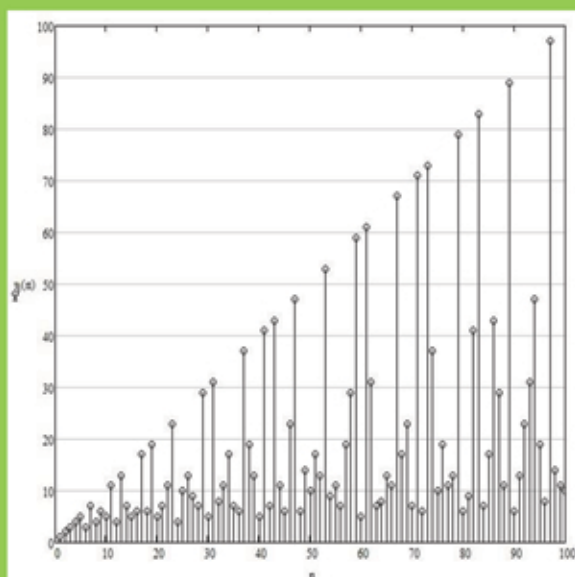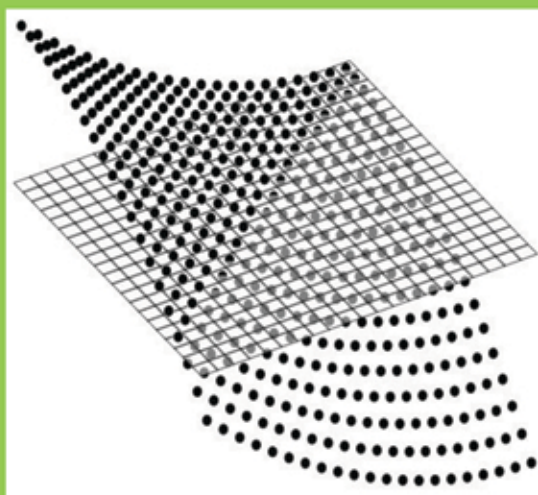# Solving Diophantine Equations

Octavian Cira and Florentin Smarandache

2014

Solving Diophantine Equations

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2

Preface

In recent times the we witnessed an explosion of Number Theory problems that are solved using mathematical software and powerful computers. The observation that the number of transistors packed on integrated circuits doubles every two years made by Gordon E. Moore in 1965 is still accurate to this day. With ever increasing computing power more and more mathematical problems can be tacked using brute force. At the same time the advances in mathematical software made tools like Maple, Mathematica, Matlab or Mathcad widely available and easy to use for the vast majority of the mathematical research community. This tools don't only perform complex computations at incredible speeds but also serve as a great tools for symbolic computation, as proving tools or algorithm design.

The online meeting of the two authors lead to lively exchange of ideas, solutions and observation on various Number Theory problems. The ever increasing number of results, solving techniques, approaches, and algorithms led to the the idea presenting the most important of them in in this volume. The book offers solutions to a multitude of η–Diophantine equation proposed by Florentin Smarandache in previous works [Smarandache, 1993, 1999b, 2006] over the past two decades. The expertise in tackling Number Theory problems with the aid of mathematical software such as [Cira and Cira, 2010], [Cira, 2013, 2014a, Cira and Smarandache, 2014, Cira, 2014b,c,d,e] played an important role in producing the algorithms and programs used to solve over 62 η–Diophantine equation. There are numerous other important publications related to Diophantine Equations

that offer various approaches and solutions. However, this book is different from other books of number theory since it dedicates most of its space to solving Diophantine Equations involving the Smarandache function. A search for similar results in online resources like *The On-Line Encyclopedia of Integer Sequences* reveals the lack of a concentrated effort in this direction.

The brute force approach for solving η–Diophantine equation is a well known technique that checks all the possible solutions against the problem constrains to select the correct results. Historically, the proof of concept was done by Appel and Haken [1977] when they published the proof for the *four color map* theorem. This is considered to be the the first theorem that was proven using a computer. The approach used both the computing power of machines as well as theoretical results that narrowed down infinite search space to 1936 map configurations that had to be check. Despite some controversy in the '80 when a masters student discovered a series of errors in the discharging procedure, the initial results was correct. Appel and Haken went on to publish a book [Appel and Haken, 1989] that contained the entire and correct prof that *every planar map is four-colorable*.

Recently, in 2014 an empirical results of Goldbach conjecture was published in Mathematics of Computation where Oliveira e Silva et al. [2013], [Oliveira e Silva, 2014], confirm the theorem to be true for all even numbers not larger than 4×10^{18}.

The use of Smarandache function η that involves the set of all prime numbers constitutes one of the main reasons why, most of the problems proposed in this book do not have a finite number of cases. It could be possible that the unsolved problems from this book could be classified in classes of unsolved problems, and thus solving a single problem will help in solving all the unsolved problems in its class. But the authors could not classify them in such classes. The interested readers might be able to do that. In the given circumstances the authors focused on providing the most comprehensive partial solution possible, similar to other such solutions in the literature like:

- Goldbach's conjecture. In 2003 Oliveira e Silva announced that all even numbers $\leq 2 \times 10^{16}$ can be expressed as a sum of two primes.

In 2014 the partial result was extended to all even numbers smaller then 4×10^{18}, [Oliveira e Silva, 2014].

- For any positive integer n, let $f(n)$ denote the number of solutions to the Diophantine equation $4/n = 1/x + 1/y + 1/z$ with x, y, z positive integers. The *Erdős-Straus conjecture*, [Obláth, 1950, Rosati, 1954, Bernstein, 1962, Tao, 2011], asserts that $f(n) \geq 1$ for every $n \geq 2$. Swett [2006] established that the conjecture is true for all integers for any $n \leq 10^{14}$. Elsholtz and Tao [2012] established some related results on f and related quantities, for instance established the bound $f(p) \ll p^{3/5} + O\big(1/\log(\log(p))\big)$ for all primes p.

