Full-State Observer Notes and Example

A. Introduction

An observer is a dynamic system that is used to estimate the state of a system or some of the states of a system. A full-state observer is used to estimate all the states of the system. The observer can be designed as either a continuous-time system or a discrete-time system. The characteristics are the same, and the design processes are at least very similar and in some cases identical. These notes will focus primarily on continuous-time observers.

Specifically, a full-state observer as discussed in ECE 521 has the following characteristics.

1) The purpose of the observer is to generate an estimate of the state $x(t)$ based on measurements of the system output $y(t)$ and the system input $u(t)$. The input and output signals are assumed to be exactly measurable—no noise or other interference.
2) The observer uses a mathematical model of the state space realization of the system. Therefore, the $\{A, B, C, D\}$ matrices are assumed to be known exactly. The possibility of modeling errors is not included in the derivation of the observer.
3) The observer is an n^{th}-order linear dynamic system, where n is the number of state variables in the system.
4) Assuming that the observer is to be used as part of a feedback control system, the estimate $\widehat{x}(t)$ will be used by the controller as if it were the true state $x(t)$. Thus, with full-state feedback control using a full-state observer, the control signal is generated from $u(t) = -K\widehat{x}(t) + Fv(t)$, where K is the $r \times n$ control gain matrix chosen to place the closed-loop eigenvalues at specified locations.
5) The observer being considered in these notes is a deterministic system. It assumes that there is no measurement noise or unmeasured disturbances acting on the system. If there are disturbances and measurement noise acting on the system, then the Kalman filter should be implemented since it uses knowledge of the statistical properties of the system in its design.

B. Model of the Observer

There are several ways to derive the state equations for the full-state observer. The approach in these notes is to model the observer state equations as a model of the actual system plus a correction term based on the measured output and the estimate of what that output is expected to be. With the actual system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t) \tag{1}$$

the observer is modeled as

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) + L\left[y(t) - \widehat{y}(t)\right], \qquad \widehat{y}(t) = C\widehat{x}(t) + Du(t) \tag{2}$$

where L is the $n \times m$ gain matrix for the observer. The state equation in (2) is seen to model the actual state equation, with the true state $x(t)$ replaced by the estimate $\widehat{x}(t)$, and a correction term which is the difference between the actual measured output $y(t)$ and its estimate $\widehat{y}(t)$. The output equation in (2) is also seen to be a model of the system's output equation, with $x(t)$ replaced by its estimate.

Substituting the expression for $\widehat{y}(t)$ into the observer's state equation yields the following alternative forms for the model of the observer.

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) + Ly(t) - LC\widehat{x}(t) - LDu(t) = (A - LC)\,\widehat{x}(t) + (B - LD)u(t) + Ly(t) \tag{3}$$

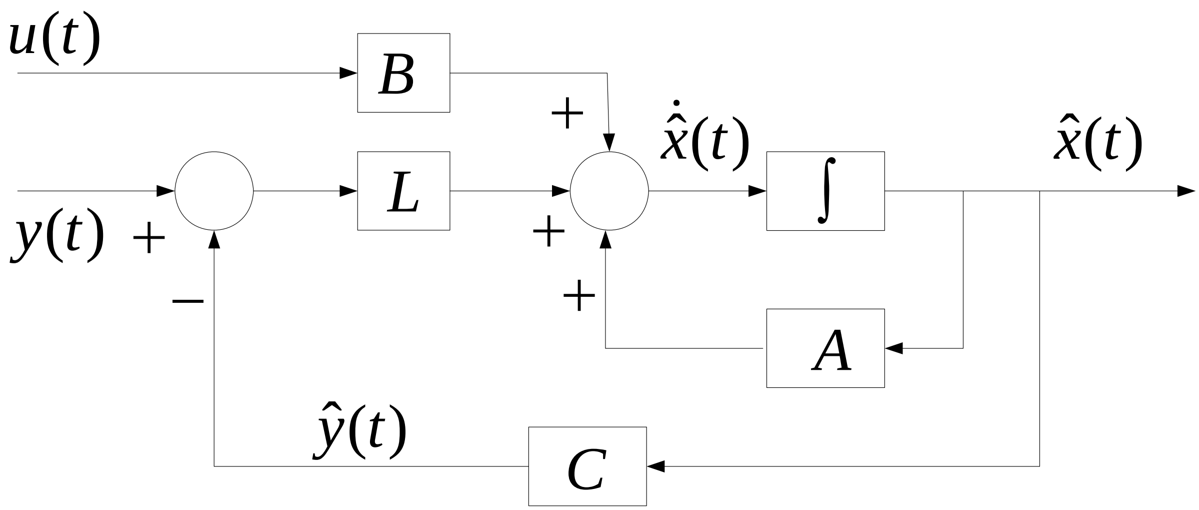
Although the D matrix explicitly appears in Eqn. (3), the D matrix has no influence on the state estimate produced by the observer. The reason for this is that $y(t) - \widehat{y}(t) = Cx(t) + Du(t) - C\widehat{x}(t) - Du(t) = C\left[x(t) - \widehat{x}(t)\right]$. Therefore, the $Du(t)$ term cancels out. If $D \neq 0$, the $-LD$ term must be included in (3) so that the cancellation does in fact take place. Figure 1 presents the block diagram for each of the observer models in the above equations, with $D = 0$.

