{R}^3$ has the eigenvalue $\lambda = 1$
- W If λ_1 with v and λ_2 with w, so is $(\lambda_1 + \lambda_2)$ a eigenvalue with eigenvector $v + w$

Let $A \in \mathbb{R}^{3\times 3}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$

- C A is diagonisable if all eigenvalues are different
- W if A is diagonisable, all eigenvalues have to be different
- W A is diagonisable if it has 3 eigenvectors They have to bo linearly independant
- C If $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 1$ and $B = A^3 - 3A^3$ then is B diagonisable
- W If $AP = PD$ and D is a diagonalmatrix then the columns of P are eigenvectors of A Only if the eigenvalues of A are on the diagonal of D

Let $A \in \mathbb{R}^{n\times n}$ be positiv definite and symmetric. Further, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues to the eigenvectors $v_1, \cdots, v_n$

- W A^2 has at least one eigenvalue with a strictly positive imaginary part
- C it holds that $\lambda_j > 0$ for all $j = 1, \cdots, n$
- W A hast at least one eigenvalue which satisfies: $geom.mult. < alg.mult$
- W The eigenvalues are pairwise distinct: $\lambda_j \ne \lambda_i$, if $j \ne i$
- C There exist positiv real numbers $\alpha > 0$ such that $v^T Av \ge v^T v$ for all $v \in \mathbb{R}^n$

Let $A \in \mathbb{E}^{2\times 2}$ with $Rank(A) = 1$ and $Trace(A) = 5$. What are the eigenvalues

- W 1 is an eigenvalue
- C 0 is an eigenvalue
- W 2 is an eigenvalue
- W -5 is an eigenvalue
- C 5 is an eigenvalue Since $det(A) = 0$ and $trace(A) = 5 \Rightarrow 0 = \lambda_1 \cdot \lambda_2$ and $5 = \lambda_1 + \lambda_2$

Decompositions

- C A matrix $A \in \mathbb{R}^{n\times n}$ with n eigenvalues has 2^n normed spectral decompositions since the sign can be changed n-times

Let $A \in \mathbb{R}^{m\times n} m \ge n$ be a matrix with rank k. Denote the QR-decomposition of A as $A = QR$, where $Q \in \mathbb{R}^{m\times k}$ has orthonormal columns, and $R \in \mathbb{R}^{k\times n}$ is an upper (right) triangular matrix. Which one of the following statements is always true?

- W Rank A $<$ Rank R
- W $QQ^T = I$
- W If A has linearly independant columns, we have Rank R = m
- C If A has linearly independant columns, we have Rank R = n

Let $A \in \mathbb{R}^{m\times n}$ with linearly independant columns and $A = Q_1 R_1 = Q_2 R_2$, two QR-Decompositions of A

- C $Q_1^T Q_2$ is orthogonal
- C $Q_1^T Q_2$ is a upper and lower triangular matrix and therefore a diagonal matrix
- W $Q_1^T Q_2 = I$
- W $Rank(R_1) = m$
- C $Rank(R_1) = n$
- W $Rank(R_2) = m$
- C $Rank(R_2) = n$
- C $Q_1^T Q_2$ is regular

Let $A, B \in \mathbb{R}^{n\times m}$, B is regular and $B = QR$

- W $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|Ax\|_2$, is a norm in $\mathbb{R}^n$
- C If $\nexists k$, such that A^k is invertible, so is A not invertible
- C $V_x \in \mathbb{R}^n, \|Ax\|_2 \le \|A\|_2 \cdot \|x\|_2$
- W $det(B) = det(R)$
- C $\|B\|_2 = \|R\|_2$
- C $g : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|Qx\|_2$, is a norm in $\mathbb{R}^n$
- W AB is regular, but BA is not necessarily
- W $\|AB\|_2 \le \|A\|_2$

Basis

- W The transformation matrix of a basis transformation between orthonormal bases is the identity matrix. The transformation matrix of base transformation between orthonormal bases is orthogonal, the identity matrix is only one possibility.
- C The inverse of the transformation matrix of a base transformation between orthonormal bases is its Hermitian transpose.
- C $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix if and only if its columns form an orthonormal basis of $\mathbb{R}$ with respect to the Euclidean scalar product.
- C The change of basis matrix is unitary (if $\mathbb{E} = \mathbb{C}$) or orthonormal (if $\mathbb{E} = \mathbb{R}$) if both bases are orthonormal.

Procedures

- W The Gram-Schmidt orthogonalization method can be used to compute an equally large set of linearly independent vectors from a set of linearly dependent vectors.
- W Let $v_1, \cdots, v_n \subset \mathbb{R}^n$ be a set of n vectors. Using the Gram-Schmidt process, we can always produce n unit-length and pairwise orthogonal vectors. Gram Schmidts needs linear independant vectors

Let $A \in \mathbb{R}^{n\times n}, m < n$. Let $Ax = b$ be a system of linear equations and let x be a solution in the least squares sense. Which statement is always correct?

