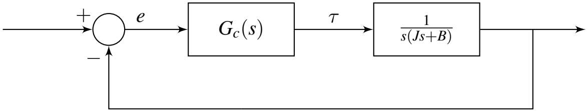
2.153 Adaptive Control
Lecture 6
Adaptive PI Control

Anuradha Annaswamy

aanna@mit.edu

- **Pset #1** out: Thu 19-Feb, **due: Fri 27-Feb**
- **Pset #2** out: Wed 25-Feb, **due: Fri 6-Mar**
- **Pset #3** out: Wed 4-Mar, **due: Fri 13-Mar**
- **Pset #4** out: Wed 11-Mar, **due: Fri 20-Mar**
- **Midterm (take home)** out: Mon 30-Mar, **due: Fri 3-Apr**

Adaptive Control of a Second-order Plant



Plant: $J\ddot{\omega} + B\dot{\omega} = \tau \qquad J > 0$

PI Control: $$G_c(s) = k_p + \frac{k_i}{s}$$

$$\tau = k_p e(t) + k_i \int e(\tau) d\tau$$

Adaptive PI Control: $$\tau = k_p(t)e(t) + k_i(t) \int e(\tau) d\tau$$

PID Control: $$G_c(s) = k_p + k_d s + \frac{k_i}{s}$$

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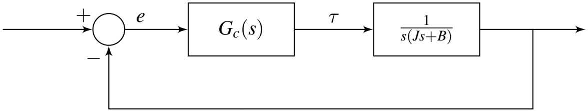
J and B are unknown. Adjust $k_p(t)$, $k_i(t)$ and $k_d(t)$ so that the closed-loop system is stable and $\lim_{t\to\infty} e(t) = 0$.

Adaptive Control of a Second-order Plant



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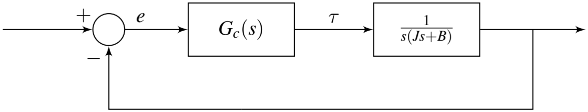

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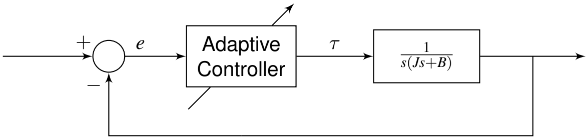

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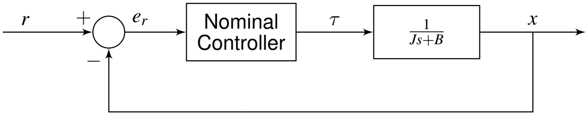
PI -Control: Algebraic Part



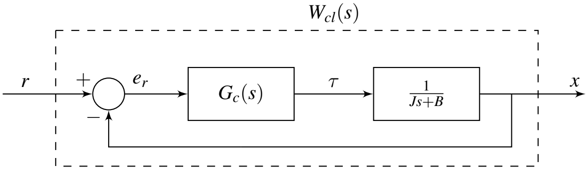
$$G_c(s) = k_p + \frac{k_i}{s}$$

Parameterize $k_p = K > 0, \quad k_i = K\lambda > 0$

Closed-loop transfer function:
$$\frac{K(s+\lambda)}{s(Js+B)+K(s+\lambda)} = \frac{K(s+\lambda)}{Js^2+s(B+K)+K\lambda}$$

Stable if $K > |B|$. Design the controller so that $x \to x_d$

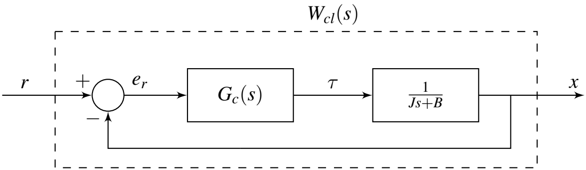
PI Control - Algebraic Part: Tracking



$$
\begin{aligned}
W_{cl}(s) &= \frac{G_c(s)}{Js + B + G_c(s)} \\
W_{cl}^{-1}(s) &= 1 + (Js + B)G_{cl}^{-1}(s)
\end{aligned}
$$

$$
\begin{aligned}
r &= W_{cl}^{-1}(s)[x_d] \\
&= x_d + \left((Js + B)G_{cl}^{-1}(s)\right)[x_d] \\
&= x_d + (Js + B)[\omega_d] = x_d + B\omega_d + J\dot{\omega}_d
\end{aligned}
$$