C. Estimation Error

The purpose of the observer is to produce an estimate of the true state $x(t)$. It is reasonable to assume that there will be some error in the estimate at the initial time, but it is hoped that the error would decrease over time. The estimation error will be defined as

$$e(t) = x(t) - \widehat{x}(t) \tag{4}$$

The error signal obeys the differential equation



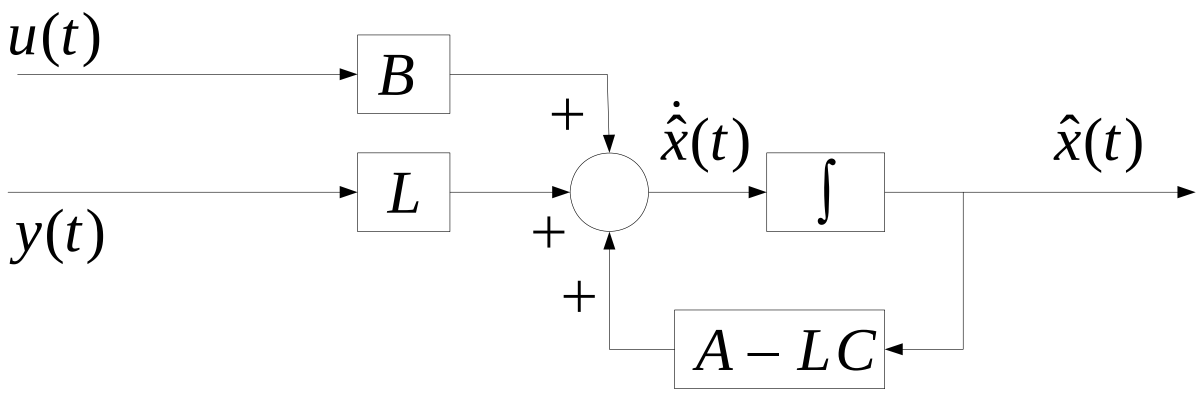


Fig. 1. Block diagrams for a continuous-time full-state observer.

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\widehat{x}}(t) = Ax(t) + Bu(t) - (A - LC)\,\widehat{x}(t) - (B - LD)u(t) - Ly(t) \\
\dot{e}(t) &= Ax(t) - (A - LC)\,\widehat{x}(t) + LDu(t) - LCx(t) - LDu(t) = (A - LC)\,x(t) - (A - LC)\,\widehat{x}(t) \\
\dot{e}(t) &= (A - LC)\,e(t) \qquad (5)
\end{aligned}
$$

Thus, the state equation for the estimation error is a homogeneous differential equation governed by the $n \times n$ matrix $A - LC$. If the gain matrix L is chosen so that the eigenvalues of $A - LC$ are strictly in the left-half of the complex plane, then the error equation is asymptotically stable, and the estimation error will decay to zero over time. If the system (A, C) is completely observable, then L can be chosen to place the eigenvalues of $A - LC$ at arbitrary locations in the plane, under the restriction that complex eigenvalues must appear in complex conjugate pairs. As long as (A, C) is at least detectable, then $A - LC$ can be made asymptotically stable[1] by choice of L.

D. Computation of Gain Matrix L

The gain matrix L of the full-state observer can be computed using any of the methods used to compute the control gain matrix K. We will assume that the system is completely observable. Therefore, the closed-loop eigenvalues of the observer can be placed at specified locations through the choice of L. For the control problem with full-state feedback, the closed-loop system matrix of interest is $A - BK$. Comparing that with the observer problem, the closed-loop system matrix is $A - LC$. The structure of those two matrices is similar; only the order of the unknown matrix differs between BK and LC. Since the

[1]Although Brogan's text only requires stability in the sense of Lyapunov, most texts require asymptotic stability of unobservable eigenvalues for the system to be detectable.

eigenvalues of a matrix and its transpose are the same, the observer problem can be formulated the same way as the control problem by considering the matrix $(A - LC)^T = A^T - C^T L^T$. Therefore, the gain matrix L can be computed using the Row-Reduced Echelon (RRE) method, Singular Value Decomposition (SVD), or the MATLAB *place* function in the same way as the control gain matrix K by replacing (A, B) by $\left(A^T, C^T\right)$. By doing this, the result from any of these methods will be the matrix L^T.

