CSE 21 HW 8

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March 12, 2025

1. Given n cords, each with an innie connector type and outie connector type, determine if it possinle to string them all together in such a wa so taht you can plug your outie USB into the innie power outlet.

a) Describe how to create a graph out of this problem such that the solution to the problem involves finding a Hamiltonian path

- This graph will be directed. Plugging cable A into cable B does not imply you can plug cable B into cable A
- The set of all verticies: V = the set of all cables
- An edge (v,w) exists if the cable v can be plugged into the cable w. (The outie connector of v = the innie connector of w.)
- Finding a Hamiltonian path from a cable with a USB innie connector to a cable with power outtie connector represents connecting cables from the phone to the wall outlet using every vertex (cable)

  If there exists a Hamiltonian path you can determine that it is possible to connect your phone to the wall using every cable.

  If there is not a Hamiltonian path you can determine that it is not possible to connect your phone to the wall using every cable.

b) Describe how to create a graph out of this problem such that the solution to the problem involves finding an Eulerian path.

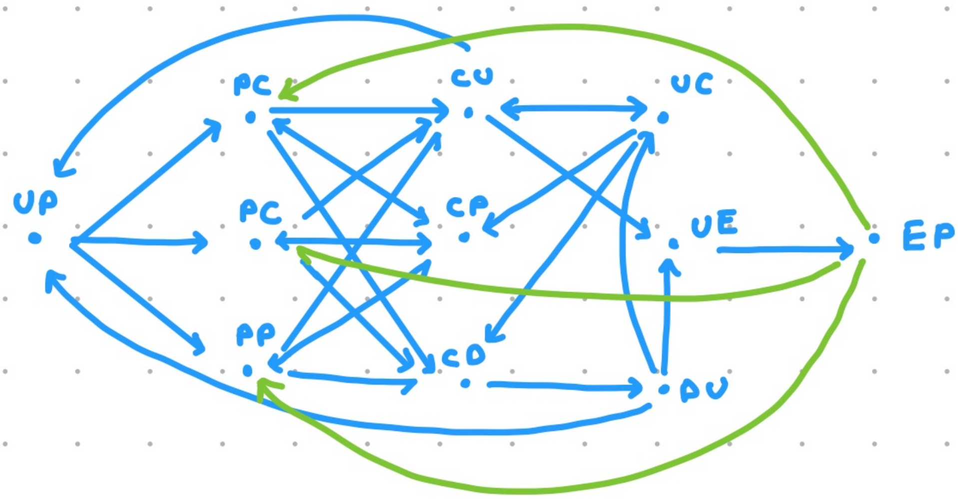
- This graph will be directed. If there is a cable with innie connect A and outie connector B does not imply there is a cable with innie connector B and outie connector A
- The set of all verticies: V = the set of all connection types
- An edge (v, w) exists if there is a cable with innie part v and outie part w.
- Finding an Eulerian path from the USB vertex to the power outlet vertex represents connecting the phone to the wall outlet using every edge (cable)

  If there exists a Eulerian path you can determine that it is possible to connect your phone to the wall using every cable

  If there is not a Eulerian path you can determine that it is not possible to connect your phone to the wall using every cable.

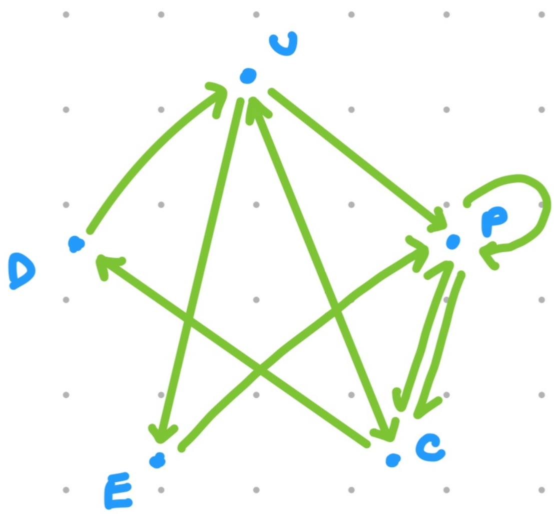
c) Draw your graph model for parts a and b on the particular set of cables described.