PI Control - Algebraic Part: Tracking



$$W_{cl}(s) = \frac{G_c(s)}{Js + B + G_c(s)}$$

$$W_{cl}^{-1}(s) = 1 + (Js + B)G_{cl}^{-1}(s)$$

$$\begin{aligned} r &= W_{cl}^{-1}(s)[x_d] \\ &= x_d + \left((Js + B)G_{cl}^{-1}(s)\right)[x_d] \\ &= x_d + (Js + B)[\omega_d] = x_d + B\omega_d + J\dot{\omega}_d \end{aligned}$$

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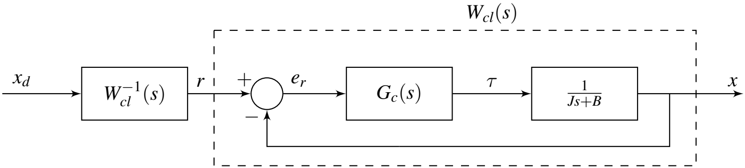

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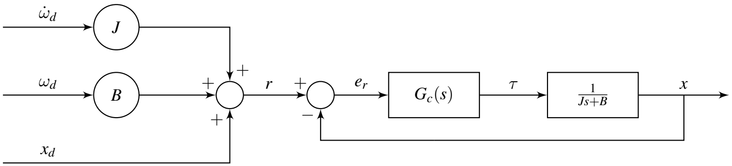
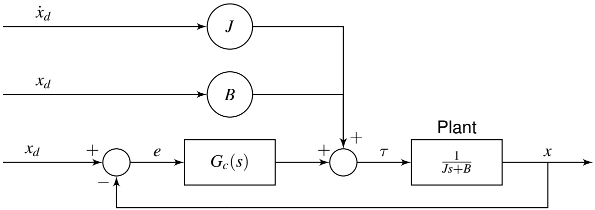
PI Control - Algebraic Part: Tracking

Using $r = J\dot{\omega}_d + B\omega_d + x_d$ the block diagram can be represented as



which can then be simplified to



PI Control - Algebraic Part: Tracking - Revised Design

J: reparametrize $G_c(s)$:

$$G_c(s) = k_p + \frac{k_i}{s} = K + \frac{K\lambda}{s} = \frac{Ks + K\lambda}{s}$$

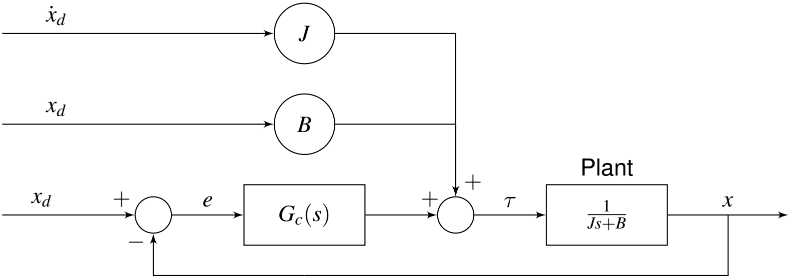
$$\text{Change to } G_c(s) = \frac{(K + J\lambda)s + K\lambda}{s}$$

$$\text{Closed-loop transfer function: } \frac{(K + J\lambda)s + K\lambda)}{s(Js + B) + (K + J\lambda)s + K\lambda)}$$

- Move B from feedforward - to feedback

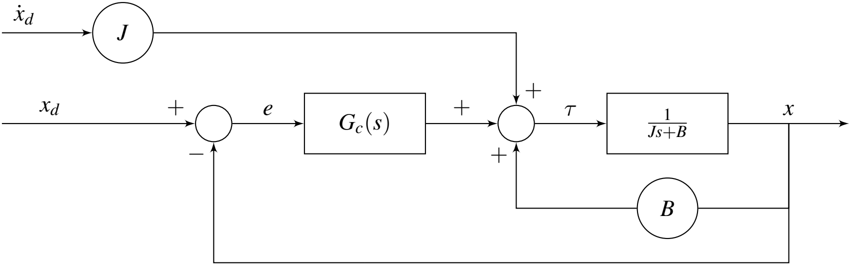
$$\text{Closed-loop transfer function: } \frac{(K + J\lambda)s + K\lambda}{Js^2 + (K + J\lambda)s + K\lambda}$$

