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Continuous-Time Adaptive Control of Systems with Unknown Backlash

Gang Tao and Petar V. Kokotović

Abstract— **This paper addresses one of the nondifferentiable nonlinearities which appear very often in industrial control applications: backlash. We first present a right inverse of backlash and then give one of its possible implementations. Characteristics like backlash are seldom known, so an adaptive version of our backlash inverse is more suitable for applications. We develop an adaptive backlash inverse scheme and apply it to feedback control of a known linear plant with an unknown backlash at its input. We use simulation results to illustrate achieved performance improvements.**

I. Introduction

Backlash characteristics are common in control system components such as mechanical connections and electromagnetic devices with hysteresis [1]–[3]. They are nondifferentiable nonlinearities and have been among the factors severely limiting the performance of feedback systems.

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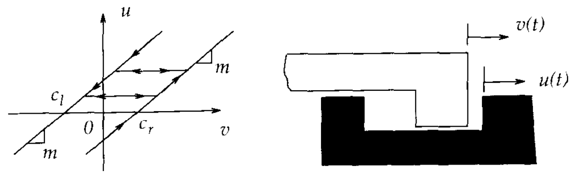

(a) (b)

Fig. 1. (a) Backlash model; (b) backlash in mechanical connections.

Adaptive control of plants with unknown backlash was recently addressed in [4] where an adaptive backlash inverse control scheme was designed in the discrete-time case. In this paper we will develop a continuous-time adaptive backlash inverse controller for plants with an unknown backlash at the input of a known linear part. A backlash element is itself a dynamic system with memory and characterized by parameters. The key feature of a backlash inverse control scheme is to use a (dynamic) backlash inverse to cancel the effect of the backlash characteristic so that a linear controller structure can be employed to achieve the control objective. Unlike in its discrete-time counterpart, in the continuous-time backlash inverse the plant output is filtered, so that both the control and its time-derivative are available. This is crucial for implementing the continuous-time backlash inverse. Because the backlash characteristics are usually poorly known and often vary with time, an adaptive scheme for updating the backlash inverse is of practical interest and is developed in this paper.

The paper is organized as follows. In Section II we present the backlash model and its inverse and give algorithms to implement the continuous-time backlash inverse. In Section III, we parameterize the backlash inverse for adaptive estimation and develop a continuous-time adaptive backlash inverse control scheme. In Section IV, we illustrate the design of an adaptive backlash inverse by an example and show performance improvements achieved with our adaptive backlash inverse controller.

II. Inverting a Backlash

In this section we present the backlash model and propose a backlash inverse scheme. A continuous-time implementation of the proposed backlash inverse is given for developing our adaptive control scheme for plants with unknown backlash in the next section.

A. Backlash Model

The backlash model and a simple backlash example are shown in Fig. 1. We see from Fig. 1(a) that the backlash characteristic $B(\cdot)$ with input $v(t)$ and output $u(t)$: $u(t) = B(v(t))$ is described by two parallel straight lines, upward and downward sides of $B(\cdot)$, connected with horizontal (inner) line segments. The upward side is active when both $v(t)$ and $u(t)$ increase: $u(t) = m(v(t) - c_r)$, $m > 0$, and $\dot{v}(t) > 0$, and the downward side is active when both $v(t)$ and $u(t)$ decrease: $u(t) = m(v(t) - c_l)$, $c_l < c_r$, and $\dot{v}(t) < 0$. The signal motion on any inner segment is characterized as $\dot{u}(t) = 0$. Mathematically, the backlash is modeled as

$$\dot{u}(t) = \begin{cases} m\dot{v}(t) & \text{if } \dot{v}(t) > 0 \text{ and } u(t) = m(v(t) - c_r), \\ & \text{or } \dot{v}(t) < 0 \text{ and } u(t) = m(v(t) - c_l) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

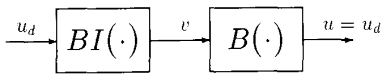

Fig. 2. Inverting a backlash.

Backlash model (2.1) appears as a first-order velocity-driven dynamical system: $v(t)$, $\dot{v}(t)$ uniquely determine $u(t)$, $\dot{u}(t)$, and the knowledge of $\dot{v}(t)$ is necessary to specify the signal motion of $B(\cdot)$ on whether a straight line or an inner segment. As an example, the dynamics of the system in Fig. 1(b) shows the backlash characteristic commonly seen in mechanical systems.