- Tutescu [1996] stated that $\eta(n) \neq \eta(n + 1)$ for any $n \in \mathbb{N}^*$. On March 3rd, 2003 Weisstein published a paper stating that all the relation is valid for all numbers up to 10^9, [Sondow and Weisstein, 2014].

- A number n is k–hyperperfect for some integers k if $n = 1 + k \cdot s(n)$, where $s(n)$ is the sum of the proper divisors of n. All k–hyperperfect numbers less than 10^{11} have been computed. It seems that the conjecture "*all k–hyperperfect numbers for odd $k > 1$ are of the form $p^2 \cdot q$, with $p = (3k + 4)/4$ prime and $q = 3k + 4 = 2p + 3$ prime*" is false [McCranie, 2000].

This results do not offer the solutions to the problems but they are important contributions worth mentioning.

The emergence of mathematical software generated a new wave of mathematical research aided by computers. Nowadays it is almost impossible to conduct research in mathematics without using software solutions such as Maple, Mathematica, Matlab or Mathcad, etc. The authors used extensively Mathcad to explore and solve various Diophantine equations because of the very friendly nature of the interface and the powerful programming tools that this software provides. All the programs presented in the following chapters are in their complete syntax as used in Mathcad. The compact nature of the code and ease of interpretation made the choice of this particular software even more appropriate for use in a written presentation of solving techniques.

iv

The empirical search programs in this book where developed and executed in Mathcad. The source code of this algorithms can be interpreted as pseudo code (the Mathcad syntax allows users to write code that is very easy to read) and thus translated to other programming languages.

Although the intention of the authors was to provide the reader with a comprehensive book some of the notions are presented out of order. For example the book the primality test that used Smarandache's function is extensively used. The first occurrences of this test preceded the definition the actual functions and its properties. However, overall, the text covers all definition and proves for each mathematical construct used. At the same time the references point to the most recent publications in literature, while results are presented in full only when the number of solutions is reasonable. For all other problems, that generate in excess of 100 double, triple or quadruple pairs, only partial results are contained in the sections of this book. Nevertheless, anyone interested in the complete list should contact the authors to obtain a electronic copy of it. Running the programs in this book will also generate the same complete list of possible solutions for any odd the problems in this book.

<div align="right">Authors</div>

Acknowledgments

Contents

List of Figures

List of Tables

Introduction

A Diophantine equation is a linear equation $ax + by = c$ where $a, b, c \in \mathbb{Z}$ and the solutions x and y are also integer numbers. This equation can be completely solved by the well known algorithm proposed by Brahmagupta [Weisstein, 2014b].

In 1900, Hilbert wondered if there is an universal algorithm that solves the Diophantine equation, but Matiyasevich [1970] proved that such an algorithm does not exist for the first order solution.

The function η relates to each natural number n the smallest natural number m such that $m!$ is a multiple of n. In this book we aim to find analytical or empirical solutions to Diophantine and η–Diophantine equation, namely Diophantine equation that contain the Smarandache's η function, Smarandache [1980b].

An analytical solution implies a general solution that completely solves the problem. For example, the general solution for the equation $a \cdot x - b \cdot y = c$, with $a, b, c \in \mathbb{N}^*$ is $x_k = b \cdot k + x_0$ and $y_k = a \cdot k + y_0$, where (x_0, y_0) is a particular solution, and k is an integer, $k \geq \max\left\{\lceil -x_0/b \rceil, \lceil -y_0/a \rceil\right\}$.

By and empirical solution we understand a set of algorithms that determine the solutions of the Diophantine equation within a finite domain of integer numbers, dubbed *the search domain* to dimension d. For example, the η–Diophantine equation $\eta(m \cdot x + n) = x$ over the valid *search domain* of dimension $d = 3$, the solutions could be the triplets $(m, n, x) \in D_c = \{1, 2, \ldots, 1000\} \times \{1, 2, \ldots 1000\} \times \{1, 2, \ldots, 999\}$.

The first chapter introduces concepts about prime numbers, primality tests, decomposition algorithms for natural numbers, counting algorithms

for all natural numbers up to a real one, etc. This concepts are fundamental for validating the empirical solutions of the η–Diophantine equations.

The second chapter introduces the function η along side its known properties. This concepts allow the description of Kempner [1918] algorithm that computes the η function. The latter sections contain the set of commands and instructions that generate the file $\eta.prn$ which contains the $\eta(n)$ values for $n = 1, 2, \ldots, 10^6$.

The third chapter describes the division functions σ_0, σ_1, usually denoted by σ, σ_2 and s. The $\sigma_0(n)$ function counts the number of divisors of n, while $\sigma(n) = \sigma_1(n)$ returns the sun of all those divisors. Consequently $\sigma_2(n)$ computed the sum of squared divisors of n while, in general $\sigma_k(n)$ add all divisors to the power of k. We call divisors of n all natural numbers that divide n including 1 and n, thus the proper divisors are considered all natural divisors excluding n itself. In this case the function $s(n) = \sigma(n) - n$ is , in fact, the sun of all proper divisors. Along side the the definition, this third chapter also contains the properties and computing algorithms that generate the files $\sigma0.prn$, $\sigma1.prn$, $\sigma2.prn$, $s.prn$ that contain all the values for functions $\sigma_0(n)$, $\sigma(n)$, $\sigma_2(n)$ and $s(n)$ for $n = 1, 2, \ldots, 10^6$. The last section describes the k–perfect numbers.