E. Example

The state space realization for this example is

$$A = \begin{bmatrix} -7 & 1 & 15 \\ -2 & 0 & 5 \\ -2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 65 \\ -5 \\ 30 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}, \quad D = 0 \tag{6}$$

The eigenvalues of A are $\lambda_{OL} = \{-1, -1 \pm j1\}$, and the desired closed-loop eigenvalues for the observer are $\lambda_{CL} = \{-5, -6, -7\}$. This is a single-output system, so the gain matrix L is unique for these eigenvalues since this is equivalent to a single-input system for the control design problem. The Row-Reduced Echelon (RRE) method will be used for the design. The matrix that will be manipulated during the RRE method is $\begin{bmatrix} \lambda I - A^T & \vdots & C^T \end{bmatrix}$.

For $\lambda = -5$

$$\begin{bmatrix} -5I - A^T & \vdots & C^T \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 1 \\ -1 & -5 & 0 & 0 \\ -15 & -5 & -9 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -0.3676 \\ 0 & 1 & 0 & 0.0735 \\ 0 & 0 & 1 & 0.7941 \end{bmatrix} \xi_1 = 0 \tag{7}$$

From the RRE matrix, it is seen that $\xi_{11} = 0.3676\xi_{14}$, $\xi_{12} = -0.0735\xi_{14}$, and $\xi_{13} = -0.7941\xi_{14}$. The value of ξ_{14} is arbitrary and will be set to 1 for this example, so that $\xi_1 = \begin{bmatrix} 0.3676 & -0.0735 & -0.7941 & 1 \end{bmatrix}^T$.

For $\lambda = -6$

$$\begin{bmatrix} -6I - A^T & \vdots & C^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & -6 & 0 & 0 \\ -15 & -5 & -10 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -0.2769 \\ 0 & 1 & 0 & 0.0462 \\ 0 & 0 & 1 & 0.5923 \end{bmatrix} \xi_2 = 0 \tag{8}$$

From the RRE matrix, it is seen that $\xi_{21} = 0.2769\xi_{24}$, $\xi_{22} = -0.0462\xi_{24}$, and $\xi_{23} = -0.5923\xi_{24}$. The value of ξ_{24} is arbitrary and will be set to -1 for this example, giving $\xi_2 = \begin{bmatrix} -0.2769 & 0.0462 & 0.5923 & -1 \end{bmatrix}^T$.

Finally, for $\lambda = -7$

$$\begin{bmatrix} -7I - A^T & \vdots & C^T \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 1 \\ -1 & -7 & 0 & 0 \\ -15 & -5 & -11 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -0.2207 \\ 0 & 1 & 0 & 0.0315 \\ 0 & 0 & 1 & 0.4685 \end{bmatrix} \xi_3 = 0 \tag{9}$$

From the RRE matrix, it is seen that $\xi_{31} = 0.2207\xi_{34}$, $\xi_{32} = -0.0315\xi_{34}$, and $\xi_{33} = -0.4685\xi_{34}$. The value of ξ_{34} is arbitrary and will be set to 1 for this example, so that $\xi_3 = \begin{bmatrix} 0.2207 & -0.0315 & -0.4685 & 1 \end{bmatrix}^T$.
The transpose of the observer gain is computed in the same fashion as the controller gain matrix K.

$$L^T = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.3676 & -0.2769 & 0.2207 \\ -0.0735 & 0.0462 & -0.0315 \\ -0.7941 & 0.5923 & -0.4685 \end{bmatrix}^{-1} = \begin{bmatrix} 431 & -105 & 208 \end{bmatrix} \tag{10}$$

Therefore, the observer gain matrix and closed-loop system matrix are

$$L = \begin{bmatrix} 431 \\ -105 \\ 208 \end{bmatrix}, \qquad A_{CL} = A - LC = \begin{bmatrix} -438 & 1 & 877 \\ 103 & 0 & -205 \\ -210 & 0 & 420 \end{bmatrix} \tag{11}$$

It is easily verified that the eigenvalues of A_{CL} are located at $\{-5, -6, -7\}$, which are the specified values.