B. Adaptive Backlash Inverse

The desired function of a backlash inverse is to cancel the damaging effects of backlash on system performance: the delay corresponding to the time needed to traverse an inner segment of $B(\cdot)$ and the information loss occurring on an inner segment when the output $u(t)$ remains constant while the input $v(t)$ continues to change. That is, given a desired signal $u_d(t)$ for $u(t)$, a backlash inverse $BI(\cdot)$ is such that $u_d(t) = B(BI(u_d(t)))$ (see Fig. 2). The following mapping $BI(\cdot) = BI(m, c_r, c_l; \cdot)$ from $u_d(t)$ to $v(t)$ defines such a backlash inverse

$$\dot{v}(t) = \begin{cases} \frac{1}{m}\dot{u}_d(t) & \text{if } \dot{u}_d(t) > 0 \text{ and } v(t) = \frac{u_d(t)}{m} + c_r, \\ & \text{or } \dot{u}_d(t) < 0 \text{ and } v(t) = \frac{u_d(t)}{m} + c_l \\ 0 & \text{if } \dot{u}_d(t) = 0 \\ g(t, t) & \text{if } \dot{u}_d(t) > 0 \text{ and } v(t) = \frac{u_d(t)}{m} + c_l \\ -g(t, t) & \text{if } \dot{u}_d(t) < 0 \text{ and } v(t) = \frac{u_d(t)}{m} + c_r \end{cases} \tag{2.2}$$

where $g(\tau, t) = \delta(\tau - t)(c_r - c_l)$ with $\delta(t)$ being the Dirac δ-function. In this definition the inverse of a horizontal segment of the backlash characteristic is a vertical jump of a distance $c_r - c_l$.

Lemma 2.1 [4]: The characteristic $BI(\cdot)$ defined by (2.2) is the right inverse of the characteristic $B(\cdot)$ defined by (2.1) in the sense: $B(BI(u_d(t_0))) = u_d(t_0) \Rightarrow B(BI(u_d(t))) = u_d(t)$, $\forall t \geq t_0$, for any piecewise continuous $u_d(t)$ and any $t_0 \geq 0$.

When the exact backlash parameters m, c_r, c_l are unknown, we use their estimates $\hat{m}(t)$, $\hat{c}_r(t)$, $\hat{c}_l(t)$ to design an adaptive backlash inverse $\widehat{BI}(\cdot) \triangleq BI(\hat{m}(t), \hat{c}_r(t), \hat{c}_l(t); \cdot)$ which is graphically depicted in Fig. 3 by two parallel straight lines and instantaneous vertical transitions between these two lines, where the downward side is $v(t) = \frac{u_d(t)}{\hat{m}(t)} + \hat{c}_l(t)$, and $\dot{u}_d(t) < 0$, and the upward side is $v(t) = \frac{u_d(t)}{\hat{m}(t)} + \hat{c}_r(t)$, and $\dot{u}_d(t) > 0$. Instantaneous vertical transitions take place whenever $\dot{u}_d(t)$ changes its sign. On these two straight lines, $\dot{v}(t) = 0$ whenever $\dot{u}_d(t) = 0$.

The (exact) backlash inverse (2.2) is also a first-order velocity-driven dynamical system: $u_d(t)$, $\dot{u}_d(t)$ uniquely determine $v(t)$, $\dot{v}(t)$, and the knowledge of $\dot{u}_d(t)$ is necessary to specify the signal motion of the backlash inverse. Its implementation requires the knowledge of $\dot{u}_d(t)$ and $u_d(t)$. In our adaptive control scheme, $u_d(t)$ is a design signal, and we will choose a linear controller structure to generate $u_d(t)$ such that $\dot{u}_d(t)$ is a linear combination of physically measured signals.

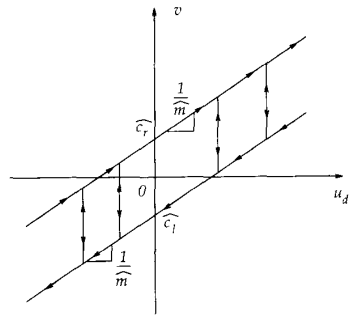

Fig. 3. Adaptive backlash inverse.

