Working cubic splines setup.

May 2024

1 Problem Setup

Aiming to solve

$$0 = \int_0^T e^{-\frac{t}{\epsilon}} \left(u_{tt} v_{tt} + \frac{\lambda}{\epsilon^2} uv \right) \, \mathrm{d}t \tag{1.1}$$

We wish to solve this with v being such that $v(0) = 0, v_t = 0$ and $u(0) = u^0, u_t(0) = u^1$. Let w be such that $w(0) = u^0, w_t(0) = u^1$. Then,

$$\tilde{u} = u - w$$

Also satisfies $\tilde{u}(0) = 0, \tilde{u}_t(0) = 0$. Say for example, taking $w = u^0 + tu^1$, we have,

$$-\frac{\lambda}{\epsilon^2} \int_0^T e^{-\frac{t}{\epsilon}} wv \, \mathrm{d}t = \int_0^T e^{-\frac{t}{\epsilon}} \left(\tilde{u}_{tt} v_{tt} + \frac{\lambda}{\epsilon^2} \tilde{u} v \right) \, \mathrm{d}t \tag{1.2}$$

We now aim to use the space of cubic splines to discretise the RHS ofeq. (1.2), namely, we write

$$\tilde{u} = \sum_{j=-1}^{N_t+1} \sigma_j \phi_j$$

Where ϕ_j is the cubic spline centered at $j\tau$. One thing that we note is that the set,

$$\{\phi_{-1}, \cdots, \phi_{N_t+1}\}$$

is not a basis for the cubic splines that satisfy 0 initial position and velocity. Now, as $\phi_2, \cdots, \phi_{N_t+1}$ all are 0 at $t = 0$, they are still in the subspace, but $\phi_{-1}, \phi_0, \phi_1$ are not, but some linear combination of them $\tilde{\phi} = \sigma_{-1}\phi_{-1} + \sigma_0\phi_0 + \sigma_1\phi_{11}$ is. For these functions we require that

$$\begin{aligned} \frac{1}{4}\sigma_{-1} + \sigma_0 + \frac{1}{4}\sigma_1 &= 0 \\ \frac{-3}{4\tau}\sigma_{-1} + \frac{3}{4\tau}\sigma_1 &= 0 \end{aligned}$$

Also imposing that $\tilde{\phi}(\tau) = 1$, we obtain that

$$\tilde{\phi} = \frac{8}{7}\phi_{-1} - \frac{4}{7}\phi_0 + \frac{8}{7}\phi_1$$

Finally, the set,

$$\{\tilde{\phi}, \phi_2, \cdots, \phi_{N_t+1}\}$$

is a basis for the cubic splines that satisfy these 0 initial conditions. We can then write,(abusing notation that $p\tilde{h}i = \phi_1$

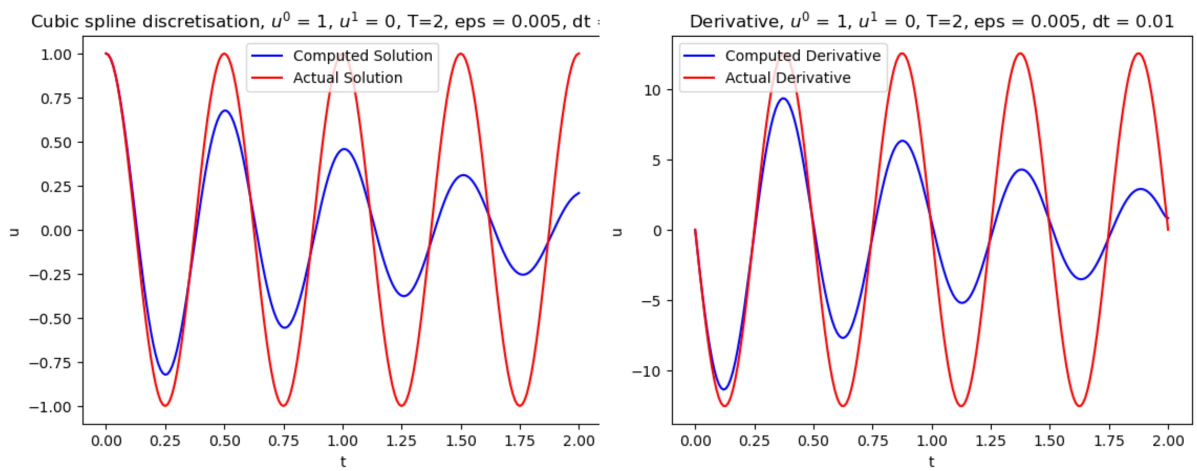
$$\tilde{u} = \sum_{j=1}^{N_t+1} \sigma_j \phi_j$$

