Boundary conditions.

May 2024

1 Problem Setup

Aiming to solve

$$0 = \int_0^T u_{tt}\left(\epsilon^2 w_{tt} + 2\epsilon w_t + w\right) + \lambda uw \, \mathrm{d}t \tag{1.1}$$

Setting $\epsilon = 0$ we obtain,

$$0 = \int_0^T u_{tt} w + \lambda uw \, \mathrm{d}t \tag{1.2}$$

We project eq. (1.2) onto a basis of cubic splines, i.e.

$$u(t) = \sum_{j=0}^{N_t} \sigma_j \psi_j(t)$$

Where,

$$\psi_j(t) = \frac{1}{4\tau^3}\left((t-(j-2)\tau)_+^3 - 4(t-(j-1)\tau)_+^3 + 6(t-j\tau)_+^3 - 4(t-(j+1)\tau)_+^3 + (t-(j+2)\tau)_+^3\right)$$

and,

$$(x)_+ = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Plugging this into eq. (1.2), we obtain

$$0 = \sum_{j=0}^{N_t} \sigma_j \int_0^T (\psi_j)_{tt} w + \lambda \psi_j w \, \mathrm{d}t$$

Testing this against a suitable trial space, also comprised of cubic splines, we obtain.

$$0 = \sum_{j=0}^{N_t} \sigma_j \int_0^T (\psi_j)_{tt} \psi_i + \lambda \psi_j \psi_i \, \mathrm{d}t, \qquad i = 0, \cdots, N_t \tag{1.3}$$

While this is the set of equations we intend to solve, the test space may require that eq. (1.3) only holds for $i \subset \{0, \cdots, N_t\}$. For example, if we assume that the trial space is such that the function and its derivative are 0 at $t = 0$, then this forces the coefficients at $i = 0, i$ all to be 0, hence the set of equations to be solved is

$$0 = \sum_{j=0}^{N_t} \sigma_j \int_0^T (\psi_j)_{tt} \psi_i + \lambda \psi_j \psi_i \, \mathrm{d}t, \qquad i = 2, \cdots, N_t \tag{1.4}$$

Along with appropriate boundary conditions for the initial data. In this case they are,

$$
\begin{aligned}
u(0) &= u^0 = \sigma_0 + \frac{1}{4}\sigma_1 \\
u_t(0) &= u^1 = \frac{3}{4\tau}\sigma_1
\end{aligned}
$$

For this system we can compute the $(N_t - 1) \times (N_t + 1)$ mass and stiffness matricies as follows. $M = (m_{i,j})_{i=2,\cdots,N_t, j=0,\cdots,N_t}$, $A = (a_{i,j})_{i=2,\cdots,N_t, j=0,\cdots,N_t}$ Where,

$$
m_{i,j} = \int_0^T \psi_j \psi_i \, \mathrm{d}t
$$

$$
a_{i,j} = \int_0^T (\psi_j)_{tt} \psi_i \, \mathrm{d}t
$$

Then, the linear system can be written in the form

$$
\begin{pmatrix} \begin{pmatrix} 1 & 1/4 \\ 0 & 3/(4\tau) \end{pmatrix} & \mathbf{0} \\ A + \lambda M & \end{pmatrix} \sigma = \mathbf{F}
$$

Where,

$$
\mathbf{F} = \begin{pmatrix} u^0 \\ u^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

1.1 Instability

This is the output of the above setup for decreasing values of dt.

