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## A Divide and Conquer Approach to Least-Squares Estimation

**The problem of estimating parameters $\theta$ which determine the mean $\mu(\theta)$ of a Gaussian-distributed observation X is considered. It is noted that the *maximum likelihood* (ML) estimate $\hat{\theta}_{ML}$, in this case the *least squares* estimate, has desirable statistical properties, but can be difficult to compute when $\mu(\theta)$ is a nonlinear function of $\theta$. An estimate, formed by combining ML estimates based on *subsections* of the data vector X, is proposed as a computationally inexpensive alternative. The main result of the paper is that this alternative estimate, termed here the *divide and conquer* (DAC) estimate, has ML performance in the small-error region when the data vector X is appropriately subdivided. As an example application, an inexpensive range-difference-based position estimator is derived, and shown via Monte-Carlo simulation to have small-error-region mean square error equal to the Cramér-Rao lower bound (CRB).**

## I. INTRODUCTION

Consider the problem of estimating parameters $\theta$ which determine the mean $\mu(\theta)$ of a Gaussian-distributed observation X:

$$\wp(\mathbf{X};\theta) = \frac{\exp\{-\frac{1}{2}[\mathbf{X}-\mu(\theta)]^{\mathrm{T}}\Sigma^{-1}[\mathbf{X}-\mu(\theta)]\}}{\det\{2\pi\Sigma\}^{1/2}} \quad (1)$$

where $\Sigma$ is the covariance of X. (This problem may be viewed as that of estimating the parameters of a deterministic signal given observations corrupted by additive Gaussian noise; it frequently occurs in the fields of statistics and signal processing, see e.g. [1].)

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The *maximum likelihood* (ML) estimate $\hat{\theta}_{ML}$ is, in this case, the *least-squares* estimate

$$\hat{\theta}_{ML} = \arg\left[\min_{\theta}\{[\mathbf{X}-\mu(\theta)]^{\mathrm{T}}\Sigma^{-1}[\mathbf{X}-\mu(\theta)]\}\right] \quad (2)$$

and is often used for its statistical properties, namely, asymptotic efficiency [2, pp. 90–106]. However, when $\mu(\theta)$ is nonlinear in $\theta$, the computation of the ML estimate $\hat{\theta}_{ML}$ (2) can be quite difficult, involving the minimization of a possibly nonconvex cost function.

It should be noted that the complexity in computing $\hat{\theta}_{ML}$ decreases with decreasing data vector length, and, occasionally, quick solutions are available when the lengths of $\theta$ and X are equal. The estimate formed by *combining* ML *estimates based on subsections of the observation vector* X then has the potential to be much less expensive to compute than the ML estimate $\hat{\theta}_{ML}$. This estimate, called here the *divide and conquer* (DAC) *estimate* $\hat{\theta}_{DAC}$, was proposed by Weinstein [3] for the problem of estimating parameters determining the covariance of zero-mean Gaussian-distributed observations. Here it is applied to the least-squares problem (2), and its statistical performance analyzed. The main question addressed is under what conditions is the DAC estimate $\hat{\theta}_{DAC}$ a good approximation of the ML estimate $\hat{\theta}_{ML}$.

## II. PRELIMINARIES

The following results and notation are needed. Given an observation X drawn from a probability density dependent on parameters $\theta$, $\wp(\mathbf{X};\theta)$, the variance of any *unbiased* estimate $\hat{\theta}(\mathbf{X})$ of parameters $\theta$ is lower bounded by the *information inequality* [1, pp. 123–130], commonly referred to as the *Cramér-Rao lower bound* (CRB):

$$\mathrm{var}\left\{\hat{\theta}(\mathbf{X})\right\} \geq \mathbf{J}^{-1}, \quad (3)$$

J being the *Fisher information matrix*, given by

$$\mathbf{J} \stackrel{\mathrm{def}}{=} E_{\theta}\left\{\frac{\partial l(\mathbf{X};\theta)}{\partial\theta}\frac{\partial l(\mathbf{X};\theta)}{\partial\theta^{\mathrm{T}}}\right\}, \quad (4)$$

where

$$l(\mathbf{X};\theta) \stackrel{\mathrm{def}}{=} \ln\wp(\mathbf{X};\theta) \quad (5)$$

is the so-called *loglikelihood function*. The Fisher information may then be thought of as a gauge of the accuracy with which a quantity can be estimated from a set of observations.