Euler's totient function also known as the φ function that counts the natural numbers less than or equal to n that are relatively prime is described in chapter 4. As an example, for $n = 12$ the relatively prime factors are 1, 5, 7, and 11 because $(1, 12) = 1^1$, $(5, 12) = 1$, $(7, 12) = 1$, and $(11, 12) = 1$, thus $\varphi(12) = 4$. The chapter also describes the most important properties of this function. The latter section of the chapter contain the algorithm that generates the file $\varphi.prn$ that contains the values of the function φ for n raging from $1, 2, \ldots, 10^6$. Also, in this chapter the describes a generalization of Euler theorem relative to the totient function φ and the algorithm the computes the pair (s, m_s) that verifies the Diophantine equation $a^{\varphi(m_s)} \equiv a^s \ (mod \ m)$, where $a, m \in \mathbb{N}^*$.

In chapter 5 we define a function L which will allow us to (separately or simultaneously) generalize many theorems from Number Theory ob-

---

[1]where (m, n) is $gcd(m, n)$ that is the greatest common divisor of n and m

tained by Wilson, Fermat, Euler, Gauss, Lagrange, Leibniz, Moser, and Sierpinski.

Various analytical solutions to Diophantine equations such as: the second degree equation, the linear equation with n unknown, linear systems, the n degree equation with one unknown, Pell general equation, and the equation $x^2 - 2y^4 = 1$. For each of this cases, in chapter six we present symbolic computation that ensure the detection of the solutions for the particular Diophantine equations.

Chapter seven describes the solutions to the η–Diophantine equations using the search algorithms in the search domains.

The Conclusions and Index section conclude the book.

Chapter 1

Prime numbers

A prime number (or a prime) is a natural number greater than 1 that has no positive divisors other than 1 and itself. A natural number greater than 1 that is not a prime number is called a composite number. For example, 7 is prime because 1 and 7 are its only positive integer factors, whereas 10 is composite because it has the divisors 2 and 5 in addition to 1 and 10. The fundamental theorem of Arithmetics, [Hardy and Wright, 2008, p. 2-3], establishes the central role of primes in the Number Theory: any integer greater than 1 can be expressed as a product of primes that is unique up to ordering. The uniqueness in this theorem requires excluding 1 as a prime because one can include arbitrarily many instances of 1 in any factorization, e.g., 5, $1 \cdot 5$, $1 \cdot 1 \cdot 5$, etc. are all valid factorizations of 5, [Estermann, 1952, Vinogradov, 1955].

The property of being prime (or not) is called primality. A simple but slow method of verifying the primality of a given number n is known as trial division. It consists of testing whether n is a multiple of any integer between 2 and $\lfloor \sqrt{n} \rfloor$. The floor function $\lfloor x \rfloor$, also called the greatest integer function or integer value [Spanier and Oldham, 1987], gives the largest integer less than or equal to x. The name and symbol for the floor function were coined by Iverson, [Graham et al., 1994]. Algorithms much more efficient than trial division have been devised to test the primality of large

numbers. Particularly fast methods are available for numbers of special forms, such as Mersenne numbers. As of April 2014, the largest known prime number $2^{57885161} - 1$ has 17425170 decimal digits [Caldwell, 2014a].

There are infinitely many primes, as demonstrated by Euclid around 300 BC. There is no known useful formula that sets apart all of the prime numbers from composites. However, the distribution of primes, that is to say, the statistical behavior of primes in the large, can be modelled. The first result in that direction is the prime number theorem, proven at the end of the 19th century, which says that the probability that a given, randomly chosen number n is prime is inversely proportional to its number of digits, or to $\log(n)$.

Many questions around prime numbers remain open, such as Goldbach's conjecture, and the twin prime conjecture, Diophantine equations that have integer functions. Such questions spurred the development of various branches of the Number Theory, focusing on analytic or algebraic aspects of numbers. Prime numbers give rise to various generalizations in other mathematical domains, mainly algebra, such as prime elements and prime ideals.

1.1   Generating prime numbers

The generation of prime numbers can be done by means of several deterministic algorithms, known in the literature, as sieves: Sieve of Eratosthenes, Sieve of Euler, Sieve of Sundaram, Sieve of Atkin, etc. In this volume we will detail only the most efficient prime number generating algorithms.

The Sieve of Eratosthenes is an algorithm that allows the generation of all prime numbers up to a given limit $L \in \mathbb{N}^*$. The algorithm was given by Eratosthenes around 240 BC.

Program 1.1. Let us consider the origin of vectors and matrices 1, which can be defined in Mathcad by assigning $ORIGIN := 1$. The Sieve of Eratosthenes in the linear variant of Pritchard, presented in pseudo code in the article [Pritchard, 1987], written in Mathcad is:

$$
CEP(L) := \begin{vmatrix} for \ k \in 1..L \\ \quad is\_prime_k \leftarrow 1 \\ k \leftarrow 2 \\ while \ k^2 \leq L \\ \quad \begin{vmatrix} j \leftarrow k^2 \\ while \ j \leq L \\ \quad \begin{vmatrix} is\_primep_j \leftarrow 0 \\ j \leftarrow j + k \end{vmatrix} \\ k \leftarrow k + 1 \end{vmatrix} \\ j \leftarrow 1 \\ for \ k \in 1..L \\ \quad \begin{vmatrix} if \ is\_prime_k = 1 \\ \quad \begin{vmatrix} prime_j \leftarrow k \\ j \leftarrow j + 1 \end{vmatrix} \end{vmatrix} \\ return \ prime \end{vmatrix}
$$

It is well known that the segmented version of the Sieve of Eratosthenes, with basic optimizations, uses $O(L)$ operations and

$$
O\left(\sqrt{L} \frac{\log(\log(L))}{\log(L)} \right)
$$

bits of memory, [Pritchard, 1987, 1994].

