Lecture 13: Passivity-based design

- Adaptation of Feedforward Gain
- Output feedback
- Strictly positive real functions (*SPR*), Kalman-Yakubovich (*KY*) Lemma
- The Augmented Error

General Case for State Feedback

Given a continuous time system and the target dynamics

$$\dot{x} = A\,x + B\,u, \qquad \dot{x}_m = A_m\,x_m + B_m\,u_c$$

With the controller and the error signals

$$u(t) = M(\theta)\,u_c(t) - L(\theta)\,x(t), \qquad e(t) = x(t) - x_m(t)$$

Assuming solvability of the model-matching problem in the nominal case $\theta = \theta^0$, the error dynamics are

$$\dot{e} = A_m\,e + \underbrace{\Big(A - A_m - B\,L\Big)\,(e + x_m(t)) + \Big(B\,M - B_m\Big)\,u_c(t)}_{\Psi(t,e)\,(\theta - \theta^0)}$$

The adaptation law can be designed using ($\Gamma = \Gamma^T > 0$)

$$V\Big(e, \theta - \theta^0\Big) = \frac{1}{2}\left[e^T\,P\,e + (\theta - \theta^0)^T\,\Gamma^{-1}\,(\theta - \theta^0)\right]$$

General Case for State Feedback (cont'd)

The time-derivative of $V = \frac{1}{2}\left[e^T P e + (\theta - \theta^0)^T \Gamma^{-1} (\theta - \theta^0)\right]$ is

$$\dot{V} = \frac{1}{2} e^T \left[P A_m + A_m{}^T P\right] e + (\theta - \theta^0)^T \Psi^T P e + (\theta - \theta^0)^T \Gamma^{-1} \dot{\theta}$$

Assuming that $P = P^T > 0$ is the solution of

$$P\, A_m + A_m{}^T P = -Q, \qquad Q = Q^T > 0$$

and taking the adaptation law as

$$\dot{\theta} = -\Gamma\, \Psi^T(t, e)\, P\, e = -\Gamma\, \Psi^T(t, e)\, P\, (x - x_m)$$

one obtains

$$\frac{d}{dt} V\Big(e(t), \theta(t) - \theta^0\Big) = -\frac{1}{2}\, e^T(t)\, Q\, e(t)$$

and concludes that the signals are bounded and $\lim_{t \to \infty} e(t) = 0$.

Adaptation of Feedforward Gain

Chaging the gain of a stable process

$$y(t) = k \cdot G_0(p) \Big(u(t)\Big) \quad \longrightarrow \quad y_m(t) = k_0 \cdot G_0(p) \Big(u_c(t)\Big)$$

can be done using the simple controller $\quad u(t) = \theta(t)\, u_c(t)$

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can be done using the simple controller $u(t) = \theta(t)\, u_c(t)$

To design an adaptation law, we compute the error signal

$$e(t) = y - y_m = k \cdot G_0(p)\left(\theta(t) - \theta^0\right) u_c(t), \qquad \theta^0 = k_0/k$$

and proceed as follows.

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and proceed as follows.

First, find a minimal realization $\{A, B, C\}$

$$G(s) = k \cdot G_0(s) = C^T (s\, I - A)^{-1} B$$

with known (independent on k) A and B.

Adaptation of Feedforward Gain (cont'd)

For the state-space model

$$\dot{x} = A\,x + B\,(\theta - \theta^0)\,u_c(t), \qquad e(t) = C^T\,x(t)$$

with a Hurwitz matrix A (since $G(s)$ is stable), one can find the solution $P = P^T > 0$ of the Lyapunov equation

$$P\,A + A^T\,P = -Q, \qquad Q = Q^T > 0$$

Adaptation of Feedforward Gain (cont'd)

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and consider the Lyapunov function

$$V\left(x, \theta - \theta^0\right) = \frac{1}{2}\left[\gamma\,x^T\,P\,x + (\theta - \theta^0)^2\right]$$

Adaptation of Feedforward Gain (cont'd)

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$$V\left(x, \theta - \theta^0\right) = \frac{1}{2}\left[\gamma\, x^T\, P\, x + (\theta - \theta^0)^2\right]$$

Taking

$$\dot{\theta} = -\gamma\, x^T\, P\, B\, u_c(t)$$

we would have

$$\dot{V} = -\frac{\gamma}{2}\, x^T\, Q\, x \quad \Rightarrow \quad x(t) \to 0 \quad \Rightarrow \quad e(t) \to 0$$

Adaptation of Feedforward Gain (cont'd)

However, the adaptation law

$$\dot{\theta} = -\gamma\, x(t)^T\, P\, B\, u_c(t)$$

cannot be used since the signal $x(t)$ is not available!
We only have $u_c(t)$ and $e(t) = y(t) - y_m(t)$.