- W The vector (bAx) is orthogonal to the row space of A.
- C The vector (bAx) is orthogonal to the column space of A. $\Rightarrow$ normal equations
- W $x$is in the null space of A.
- W The solution x does not always exist

Kernel/Image

- W If the nullspace of an 8×7 matrix is 5-dimensional, the rowspace has dimension 3 $n - dim(Ker(A)) = 7 - 5 = 2 = dim(Im(A))$

Let $A \in \mathbb{R}^{m\times n}$ be such that $Ax = 0$ has only the trivial solution. Then it holds that:

- C $dimIm(A) = n$
- W $dimIm(A) = 1$
- C $dimKer(A) = 0$
- W $dimKer(A) = 1$ The kernel of A is exactly the solution set of the system of equations $Ax = 0$. Since $Ax = 0$ has only the trivial solution, $dimKer(A) = 0$. Furthermore, it holds that $dimKer(A) + dimIm(A) = n$. Therefore $dimIm(A) = n$.

Which of the following statements with $A \in \mathbb{R}^{n\times n}$ is generally true

- C $im(A) = im(2A)$
- C $ker(A))ker(2A)$
- W $im(A) = im(A^2)$
- W $im(A) = im(A + I)$
- W $im(A) = im(A^T)$
- W $ker(A))ker(A^2)$
- W $ker(A))ker(A + I)$
- W $ker(A))ker(A^T)$

Proofs

1) Prove that $A^H A$ and AA^H have the same eigenvalues

   We have that $A^H A$ and AA^H are similar with $T = A$ and $T^{-1} = A^H$
   Therefore by Satz 9.7 they have the same eigenvalues (and also the same trace and det)

2) Let $Q \in \mathbb{E}^{n\times n}$ be an orthogonal matrix. Prove that, if n is odd, that at least one of the matrices $(Q + I)$ and $(Q - I)$ singular.

   Let $\mathcal{K}_Q(x)$ be the characteristic polynomial of Q.
   λ is an eigenvalue of $Q \Leftrightarrow \lambda$ is a root of $\mathcal{K}_Q(x)$
   Q is orthogonal $\Rightarrow |\lambda| = 1$
   By Lemma 9.2 we have: λ is eigenvalue
   $\Rightarrow (A - \lambda I)$ is singular
   therefore with $\lambda = \pm 1$ at least one of them has to be singular

3) Prove that for an orthogonal matrix Q it holds that $\|Qx\|_2 = \|x\|_2$

   $$\|Qx\|_2 \overset{1}{=} \sqrt{\langle Qx, Qx\rangle} \overset{2}{=} \sqrt{(Qx)^T Qx} \overset{S.2.6}{=} \sqrt{x^T Q^T Qx} \overset{S.2.20}{=} \sqrt{x^T Ix} \overset{2}{=} \sqrt{\langle x, x\rangle} \overset{1}{=} \|x\|_2$$
   1 = def of norm, 2 def of scalar product

4) Prove that, if λ is an eigenvalue of orthogonal Q then $\lambda = \pm 1$

   $$Qv = \lambda v(1) \Leftrightarrow \|Qv\| = \|\lambda v\| \overset{2}{\Leftrightarrow} \|v\| = \|\lambda v\| \overset{N2}{\Leftrightarrow} \|v\| = |\lambda| \|v\| \Leftrightarrow |\lambda| = 1$$
   1 = def eigenvalue, 2 = as proven before, N2 = norm is homogeneous

5) Prove that for a arbitrary matrix A with its eigenvalue λ it holds that $(A - \lambda I)$ is singular

   Let v be the eigenvector to the coresponding λ in $(A - \lambda I)$
   $$(A - \lambda I)v = (Av - \lambda Iv) = Av - \lambda v \overset{1}{=} \lambda v - \lambda v = 0v$$
   As v is eigenvector we have
   $$v \ne 0(2) \Rightarrow \lambda = 0 \overset{L.9.6}{\Leftrightarrow} (A - \lambda I) \text{ is singular}$$
   1 = as v is eigenvalue, 2 = def eigenvalue

6) Let $A \in \mathbb{R}^{n\times n}$ be a real matrix and $x \in \mathbb{R}^n$ be a vector.Prove that: $\|Ax\|_2 \ge \sigma_{min} \|x\|_2$. Where σ_{min} is the smallest singular value of A

   Let $A = U\Sigma V^T$ be the SVD of A
   $$\|Ax\|_2 = \left\|U\Sigma V^T x\right\|_2 \overset{1}{=} \left\|\Sigma V^T x\right\|_2 \overset{2}{\ge} \left\|\Sigma_{min} V^T x\right\|_2 = \left\|\sigma_{min} I V^T x\right\|_2 \overset{N2}{=} |\sigma_{min}| \left\|IV^T x\right\|_2 = \left\|V^T x\right\|_2 \overset{3}{=} |\sigma_{min}| \|x\|_2$$
   1 = U is orthogonal and *proof 3*, 2 = $\Sigma_{min} = diag(\sigma_{min} \cdots)$, 3 = V^T is orthogonal and *proof 3*

7) Prove that for a regular matrix A the two SLE $Ax = b$ and $A^T Ax = A^T b$ yield the same solution

   $$A^T Ax = A^T b \overset{1}{\Rightarrow} A^{-T} A^T Ax = A^{-T} A^T b \Rightarrow Ax = b$$
   1 = Since A is regular
   $det(A) \ne 0 \overset{S.8.9}{\Rightarrow} det(A^T) \ne 0$, hence A^T is regular as well

8) For $A \in \mathbb{R}^{n\times n}$ holds that $A = A^T$. Prove that all eigenvalues of $A^{2k} k \in \mathbb{N}$ are not negative.