The linear variant of the Sieve of Eratosthenes implemented by Pritchard, given by the code 1.1, has the inconvenience that is repeats uselessly operations. For example, for $L = 25$, in table (1.1) is given the binary vector is_prime which contains at each position the values 1 or 0. On the first line is the index of the vector.

1. Initially, all the positions of vector is_prime have the value 1.

2. For $q = 2$ the algorithm puts 0 on all the positions is_prime_k multiple of 2, for $k \geq q^2 = 4$.

3. For $q = 3$ the algorithm puts 0 on all the positions is_prime_k multiple of 3, for $k \geq q^2 = 9$, which means positions 9, 12, 15, 18, 21 and 24

$q \backslash is\_prime$	1	2	3	4	5	6	7	8	9	10	11	12
	0	1	1	1	1	1	1	1	1	1	1	1
2				0		0		0		0		0
3									0			0
4												
5												
	0	1	1	0	1	0	1	0	0	0	1	0

13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1	1	1	1	1	1	1	1	1	1	1
	0		0		0		0		0		0	
		0			0			0			0	
			0				0				0	
												0
1	0	0	0	1	0	1	0	0	0	1	0	0

Table 1.1: The vector is_prime in the code 1.1

but positions 12, 18 and 24 were already annulated in the previous step.

4. For $q = 4$ the algorithm puts 0 on all the positions is_prime_k multiple of 4, for $k \geq q^2 = 16$, which means positions 16, 20, 24, but these positions were annulated also in the second step, and on position 24 is taken 0 for the third time.

5. For $q = 5$ one takes $is\_prime_{q^2} = 0$.

Eventually, vector is_prime is read. The index of vector is_prime, which has the value 1, is a prime number. If we count the number of attributing the value 0, we remark that this operation was made 21 time. It is obvious that these repeated operations make the algorithm less efficient.

Program 1.2. This program is a better version of program 1.1 because it puts 0 only on the odd positions of the vector is_prime.

$$CEPi(L) := \begin{vmatrix} for\ k \in 3, 5..L \\ \quad is\_prime_k \leftarrow 1 \\ for\ k \in 3, 5..floor(\sqrt{L}) \\ \quad for\ j \in k^2, k^2 + 2k..L \\ \quad\quad is\_prime_j \leftarrow 0 \\ prime_1 \leftarrow 2 \\ j \leftarrow 2 \\ for\ k \in 1, 3..L \\ \quad if\ is\_prime_k = 1 \\ \quad\quad \begin{vmatrix} prime_j \leftarrow k \\ j \leftarrow j + 1 \end{vmatrix} \\ return\ prime \end{vmatrix}$$

Program 1.3. This program is a better version of program 1.2 because it uses a minimal memory space.

$$CEPm(L) := \begin{vmatrix} \lambda \leftarrow floor\left(\frac{L}{2}\right) \\ for\ k \in 1..\lambda \\ \quad is\_prime_k \leftarrow 1 \\ for\ k \in 3, 5..floor(\sqrt{L}) \\ \quad for\ j \in k^2, k^2 + 2k..L \\ \quad\quad is\_prime_{\frac{j-1}{2}} \leftarrow 0 \\ prime_1 \leftarrow 2 \\ j \leftarrow 2 \\ for\ k \in 1..\lambda - 1 \\ \quad if\ is\_prime_k = 1 \\ \quad\quad \begin{vmatrix} prime_j \leftarrow 2 \cdot k + 1 \\ j \leftarrow j + 1 \end{vmatrix} \\ return\ prime \end{vmatrix}$$

Even the execution time of the program 1.3 is a little longer than of the program 1.2, the best linear variant of the Sieve of Eratosthenes is the pro-

gram 1.3, as it provides an important memory economy (11270607 memory locations instead of 21270607, the amount of memory locations used by programs 1.1 and 1.2).

Program 1.4. The program for the Sieve of Eratosthenes, Pritchard variant, was improved in order to allow the number of repeated operations to diminish. The improvement consists in the fact that attributing 0 is done for only odd multiples of prime numbers. The program has a restriction, but which won't cause inconveniences, namely L must be a integer greater than 14.

$$
CEPb(L) := \left|
\begin{array}{l}
for\ k \in 3,5..L \\
\quad is\_prime_k \leftarrow 1 \\
prime \leftarrow (2\ 3\ 5\ 7)^{\mathrm{T}} \\
i \leftarrow last(prime) + 1 \\
for\ j \in 9,15..L \\
\quad is\_prime_j \leftarrow 0 \\
k \leftarrow 3 \\
s \leftarrow (prime_{k-1})^2 \\
t \leftarrow (prime_k)^2 \\
while\ t \le L \\
\quad \left|
\begin{array}{l}
for\ j \in t, t+2 \cdot prime_k..L \\
\quad is\_prime_j \leftarrow 0 \\
for\ j \in s+2, s+4..t-2 \\
\quad if\ is\_prime_j = 1 \\
\quad \quad \left|
\begin{array}{l}
prime_i \leftarrow j \\
i \leftarrow i+1
\end{array}
\right. \\
s \leftarrow t \\
k \leftarrow k+1 \\
t \leftarrow (prime_k)^2
\end{array}
\right. \\
for\ j \in s+2, s+4..L \\
\quad if\ is\_prime_j = 1 \\
\quad \quad \left|
\begin{array}{l}
prime_i \leftarrow j \\
i \leftarrow i+1
\end{array}
\right. \\
return\ prime
\end{array}
\right.
$$

We remark that it is not necessary to put 0 on each positions $(prime_k)^2 + prime_k$, as in the original version of the program 1.1, because the sum of two odd numbers is an even number and the even positions are not considered. In this moment of the program we are sure that the positions from $(prime_{k-1})^2 + 2$ to $(prime_k)^2 - 2$ of the vector *is_prime* (from 2 in 2) which were left on 1 (which means that their indexes are prime numbers), can be added to the prime numbers vector. Hence, instead of building the vector *prime* at the end of the markings, we do it in intermediary steps. The advantage consists on the fact that we have a list of prime numbers which can be used to obtain the other primes, up to the given limit L.