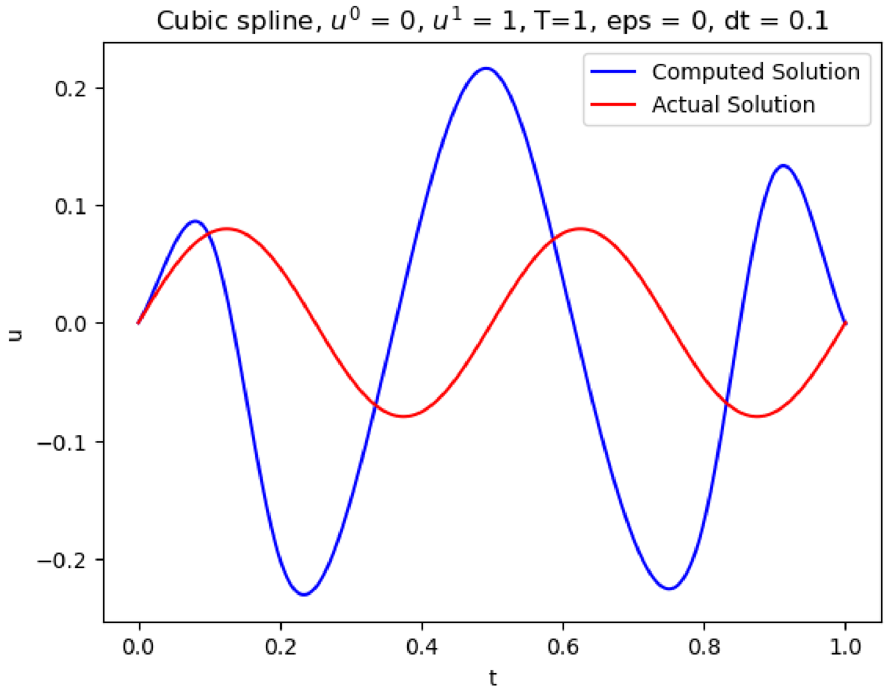

Figure 1: $\tau = 0.1$

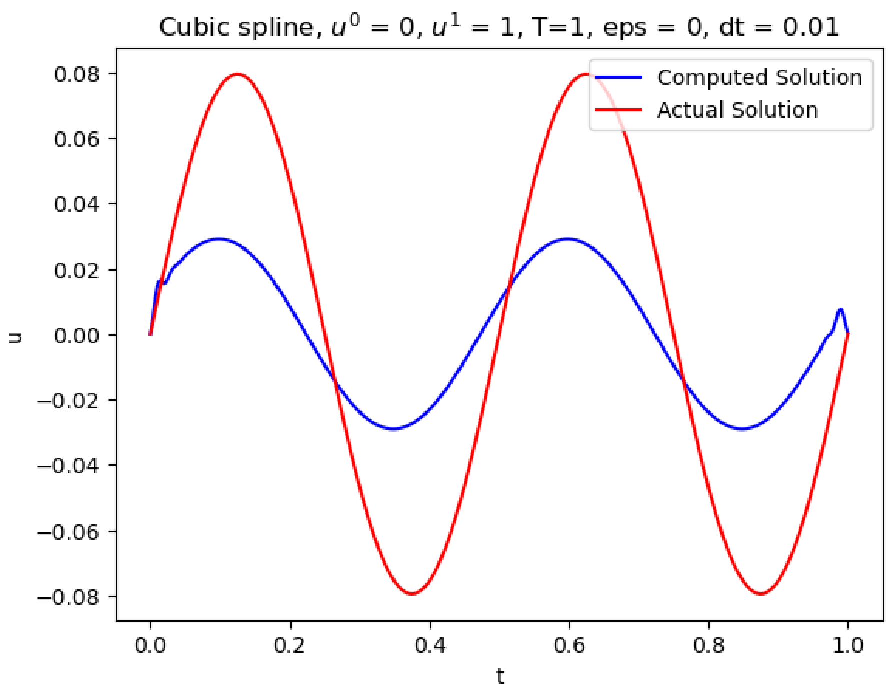

Figure 2: $\tau = 0.01$

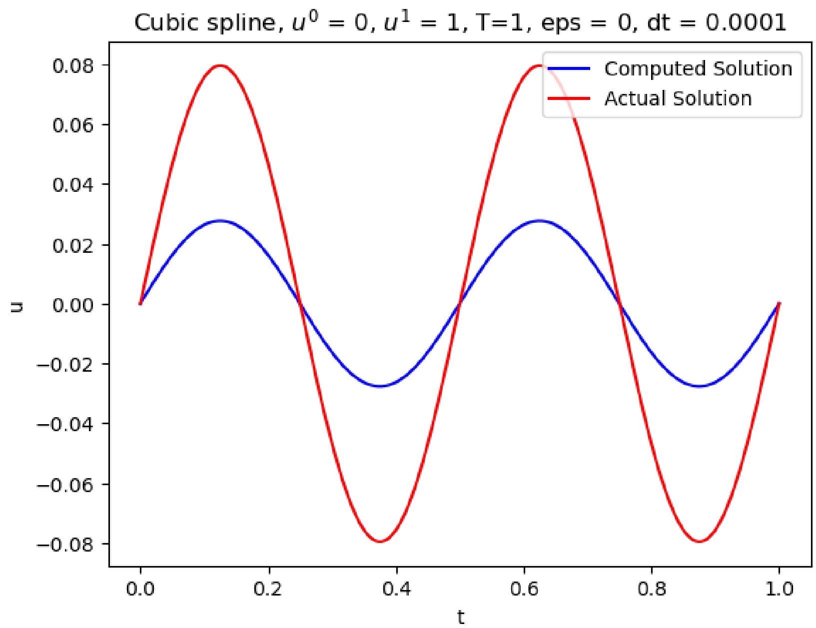

Figure 3: $\tau = 0.0001$

While these images seem to be ok, the real issue lies within jumps in the derivative, as shown here.

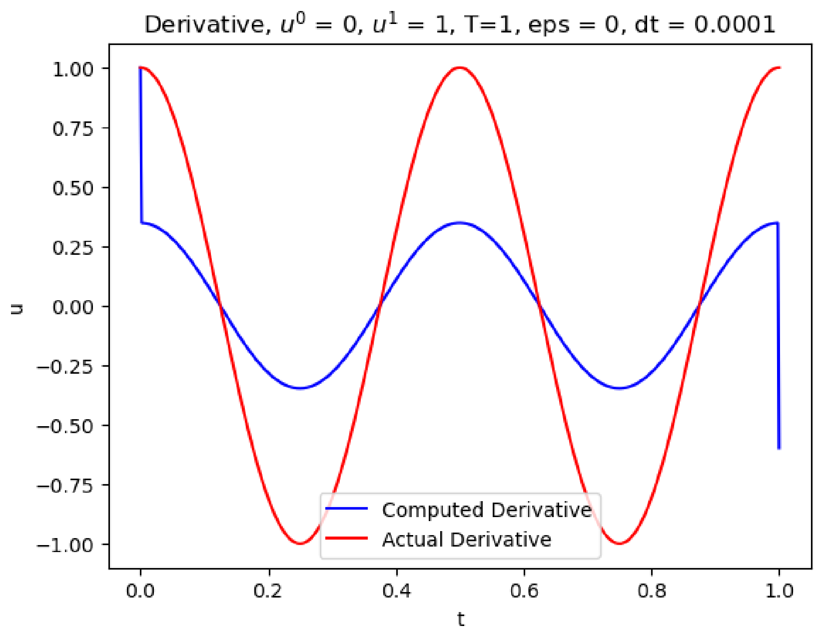

Figure 4: Derivative with $\tau = 0.0001$