A bound-achieving estimate $\hat{\theta}(\mathbf{X})$, if one exists, satisfies [2, pp. 74–76]

$$\hat{\theta}(\mathbf{X}) = \theta + \mathbf{J}^{-1}\frac{\partial l(\mathbf{X};\theta)}{\partial\theta}. \quad (6)$$

Subject to certain regularity conditions on $\wp(\mathbf{X};\theta)$, the ML estimate $\hat{\theta}_{ML}$ [the value of $\theta$ maximizing

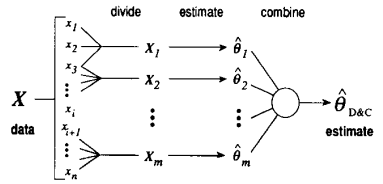


Fig. 1. DAC estimate formation. The DAC estimate is a combination of estimates based on subsections of the data.

the loglikelihood function $l(\mathbf{X};\theta)$] will be unbiased and achieve the CRB in the limit of large Fisher information [2, pp. 90–106]. Accordingly, for $\mathbf{J}$ sufficiently large, we have from (6) the following so-called *stochastic approximation*:

$$\hat{\theta}_{\mathrm{ML}} \approx \theta + \mathbf{J}^{-1}\frac{\partial l(\mathbf{X};\theta)}{\partial \Theta}, \tag{7}$$

valid for $\theta$ near $\hat{\theta}_{\mathrm{ML}}$.

In the case of Gaussian-distributed observations [$\wp(\mathbf{X};\theta)$ given by (1)], we have

$$l(\mathbf{X};\theta) = k - \tfrac{1}{2}[\mathbf{X}-\mu(\theta)]^{\mathrm{T}}\Sigma^{-1}[\mathbf{X}-\mu(\theta)] \tag{8}$$

where $k = -\frac{1}{2}\ln\det\{2\pi\Sigma\}$, and

$$\mathbf{J} = \frac{\partial \mu^{\mathrm{T}}}{\partial \theta}\Sigma^{-1}\frac{\partial \mu}{\partial \theta^{\mathrm{T}}}. \tag{9}$$

## III. DIVIDE AND CONQUER

The divide and conquer estimate is formed by combining maximum likelihood parameter estimates based on subsections of the data vector. More specifically (see Fig. 1), the observation $\mathbf{X}$ is partitioned into $m$ possibly overlapping subvectors $\mathbf{X}_i$, so that each element of $\mathbf{X}$ is represented in some $\mathbf{X}_i$. Each subvector is used to estimate, via maximum likelihood, parameters $\theta_i$, given by

$$\theta_i = \mathbf{S}_i\theta, \tag{10}$$

where $\mathbf{S}_i$ is a *selection matrix* (the identity matrix $\mathbf{I}$ with the appropriate rows removed). The estimates, denoted by $\hat{\theta}_i$, are linearly combined to form $\hat{\theta}_{\mathrm{DAC}}$, the divide and conquer estimate:

$$\hat{\theta}_{\mathrm{DAC}} = (\mathbf{S}^{\mathrm{T}}\mathbf{W}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{W}\hat{\vartheta}, \tag{11}$$

where $\hat{\vartheta}$ and $\mathbf{S}$ are concatenations of the $\hat{\theta}_i$ and the $\mathbf{S}_i$,

$$\hat{\vartheta} \overset{\mathrm{def}}{=} \begin{bmatrix}\hat{\theta}_1\\ \vdots \\ \hat{\theta}_m\end{bmatrix}, \qquad \mathbf{S} \overset{\mathrm{def}}{=} \begin{bmatrix}\mathbf{S}_1\\ \vdots \\ \mathbf{S}_m\end{bmatrix}, \tag{12}$$

and $\mathbf{W}$ is a positive-definite weighting matrix, chosen to minimize the mean square error of the resulting estimate.