Program 1.5. The program that improves the program CEPb by halving the used memory space.

$$CEPbm(L) := \begin{vmatrix} \lambda \leftarrow floor\left(\frac{L}{2}\right) \\ for\ k \in 1..\lambda \\ \quad is\_prime_k \leftarrow 1 \\ prime \leftarrow (2\ 3\ 5\ 7)^{\mathrm{T}} \\ i \leftarrow last(prime) + 1 \\ for\ j \in 4, 7..\lambda \\ \quad is\_prime_j \leftarrow 0 \\ k \leftarrow 3 \\ s \leftarrow (prime_{k-1})^2 \\ t \leftarrow (prime_k)^2 \\ while\ t \leq L \\ \quad \begin{vmatrix} for\ j \in t, t + 2 \cdot prime_k..L \\ \quad is\_prime_{\frac{j-1}{2}} \leftarrow 0 \\ for\ j \in s + 2, s + 4..t - 2 \\ \quad if\ is\_prime_{\frac{j-1}{2}} = 1 \\ \quad \quad \begin{vmatrix} prime_i \leftarrow j \\ i \leftarrow i + 1 \end{vmatrix} \\ s \leftarrow t \\ k \leftarrow k + 1 \\ t \leftarrow (prime_k)^2 \end{vmatrix} \end{vmatrix}$$

$$\left| \begin{array}{l} for\ j \in s+2, s+4..L \\ \quad if\ is\_prime_{\frac{j-1}{2}}=1 \\ \qquad \left| \begin{array}{l} prime_i \leftarrow j \\ i \leftarrow i+1 \end{array} \right. \\ return\ prime \end{array} \right.$$

The performances of the 5 programs can be observed on the following execution sequences (the call of the programs have been done on the same computer and in similar conditions):

1. Call of the program CEP1.1, i.e. the Sieve of Eratosthenes in the linear version of Pritchard

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CEP(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 28.238s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999 \ ,$$

2. Call of the program CEPm1.3,

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CEPm(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 10.920s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999 \ .$$

3. Call of the program CEPi1.2,

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CEPi(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 7.231s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999 \ ,$$

4. Call of the program CEPb1.4,

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CEPb(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 5.064s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999$$

5. Call of the program CEPbm1.5,

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CEPb(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 7.133s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999$$

In the comparative table 1.2 are presented the attributions of 0, the execution times on a computer with a processor Intel of 2.20GHz with RAM of 4.00GB (3.46GB usable) and the number of memory units for the programs 1.1, 1.3, 1.2, 1.4 and 1.5.

program	Attributions of 0	Execution time	Memory used
1.1	71 760 995	28.238 sec	21 270 607
1.3	35 881 043	10.920 sec	11 270 607
1.2	35 881 043	7.231 sec	21 270 607
1.4	18 294 176	5.064 sec	21 270 607
1.5	18 294 176	7.133 sec	11 270 607

Table 1.2: Comparative table

The Sieve of Sundaram is a simple deterministic algorithm for finding the prime numbers up to a given natural number. This algorithm was presented by Sundaram and Aiyar [1934]. As it is known, the Sieve of Sundaram uses $O(L\log(L))$ operations in order to find the prime numbers up to L. The algorithm of the Sieve of Sundaram in pseudo code Mathcad is:

$$
CS(L) := \left|
\begin{array}{l}
m \leftarrow floor\left(\dfrac{L}{2}\right) \\
for\ k \in 1..m \\
\quad is\_prime_k \leftarrow 1 \\
for\ k \in 1..m \\
\quad for\ j \in 1..ceil\left(\dfrac{m-k}{2 \cdot k + 1}\right) \\
\qquad is\_prime_{k+j+2 \cdot k \cdot j} \leftarrow 0 \\
prime_1 \leftarrow 2 \\
j \leftarrow 1 \\
for\ k \in 1..m \\
\quad if\ is\_prime_k = 1 \\
\qquad \left|
\begin{array}{l}
j \leftarrow j + 1 \\
prime_j \leftarrow 2 \cdot k + 1
\end{array}
\right.
\end{array}
\right.
$$
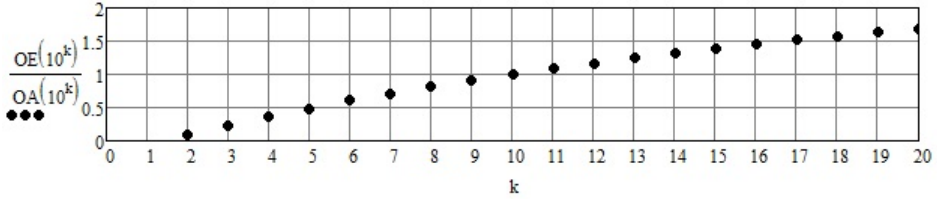
Figure 1.1: The ratio of the numbers of operations

$$\Big| return\ prime$$

The Call of the program CS

$$L := 2 \cdot 10^7 \quad t_0 := time(0) \quad p := CS(L) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = 32.706s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999$$

Until recently, i.e. till the appearance of the Sieve of Atkin, [Atkin and Bernstein, 2004], the Sieve of Eratosthenes was considered the most efficient algorithm that generates all the prime numbers up to a limit L. The figure 1.1 emphasize the graphic representation of the ratio between the number of operations needed for the Sieve of Eratosthenes, $OE(L) := O(L \cdot \log(\log(L)))$, and the number of operations needed for the Sieve of Atkin, $OA(L) := O(L/\log(\log(L)))$, for $L = 10^2, 10^3, \ldots, 10^{20}$. In this figure one can see that the Sieve of Atkin is better (relative to the number of operations needed by the program) then the Sieve of Eratosthenes, for $L > 10^{10}$.

