# Cubic Splines on the 0D wave equation.

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## 1 Problem Setup

We begin with the continious functional

$$I_\epsilon(u) = \int_0^T e^{-\frac{t}{\epsilon}} \left( ||u_{tt}||^2_{L^2(\Omega)} + \frac{1}{\epsilon^2} ||\nabla u||^2_{L^2(\Omega)} \right) \, \mathrm{d}t$$

Solving for a minimiser of this functional, we obtain the E-L equation,

$$0 = \int_0^T e^{-\frac{t}{\epsilon}} \left( \langle u_{tt}, v_{tt} \rangle_{L^2(\Omega)} + \frac{1}{\epsilon^2} \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} \right) \, \mathrm{d}t \tag{1.1}$$

We use 1.1 as the bilinear form and use finite elements to discritise in time. Importantly, the function $v$ is chosen such that $v(0) = v'(0) = 0$.

## 2 Time discretisation

We solve 1.1 in 0D, by considering

$$0 = \int_0^T e^{-\frac{t}{\epsilon}} \left( u_{tt} v_{tt} + \frac{\lambda}{\epsilon^2} uv \right) \, \mathrm{d}t \tag{2.1}$$

We discretise the system by solving 2.1 on a finite dimensional subspace, namely the space of cubic splines.

To get the basis, we use the following theorem For $n \geq 1$, the function $S_{(n)}$ defined by

$$S_{(n)}(t) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x - kh)^n_+$$

Where,

$$(x-a)^n_+ = \begin{cases} (x-a)^n \text{ if } x \geq a \\ 0 \text{ if } x < a \end{cases}$$

In particular, the family of functions

$$\psi_k(t) = \frac{1}{4\tau^3} S_{(3)}(t - (k-2)\tau)$$

is a basis for the cubic splines.

Unwinding the definitions, this means that

$$\psi_k(t) = \frac{1}{4\tau^3}\left((t-(k-2)\tau)_+^3 - 4(t-(k-1)\tau)_+^3 + 6(t-k\tau)_+^3 - 4(t-(k+1)\tau)_+^3 + (t-(k+2)\tau)_+^3\right).$$

Let $\tau$ be the step size, and define $N_t = \frac{T}{\tau}$. Then the set,

$$\{\psi_0, \psi_1, \cdots, \psi_{N_t}\}$$

is a $N_t + 1$ dimensional basis for the set of cubic splines on $[0, T]$. **Note:** In the cases where the basis functions have support outside the interval $[0, T]$, we set them to be 0.