Adaptation of Feedforward Gain (cont'd)

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Suppose the matrix Q is chosen such that

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Than, we have

$$e(t) = C^T\, x(t) = x(t)^T\, C = x(t)^T\, P\, B$$

and the adaptation law can be rewritten as

$$\dot{\theta} = -\gamma\, e(t)\, u_c(t)$$

Remark: the MIT rule gives:

$$\dot{\theta} = -\gamma\, e(t)\, y_m(t)$$

SPR and KY lemma

Definition: A rational transfer function $G(s)$ is *positive real* (PR) if

$$\mathrm{Re}\{G(s)\} \geq 0 \qquad \text{for} \qquad \mathrm{Re}\{s\} \geq 0$$

and *strictly positive real* (SPR) if

$$\mathrm{Re}\{G(s - \varepsilon)\} \geq 0 \qquad \text{for} \qquad \mathrm{Re}\{s\} \geq 0$$

for some $\varepsilon > 0$.

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for some $\varepsilon > 0$.

Lemma (Kalman-Yakubovich): The transfer function

$$G(s) = C^T (s\,I - A)^{-1} B$$

where (A, B) is controllable and (A, C^T) is observable, is SPR if and only if there exist $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$P\,A + A^T P = -Q, \qquad P\,B = C$$

Strictly Positive Real functions

Definition: A rational transfer function $G(s)$ is (SPR) if

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Lemma: The rational transfer function $G(s)$ with real coefficients is SPR if and only if

(i) There are no poles in the closed right-half plane (stable).

(ii) $G(\infty) > 0$ OR $G(\infty) = 0$ & $\lim_{\omega\to\infty} \omega^2 \,\mathbf{Re}\{G(j\omega)\} > 0$.

(iii) $\mathbf{Re}\{G(j\,\omega)\} \geq 0$ for $\omega \in \mathbb{R}$.

Strictly Positive Real functions

Definition: A rational transfer function $G(s)$ is (SPR) if

$$\mathbf{Re}\{G(s-\varepsilon)\} \geq \mathbf{0} \qquad \text{for} \qquad \mathbf{Re}\{s\} \geq \mathbf{0}$$

for some $\varepsilon > \mathbf{0}$.

Lemma: The rational transfer function $G(s)$ with real coefficients is SPR if and only if

(i) There are no poles in the closed right-half plane (stable).

(ii) $G(\infty) > \mathbf{0}$ OR $G(\infty) = \mathbf{0}$ & $\lim_{\omega\to\infty} \omega^2\,\mathbf{Re}\{G(j\omega)\} > \mathbf{0}$.

(iii) $\mathbf{Re}\{G(j\,\omega)\} \geq \mathbf{0}$ for $\omega \in \mathbb{R}$.

Remark: If the (scalar) rational transfer function $G(s)$ with real coefficients is SPR, than

1. The Nyquist plot lies in the closed right-half plane.
2. The relative degree is zero or one.

Passivity

<u>Definition:</u> A SISO time-varying nonlinear system

$$\dot{x} = f(t, x, u), \qquad y = h(t, x, u)$$

is called passive if there is a positive semidefinite storage function $V(t, x) \geq 0$ such that

$$u\, h(t, x, u) = \boxed{u(t)\, y(t) \geq \dot{V}} = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) \cdot f(t, x, u)$$

or

$$\int_0^\tau u(s)\, y(s)\, ds \geq V\Big(\tau, x(\tau)\Big) - V\Big(0, x(0)\Big)$$

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Lemma: If the transfer function $G(s) = C^T (s\, I - A)^{-1} B$ with real coefficients is SPR, then its minimal state-spase realizatition defines a passive system.

Remark: A mechanical system with u as the generalized force and y as the generalized velocity defines a passive system with total energy as the storage function.