With $u_d(t)$ and $\dot{u}_d(t)$ available, introducing

$$\omega(t) = \begin{cases} c_r & \text{for } \dot{u}_d(t) > 0 \\ c_l & \text{for } \dot{u}_d(t) < 0 \\ \omega(t_-) & \text{for } \dot{u}_d(t) = 0 \end{cases} \tag{2.3}$$

where $\omega(t) = \omega(t_-)$ means $\dot{\omega}(t) = 0$, we implement the backlash inverse $v(t) = BI(u_d(t))$ as

$$v(t) = \frac{1}{m} u_d(t) + \omega(t). \tag{2.4}$$

The signal $\omega(t)$ in this implementation is discontinuous when $\dot{u}_d(t)$ changes its sign. Any of the following $\bar{\omega}(t)$'s can be used as a continuous approximation of $\omega(t)$ as shown in (2.5), found at the bottom of the page, with $\beta_r > 0$, $\beta_l > 0$, $f_r(\cdot)$, $f_l(\cdot)$ being continuous and monotonically increasing functions such that $f_r(0) = c_l$, $f_l(0) = c_r$, $f_r(\beta_r) = c_r$, $f_l(-\beta_l) = c_l$

$$\dot{\bar{\omega}}(t) = -\alpha\bar{\omega}(t) + \alpha\omega(t), \quad \alpha > 0, \quad \bar{\omega}(0) \in [c_l, c_r]. \tag{2.6}$$

These definitions of $\bar{\omega}(t)$ have the following properties:

i) $|\omega(t) - \bar{\omega}(t)| \leq \max\{c_r, |c_l|\}$ for $\bar{\omega}(t)$ in (2.5), and $|\omega(t) - \bar{\omega}(t)| \leq c_r - c_l$ for $\bar{\omega}(t)$ in (2.6);

ii) $\bar{\omega}(t) \to \omega(t)$, as $\beta_r \to 0$, $\beta_l \to 0$, or $\alpha \to \infty$, for $\dot{u}_d \neq 0$;

iii) For given design parameters β_r, β_l, or α, the signal $\bar{\omega}(t)$ in (2.5) becomes closer to $\omega(t)$ if $\dot{u}_d(t)$ has a larger magnitude, while $\bar{\omega}(t)$ in (2.6) becomes closer to $\omega(t)$ if $\dot{u}_d(\tau)$ has not changed sign over $[t - T, t]$ for a larger $T > 0$.

Property i) implies that an approximate backlash inverse introduces a bounded error in $v(t)$, while properties ii) and iii) characterize the qualitative accuracy of approximation.

Other ways to approximate the backlash inverse $BI(\cdot)$ defined by (2.4) are to replace the vertical jumps between its upward and downward lines by continuous curves with bounded gains. For example, a vertical transition is replaced with a line segment which links two sides of $BI(\cdot)$ and has a slope of a positive and finite value.

For an adaptive backlash inverse $BI(\cdot)$, we replace the parameters m, c_r, c_l in (2.3), (2.4) by their estimates $\hat{m}$, $\hat{c}_r$, $\hat{c}_l$ obtained from an adaptive update law, that is

$$v(t) = \widehat{BI}(u_d(t)) = \frac{1}{\hat{m}} u_d(t) + \hat{\omega}(t), \quad \hat{\omega}(t) = \begin{cases} \hat{c}_r & \text{for } \dot{u}_d(t) > 0 \\ \hat{c}_l & \text{for } \dot{u}_d(t) < 0 \\ \hat{\omega}(t_-) & \text{for } \dot{u}_d(t) = 0. \end{cases} \tag{2.7}$$

$$\bar{\omega}(t) = \begin{cases} c_r & \text{for } 0 \leq \dot{u}_d(t) < \beta_r \text{ and } \bar{\omega}(t_-) = c_r, \text{ or } \dot{u}_d(t) \geq \beta_r \\ c_l & \text{for } -\beta_l < \dot{u}_d(t) \leq 0 \text{ and } \bar{\omega}(t_-) = c_l, \text{ or } \dot{u}_d(t) \leq -\beta_l \\ f_r(\dot{u}_d(t)) & \text{for } 0 < \dot{u}_d(t) < \beta_r \text{ and } \bar{\omega}(t_-) < c_r \\ f_l(\dot{u}_d(t)) & \text{for } -\beta_l < \dot{u}_d(t) < 0 \text{ and } \bar{\omega}(t_-) > c_l \end{cases} \tag{2.5}$$

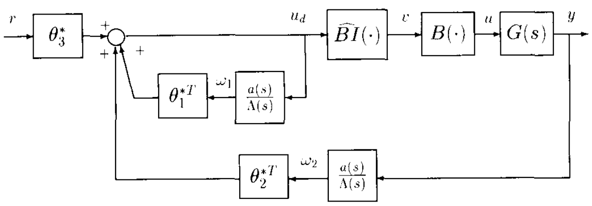

Fig. 4. Adaptive backlash inverse controller structure.