### A. Estimate Mean Square Error

In this subsection, expressions for the bias and covariance of the DAC estimate are developed for the case of Gaussian observations [$\wp(\mathbf{X};\theta)$ given by (2)].

We need the expected values and cross covariances of the subvector-based ML estimates $\hat{\theta}_i$. Denoting by $\mathbf{J}_i$ the Fisher information in $\mathbf{X}_i$ relative to parameters $\theta_i$, the stochastic approximation (7), via linearity of expectation, leads to the small-error-region unbiasedness of $\hat{\theta}_i$:

$$E_\theta\{\hat{\theta}_i\} \approx E_\theta\left\{\theta_i - \mathbf{J}_i^{-1}\frac{\partial l(\mathbf{X}_i;\theta_i)}{\partial \theta_i}\right\} = \theta_i, \tag{13}$$

where $\partial l(\mathbf{X}_i;\theta_i)/\partial\theta_i = (\partial\mu_i^{\mathrm{T}}/\partial\theta_i)\Sigma_i^{-1}[\mathbf{X}_i - \mu_i]$ and $E_\theta\{\mathbf{X}_i\} = \mu_i$ have been used. The small-error-region cross covariance between $\hat{\theta}_i$ and $\hat{\theta}_j$, denoted $(\Sigma_{\hat{\vartheta}})_{ij}$, is also found using the stochastic approximation:

$$\begin{aligned}(\Sigma_{\hat{\vartheta}})_{ij} &\overset{\mathrm{def}}{=} E_\theta\left\{\left[\hat{\theta}_i - E_\theta\{\hat{\theta}_i\}\right]\left[\hat{\theta}_j - E_\theta\{\hat{\theta}_j\}\right]^{\mathrm{T}}\right\}\\ &\approx \mathbf{J}_i^{-1}E_\theta\left\{\frac{\partial l(\mathbf{X}_i;\theta_i)}{\partial\theta_i}\frac{\partial l(\mathbf{X}_j;\theta_j)}{\partial\theta_j^{\mathrm{T}}}\right\}\mathbf{J}_j^{-1}\\ &= \mathbf{J}_i^{-1}\frac{\partial\mu_i^{\mathrm{T}}}{\partial\theta_i}\Sigma_i^{-1}\Sigma_{ij}\Sigma_j^{-1}\frac{\partial\mu_j}{\partial\theta_j^{\mathrm{T}}}\mathbf{J}_j^{-1}\end{aligned} \tag{14}$$

where $\Sigma_{ij}$ is the $ij$th block of $\Sigma$.

We are now in a position to compute the DAC estimate bias and variance, as well as the weighting matrix $\mathbf{W}$. The DAC estimate is easily seen to be unbiased: $E_\theta\{\hat{\theta}_{\mathrm{DAC}}\} = (\mathbf{S}^{\mathrm{T}}\mathbf{W}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{W}E_\theta\{\hat{\vartheta}\}$. But $E_\theta\{\hat{\vartheta}\} = \mathbf{S}\theta$, and

$$E_\theta\{\hat{\theta}_{\mathrm{DAC}}\} = \theta. \tag{15}$$

Using (11) and (14), the DAC estimate covariance $\Sigma_{\hat{\theta}_{\mathrm{DAC}}}$ is easily evaluated.

$$\begin{aligned}\Sigma_{\hat{\theta}_{\mathrm{DAC}}} &= (\mathbf{S}^{\mathrm{T}}\mathbf{W}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{W}\Sigma_{\hat{\vartheta}}\mathbf{W}\mathbf{S}(\mathbf{S}^{\mathrm{T}}\mathbf{W}\mathbf{S})^{-1}\\ &\geq (\mathbf{S}^{\mathrm{T}}\Sigma_{\hat{\vartheta}}^{-1}\mathbf{S})^{-1},\end{aligned} \tag{16}$$

with equality holding when $\mathbf{W} = \Sigma_{\hat{\vartheta}}^{-1}$. It should be noted that, since (7) was used in deriving (13) and (14), implicit in the bias and variance expressions above is the assumption that *each* $\mathbf{J}_i$ is sufficiently large.

### B. Special Cases

Here, the DAC estimate mean square errror is shown to approach the CRB in two important special cases: 1) independent subvectors $\mathbf{X}_i$, and 2) parameter-vector-length subvectors $\mathbf{X}_i$.