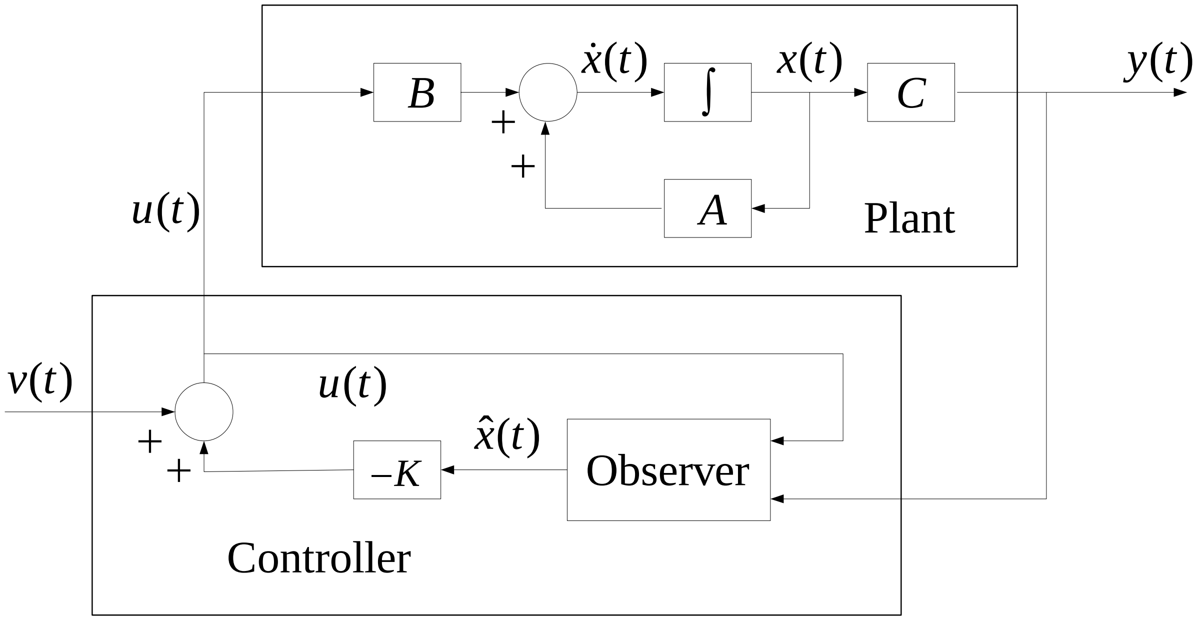


Fig. 2. Full-state observer with full-state control gain matrix.

F. Placement of the Observer Eigenvalues

Since the eigenvalues of $A - LC$ determine the speed at which the estimation error $e(t)$ decays, it is logical to make the real parts of those eigenvalues as negative as possible. That will force the error to decay very rapidly. However, there is an problem with this logic when there are modeling errors to be considered. In actual applications, the values in the $\{A, B, C, D\}$ matrices may not be known exactly. Research has shown that in order for the observer to be robust against modeling errors, as well as causing the estimation error to decay rapidly, a different approach is required.

If the original system has m_1 stable zeros, then m_1 of the observer's eigenvalues should be placed at the values of those stable zeros. The remaining eigenvalues of the observer may be placed well into the left-half plane, but at locations that are equidistant from the origin in what is known as the Butterworth configuration.[2]

G. Full-State Feedback and Full-State Observer

When the full-state observer is used in conjunction with the feedback control gain matrix K, the result is an n^{th}order dynamic controller. The system's input and output signals are measured by the observer, and an estimate of the complete state is generated. This estimate is used by the control gain matrix as if it were the true state, and the control law is $u(t) = -K\widehat{x}(t) + Fv(t)$. With this controller, the dimension of the complete system is $2n$. Figure 2 shows the the complete configuration of the plant and controller.

The control gain matrix K was chosen to place the n closed-loop eigenvalues of the system at specified locations under the assumption of full-state feedback. The observer gain matrix L was chosen to place the n closed-loop observer eigenvalues at selected locations. Due to the Separation Principle, the complete set of $2n$ closed-loop eigenvalues of the system are still located at the desired locations obtained through the K and L matrices. The total closed-loop characteristic equation is given below.

$$\Delta_{total}(\lambda) = |\lambda I - A + BK| \cdot |\lambda I - A + LC| = \Delta_{cont}(\lambda) \cdot \Delta_{obsv}(\lambda) \tag{12}$$

Thus, there is a separation between the control design and the observer design. Each of the gain matrices, K and L, can be computed to place the corresponding eigenvalues at specific locations, and neither of the designs has any effect on the other in terms of those eigenvalue locations.

[2]*Modern Control Theory*, 3rd Edition, William Brogan, Prentice Hall, 1991, pp. 532–533.