Program 1.6. The Sieve of Atkin in pseudo code presented in Mathcad is:

$$Atkin(L) := \Big| \begin{array}{l} for\ k \in 5..L \\ \quad is\_prime_k \leftarrow 0 \\ for\ x \in 1..\sqrt{L} \\ \quad \Big| for\ y \in 1..\sqrt{L} \\ \quad \quad \Big| n \leftarrow 4x^2 + y^2 \\ \quad \quad if\ n \leq L \wedge \big(mod(n,12){=}1 \vee mod(n,12){=}5\big) \end{array}$$

$$is\_prime_n \leftarrow \neg is\_prime_n$$
$$n \leftarrow 3x^2 + y^2$$
$$if\ n \leq L \wedge mod(n, 12)\text{=}7$$
$$is\_prime_n \leftarrow \neg is\_prime_n$$
$$n \leftarrow 3x^2 + y^2$$
$$if\ x \neq y \wedge n \leq L \wedge mod(n, 12)\text{=}11$$
$$is\_prime_n \leftarrow \neg is\_prime_n$$
$$for\ n \in 5..\sqrt{L}$$
$$if\ is\_prime_n$$
$$for\ k \in 1.. \left\lfloor \frac{L}{n^2} \right\rfloor$$
$$is\_prime_{k \cdot n^2} \leftarrow 0$$
$$prime_1 \leftarrow 2$$
$$prime_2 \leftarrow 3$$
$$j \leftarrow 3$$
$$for\ n \in 5..L$$
$$if\ is\_prime_n$$
$$prime_j \leftarrow n$$
$$return\ prime$$

As it is known, this algorithm uses only $O(L/\log(\log(L)))$ simple operations and $O(L^{1/2+o(1)})$ memory locations, [Atkin and Bernstein, 2004].

Our implementation, in Mathcad, of Atkin's algorithm contains some remarks that make more performance program than the original algorithm.

1. Except 2 all even numbers are not prime, it follows that, with the initialization $is\_prime_{2k} \leftarrow 0$ for $k \in \{2, 3, \ldots, L/2\}$, there is no need to change the values of these components. Consequently, we will change only the odd components.

2. If j is odd then $4k^2 + j^2$ is always odd. It follows that the sequence

$$j \in \left\{1, 3.. \left\lfloor \sqrt{L} \right\rfloor\right\} \ \text{ and } \ k \in \left\{1, 2.. \left\lfloor \frac{\sqrt{L - j^2}}{2} \right\rfloor\right\} \tag{1.1}$$

assures that the number $4k^2 + j^2$ is always odd.

3. If j and k have different parities, Then $3k^2 + j^2$ is odd. Then the sequence

$$j \in \left\{ 1, 2, .. \left\lfloor \sqrt{L} \right\rfloor \right\}$$

$$\text{and } k_0 = mod(j, 2) + 1 , \quad k \in \left\{ k_0, k_0 + 2.. \left\lfloor \sqrt{\frac{L - j^2}{3}} \right\rfloor \right\} \quad (1.2)$$

assures that $3k^2 + j^2$ is odd.

4. If $k > j$ and k and j have different parities, then $3k^2 - j^2$ is odd. Then the sequence

$$j \in \left\{ 1, 2, .. \left\lfloor \sqrt{L} \right\rfloor \right\} \text{ and } k \in \left\{ j + 1, j + 3.. \left\lfloor \sqrt{\frac{L + j^2}{3}} \right\rfloor \right\} \quad (1.3)$$

assures that $3k^2 - j^2$ is odd.

5. Similarly as in 1, we will eliminate only the perfect squares for odd numbers ≥ 5, because only these are odd.

Program 1.7. AO program (Atkin optimized) of generating prime numbers up to L.

$$AO(L) := \begin{vmatrix} is\_prime_L \leftarrow 0 \\ \lambda \leftarrow floor\left(\sqrt{L}\right) \\ for\ j \in 1..ceil(\lambda) \\ \quad \begin{vmatrix} for\ k \in 1..ceil\left(\frac{\sqrt{L-j^2}}{2}\right) \\ \quad \begin{vmatrix} n \leftarrow 4k^2 + j^2 \\ m \leftarrow mod(n, 12) \\ is\_prime_n \leftarrow \neg is\_prime_n\ if\ n \leq L \wedge (m{=}1 \vee m{=}5) \end{vmatrix} \\ for\ k \in 1..ceil\left(\sqrt{\frac{L-j^2}{3}}\right) \end{vmatrix} \end{vmatrix}$$

$$\left|\left|\left|\begin{array}{l} n \leftarrow 3k^2 + j^2 \\ is\_prime_n \leftarrow \neg is\_prime_n \; if \; n \leq L \wedge mod(n, 12){=}7 \end{array}\right.\right.\right.$$

$$\left.\left|for \; k \in j+1..ceil\left(\sqrt{\frac{L+j^2}{3}}\right)\right.\right.$$

$$\left|\left|\begin{array}{l} n \leftarrow 3k^2 - j^2 \\ is\_prime_n \leftarrow \neg is\_prime_n \; if \; n \leq L \wedge mod(n, 12){=}11 \end{array}\right.\right.$$

$$for \; j \in 5, 7..\lambda$$
$$\quad for \; k \in 1, 3..\frac{L}{j^2} \; if \; is\_prime_j$$
$$\quad\quad is\_prime_{k \cdot j^2} \leftarrow 0$$

$$prime_1 \leftarrow 2$$
$$prime_2 \leftarrow 3$$
$$for \; n \in 5, 7..L$$
$$\quad if \; is\_prime_n$$
$$\quad\quad \left|\begin{array}{l} prime_j \leftarrow n \\ j \leftarrow j+1 \end{array}\right.$$
$$return \; prime$$

In this program function $ceil$ was used (which means $\lceil \cdot \rceil$) instead of function $floor$ (which means $\lfloor \cdot \rfloor$) in formulas (1.1), (1.2) and (1.3), in order to avoid errors of floating comma which could determine the loss of cases at limit L, for example, when L is a perfect square.