   Lets define the SVD as follows: $A = V\Sigma V^{-1}$
   Further we know A is orthogonal
   $A = V\Sigma V^{-1} \overset{S.2.20}{\Rightarrow} A = V\Sigma V^T$
   One has to prove inductively that $A^n = V\Sigma^n V^T$
   B.C for $n = 1$ it holds that: $A = V\Sigma^1 V^T$
   I.H Assume it holds for any $k \in \mathbb{N}$

I.S $k \mapsto k+1$: $A^{k+1} = AA^k \overset{\text{I.H}}{=} AV\Sigma^k V^T \overset{\text{n=1}}{=}$
$V\Sigma V^T V\Sigma^k V^T \overset{\text{S.2.20}}{=} V\Sigma\Sigma^k V^T = V\Sigma^{k+1}V^T$
The eigenvalue for A^n can be found in the diagonal of Σ^n. With σ_q as the original eigenvalues of A, one can write:
$\Sigma^2 k = diag(\sigma_1^{2k} \cdots \sigma_n^{2k})$ **Therefore no eigenvalue can be negative.**

9) Prove that $A^H Ax = A^H b$ has infinitely many solutions.

$A^H Ax = A^H b \Rightarrow A^H(Ax - b) = 0 \Rightarrow$

$A^H(Ax - (b_\perp + b_\|)) = A^H((Ax - b_\perp) - b_\|) \overset{1}{=}$

$A^H(-b_\perp) \Rightarrow -(A^H b_\perp) \overset{2}{=} 0$

1 as $b_\perp \in \mathcal{R}(A)$ it follows: $Ax - b_\perp = 0$
2 $b \in \mathcal{N}(A^H)$

Therefore at least one solution exists. Since $rank(A) < n \Leftrightarrow dim(\mathcal{N}(A)) > 0$ and as every solution of the system is in $S_p + \alpha S_h$ for any α. Where S_p is a arbitrary particular solution and S_h is the homogeneous solution.

9) Prove Satz 9.7. Namely, proof for any two similar matrices that the characteristict polynomial and det is equal

Assume A and C are similar. Therefore we have that $C = T^{-1}AT$.

Characterstic Polynomial: $\mathcal{X}_C(\lambda) \overset{1}{=} (C - \lambda I) \overset{2}{=}$

$det(T^{-1}(AT - \lambda T)) \overset{\text{S.8.7}}{=} det(T^{-1})det(AT - \lambda T) =$

$det(AT - \lambda T)det(T^{-1}) \overset{\text{S.8.7}}{=} det((AT - \lambda T)T^{-1}) =$

$det(ATT^{-1} - \lambda TT^{-1}) = det(A - \lambda I) \overset{1}{=} \mathcal{X}_A(\lambda)$

1 = def of characteristic polynomial
2 = since $C = T^{-1}AT$

Det:

$det(C) \overset{1}{=} det(T^{-1}AT) \overset{\text{S.8.7}}{=}$
$det(T^{-1}) \cdot det(A) \cdot det(T) =$

$det(T^{-1}) \cdot det(T) \cdot det(A) \overset{\text{C.8.8}}{=} 1 \cdot det(A) = det(A)$

1 = def of $C = T^{-1}AT$

10) let V and W be two vectors spaces. Let $\phi : V \mapsto W$ be a linear mapping. Show, that $Im(\phi)$ is a subspace of W

1 $im(\phi)$ isn't empty: $0 = \phi(0)$ since ϕ is linear
2 For $x, y \in Im(\phi)$ holds that $x + y \in im(\phi)$: $\exists a, \exists b$ such that $\phi(a) = x$ and $\phi(b) = y$ Therefore, $x + y = \phi(a) + \phi(b) = \phi(a + b) \in Im(\phi)$ since ϕ is linear
3 For $x \in Im()$ and $\alpha \in \mathbb{R}$ holds that $\alpha x \in Im()$: $\exists a$, such that $\phi(a) = x$ Therefore, $\alpha x = \alpha\phi(a) = \phi(\alpha a) \in Im(\phi)$ since ϕ is linear

11) Let $B = (1, x, x^2)$ and $B' = (x+1, x-1, x^2)$.The columns (spalten) of T are the elements of B' in the basis B Then $T_{B' \to B} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by inverting $T_{B \to B'}$ we get $T_{B' \to B} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The mapping matrix D' is then given by

$D' = T_{B \to B'} D T_{B' \to B} = \begin{pmatrix} 0.5 & 0.5 & 1 \\ -0.5 & -0.5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$