Passivity interpretation

The closed-loop system with adaptation of the feedforward gain

$$e(t) = y - y_m = G(p)\left(\theta - \theta^0\right) u_c(t), \qquad \theta = -\frac{\gamma}{p}\, e(t)\, u_c(t)$$

Passivity interpretation

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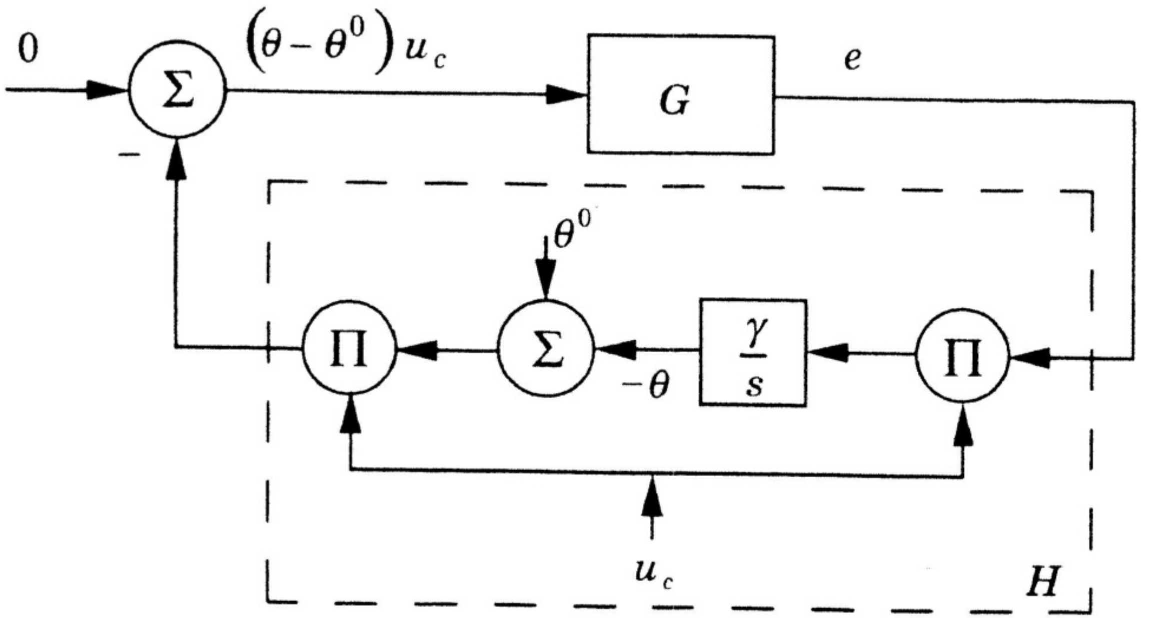

Figure 5.18 Representation of the system with adjustable feedforward gain when using the control law of Eq. 5.40. Compare with Fig. 5.14(b).

Passivity interpretation (cont'd)

Let us show that this is a feedback connection of two passive systems.

We just need to verify that H-subsystem

$$\dot{z} = \gamma\, u_c(t)\, e, \qquad \xi = u_c(t)\, z$$

is passive with the output ξ and the input e:

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$$V_H(z) = \frac{1}{2\,\gamma}\, z^2$$

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Passivity interpretation (cont'd)

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The fact that $e(t) \to 0$ can be shown using the Lyapunov function

$$V(x,z) = \frac{\gamma}{2}\, x^T P\, x + V_H(z) \quad \Rightarrow \quad \dot{V} \leq -x^T Q\, x$$

with any other passive system defining an adaptation law!

Relaxing the SPR requirement

Suppose $G(s) = \dfrac{B(s)}{A(s)}$ is

- stable, i.e. $A(s)$ is Hurwitzh,
- minimum phase, i.e. $B(s)$ is Hurwitzh as well,
- has relative degree one, i.e. $\deg\{B\} = \deg\{A\} - 1$,

but it is NOT SPR.

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- has relative degree one, i.e. $\deg\{B\} = \deg\{A\} - 1$,

but it is NOT SPR.

Let us use passivity arguments to design an adaptation law for this case.

First, we construct a polynomial $C(s)$ such that

- $\deg\{C\} = \deg\{A\} - 1$,
- the transfer function $\dfrac{C(s)}{A(s)}$ is SPR.

Relaxing the SPR requirement (cont'd)

Introduce the canonical realization of

$$1/A(s) = 1/(s^n + a_1\, s^{n-1} + \cdots + a_n)$$

$$\dot{z} = A\, z + B\, v = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} v$$

Relaxing the SPR requirement (cont'd)

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Choose any $Q = Q^T > 0$ and solve

$$P\,A + A^T\,P = -Q, \qquad P = P^T > 0$$

Relaxing the SPR requirement (cont'd)

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$$P\, A + A^T\, P = -Q, \qquad P = P^T > 0$$

Choose the matrix $C = P\, B$, i.e.

$$C^T = B^T\, P = [p_{11}, \ldots, p_{1n}] = \text{the first raw of } P$$

and take $\qquad C(s) = p_{11}\, s^{n-1} + \cdots + p_{1n}.$

Relaxing the SPR requirement (cont'd)

With the constructed polynomial $C(s)$ let us define the filter

$$G_c(s) = \frac{C(s)}{B(s)}$$

Relaxing the SPR requirement (cont'd)

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$$G_c(s) = \frac{C(s)}{B(s)}$$

which is

- proper: $\deg\{B\} = \deg\{A\} - 1$ (relative degree one) and $\deg\{C\} = \deg\{A\} - 1$,
- stable: $B(s)$ is stable (minimum phase property).

