CSE 21 Hw 2

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1. (a) Suppose you are a painter and you have 18 different paintings and there are 6 different galleries that are interested in showing your paintings

i) How many ways can you distribute not necessarily all 18 paintings to the 6 galleries.

$$\sum_{k=0}^{18} \binom{18}{k} 6^k$$

- $\sum_{k=0}^{18}$ *: Iterate through all the possible number of paintings to distribute*
- $\binom{18}{k}$ *: Number of ways to select k paintings*
- 6^k *: number of ways to distribute k paintings to 6 galleries*

ii) How many ways can you distribute all 18 paintings to the 6 galleries so that each gallery gets at least one painting?

$$\sum_{k=0}^{6} (-1)^{6-k} \binom{6}{k} k^{18}$$

The function mapping all 18 paintings to the 6 galleries must be onto, which requires the use of principle inclusion exclusion.

iii) How many ways can you distribute all 18 paintings to the 6 galleries if at least 1 gallery gets exactly 9 paintings?

$$\binom{18}{9} \cdot \binom{6}{1} \cdot 5^9$$

- $\binom{18}{9}$ *: Choose 9 paintings for the one gallery*
- $\binom{6}{1}$ *: Pick the gallery to receive the 9 paintings*
- 5^9 *: Distribute the Remaining Paintings*

1. (b) Suppose you are a woodcut printmaker and you have 18 identical woodcut prints and there are 6 different galleries that are interested in showing your prints

i) How many ways can you distribute not necessarily all 18 identical prints to the 6 galleries?

$$\sum_{k=0}^{18} \binom{k+5}{5}$$

- $\sum_{k=0}^{18}$ *: Iterate through all the possible number of prints to distribute*
- $\binom{k+5}{5}$ *: Number of ways to distribute k identical prints to 6 places*

ii) How many ways can you distribute all 18 identical prints to the 6 galleries so that each gallery gets at least one print?

$$\binom{18-6+5}{5} = \binom{17}{5}$$

- $\binom{17}{5}$ *: Distribute 6 of the prints to each of the galleries, then distribute the remaining among the 6 galleries*

ii) How many ways can you distribute all 18 identical prints to the 6 galleries if at least 1 gallery gets exactly 9 prints?

$$\binom{6}{1} \cdot \binom{9+4}{4}$$

- $\binom{6}{1}$ *: Number of ways to choose the gallery to receive 9 prints*
- $\binom{13}{4}$ *: Number of ways to distribute the remaining prints to the remaining galleries*

2. Suppose you are traveling from the bottom-left corner of a 10 x 12 grid of blocks and you wish to get to the top-right corner only using up and right movements

a) How many paths are there from the bottom-left corner to the top-right corner using right and up moves

$$\binom{10+12}{10} = \frac{22!}{10! \cdot 12!}$$

- 22 *: Number of movement you must take to get from the start point to the end point.*
- 10 *: Number of different places you can choose to move up*

b) How many paths are there from the bottom-left corner to the top-right corner using right and up moves that avoid passing through either blue dot?

$$\binom{22}{10} - (D_1 + D_2) + D_3$$

$$\begin{aligned}
D_1 &= \binom{3+4}{3} \cdot \binom{9+6}{6} = \binom{7}{3}\binom{15}{6} \qquad (1)\\
D_2 &= \binom{7+7}{7} \cdot \binom{5+3}{3} = \binom{14}{7}\binom{8}{3} \qquad (2)\\
D_3 &= \binom{7}{3} \cdot \binom{4+3}{3} \cdot \binom{8}{3} = \binom{7}{3}\binom{7}{3}\binom{8}{3} \qquad (3)
\end{aligned}$$

- $\binom{22}{10}$ *: Total number of paths*
- D_1 *: Number of paths going through at least* P_1
- D_2 *: Number of paths going through at least* P_2
- D_3 *: Number of paths going through both* P_1 *and* P_2