A continuous approximation of $\widehat{BI}(\cdot)$ is

$$v(t) = \frac{1}{\hat{m}} u_d(t) + \bar{\hat{\omega}}(t) \tag{2.8}$$

where $\bar{\hat{\omega}}(t)$ has the same form as $\bar{\omega}(t)$ given in (2.5) or (2.6) except that the parameters c_r, c_l are replaced by their estimates $\hat{c}_r(t)$, $\hat{c}_l(t)$ respectively.

An adaptive backlash inverse $\widehat{BI}(\cdot)$ is to be used as a part of our controller structure developed in Section III for plants with an unknown backlash $B(\cdot)$.

III. Adaptive Backlash Inverse Control

We consider the plant with a linear part $G(s)$ and a backlash nonlinearity $B(\cdot)$ at its input

$$y(t) = G(s)[u](t), \quad u(t) = B(v(t)) \tag{3.1}$$

where $G(s) = k_p \frac{Z(s)}{R(s)}$, $Z(s)$, $R(s)$ are monic polynomials in s of degrees m, n, respectively, and k_p is a nonzero constant gain. In our control problem, the input $u(t)$ to the linear part is not accessible for measurement, and the backlash $B(\cdot)$ is unknown while the linear part $G(s)$ is known. Our objective is to design a feedback control $v(t)$ using the measured plant output $y(t)$ to achieve global stabilization and close tracking of a reference signal $y_m(t)$ by $y(t)$.

Our control strategy is to use a linear model reference controller structure plus an adaptive backlash inverse to be placed at the input of the backlash $B(\cdot)$.

To employ a model reference approach, we characterize the reference signal $y_m(t)$ as: $y_m(t) = \frac{1}{R_m(s)}[r](t)$, where $R_m(s)$ is a stable polynomial and $r(t)$ is bounded and piecewise differentiable, and make the following assumptions for $R_m(s)$ and $G(s)$:

A1) $Z(s)$ is a Hurwitz polynomial;

A2) The order of $R_m(s)$ is equal to the relative degree n^* of $G(s)$.

To implement an adaptive backlash inverse, we assume

A3) $m \geq m_0$ for some known $m_0 > 0$, and $c_l \leq 0 \leq c_r$.

This assumption is used to project the estimates $\hat{m}$, $\hat{c}_r$, $\hat{c}_l$ such that $\hat{m} \geq m_0$, $\hat{c}_r \geq 0$, $\hat{c}_l \leq 0$.

A. Controller Structure

Our adaptive backlash inverse controller structure is shown in Fig. 4, in which the desired control signal $u_d(t)$ is generated by

$$u_d(t) = \theta_1^{*T}\omega_1(t) + \theta_2^{*T}\omega_2(t) + \theta_3^* r(t) \tag{3.2}$$

where $\omega_1(t) = \frac{a(s)}{\Lambda(s)}[u_d](t)$, $\omega_2(t) = \frac{a(s)}{\Lambda(s)}[y](t)$, $a(s) = (1, s, \cdots, s^{n-1})^T$, $\theta_1^* = (\theta_{11}^*, \cdots, \theta_{1n}^*) \in R^n$, $\theta_2^* = (\theta_{21}^*, \cdots, \theta_{2n}^*) \in R^n$, $\theta_3^* \in R$, and $\Lambda(s)$ is a chosen Hurwitz polynomial of degree n.

By assumption, $G(s)$ is known, so we can solve the Diophantine equation

$$\theta_1^{*T} a(s)R(s) + \theta_2^{*T} a(s)k_p Z(s) = \Lambda(s)(R(s) - k_p\theta_3^* Z(s)R_m(s)), \quad \theta_3^* = k_p^{-1} \tag{3.3}$$

for θ_1^*, θ_2^*, θ_3^* to implement the linear controller (3.2).