In the case of cubic splines, this is an example of one,

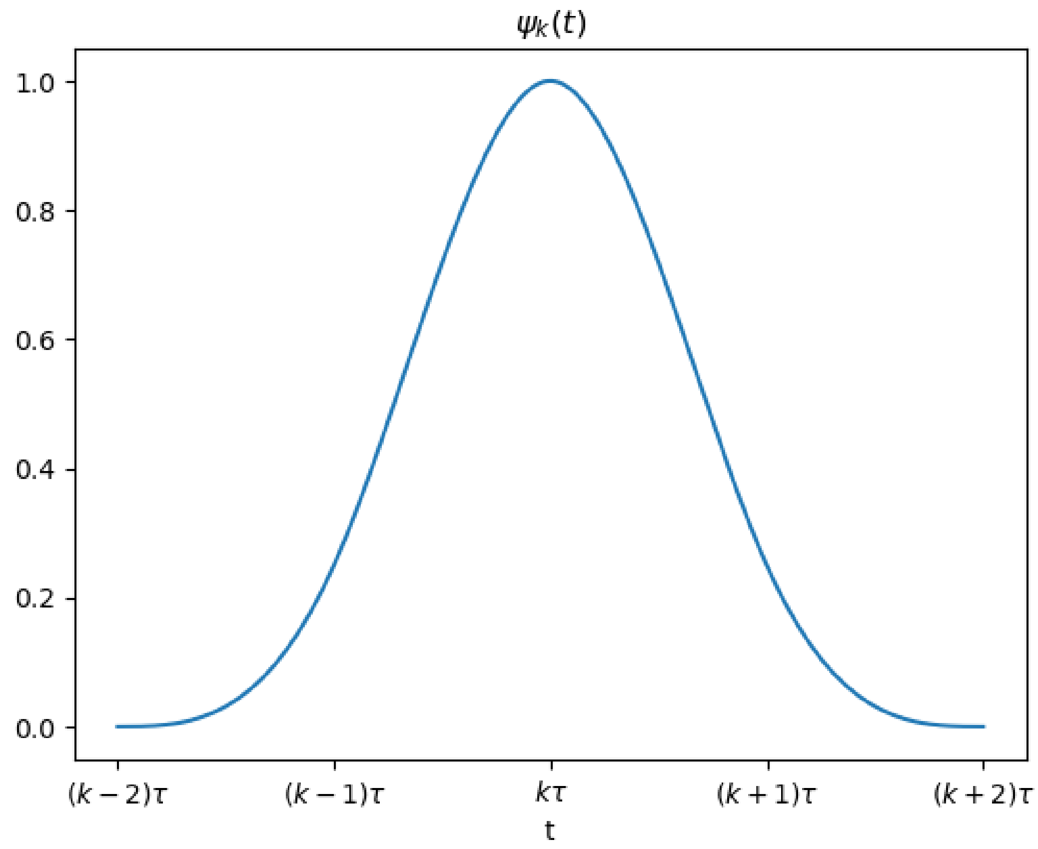


Figure 1: Plot of an example of a cubic spline.

We must now consider the trial and test spaces. The Euler-Lagrange equation was obtained by assuming that our test functions v are such that $v(0) = v'(0) = 0$, whereas the trial space is such that $u(0) = u^0, u_t(0) = u^1$. Then, to enforce the initial conditions, we require that, by writing $u(t) = \sum_{j=0}^{N_t} \sigma_j \psi_j(t)$

$$\begin{aligned} u^0 &= \sum_{j=0}^{N_t} \sigma_j \psi_j(0) \\ &= \sigma_0 \psi_0(0) + \sigma_1 \psi_1(0) \\ &= \sigma_0 + \frac{1}{4}\sigma_1 \end{aligned}$$

Also,

$$\begin{aligned} u^1 &= \sum_{j=0}^{N_t} \sigma_j' \psi_j(0) \\ &= \sigma_0 \psi_0'(0) + \sigma_1 \psi_1'(0) \\ &= \frac{3}{4\tau}\sigma_1 \end{aligned}$$

Therefore, the trial function can be written in the form

$$u(t) = \sum_{j=0}^{N_t} \sigma_j \psi_j(t)$$

Where $\sigma_0, \sigma_1$ satisfy the above.

For the trial functions, we require that $v(0) = v_t(0) = 0$, in the above situation, this forces $\sigma_0 = \sigma_1 = 0$. Hence, any function in the trial space can be written as

$$v(t) = \sum_{i=2}^{N_t} \alpha_i \psi_i(t)$$

The consequence is that the set

$$\{\psi_2, \cdots, \psi_{N_t}\}$$

is a basis for the test space.

Testing 2.1 with this basis, we obtain the following $N_t - 1$ equations.

$$0 = \int_0^T e^{-\frac{t}{\epsilon}} \left( u_{tt}\psi_{i,tt} + \frac{\lambda}{\epsilon^2} u \psi_i \right) \, \mathrm{d}t, \qquad \text{for } i = 2, \cdots, N_t$$

Finally, using the cubic spline projection for $u$, we obtain

$$0 = \sum_{j=0}^{N_t} \sigma_j \int_0^T e^{-\frac{t}{\epsilon}} \left( \psi_{j,tt}\psi_{i,tt} + \frac{\lambda}{\epsilon^2} \psi_j \psi_i \right) \, \mathrm{d}t, \qquad \text{for } i = 2, \cdots, N_t \tag{2.2}$$

These $N_t - 1$ equations along with the 2 initial conditions give the full $N_t - 1$ equations to solve for. Defining our equivalent Mass and stiffness matrices in this setup, we have

$$M = (m_{i,j})_{i,j=2,\cdots,N_t} = \int_0^T e^{-\frac{t}{\epsilon}} \psi_j \psi_i \, \mathrm{d}t$$
$$A = (a_{i,j})_{i,j=2,\cdots,N_t} = \int_0^T e^{-\frac{t}{\epsilon}} \psi_{j,tt} \psi_{i,tt} \, \mathrm{d}t$$

Then, the system "can" be written in matrix vector form as

$$\left( A + \frac{\lambda}{\epsilon^2} M \right) \sigma = 0$$

In the implementation, what is done is the following linear system is solved,

$$L\sigma = F$$

where,

$$L = \begin{pmatrix} A_{IC} & 0 \\ A_{IC,int} & A_{int} \end{pmatrix}$$

$$F = \begin{pmatrix} u^0 \\ u^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A_{IC} = \begin{pmatrix} 1 & \frac{1}{4} \\ 0 & \frac{3}{4\tau} \end{pmatrix}$$

$$A_{int} = A + \frac{\lambda}{\epsilon^2} M$$

Finally, $A_{IC,int}$ is determined by the non-zero values of 2.2 for $j = 0, j = 1$. In particular, there are 5 non-zero entries of $A_{IC,int}$.