CSE 21 HW 4

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Februrary 5, 2025

1. Consider the set H of all non-negative integer solutions to the equation:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30$$

You wish to pick one of the elements of H at random and the way you decide to do it is the following: Initialize a 6-tuple to be all zeros (0,0,0,0,0,0) Roll a fair 6 sided die 30 times. For each roll that shows a 1, increment the 1st position, for each roll that shows a 2, increment the 2nd position and so on

a) What is the expected number of entries equal to zero using this sampling method?

$$X_i = \begin{cases} 1 \text{ if entry } a_i = 0. \text{ (You roll none of i)} \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} E(X) &= \sum_{k=1}^{6} E(X_i) && (1) \\ &= \sum_{k=1}^{6} p(x = r) && (2) \\ &= \sum_{k=1}^{6} \frac{5^{30}}{6^{30}} = 6 \cdot \frac{5^{30}}{6^{30}} && (3) \\ &\approx 0.0253 && (4) \\ &= 0 && (5) \end{aligned}$$

Using this distribution method, you can expect 0 $a_i = 0$

b) What is the expected number of entries equal to zero using a uniform sampling method?

$$\begin{aligned} E(X) &= \sum_{k=1}^{6} E(X_i) && (6) \\ &= \sum_{k=1}^{6} \frac{\binom{30+5-1}{5-1}}{\binom{30+6-1}{6-1}} && (7) \\ &= \sum_{k=1}^{6} \frac{\binom{34}{4}}{\binom{35}{5}} && (8) \\ &\approx 0.871 && (9) \\ &= 1 && (10) \end{aligned}$$

Using this distribution method, you can expect 1 $a_i = 0$

c) What is the expected number of entries equal to 1 using this sampling method?

$$X_i = \begin{cases} 1 \text{ if entry } a_i = 1. \text{ (You roll only one of i)} \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} E(X) &= \sum_{k=1}^{6} E(X_i) && (11) \\ &= \sum_{k=1}^{6} \frac{\binom{30}{1} \cdot 5^{29}}{6^{30}} && (12) \\ &= 6 \cdot \frac{30 \cdot 5^{29}}{6^{30}} && (13) \\ &\approx 0.152 && (14) \\ &= 0 && (15) \end{aligned}$$

Using this distribution method, you can expect at 0 $a_i = 1$

d) What is the expected number of entries equal to 1 using a uniform sampling method?

$$\begin{aligned} E(X) &= \sum_{k=1}^{6} E(X_i) && (16) \\ &= \sum_{k=1}^{6} \frac{\binom{30-1+5-1}{5-1}}{\binom{30+6-1}{6-1}} && (17) \\ &= 6 \cdot \frac{\binom{33}{4}}{\binom{35}{5}} && (18) \\ &\approx 0.756 && (19) \\ &= 1 && (20) \end{aligned}$$

Using this distribution method, you can expect at 1 $a_i = 1$

2. Recall MinSort and BubbleSort from class. For these problems, assume that the input is a permutation of length n of distinct integers chosen uniformly at random.

a) Use linearity of expectation to compute the expected number of swaps for MinSort on an input ($a_1 ... a_n$) of distinct integers chosen uniformly at random

$$X_i = \begin{cases} 1 \text{ if } a_1 \text{ is in the wrong place (a swap occours)} \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} E(X) &= \sum_{k=1}^{n-1} E(X_i) = \sum_{k=1}^{n-1} 1 - (\frac{1}{n-k+1}) && (21) \\ &= \sum_{k=1}^{n-1} (\frac{n-k}{n-k+1}) && (22) \end{aligned}$$

b) Use linearity of expectation to compute the expected number of swaps for BubbleSort on an input ($a_1 ... a_n$) of distinct integers chosen uniformly at random

$$X_i = \textit{Expected number of swaps for } a_i = \sum_{j=1}^{i-1} X_j$$

$$X_j = \begin{cases} 1 \text{ if } a_j > a_i \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} E(X_i) &= \sum_{j=1}^{i-1} E(X_j) = \sum_{j=1}^{i-1} \frac{n-i}{n} && (23) \\ & && (24) \end{aligned}$$

$\frac{n-i}{n}$ *is the probability that some a_j before a_i is great than a_i*

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{n} E(X_i) && (25)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{k-1}\frac{n-k}{n} && (26)\\
&= \sum_{k=1}^{n}(k-1)(\frac{n-k}{n}) && (27)\\
&= \frac{-(n+1)(n+2)}{6}+\frac{(n+1)(n+1)}{2}-n && (28)\\
&= \frac{1}{6}n^2-\frac{1}{2}n+\frac{1}{3} && (29)
\end{aligned}
$$

3. For each algorithm, compute the exact number of times the algorithm prints in terms of n and compute the runtime in terms of θ, where n $\geq$ 3.

a)

$$
\begin{aligned}
&= \sum_{i=1}^{n} 2i - 1 && (30)\\
&= 2 \cdot (\sum_{i=1}^{n} i) - n && (31)\\
&= n(n+1) - n && (32)\\
&= n^2 && (33)\\
& && (34)
\end{aligned}
$$

$$f(n) = \theta(n^2)$$

b) if n is even:

$$2\sum_{i=1}^{n/2} i = n(n-1)$$

if n is odd:

$$2(\sum_{i=1}^{n/2} i) - 1 = n(n-1) - 1$$

in both cases $f(n) = \theta(n^2)$

c)

$$
\begin{aligned}
&= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} (n-j) && (35)\\
&= \sum_{i=1}^{n-2} n(n-i-1) - (\frac{(n-1)(n)}{2} - \frac{i(i+1)}{2}) && (36)\\
&= \sum_{i=1}^{n-2} \frac{n^2-2n+n}{2} - ni + \frac{1}{2}(i^2+i) && (37)\\
&= \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n && (38)
\end{aligned}
$$

$$f(n) = \theta(n^3)$$

4. Let f : R $\rightarrow$ R be a continuous increasing function. Then we can use a version of binary search to solve for the "root" of the function f (value x such that f(x) = 0.) This algorithm will take the input f, starting value n such that 0 < r < n where r is the root of the function and an error term E.

a) Trace through the algorithm on the input FunctionSolverp(ln(x) + x^2*, 2, 0.03125).*

iteration	*hi - lo > 2E*	*f(v)*	*f(v)__0*	*lo*	*hi*	*v*
0				0	2	1
1	TRUE	1	>	0	1	0.5
2	TRUE	-0.443	<	0.5	1	0.75
3	TRUE	0.2748	>	0.5	0.75	0.625
4	TRUE	-0.07938	<	0.625	0.75	0.6875
5	TRUE	0.0979	>	0.625	0.6875	0.656250
6	FALSE					

b) if $n = 2^j$ *and* $E = 2^k$ *for some integers k and j such that k <j, how many iterations does the algorithm go through and why?.*

For every v > 1, f(v) = ln(v) + $v^2 > 0$.

Specifically, this happens when v = 2^i for any i $\in \mathbb{Z}, i > 0$

This means that the lowerBound (lo), will always stay 0 (since f(v) > 0), and that the upperBound (hi) will be divided by 2 each iteration. Observe:

iteration	*hi - lo > 2E*	*f(v)__0*	*lo*	*hi*	*v*
0			0	2^j	2^{j-1}
1	$2^j > 2^{k+1}$	>	0	2^{j-1}	2^{j-2}
2	$2^{j-1} > 2^{k+1}$	>	0	2^{j-2}	2^{j-3}
...					

This will end once $2^{j-n} \leq 2^{k+1}$

This happens when $n = j - k - 1$