If the subvectors $\mathbf{X}_i$ are uncorrelated, that is, $\Sigma_{ij} = 0$, $i \neq j$, then $\Sigma_{\hat{\vartheta}}$ is block diagonal with $i$th block $\mathbf{J}_i$.

The optimally weighted (i.e., $\mathbf{W} = \Sigma_{\hat{\vartheta}}^{-1}$) DAC estimate then has covariance

$$\Sigma_{\hat{\theta}_{\mathrm{DAC}}} = (\mathbf{S}^{\mathrm{T}}\Sigma_{\hat{\vartheta}}^{-1}\mathbf{S})^{-1} = \left(\sum_{i=1}^{m}\mathbf{S}_i^{\mathrm{T}}\mathbf{J}_i^{-1}\mathbf{S}_i\right)^{-1} \tag{17}$$

which is recognized as the CRB. This result is not entirely expected. Recall that when the subvectors $\mathbf{X}$ are uncorrelated, the loglikelihood function is the sum of the subvector loglikelihood functions. Now, (17) states that an average of the subvector loglikelihood minimizers behaves like the ML estimate—the minimizer of the sum of the subvector loglikelihoods. What makes (17) possible is the assumption of large Fisher information, leading to the stochastic approximation (7) which *linearly* relates the data $\mathbf{X}_i$ and the estimates $\hat{\theta}_i$.

If the subvectors and resulting parameter vector estimates are of equal length, then the matrix $\partial\mu_i^{\mathrm{T}}/\partial\theta_i$ is square and invertable. (Invertability follows from the assumed nonsingularity of the Fisher information.) The $ij$th block of $\Sigma_{\hat{\vartheta}}$ is then

$$\begin{aligned}(\Sigma_{\hat{\vartheta}})_{ij} &= \mathbf{J}_i^{-1}\frac{\partial\mu_i^{\mathrm{T}}}{\partial\theta_i}\Sigma_i^{-1}\Sigma_{ij}\Sigma_j^{-1}\frac{\partial\mu_i}{\partial\theta_i^{\mathrm{T}}}\mathbf{J}_j^{-1} \\ &= \left(\frac{\partial\mu_i^{\mathrm{T}}}{\partial\theta_i}\right)^{-1}\Sigma_{ij}\left(\frac{\partial\mu_i}{\partial\theta_i^{\mathrm{T}}}\right)^{-1}\end{aligned} \tag{18}$$

and

$$\Sigma_{\hat{\vartheta}} = \Lambda_{\partial}^{-\mathrm{T}}\Sigma\Lambda_{\partial}^{-1} \tag{19}$$

where $\Lambda_{\partial}$ is the block diagonal matrix with $i$th block $\partial\mu_i/\partial\theta_i^{\mathrm{T}}$. Assuming $\mathbf{W} = \Sigma_{\hat{\vartheta}}^{-1}$, the DAC estimate covariance can be written as

$$\begin{aligned}\Sigma_{\hat{\theta}_{\mathrm{DAC}}} &= \left[\mathbf{S}^{\mathrm{T}}(\Lambda_{\partial}^{-\mathrm{T}}\Sigma\Lambda_{\partial}^{-1})^{-1}\mathbf{S}\right]^{-1} \\ &= [\mathbf{S}^{\mathrm{T}}\Lambda_{\partial}^{\mathrm{T}}\Sigma^{-1}\Lambda_{\partial}\mathbf{S}]^{-1}.\end{aligned} \tag{20}$$

Using $\mathbf{S}^{\mathrm{T}}(\partial\mu_i^{\mathrm{T}}/\partial\theta_i) = \partial\mu_i^{\mathrm{T}}/\partial\theta$, we have

$$\Sigma_{\hat{\theta}_{\mathrm{DAC}}} = \left[\frac{\partial\mu^{\mathrm{T}}}{\partial\theta}\Sigma^{-1}\frac{\partial\mu}{\partial\theta^{\mathrm{T}}}\right]^{-1} \tag{21}$$

which is the CRB, cf. (9). The result (21) again hinges on the assumption of large Fisher information and the stochastic approximation. With errors in the estimates $\hat{\theta}_i$ being linearly related to errors in the data $\mathbf{X}_i$ via (7), when the $\hat{\theta}_i$ and $\mathbf{X}_i$ have the same length, the appropriately weighted average of the $\hat{\theta}_i$ will also minimize the sum of square errors in $\mathbf{X}$, as errors in each component of $\mathbf{X}$ are represented in a $\hat{\theta}_i$.