PI Control - Algebraic Part: Tracking - Revised Design



$$G_c(s) = \frac{(K + J\lambda)s + K\lambda}{s}$$

PI Control - Algebraic Part: Tracking - Revised Design



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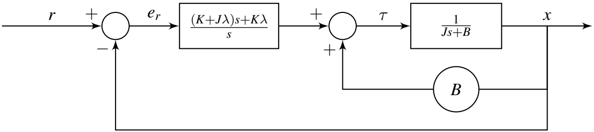


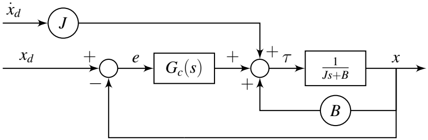
$$W_{cl}(s): \quad \frac{(K+J\lambda)s+K\lambda}{Js^2+(K+J\lambda)s+K\lambda}$$

$$\omega_n = \sqrt{\frac{K\lambda}{J}}$$

$$\zeta = \frac{K+J\lambda}{2J}\cdot\sqrt{\frac{J}{K\lambda}} = \frac{K+J\lambda}{2\sqrt{JK\lambda}}$$

- Less sensitive to uncertainties in J
- Less sensitive to uncertainties in B

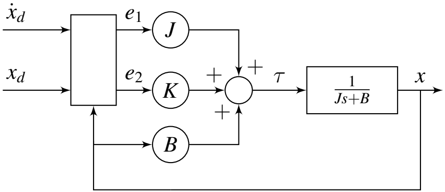
PI Control - Algebraic Part: Tracking - Complete Design



$$G_c(s) = \frac{(K+J\lambda)s+K\lambda}{s}, W_{cl}(s) = \frac{(K+J\lambda)s+K\lambda}{Js^2+(K+J\lambda)s+K\lambda}$$

$$\begin{aligned} r &= W_{cl}^{-1}(s)[x_d] \\ &= x_d + \left((Js)G_{cl}^{-1}(s)\right)[x_d] \\ &= x_d + J\dot{\omega}_d \\ \tau &= Bx + J\dot{x}_d + G_c(s)[e] \\ &= Bx + J\dot{x}_d + (K+J\lambda)e + K\lambda\int e(\tau)d\tau \\ &= J(\dot{x}_d+\lambda e) + Bx + K\left(e+\lambda\int e(\tau)d\tau\right) = \theta^{*T}\phi(t) \end{aligned}$$

PI Control - Algebraic Part: Tracking - Complete Design



$$\begin{aligned} \tau &= Bx + J\dot{x}_d + G_c(s)[e] \\ &= Je_1(t) + Bx(t) + Ke_2(t) = \theta^{*T}\phi(t) \\ e_1 &= (\dot{x}_d + \lambda e)\,, \quad e_2 = \left(e + \lambda \int e(\tau)d\tau\right) \end{aligned}$$

$$\phi = \begin{bmatrix} e_1 & x & e_2 \end{bmatrix}^\top, \quad \theta^* = \begin{bmatrix} J & B & K \end{bmatrix}^\top$$

Adaptive PI control:

$$\tau = \widehat{J}(t)e_1 + \widehat{B}(t)x + Ke_2$$

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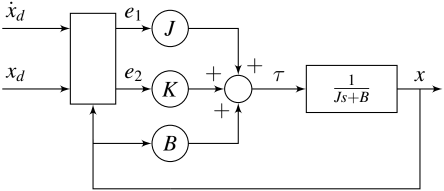

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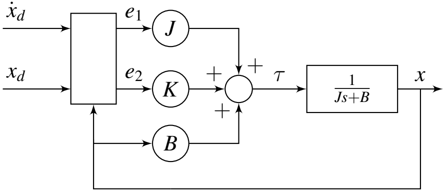

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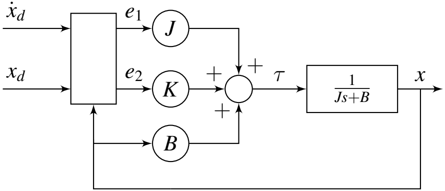