Subtracting the number of paths that at least go through P_1 and the number of paths that at least go through P_1 under counts the final value, since they both count the paths that go through both. Thus, you have to add back the number of paths that go through both

3. Compute the number of integer solutions for the equation

$$a_1 + a_2 + a_3 + a_4 = 26$$

a) $i \leq a_i$

$$\binom{26 - 10 + 3}{3} = \binom{19}{3}$$

- $\binom{19}{3}$ *: 1 + 2 + 3 + 4 = 10, is the total amount that needs to be distributed to the various* a_i*. The rest can be randomly distributed.*

b) $0 \leq a_i$ *for* $a_i \in \{1, 2, 3, \}$, $5 \leq a_4 \leq 10$

$$\binom{26 - 5 + 3}{3} - \binom{26 - 11 + 3}{3}$$
$$= \binom{24}{3} - \binom{18}{3}$$

- $\binom{24}{3}$ *: Number of solutions for* $a_4 \geq 5$
- $\binom{18}{3}$ *: Number of solutions for* $a_4 > 10$

c) $0 \geq a_i \geq 7$

$$= \sum_{k=0}^{4} (-1)^{4-k} \binom{4}{k} \binom{26 - 8k + 3}{3} \tag{4}$$

$$= \sum_{k=0}^{4} (-1)^{4-k} \binom{4}{k} \binom{29 - 8k}{3} \tag{5}$$

This solution uses the principle of inclusion/exclusion. It first counts all possible solutions $\binom{29}{3}$. Then for each a_i, subtracts the solutions where at least $a_i > 7$. Then it adds back solutions where for each pair a_i, a_j, both are greater than 7, and so on.

- $\sum_{k=0}^{4} (-1)^{4-k}$ *: Iterates through the number of* a_i *terms that are > 7. (For example, picking two* a_i, a_j*, and counting all the solutions where at least both are > 7)*
- $(-1)^{4-k}$ *: Oscillates between adding and subtracting the number of solutions per the principle of inclusion / exclusion*
- $\binom{4}{k}$ *: Number of ways to pick k a's out of the 4 total*
- $\binom{29-8k}{3}$ *: Uses 8k for the a's that are > 7, then distributes the remaining amount (26 - 8k) among the 4 a's*

4.

a) For integers $n \geq k \geq 2$, consider the identity:

$$\binom{n}{k}k = \binom{n-1}{k-1}n$$

- *LHS : Counts the number length n strings using three chars: $\{a_1, a_2, a_3\}$. In this string, a_1 is used exactly once, a_2 appears exactly k - 1 times, and the rest is a_3*
  PROCEDURE : Create a length n string of all a_3. Choose k out of the n spots to be a_2, then out of those k spots, pick one to replace with a_3

- *RHS : Counts the same set.*
  PROCEDURE : Create a length n string of all a_3. Choose k-1 out of the first n-1 spots to be a_2. Then out of the n spots, choose a character. If it is a_1, replace it with a_3. If it is a_2, replace it with a_3 and make the nth spot a_2

b) For integers $n \geq k \geq 0$, consider the identity:

$$\binom{n+k-1}{k-1} = \sum_{j=0}^{k-1}\binom{k}{j}\binom{n-1}{k-1-j}$$

- *LHS : Counts the number of ways to distribute n indistinguishable objects among k unique groups*

- *RHS : Counts the same set by counting all the different distributions of containers with 0 objects.*

  j = 0:
  - $\binom{k}{0}$: number of ways to pick 0 containers to have 0 objects.
  - $\binom{n-1}{k-1} = \binom{n+k-1-k}{k-1}$: number of ways to distribute n objects to k containers such that each gets at least one

  j = 1:
  - $\binom{k}{1}$: Number of ways to pick 1 container to have 0 objects
  - $\binom{n-1}{k-2} = \binom{n+(k-1-j)-(k-j)}{k-2}$: Number of ways to distribute n objects to k - 1 containers such that each gets at least one.

  continues for all $j \leq$ k - 1