CSE 21 HW 8

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1. Given n cords, each with an innie connector type and outie connector type, determine if it possinle to string them all together in such a wa so taht you can plug your outie USB into the innie power outlet.

a) Describe how to create a graph out of this problem such that the solution to the problem involves finding a Hamiltonian path

- This graph will be directed. Plugging cable A into cable B does not imply you can plug cable B into cable A
- The set of all verticies: V = the set of all cables
- An edge (v,w) exists if the cable v can be plugged into the cable w. (The outie connector of v = the innie connector of w.)
- Finding a Hamiltonian path from a cable with a USB innie connector to a cable with power outtie connector represents connecting cables from the phone to the wall outlet using every vertex (cable)

  If there exists a Hamiltonian path you can determine that it is possible to connect your phone to the wall using every cable.

  If there is not a Hamiltonian path you can determine that it is not possible to connect your phone to the wall using every cable.

b) Describe how to create a graph out of this problem such that the solution to the problem involves finding an Eulerian path.

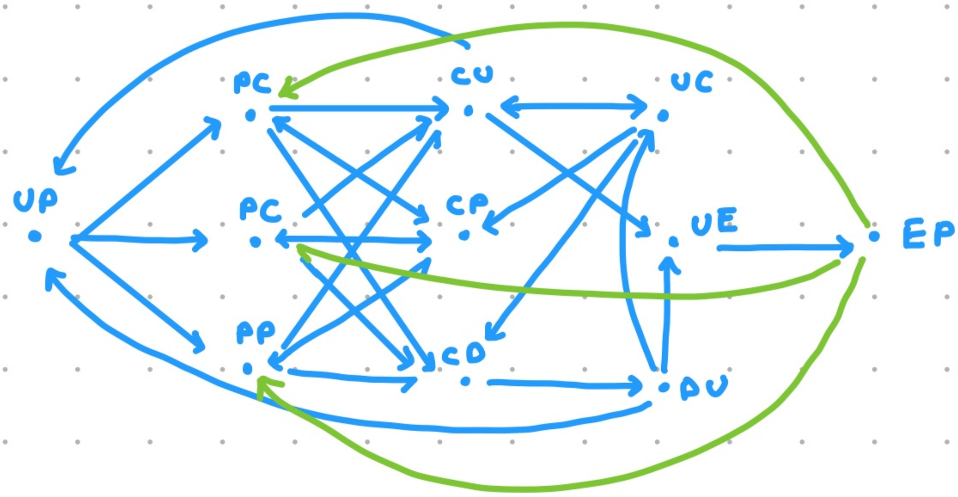
- This graph will be directed. If there is a cable with innie connect A and outie connector B does not imply there is a cable with innie connector B and outie connector A
- The set of all verticies: V = the set of all connection types
- An edge (v, w) exists if there is a cable with innie part v and outie part w.
- Finding an Eulerian path from the USB vertex to the power outlet vertex represents connecting the phone to the wall outlet using every edge (cable)

  If there exists a Eulerian path you can determine that it is possible to connect your phone to the wall using every cable

  If there is not a Eulerian path you can determine that it is not possible to connect your phone to the wall using every cable.

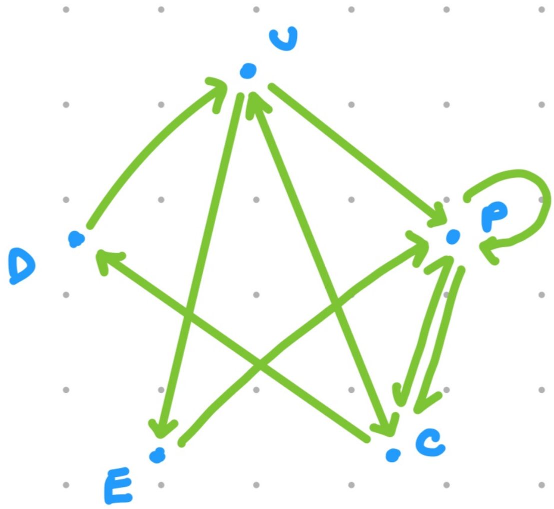
c) Draw your graph model for parts a and b on the particular set of cables described.