Relaxing the SPR requirement (cont'd)

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which is

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Since

$$G_c(s)\,G(s) = \frac{C(s)}{B(s)}\,\frac{B(s)}{A(s)} = \frac{C(s)}{A(s)} \qquad \text{is SPR}$$

let

$$e_c(t) = G_c(p)\,e(t) = G_c(p)\,G(p)\,(\theta - \theta^0)\,u_c(t)$$

and take

$$\dot\theta = -\gamma\,e_c(t)\,u_c(t).$$

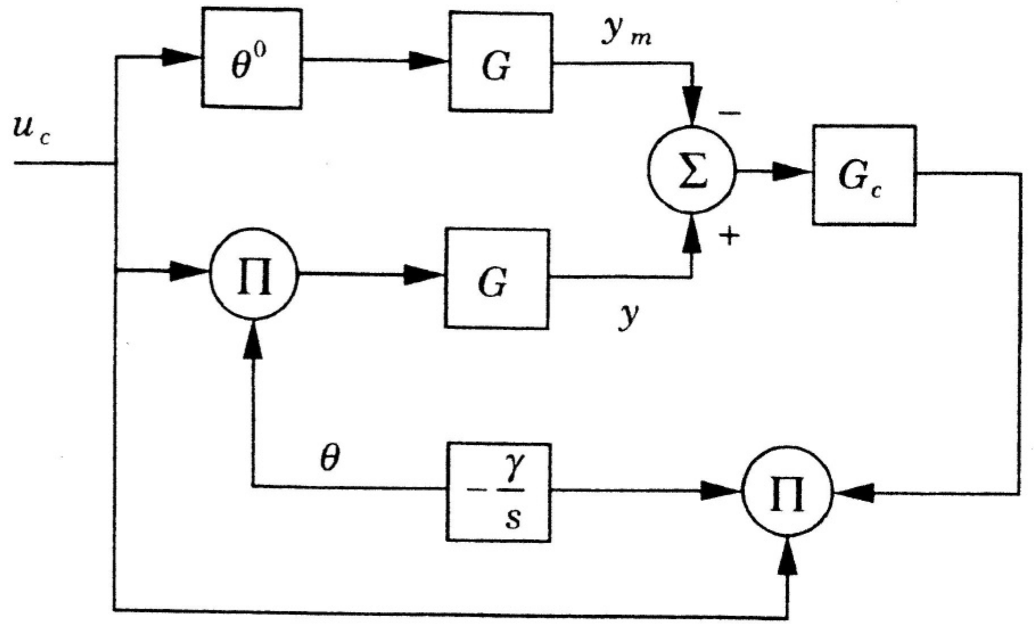

Figure 5.19 A stable parameter adjustment law is obtained if GG_c is SPR.

As before, all the signals will be bounded and $\lim_{t\to\infty} e_c(t) = 0$.

Since $G_c(s)$ is minimum phase, we conclude that $\lim_{t\to\infty} e(t) = 0$ as well.

Relaxing the relative degree restriction

In the general case of stable $G(s)$, let

$$G(s) = G_1(s)\, G_2(s)$$

where $G_1(s)$ is SPR and can be taken as $G_1(s) = 1$.

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where $G_1(s)$ is SPR and can be taken as $G_1(s) = 1$.

$$e(t) = G\left(\theta - \theta^0\right) u_c(t) = G_1\, G_2 \left(\theta - \theta^0\right) u_c(t)$$

can be rewritten as

$$e(t) = G_1 \left[(\theta - \theta^0) G_2 u_c\right] - G_1 \left[(\theta - \theta^0) G_2 u_c - G_2 (\theta - \theta^0) u_c\right]$$

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Let us introduce augmented error

$$\varepsilon(t) = \left(\theta(t) - \theta^0\right)\, G_2(p)\, u_c(t)$$

and error augmentation

$$\eta(t) = G_1(p)\left[(\theta(t) - \theta^0) G_2(p) u_c(t)\right] - G(p)(\theta(t) - \theta^0) u_c(t)$$