Graph for part a, using Hamiltonian Path to solve



Note: The first letter of a vertex is the innie connector, the second is the outtie connector

Graph for part b, using Eulerian Path to solve



2. A binomial Tree is a special kind of rooted tree used for various data structures in computer science. A level d binomial tree cna be defined recursively.

a) What is the hieght of a degree d binomial tree? Prove your result by induction on d.

$$D(d) = 1 + D(d-1); D(0) = 0$$
$$D(d) = d$$

- Base Case:
  D(0) = 0. A single node has height 0.

- Induction Step. For some t > 0, assume D(t) = t. Show D(t) = t + 1
  D(t + 1) = 1 + D(t + 1 - 1) = t + 1

b) Write a recurrence relation for the number of nodes N(d) in a binomial tree of degree d.

$$N(d) = 1 + \sum_{i=1}^{n} N(d-i); N(0) = 1$$

c) Use the guess-and-check method to guess a formula for N(d). Prove that your formula holds bu induction on d.

$$N(d) = \{1, 2, 4, 8, ...\}$$
$$= 2^d$$

Proof by Induction:

- Base Case:
  N(0) = 1 = 2^0 = *Number of nodes on a zero tree*

- Induction Step:
  $\forall t > 0$ assume $N(t) = 2^t$. Show $N(t+1) = 2^{t+1}$

$$\begin{aligned} N(t+1) &= 1 + N(t) + N(t-1) + \ldots \\ &= 1 + \sum_{i=0}^{t} 2^i \\ &= 1 + \frac{s^{t+1} - 1}{2 - 1} \\ &= 2^{t+1} \end{aligned}$$

3. Prove that for any integer $n \geq 1$ any tournament graph on n verticies has a hamiltonian path.

- Base Case:
  If the tournament has 1 player then the graph with only 1 vertex has a trivial hamiltonian path.

- Inductive Step:
  Let n be an arbitrary integer such that $n > 1$. Assume that any tournament with n - 1 players, the corresponding graph has a hamiltonian path.

  Consider an arbitrary tournament T with n players $\{1, ..., n\}$. Then if you consider the tournament T' involving only {1,...n - 1 then by the inductive hypothesis, there is a hamiltonian path $(a_1, ..., a_{n-1})$ in the corresponding graph.

  - Case 1: Player n wins against player a_1. Then T has the hamiltonian path $(n, a_1, ..., a_{n-1})$
  - Case 2: Player a_{n-1} wins against player n. Then T has the hamiltonian path $(a_1, ..., a_{n-1}, n)$
  - Case 3: Player n loses against player a_1 and player n wins against player a_{n-1}. Then T has the hamiltonian path $(a1, a2, ..., a_{i-1}, n, a_i, ...a_{n-1})$

    Where a_i is the first in $(a_2, ...a_{n-1})$ that n wins.

4. Recall that adjecncy matricies of simple direct graphos are matricies consisting of 0s and 1s such that there are no 1s down the diagonal. For the questions below suppose A is the adjency matrix of a simple directed graph G with n verticies.

a) Consider the graph and its adjency matrix M. Compute M^2 *and* M^3

$$M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad M^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b) Let A be an adjacency matrix of a simple directed graph. What does it mean to have non-zero entries in the diagonal of M^3*.*

If position (i, i) in an adjacency matrix is non zero it means there is a circuit of length 3, starting and ending at vertex i.

c) Provide a justification for the following statement: Suppose that A is the adjacency matrix of a DAG. There exists some $t \geq 2$ *such that* A^t *is the zero matrix.*

If there are no cycles in the graph, then every path in the graph must be finite. Let L be the longest path in the graph. Then there are no length $t > L$ paths between any two verticies. Thus the adjacency matrix will be the zero matrix.