1. Call of the program 1.6 the Sieve of $Atkin$

$$L := 2 \cdot 10^7 \;\; t_0 := time(0) \;\; p := Atkin(L) \;\; t_1 := time(1)$$

$$(t_1 - t_0)s = 23.531s \;\; p_{last(p)} = 19999999 \;\; last(p) = 1270607 \;,$$

2. Call of the program 1.7 the optimized Sieve of $Atkin$

$$L := 2 \cdot 10^7 \;\; t_0 := time(0) \;\; p := AO(L) \;\; t_1 := time(1)$$

$$(t_1 - t_0)s = 19.45s \;\; p_{last(p)} = 19999999 \;\; last(p) = 1270607 \;,$$

There exists an implementation for the Sieve of Atkin, due to Bernstein [2014] under the name *Primgen*. *Primegen* is a library of programs for fast

generating prime numbers, increasingly. *Primegen* generates all 50847534 prime numbers up to 10^9 in only 8 seconds on a computer with a Pentium II-350 processor. *Primegen* can generate prime numbers up to 10^{15}.

1.2   Primality tests

A central problem in the Number Theory is to determine weather an odd integer is prime or not. The test than can establish this is called primality test.

Primality tests can be deterministic or non-deterministic. The deterministic ones establish exactly if a number is prime, while the non-deterministic ones can falsely determine that a composite number is prime. These test are much more faster then the deterministic ones. The numbers that pass a non-deterministic primality test are called *probably prime* (this is denoted by *prime?*) until their primality is deterministically proved. A list of *probably prime* numbers are Mersenne's numbers, [Caldwell, 2014b]:

$$M_{43} = 2^{30402457} - 1, \text{Dec. 2005 – Curtis Cooper and Steven Boone,}$$

$$M_{44} = 2^{32582657} - 1, \text{Sept. 2006 – Curtis Cooper and Steven Boone,}$$

$$M_{45} = 2^{37156667} - 1, \text{Sept. 2008 – Hans-Michael Elvenich,}$$

$$M_{46} = 2^{42643801} - 1, \text{Apr. 2009 – Odd Magnar Strindmo,}$$

$$M_{47} = 2^{43112609} - 1, \text{Aug. 2008 – Edson Smith,}$$

$$M_{48} = 2^{57885161} - 1, \text{Jan. 2013 – Curtis Cooper.}$$

1.2.1   The test of primality η

As seen in Theorem 2.3, we can use as primality test the computing of the value of η function. For $n > 4$, if relation $\eta(n) = n$ is satisfied, it follows that n is prime. In other words, the prime numbers (to which number 4 is added) are fixed points for η function. In this study we will use this primality test.

Program 1.8. The program for η primality test. The program returns the value 0 if the number is not prime and the value 1 if the number is prime. File $\eta.prn$ is read and assigned to vector η .

$$ORIGIN := 1 \quad \eta := READPRN(" \dots \backslash \eta.prn")$$

$$Tp\eta(n) := \begin{vmatrix} return \ "Error \ n < 1 \ or \ not \ integer" \ if \ n < 1 \lor n \neq trunc(n) \\ if \ n > 4 \\ \quad \begin{vmatrix} return \ 0 \ if \ \eta_n \neq n \\ return \ 1 \ otherwise \end{vmatrix} \\ otherwise \\ \quad \begin{vmatrix} return \ 0 \ if \ n{=}1 \lor n{=}4 \\ return \ 1 \ otherwise \end{vmatrix} \end{vmatrix}$$

By means of the program 1.8 was realized the following test.

$$n := 499999 \ \ k := 1..n \ \ v_k := 2 \cdot k + 1$$

$$last(v) = 499999 \ \ v_1 = 3 \ \ v_{last(v)} = 999999$$

$$t_0 := time(0) \ \ w_k := Tp\eta(v_k) \ \ t_1 := time(1)$$

$$(t_1 - t_0)sec = 0.304s \ \ \sum w = 78497 \ .$$

The number of prime numbers up to 10^6 is 78798, and the sum of non-zero components (equal to 1) is 78797, as 2 was not counted as prime number because it is an even number. We remark that the time needed by the primality test of all odd numbers is $0.304s$ a much more better time than the $8s$ necessary for the primality test 1.11 on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

1.2.2 Deterministic tests

Proving that an odd number n is prime can be done by testing sequentially the vector p that contains prime numbers.

The browsing of the list of prime numbers can be improved by means of the function that counts the prime numbers [Weisstein, 2014e]. Traditionally, by $\pi(x)$ is denoted the function that indicates the number of prime

numbers $p \leq x$, [Shanks, 1962, 1993, p. 15]. The notation for the function
that counts the prime numbers is a little bit inappropriate as it has nothing
to do with π, The universal constant that represents the ratio between the
length of a circle and its diameter. This notation was introduced by the
number theorist Edmund Landau in 1909 and has now become standard,
[Landau, 1958] [Derbyshire, 2004, p. 38]. We will give a famous result
of Rosser and Schoenfeld [1962], related to function $\pi(x)$. Let functions
π_s, $\pi_d : (1, +\infty) \to \mathbb{R}_+$ given by formulas

$$\pi_s(x) = \frac{x}{\ln(x)} \left(1 + \frac{1}{2\ln(x)}\right) \tag{1.4}$$

and

$$\pi_d(x) = \frac{x}{\ln(x)} \left(1 + \frac{3}{2\ln(x)}\right) \ . \tag{1.5}$$

Theorem 1.9. *Following inequalities*

$$\pi_s(x) < \pi(x) < \pi_d(x) \ , \tag{1.6}$$

*hold, for all $x > 1$, the right side inequality, and for all $x \geq 59$ the left side
inequality.*