Relaxing the relative degree restriction

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Adaptation with augmented error

The adpatation law can be taken as

$$\dot{\theta} = -\gamma\,\varepsilon(t)\ \left(G_2(p)\,u_c(t)\right)$$

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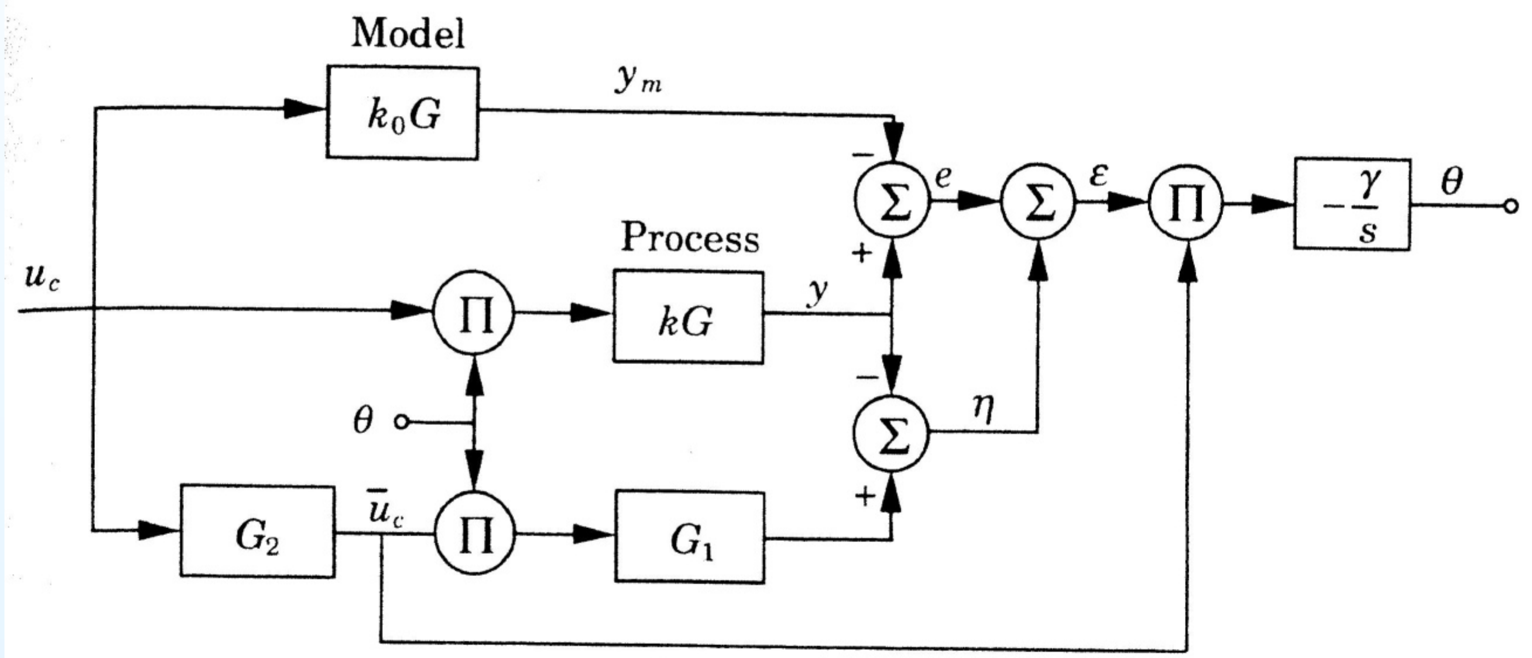

Figure 5.20 Block diagram of a model-reference adaptive system based on the augmented error.

It can be shown that with "nice" $u_c(t)$: $e(t) \to 0$.

Next Lecture / Assignments:

Next meeting (May 27, 15:00-17:00, in A206Tekn): Recitations.

Homework problems:

- Which of the transfer functions: $G_1(s) = 1/s$, $G_2(s) = 1/(s+a)$ with $a > 0$, $G_2(s) = 1/(s^2+s+1)$ are PR / SPR?
- Proof that the adaptation law for the problem of adjusting a gain of an SPR transfer function can be taken as a PI law:

$$\theta(t) = -\gamma_1\, u_c(t)\, e(t) - \gamma_2 \int_0^t u_c(s)\, e(s)\, ds$$

with $\gamma_1, \gamma_2 > 0$.