The key feature of our controller structure is the use of the adaptive backlash inverse $\widehat{BI}(\cdot)$ for generating the control signal $v(t)$

$$v(t) = \widehat{BI}(u_d(t)). \tag{3.4}$$

The parameters of $\widehat{BI}(\cdot)$ will be updated from an adaptive law. With linear controller (3.2), the adaptive backlash inverse (3.4) is implementable because $\dot{u}_d(t)$ depends only on the measured $y(t)$, control $u_d(t)$ and given $\dot{r}(t)$.

To show that our controller structure achieves the model-plant matching when the plant (3.1) is known, we note that (3.3) implies that

$$u(t) = \theta_1^{*T}\frac{a(s)}{\Lambda(s)}[u](t) + \theta_2^{*T}\frac{a(s)}{\Lambda(s)}[y](t) + \theta_3^* R_m(s)[y](t) \tag{3.5}$$

and that $\widehat{BI}(\cdot) = BI(\cdot)$ implies that $u(t) = u_d(t)$. Hence, with $\widehat{BI}(\cdot) = BI(\cdot)$ used in (3.4) and (3.5) substituted in (3.2), we see that $R_m(s)[y - y_m](t) = 0$ and the closed-loop system poles are the zeros of $Z(s)$, $\Lambda(s)$, and $R_m(s)$ which are stable from A1) and our design for $\Lambda(s)$, $R_m(s)$.

$\widehat{BI}(\cdot) = BI(\cdot)$, however, cannot be realized because the parameters of $B(\cdot)$ are unknown. We need to use an adaptive law to update $\widehat{BI}(\cdot)$ on line so that the desired stabilization and tracking properties are preserved.

B. Parameterization of Adaptive Backlash Inverse

To develop an adaptive law for updating the backlash parameter estimates $\hat{m}(t)$, $\hat{c}_r(t)$, $\hat{c}_l(t)$, we first parameterize the control error $u(t) - u_d(t)$ from using the adaptive backlash inverse (2.7).

Let $\chi[X](t)$ be the indicator function of $X(t)$: $\chi[X](t) = 1$ if $X(t)$ is true, $\chi[X](t) = 0$ otherwise. Defining $\widehat{\chi}_r(t) = \chi[u_d(t), v(t)$ on the upward side of $\widehat{BI}(\cdot)](t)$, $\widehat{\chi}_l(t) = 1 - \widehat{\chi}_r(t)$, $\chi_r(t) = \chi[v(t), u(t)$ on the upward side of $B(\cdot)](t)$, $\chi_l(t) = \chi[v(t), u(t)$ on the downward side of $B(\cdot)](t)$, $\chi_s(t) = \chi[v(t), u(t)$ on an inner segment of $B(\cdot)](t)$, we express the adaptive backlash inverse output $v(t)$ and the backlash output $u(t)$ as

$$v(t) = \frac{\widehat{\chi}_r(t)}{\hat{m}(t)}(u_d(t) + \hat{m}(t)\hat{c}_r(t)) + \frac{\widehat{\chi}_l(t)}{\hat{m}(t)}(u_d(t) + \hat{m}(t)\hat{c}_l(t)) \tag{3.6}$$

$$u(t) = \chi_r(t)m(v(t) - c_r) + \chi_l(t)m(v(t) - c_l) + \chi_s(t)u_s, \quad \frac{u_s}{m} + c_l < v(t) < \frac{u_s}{m} + c_r \tag{3.7}$$

where u_s is a generic constant corresponding to the value of $u(t)$ at any active inner segment.

Using (3.6) and (3.7), we have the relationship between $u(t)$ and $u_d(t)$

$$u(t) = u_d(t) + \widehat{\chi}_r(t)(m(v(t) - c_r) - \hat{m}(t)v(t) + \hat{m}(t)\hat{c}_r(t)) + \widehat{\chi}_l(t)(m(v(t) - c_l) - \hat{m}(t)v(t) + \hat{m}(t)\hat{c}_l(t)) + d_0(t) \tag{3.8}$$

where $d_0(t) = (\chi_r(t) - \widehat{\chi}_r(t))m(v(t) - c_r) + (\chi_l(t) - \widehat{\chi}_l(t))m(v(t) - c_l) + \chi_s(t)u_s \in L_\infty$ [4].