In summary, when the data $\mathbf{X}$ are partitioned either into uncorrelated segments or segments with length equal to the length of the parameters they estimate, and the segments individually have large enough Fisher information, the DAC estimates are approximately unbiased with covariance approaching the CRB.

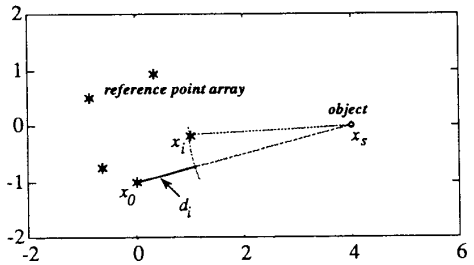


Fig. 2. RD measurement notation. The RD between reference points 0 and $i$, $d_i$, is the difference in object-reference point distances for reference points 0 and $i$.

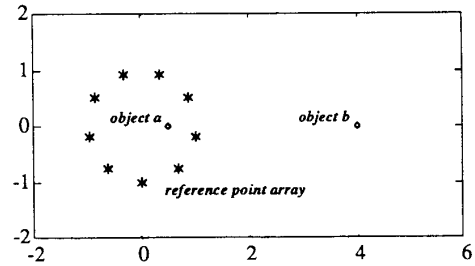


Fig. 3. Object-array geometry used in Monte-Carlo simulations.

## IV. RANGE-DIFFERENCE-BASED LOCALIZATION

An example application of the DAC estimate is given below. A problem which arises in the fields of navigation and acoustics is that of determining the position of an object given measurements of its *range difference* (RD) to a set of *reference points* [10, 11]. Denote by $\mathbf{d}(\mathbf{x}_s)$ the vector of *range differences*, the $i$th RD being defined by

$$d_i(\mathbf{x}_s) \stackrel{\mathrm{def}}{=} \|\mathbf{x}_s - \mathbf{x}_i\| - \|\mathbf{x}_s - \mathbf{x}_0\|. \tag{22}$$

Here $\mathbf{x}_s$ is the *object* position, and $\mathbf{x}_i$, $i = 0,\ldots,n$ are reference point positions, as shown in Fig. 2. The vector of *observed range differences* $\tilde{\mathbf{d}}$ is modeled as being drawn from a Gaussian density with mean $\mathbf{d}(\mathbf{x}_s)$ and covariance $\Sigma_d$ (see [11, §3.1]).

The RD-based localization problem is an instance of (1) for which the ML estimate (2) is (in general) very costly to compute directly [10], but available in closed form for the case of $p$ RDs determining a $p$-dimensional position [4–9]. Accordingly, the RD-based localization problem is a natural one for the divide and conquer approach. By dividing the measured RDs $\tilde{\mathbf{d}}$ into $p$-long segments, the quick $p$-RDs-to-position solutions [4–9] yield an inexpensively computed DAC estimate.

To evaluate the statistical performance of the DAC estimate, 100 trial Monte-Carlo simulations were run using the nine-sensor array and the two object positions shown in Fig. 3. The RD covariance used was of the form (see e.g. [3, page 950])

$$\Sigma_d = \sigma_d^2[\mathbf{I} + \mathbf{1}\mathbf{1}^{\mathrm{T}}] \tag{23}$$