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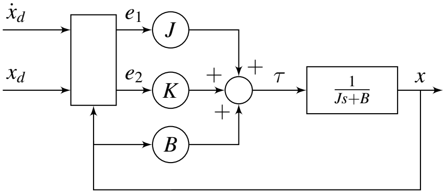

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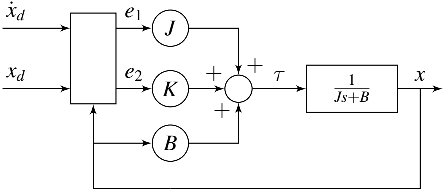

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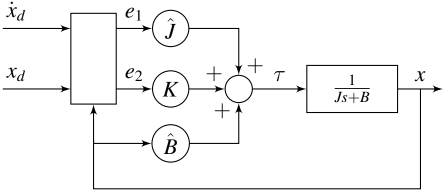

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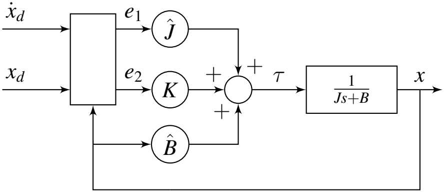
Adaptive PI Control

$$\tau = \widehat{J}(t)e_1 + \widehat{B}(t)x + Ke_2$$

Plant+controller:

$$\begin{aligned}
\dot{x} &= \frac{1}{J}\left(-Bx + \tau\right) \\
&= \frac{1}{J}\left(-Bx + \widehat{J}(t)e_1 + \widehat{B}(t)x + Ke_2\right) \\
e_2 &= \left(e + \lambda \int e(\tau)d\tau\right) \\
\dot{e}_2 &= \dot{e} + \lambda e = \dot{x}_d - \dot{x} + \lambda e \\
&= \dot{x}_d - \frac{1}{J}\left(\widehat{J}e_1 + \widetilde{B}x + Ke_2\right) + \lambda e \\
&= \left(1 - \frac{\widehat{J}}{J}\right)e_1 - \frac{1}{J}\left(\widetilde{B}x + Ke_2\right) \\
&= -\frac{K}{J}e_2 + \frac{1}{J}\left(-\widetilde{J}e_1 - \widetilde{B}x\right) - \text{Error Model 3}
\end{aligned}$$

Adaptive PI Control - Stability



$$\tau = \widehat{J}(t)e_1 + \widehat{B}(t)x + Ke_2$$

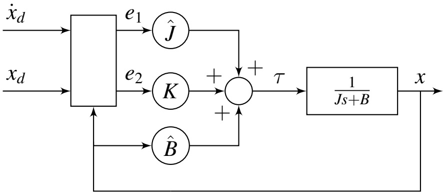
Error Equation: $\dot{e}_2 = -\frac{K}{J}e_2 + \frac{1}{J}\left(-\widetilde{J}e_1 - \widetilde{B}x\right)$

Adaptive Law: $\dot{\widetilde{J}} = \gamma_1 e_2 e_1, \qquad \dot{\widetilde{B}} = \gamma_1 e_2 x$

Lyapunov function: $V = \frac{1}{2}\left(e_2^2 + \frac{1}{J}\left(\frac{\widetilde{J}^2}{\gamma_1} + \frac{\widetilde{B}^2}{\gamma_2}\right)\right)$

$$\dot{V} = -\frac{K}{J}e_2^2$$

Adaptive PI Control - Stability and Asymptotic Tracking



- $e_2, \widetilde{J}, \widetilde{B} \in \mathcal{L}_\infty \implies \dot{e}_2 \in \mathcal{L}_\infty$
- $e_2 \in \mathcal{L}_\infty \implies e \in \mathcal{L}_\infty$
- $e \in \mathcal{L}_\infty \implies e_1 \in \mathcal{L}_\infty$
- Therefore $\dot{e}_2 \in \mathcal{L}_\infty; e_2 \in \mathcal{L}_2$
- $\implies \lim_{t\to\infty} e_2(t) = 0 \implies \lim_{t\to\infty} e(t) = 0$