Graph for part a, using Hamiltonian Path to solve



Note: The first letter of a vertex is the innie connector, the second is the outtie connector

Graph for part b, using Eulerian Path to solve



2. A binomial Tree is a special kind of rooted tree used for various data structures in computer science. A level d binomial tree cna be defined recursively.

a) What is the hieght of a degree d binomial tree? Prove your result by induction on d.

$$D(d) = 1 + D(d-1); D(0) = 0$$
$$D(d) = d$$

- Base Case:
  D(0) = 0. A single node has height 0.

- Induction Step. For some t > 0, assume D(t) = t. Show D(t) = t + 1
  D(t + 1) = 1 + D(t + 1 - 1) = t + 1

b) Write a recurrence relation for the number of nodes N(d) in a binomial tree of degree d.

$$N(d) = 1 + \sum_{i=1}^{n} N(d-i); N(0) = 1$$

c) Use the guess-and-check method to guess a formula for N(d). Prove that your formula holds by induction on d.

$$N(d) = \{1, 2, 4, 8, ...\}$$
$$= 2^d$$

Proof by Induction:

- Base Case:
  N(0) = 1 = 2^0 *= Number of nodes on a zero tree*

- Induction Step (using Strong Induction):
  $\forall t > 0$ assume $N(t) = 2^t$. Show $N(t+1) = 2^{t+1}$

$$\begin{aligned} N(t+1) &= 1 + N(t) + N(t-1) + \ldots \\ &= 1 + \sum_{i=0}^{t} 2^i \\ &= 1 + \frac{s^{t+1} - 1}{2 - 1} \\ &= 2^{t+1} \end{aligned}$$

3. Prove that for any integer $n \geq 1$ any tournament graph on n verticies has a hamiltonian path.

- Base Case:
  If the tournament has 1 player then the graph with only 1 vertex has a trivial hamiltonian path.

- Inductive Step:
  Let n be an arbitrary integer such that $n > 1$. Assume that for any tournament with n - 1 players, the corresponding graph has a hamiltonian path.

  Consider an arbitrary tournament T with n players $\{1, ..., n\}$. Then if you consider the tournament T' involving only {1,...n - 1}, then by the inductive hypothesis, there is a hamiltonian path $(a_1, ..., a_{n-1})$ in the corresponding graph.

  - Case 1: Player n wins against player a_1. Then T has the hamiltonian path $(n, a_1, ..., a_{n-1})$
  - Case 2: Player a_{n-1} wins against player n. Then T has the hamiltonian path $(a_1, ..., a_{n-1}, n)$
  - Case 3: Player n loses against player a_1 and player n wins against player a_{n-1}. Then T has the hamiltonian path $(a1, a2, ..., a_{i-1}, n, a_i, ...a_{n-1})$

    Where a_i is the first in $(a_2, ...a_{n-1})$ that n wins against.

4. Recall that adjecncy matricies of simple direct graphos are matricies consisting of 0s and 1s such that there are no 1s down the diagonal. For the questions below suppose A is the adjency matrix of a simple directed graph G with n verticies.

a) Consider the graph and its adjency matrix M. Compute M^2 and M^3

$$M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad M^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b) Let A be an adjacency matrix of a simple directed graph. What does it mean to have non-zero entries in the diagonal of M^3.

If position (i, i) in an adjacency matrix is non zero it means there is a circuit of length 3, starting and ending at vertex i.

c) Provide a justification for the following statement: Suppose that A is the adjacency matrix of a DAG. There exists some $t \geq 2$ such that A^t is the zero matrix.

If there are no cycles in the graph, then every path in the graph must be finite. Let L be the longest path in the graph. Then there are no length $t > L$ paths between any two verticies. Thus the adjacency matrix will be the zero matrix.