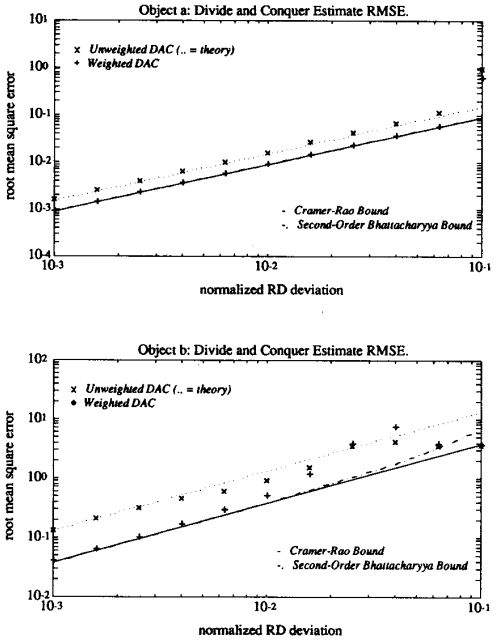


Fig. 4. Monte-Carlo simulation results.

$\sigma_d$ being termed the normalized RD deviation. The DAC estimates were based on the solution given in [8], and the weighting matrix $\mathbf{W} = \boldsymbol{\Sigma}_{\vartheta}^{-1}$ (used in the optimally weighted DAC estimate) was derived based on the unweighted (that is, $\mathbf{W} = \mathbf{I}$) DAC position estimate.[1]

In Fig. 4, the DAC estimate *sample root mean square error*, the square root of the trace of the sample mean square error, is plotted as a function of $\sigma_d$ for object positions *a* and *b*. Also plotted are the CRB and the *second-order Bhattacharyya bound* ($BHB_2$) [1, p. 131]. The $BHB_2$ is used here as an indicator of size of J. When $BHB_2 \approx$ CRB, we say the estimators are operating in the *small-error region*; conversely, when $BHB_2 >$ CRB, the estimators are said to be working in the *large-error region*. Note that for object *a*, the small-error region extends over the entire range of $\sigma_d$ tested; for object *b*, the large-error region begins roughly at $\sigma_d \approx 0.02$.

Concentrating first on object *a*, note that the DAC estimates have sample root mean square error accurately predicted by (16), with the optimally weighted DAC estimate achieving the CRB. In the case of object *b*, the DAC estimate sample mean square errors follow (16) only in the small-error region. The three-element arrays used in forming the DAC estimate will enter their large-error region at smaller values of $\sigma_d$ than will the nine-element array (since Fisher information increases with the number of reference points). Accordingly, we expect the weighted DAC estimate to deviate from the CRB at a smaller value of $\sigma_d$ than does the $BHB_2$, as noted in Fig. 4.

[1]On occasion, there can be two distinct $p$-space positions matching a given set of $p$ RDs [4, p. 842], and the "true" object positions must be separated from the artifacts via some sort of clustering method before the DAC estimate is formed. The DAC estimates formed here were based on the cluster with minimum covariance.

## V. SUMMARY

The least squares problem of estimating parameters determining the mean of a Gaussian-distributed observation was studied. The DAC estimate was introduced as a means for trading between computational complexity and estimate accuracy. Small-error region expressions for the bias and variance of the DAC estimate $\hat{\boldsymbol{\theta}}_{DAC}$ were developed. The DAC estimate $\hat{\boldsymbol{\theta}}_{DAC}$ was shown to be *unbiased with CRB variance* in the small-error region if either 1) uncorrelated observation subvectors, or 2) subvectors having length equal to the number of parameters they estimate were used. Otherwise the DAC estimate has mean square error exceeding the CRB. Finally, an inexpensive, accurate RDs-to-position estimate was derived based on the DAC estimate.

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## Corrections to "Analysis of Some Modified Ordered Statistic CFAR: OSGO and OSSO CFAR"[1]

On page 198 the caption for Fig. 1 should be corrected to read as follows;

Fig. 1. Modified CFAR processor block diagram.

$$f(X1, X2, \ldots, X(M/2)) = f(X(M/2+1), \ldots, XM) = (2/M) \sum_{i=1}^{M/2} Xi$$

for a CA CCFAR (CAGO or CASO).

$f(X1, X2, \ldots, X(M/2)) = f(X(M/2+1), \ldots, XM)$ = single value $Xk$ selected from OS 1 to $M/2$

On page 202 the authors Postal Zone Code should be changed from 08014 to 08080.

[1] Elias-Fusté et al., *IEEE Trans. Aerosp. Electr. Syst.*, **26**, January 1990, 197–202.