CSE 101 Homework 2

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1.

2. **Algorithm.**

$count \leftarrow 0$;
for *each unordered edge* $\{u, v\}$ *with* $A[u][v] = 1$ **do**
    $S \leftarrow \{w \in V : A[u][w] = 1 \text{ and } A[v][w] = 1\}$;
    $t \leftarrow 0$;
    for *each unordered pair* $\{x, y\} \subseteq S$ **do**
        if $A[x][y] = 1$ **then**
            $t \leftarrow t + 1$;
        end
    end
    $count \leftarrow count + t$;
end
return $count/6$;

Correctness.

Base Case: If all 6 edges are present, there is exactly 1 4-clique. For each edge (u, v), the common neighbor set S has the other 2 vertices. Those two vertices are adjacent, so we add 1 for each of the 6 edges. Total count = 6; dividing by 6 gives 1, which is correct.

Inductive Hypothesis: Assume the algorithm correctly counts all 4-cliques in any undirected graph with $n = k$ vertices.

Inductive Step: Consider a graph G with $k + 1$ vertices. Let v_{k+1} be the newly added vertex. We can divide 4-cliques in G into two types:

(a) 4-cliques not containing v_{k+1}: These exist entirely in the subgraph with k vertices. By the inductive hypothesis, the algorithm counts them correctly.

(b) 4-cliques containing v_{k+1}: Suppose v_{k+1} is part of some 4-clique $\{v_{k+1}, a, b, c\}$. For each edge among these 4 vertices, the intersection $N(a) \cup N(b)$ includes both v_{k+1} and c, and since they are connected, the inner loop adds 1. Each 4-clique contributes 6 such additions overall (once per edge). Thus, each new 4-clique is included and counted 6 times. Dividing the final sum by 6 again gives the correct total number of 4-cliques.

Therefore, the algorithm correctly counts all 4-cliques in any undirected graph.

Time Complexity.

For each edge $\{u, v\}$, finding $S = N(u) \cup N(v)$ costs $O(|V|)$. Checking all pairs inside S costs $O(|S|^2)$. Summed over all edges, the total time is $O((|V| + |E|)^2)$.

3. Let G be a directed graph that is not strongly connected. We want to make it strongly connected by adding a new vertex u and as few edges as possible from u to vertices in G and from vertices in G to u.

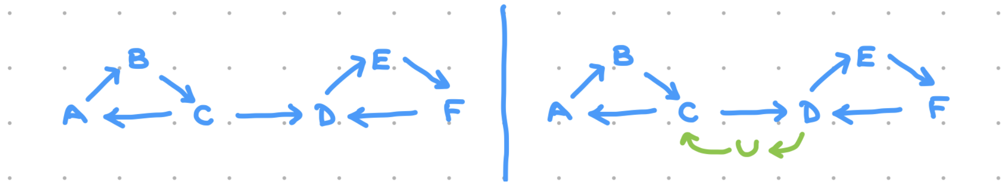
   (a) Can we ever make G strongly connected by adding u and a single edge in or out of u? Explain your answer.

   No. Strong Connected requires that $\forall u \in V$ there is a path to and from u. If we add one edge from u, it becomes a source and there is no way of reaching u. If we add an edge to u, it becomes a sink and there is no way of leaving u. Thus G will not be strongly connected.

   (b) Give an example of a directed graph with more than one strongly connected component where we can make it strongly connected by adding u and two edges in or out of u?

   $$G : (A, B), (B, C), (C, A), (C, D), (D, E), (E, F)$$

   Add the vertex u and edges $(D, u), (u, C)$ to make G strongly connected.



   (c) Give a characterization (an if and only if condition) of the minimum number of edges we must add, in terms of the strongly connected components of G.

   The minimum number of edges E_m is given by

   $$E_m = S + K$$

   where K is the number of sink SCCs, and S is the number of source SCCs.

   (d) Describe how to use an algorithm from class to compute this number.

      i. Run the Tarjan-Koseraju algorithm to find all SCCs within the graph.
      ii. Check whether each SCC is a sink in G by running a DFS algorithm on it. If a node outside the SCC is reachable, it is not a sink. If the only nodes that are reachable are those in the SCC, it is a sink.
      iii. Repeat items i, ii for the reverse graph to find its sinks, and thus the regular graph's sources.

(e) How long would this algorithm take to do this? Explain.

The algorithm runs in $O(|V|(|V| + |E|))$

i. The two Tarjan-Koseraju algorithms run in $O(|V + |E|)$ time.
ii. Each DFS call runs in $O(|V| + |E|)$, and is run for every vertex at most twice.
iii. Thus the total algorithm takes $O(|V|(|V| + |E|))$

4.