CSE 21 Hw 1

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January 13, 2025

1. (a) Use regular induction to show that for all n $\geq$ 1 that

$$\sum_{k=1}^{n} \frac{1}{(k)(k+1)} = \frac{n}{n+1}$$

i) Prove the base case: (n = 1)

$$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2} = \frac{1}{1+1}$$

ii) Assume for some n = j

$$\sum_{k=1}^{j} \frac{1}{k(k+1)} = \frac{j}{j+1}$$

and show that it holds for j + 1

$$\begin{aligned}
&\sum_{k=1}^{j+1} \frac{1}{k(k+1)} && (1)\\
= \quad & \frac{1}{(j+1)(j+2)} + \frac{j}{j+1} && (2)\\
= \quad & \frac{1+j(j+2)}{(j+1)(j+2)} && (3)\\
= \quad & \frac{(j+1)^2}{(j+1)(j+2)} && (4)\\
= \quad & \frac{(j+1)}{(j+2)} && (5)
\end{aligned}$$

(b) Use regular induction to show that for all n $\geq$ 0 that

$$\sum_{k=0}^{n} 2^k < 2^{n+1}$$

i) Prove the base case: (n = 0)

$$\sum_{k=0} n2^k = 1 < 2$$

ii) Assume for some n = j

$$\sum_{k=0} n2^k = 2^{n+1}$$

and show that it holds for j + 1

$$
\begin{aligned}
& \sum_{k=0}^{k} 2^{j+1} && (6) \\
< \;& 2^{j+1} + 2^{j+1} && (7) \\
= \;& 2 * 2^{j+1} = 2^{j+2} && (8)
\end{aligned}
$$

2. A password is a string over the alphabet of the 26 uppercase letters, the 26 lowercase letters, and the 10 digits

(a) How many 8-char passwords start and end with a digit?

$$k = 10 \cdot (26 + 26 + 10)^6 \cdot 10 = 10^2 \cdot 62^6$$

- 10 *: number of ways to start the string with a number*
- $(26 + 26 + 10)^6$ *number of 6-char strings*
- 10 *: number of ways to end the string with a number*

(b) How many 8-character passwords have exactly one uppercase and exactly one lowercase letter? (and the rest are digits)

$$k = 26^2 \cdot \binom{8}{2} \cdot 10^6$$

- 26^2 *: Number of ways to choose an uppercase and lower case letter*
- $\binom{8}{2}$ *: Number of ways to arrange an 2 elements in the string*
- 10^6 *: number of 6 char strings with only digits*

(c) How many 8-character passwords avoids having the word COUNT (with any combinationof upper and lowercase letters)

$$k = (26 + 26 + 10)^8 - 2^5$$

- $(26 + 26 + 10)^8$ *: Total number of length 8 strings*
- (2^5) *: number of case permutations of the word COUNT*

(d) How many 8-character passwords consist of three different characters with 3 copies of two chracters and 2 copies of the other?

$$k = \binom{62}{3} \cdot \binom{3}{2} \cdot \left(\frac{8!}{3!3!2!}\right)$$

- $\binom{62}{3}$ *: Number of ways to choose 3 characters from the alphabet*
- $\binom{3}{2}$ *: Number of ways to pick 2 characters from 3 characters*
- $\frac{8!}{3!3!2!}$ *: Number of anagrams of the 3 characters (given the configuration specified by the problem)*

(e) How many 8-character passwords consist of 8 different letters that are in alphabetical order such that each letter can be uppercase or lowercase?

$$k = \binom{26}{8} \cdot 1 \cdot 2^8$$

- $\binom{26}{8}$ *: Number of ways to choose pick 8 unique characters from the english alphabet*
- 1 *:Number of ways to arrange them in correct alphabetical order*
- 2^8 *: Number of combinations of upper and lowercase characters.*

3. For each expression, describe a set of objects that is counted by the expression and include your reasoning

(a)

$$8 \cdot 26 \cdot (26 + 10)^7$$

A string, starting with a number 0 - 7, followed by an english character, followed by 7 characters, either number or letter

- 8 *: States the string must start with a digit belong to* { *0, 1, 2, 3, 4, 5, 6, 7* }
- 26 *: States the second element in the string must be a letter in the english alphabet*
- 36^7 *: Number of length 7 strings using numbers and letters*

(b)

$$10^2(26 + 26)^6 + 26^2(10 + 26)^6$$

the number of

a) strings of length 8 that start with 2 numbers, and the rest are upper/lowercase letters

b) strings of length 8 that start with 2 uppercase letters, and the rest are lowercase letters or numbers

- 10^2 *: States that the string (a) must start with 2 numbers*
- $(26 + 26)^6$ *: States the remainder is a length 6 string with upper and lowercase letters.*
- 26^2 *: States that the string (b) must start with 2 uppercase letters*
- $(10 + 26)^6$ *: States that the remainder is a length 6 string with lowercase letters and numbers*

(c)

$$(26 + 26 + 10)^8 - 10^8$$

Number of length 8 strings with at least one letter

- $(26 + 26 + 10)^8$ *: Number of length 8 strings*
- 10^8 *: Number of length 8 strings without a letter*

(d)

$$(4!)2^4$$

Number of length 8 strings starting with an anagram of 4 unique characters, and ending with a length 4 binary string

- $4!$ *: Number of Anagrams of 4 unique characters*
- 2^4 *: Number of length 4 binary strings*

4. (a) How many different ways are tehre to arrange the letters in DISCRETEMATHEMATICS

$$k = \frac{18!}{2!2!2!3!3!2!2!}$$

I : 2, S : 2, C : 2, E : 3, T : 3, M : 2, A : 2

(b) How many different ways are there to color the 12 verticies of a hexagonal prism with 12 different colors?

For each coloring, there are
6 rotations along the y axis
+ 2 rotations along the x axis
+ 2 rotations along the z axis
= 10 total rearangements of one color configurations

Number of unique colorings = $\frac{12!}{24}$

(c) How many different ways are there to color the 6 edges of a tetrahedron with 6 colors?

For each coloring, there are
3 distinct rotations along the y-axis
3 distinct rotations along the z-axis

Number of unique colorings = $\frac{6!}{12}$