Subbing these into eq. (1.2), we obtain the system of equations,

$$\sum_{j=1}^{N_t+1} \sigma_j \int_0^T e^{-\frac{t}{\epsilon}} (\phi_j)_{tt} (\phi_i)_{tt} + \frac{\lambda}{\epsilon^2} \phi_j \phi_i \; \mathrm{d}t = -\int_0^T e^{-\frac{t}{\epsilon}} w \phi_i \; \mathrm{d}t, \qquad i = 1, \cdots, N_t + 1$$

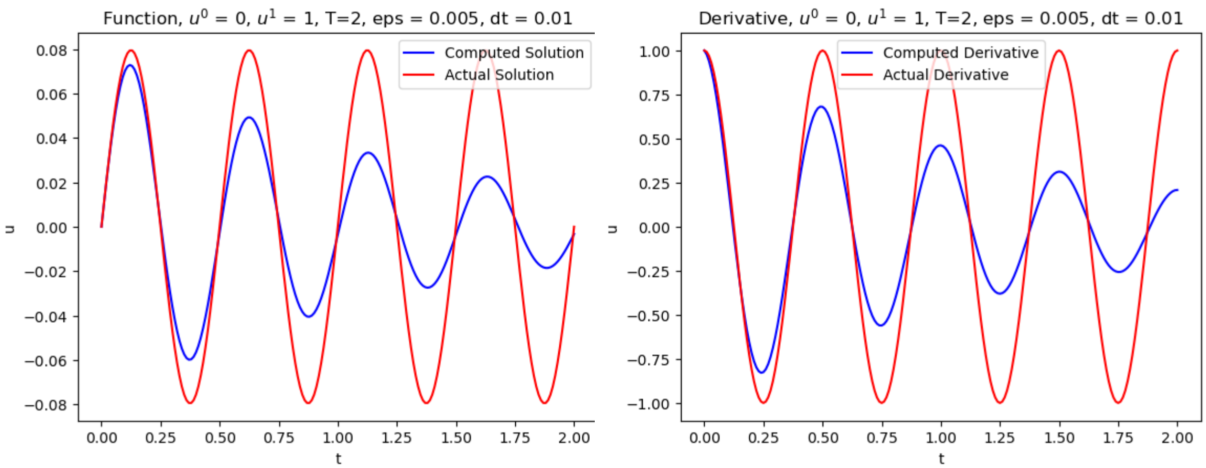
$$0 = \sum_{j=0}^{N_t} \sigma_j \int_0^T (\psi_j)_{tt} w + \lambda \psi_j w \; \mathrm{d}t$$

1.1 Experiments



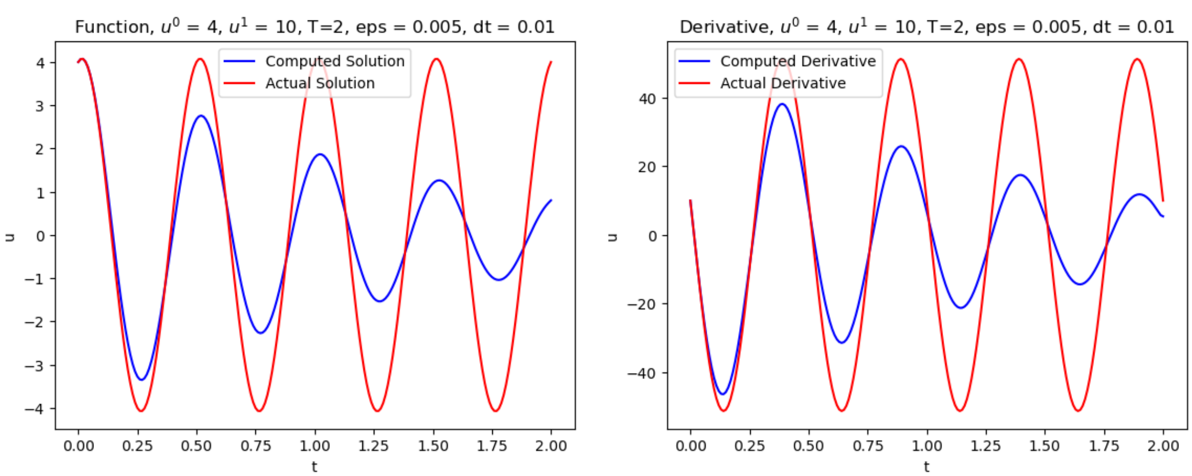
(a) Function (b) Derivative

Figure 1: $u^0 = 1$,$u^1 = 0$



(a) Function (b) Derivative

Figure 2: $u^0 = 0, u^1 = 1$



(a) Function (b) Derivative

Figure 3: $u^0 = 4, u^1 = 10$