If adaptive backlash inverse (2.7) is approximated by (2.8), then, with the definitions

$$\widehat{\chi}_r(t) = \begin{cases} 1 & \text{for } \dot{u}_d(t) > 0 \\ \widehat{\chi}_r(t_-) & \text{for } \dot{u}_d(t) = 0\,, \\ 0 & \text{for } \dot{u}_d(t) < 0 \end{cases} \quad \widehat{\chi}_l(t) = 1 - \widehat{\chi}_r(t)$$

we still have (3.8) except that $d_0(t)$ will contain, in addition to the above expression, a bounded disturbance proportional to the approximation error $|\omega(t) - \bar{\omega}(t)|$ (see Section II-B).

Letting $\widehat{mc_r}(t) = \hat{m}(t)\hat{c}_r(t)$, $\widehat{mc_l}(t) = \hat{m}(t)\hat{c}_l(t)$ and defining $\theta_b^* = (mc_r, m, mc_l)^T$, $\theta_b(t) = (\widehat{mc_r}(t), \hat{m}(t), \widehat{mc_l}(t))^T$, $\omega_b(t) = (\hat{\chi}_r(t), -v(t), \hat{\chi}_l(t))^T$, from (3.8), we obtain

$$u(t) - u_d(t) = (\theta_b(t) - \theta_b^*)^T\omega_b(t) + d_0(t). \tag{3.9}$$

This expression is crucial for establishing an error model suitable for parameter adaptation.

C. Adaptive Law

When $\widehat{BI}(\cdot) \neq BI(\cdot)$, from (3.2), (3.5), (3.9), we obtain

$$\begin{aligned} e(t) &= y(t) - y_m(t) \\ &= H(s)[(\theta_b - \theta_b^*)^T\omega_b](t) + H(s)[d_0](t), \\ H(s) &= \frac{k_p}{R_m(s)}\left(1 - \theta_1^{*T}\frac{a(s)}{\Lambda(s)}\right). \end{aligned} \tag{3.10}$$

Tracking error equation (3.10), which is familiar from adaptive linear control theory [5], [6], suggests the following update law for the estimates of the backlash parameters

$$\dot{\theta}_b(t) = -\frac{\Gamma\zeta(t)(e(t) + \xi(t))}{z(t)} + f_b(t) \tag{3.11}$$

where $\Gamma = \Gamma^T > 0$, $\zeta(t) = H(s)[\omega_b](t)$, $\xi(t) = \theta_b^T(t)\zeta(t) - H(s)[\theta_b^T\omega_b](t)$, $z(t) = 1 + \zeta^T(t)\zeta(t) + \xi^2(t)$, and $f_b(t)$ is a σ-modification signal designed as in [6] for robustness with respect to bounded "disturbance" $d_0(t)$. With θ_b updated from (3.11), we obtain the estimates: $\hat{c}_r = \frac{\widehat{mc_r}}{\hat{m}}$, $\hat{c}_l = \frac{\widehat{mc_l}}{\hat{m}}$ and implement the adaptive backlash inverse (2.7).

Although not indicated in (3.11), a projection of $\theta_b(t)$ is implemented to ensure that $m(t) \geq m^0 > 0$, $\widehat{mc_r}(t) \geq 0$, $\widehat{mc_l}(t) \leq 0$, according to A3).

Since $d(t)$ is bounded, robust adaptive control theory [5], [6] shows that with the σ-modification $f_b(t)$ all closed-loop signals are bounded and $\int_{t_1}^{t_2} e^2(t)dt \leq c_1 + c_2\int_{t_1}^{t_2} d_0^2(t)dt$ for some $c_1, c_2 > 0$, and any $t_2 \geq t_1 > 0$.

Our simulation results show that the above adaptive backlash inverse controller ensures very small tracking errors in spite the presence of backlash uncertainties. Analytical proof of the convergence of the tracking error to zero, however, is not available.

IV. Simulations

As an illustrative example, we consider plant (3.1) with an unknown backlash $B(\cdot)$ and known $G(s) = \frac{1}{(s-1.5)(s+1)}$.

Controller structure (3.2) is

$$u_d(t) = \frac{\theta_{11}^* + \theta_{12}^* s}{(s+2)^2}[u_d](t) + \frac{\theta_{21}^* + \theta_{22}^* s}{(s+2)^2}[y](t) + \theta_3^* r(t).$$

With the choice of the reference model: $W_m(s) = \frac{1}{(s+2)(s+3)}$, from (3.3), we calculate the controller parameters as: $\theta_{11}^* = -32.25$, $\theta_{12}^* = -5.5$, $\theta_{21}^* = -78.375$, $\theta_{22}^* = -76.375$, $\theta_3^* = 1$. The knowledge of $u_d(t)$ and $\dot{u}_d(t)$ is used for implementing the adaptive backlash inverse $\widehat{BI}(\cdot)$ in (2.7) or (2.8) whose parameters are adaptively updated from (3.11). The signal $v(t)$ from $\widehat{BI}(\cdot)$ is then applied to the plant: $y(t) = G(s)[B(v)](t)$.

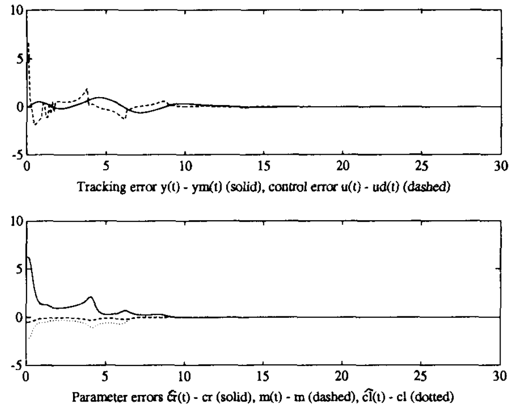

Fig. 5. System responses with an adaptive backlash inverse.

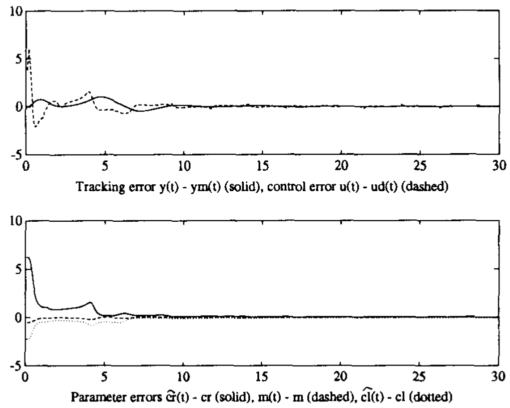

Fig. 6. System responses with an approximate adaptive backlash inverse.

With unknown $B(\cdot)$: $m = 0.7$, $c_r = 1.3$, $c_l = -0.5$, that is, $\theta_b^* = (0.91, 0.7, -0.35)^T$, we simulated the closed-loop system with $\Gamma = 5I$, $\theta_b(0) = (1.51, 0.2, -0.55)^T$ for adaptive law (3.11). Fig. 5 shows the system responses for $r(t) = 10\sin 1.3t$ with the adaptive backlash inverse (2.7). Fig. 6 shows the results with the approximate continuous adaptive backlash inverse (2.8) where $\bar{\omega}(t)$ is generated similarly to (2.6) with $\alpha = 6$. We see that in both cases the tracking errors converge to very small values and so do the control and parameter errors. As a comparison, simulations were also performed for the closed-loop system with the fixed controller (4.1) only: $v(t) = u_d(t)$, or with (4.1) plus an adaptive gain: a degenerate adaptive backlash inverse $\widehat{BI}(\cdot)$ with $\hat{c}_r(t) = \hat{c}_l(t) = 0$. In the latter cases, the tracking error remains large for large t.

V. Concluding Remarks

We have developed a continuous-time adaptive backlash inverse controller for plants which have a known linear part and an unknown backlash at its input. Our controller uses a linear controller structure and an adaptive backlash inverse, which achieves plant-model matching when the backlash is known and guarantees global signal boundedness and promises to improve system performance when the backlash is unknown.

We have not included unmodeled dynamics and disturbances in the plant model. Since an adaptive backlash inverse only introduces a bounded disturbance, the adaptive backlash inverse controller is in fact robust with respect to bounded output disturbance and can be redesigned with normalizing signals [6] to ensure robustness in the presence of certain unmodeled dynamics, leading to a mean tracking error of the order of disturbances or/and unmodeled dynamics.

In the presence of actuator dynamics prior to the backlash, the adaptive backlash inverse control problem is more difficult because it requires that the backlash be inverted through a dynamic block. This problem is currently under investigation.

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On Computing the Maximal Delay Intervals for Stability of Linear Delay Systems

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***Abstract*— This note is concerned with stability properties of linear time-invariant delay systems. We consider delay systems of both retarded and neutral types expressed in state-space forms. Our main goal is to provide a computation-oriented method for computing the maximal delay intervals over which the systems under consideration maintain stability. Our results show that this can be accomplished by computing the generalized eigenvalues of certain frequency-dependent matrices. Based on these results, we also state a necessary and sufficient condition concerning stability independent of delay for each of the retarded and neutral systems. Our results can be readily implemented and appear suitable for analyzing systems with high dimensions and many delay units.**

I. Introduction

Stability in time-delay systems has been a much studied problem for decades; a thorough discussion on this problem may be found in the monographs [1], [8], [10], [19], [20], and [24] and the references therein. Many results have been reported on this subject (see, e.g., [3], [2], [6], [9], [11], [14]–[18], [21]–[23]), particularly for linear time-invariant delay systems, modeled either as high-order scalar differential-difference equations or as a set of first-order differential-difference equations in state-space forms. A specific notion concerning stability of delay systems is stability independent of delay [15]. A system is said to be stable independent of delay if it is stable when delay parameters assume all nonnegative values.

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As noted by several authors [6], [21], the notion of stability independent of delay may be overly restrictive. It may often be the case that some systems are stable only for a bounded interval of delay values, or for some problems it may suffice to just determine whether stability is present for a finite range of delay values. As such, it appears useful to find the largest interval over which the underlying system is stable. Issues of this kind have been addressed in [12], [13], [25], [6], and [21] and are precisely the topic in the present note.

Our purpose in this paper is to furnish several new results concerning the aforementioned issues, and we shall consider both retarded and neutral linear time-invariant delay systems described in state-space forms. As the main contribution of this paper, we develop for each class of these systems a method for computing the maximal delay interval over which the systems are stable. The main computation in this method involves computing the generalized eigenvalues of certain frequency-dependent matrices. As a by-product of this method, we also obtain necessary and sufficient conditions for stability independent of delay, and these conditions may be checked by computing the spectral radii of certain frequency-dependent matrices. Our results differ from the previous methods in that they are directly applicable to systems given in state-space form, and they provide systematic, computation-oriented tests that can be implemented efficiently due to the ease of computing generalized eigenvalues and spectral radii. In contrast, the analytical tests of [12], [13], [25], and [6] deal with largely systems given as scalar differential-difference equations (or two-variable polynomials) and appear to be more amenable to analysis "by hand." The implication here is twofold. First, though it is always possible to transform the problems considered herein to one in [12], [13], [25], and [6], our method avoids completely the complexity that will necessarily result from such transformations. Furthermore, due to its computational nature, our method is suitable for analyzing systems with high dimensions and many delay units.

The notation used throughout this paper is as follows. We denote the open right-half plane by $\mathbf{C}_+ := \{s: \mathrm{Re}(s) > 0\}$ and the closed right-half plane by $\bar{\mathbf{C}}_+$. For a real constant σ, we define a shifted open right-half plane $\mathbf{C}_{\sigma+} := \{s: \mathrm{Re}(s) > \sigma\}$ and denote the corresponding shifted closed right-half plane by $\bar{\mathbf{C}}_{\sigma+}$. For a matrix A, we denote its ith eigenvalue by $\lambda_i(A)$ and its spectral radius by $\rho(A)$. For a matrix pair (A, B), we denote its ith generalized eigenvalue [7] by $\lambda_i(A, B)$. Furthermore, we define

$$\rho_m(A, B) := \min\{|\lambda|: \det(A - \lambda B) = 0\}.$$

Note that the number of bounded generalized eigenvalues for (A, B) is at most equal to the rank of B. Also, if the rank of B is constant, then $\lambda_i(A, B)$ is continuous with respect to the elements of A and B. Finally, a matrix is said to be stable if all its eigenvalues are in the open left-half plane.

Partial results in this paper have been presented previously in [4].

II. Retarded Delay Systems

The delay systems considered in this section are governed by the linear differential-difference equation

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - kh), \quad h \geq 0 \tag{2.1}$$