Proof. See [Rosser and Schoenfeld, 1962, T. 1].   □

Let functions f, π_m, $\pi_M : \mathbb{N}^* \to \mathbb{N}^*$ be defined by formulas:

$$f(n) = \left\lfloor \frac{n}{\ln(n)} \left(1 + \frac{1}{2\ln(n)}\right) \right\rfloor \ ,$$

$$\pi_m(n) = \begin{cases} f(n) - 2 & \text{if } n < 11 \\[2mm] f(n) - 1 & \text{if } 11 \leq n \leq 39 \\[2mm] f(n) & \text{if } n > 39 \end{cases} \ , \tag{1.7}$$

$$\pi_M(n) = \left\lceil \frac{n}{\ln(n)} \left(1 + \frac{3}{2\ln(n)}\right) \right\rceil \ , \tag{1.8}$$

where $\lfloor \cdot \rfloor$ is the lower integer part function and $\lceil \cdot \rceil$ is the upper integer part function. As a consequence of Theorem 1.9 we have

Theorem 1.10. *Following inequalities*

$$\pi_m(n) < \pi(x) < \pi_M(n) \tag{1.9}$$

hold, for all $n \in 2\mathbb{N}^* + 1$, where by $2\mathbb{N}^* + 1$ is denoted the set of natural odd numbers.

Proof. As function $\pi_d(n) \le \pi_M(n)$ for all $n \in \mathbb{N}^*$, it results, according to Theorem 1.9, that the right side inequality is true for all $n \in \mathbb{N}^*$, hence, also for $n \in 2\mathbb{N}^* + 1$.
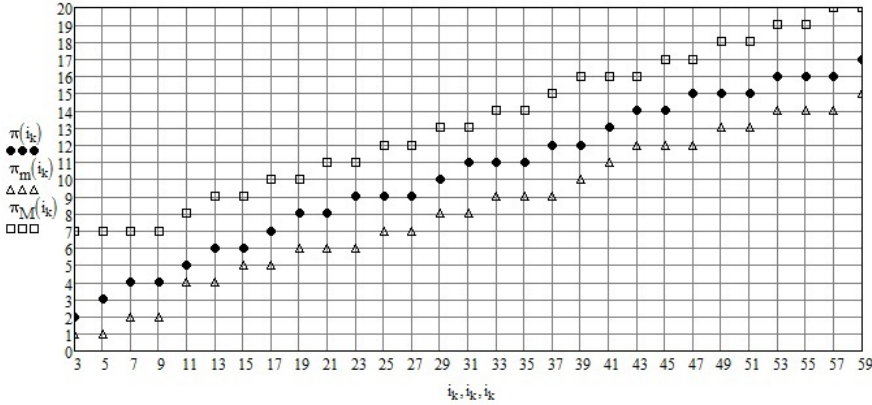
Figure 1.2: Functions $\pi_M(n)$, $\pi(n)$ and $\pi_m(n)$

As $\pi_m(n) \le \pi_s(n)$ for all $n \in \mathbb{N}^*$, and the left side inequality (1.6) holds for all $n \ge 59$, it follows that the left side inequality (1.9) holds for all $n \ge 59$.

For $n \in \{3, 5, 7, \ldots, 59\}$ we have:

$$
\begin{aligned}
\pi(3) - \pi_m(3) &= 1 & \pi(31) - \pi_m(31) &= 2 \\
\pi(5) - \pi_m(5) &= 1 & \pi(33) - \pi_m(33) &= 2 \\
\pi(7) - \pi_m(7) &= 2 & \pi(35) - \pi_m(35) &= 1 \\
\pi(9) - \pi_m(9) &= 1 & \pi(37) - \pi_m(37) &= 2 \\
\pi(11) - \pi_m(11) &= 1 & \pi(39) - \pi_m(39) &= 1 \\
\pi(13) - \pi_m(13) &= 1 & \pi(41) - \pi_m(41) &= 1 \\
\pi(15) - \pi_m(15) &= 1 & \pi(43) - \pi_m(43) &= 2 \\
\pi(17) - \pi_m(17) &= 1 & \pi(45) - \pi_m(45) &= 1 \\
\pi(19) - \pi_m(19) &= 2 & \pi(47) - \pi_m(47) &= 2 \\
\pi(21) - \pi_m(21) &= 1 & \pi(49) - \pi_m(49) &= 1 \\
\pi(23) - \pi_m(23) &= 2 & \pi(51) - \pi_m(51) &= 1 \\
\pi(25) - \pi_m(25) &= 2 & \pi(53) - \pi_m(53) &= 1 \\
\pi(27) - \pi_m(27) &= 1 & \pi(55) - \pi_m(55) &= 1 \\
\pi(29) - \pi_m(29) &= 2 & \pi(57) - \pi_m(57) &= 1 \\
 & & \pi(59) - \pi_m(59) &= 1
\end{aligned}
\tag{1.10}
$$

we analyze table 1.10 (see also 1.2) we can say that the left side inequality (1.9) holds for all $n \in 2\mathbb{N}^* + 1$.   □

Theorem 1.10 allows us to find a lower and an upper margin for the number of prime numbers up to the given odd number. Using the bisection method, one can efficiently determine if the given odd numbers is in the list of prime numbers or not.

The function that counts the maximum number of necessary tests for the bisection algorithm to decide if number N is prime, is given by the formula:

$$n_t(N) = \lceil \log_2 \left(\pi_M(N) - \pi_m(N) \right) \rceil \tag{1.11}$$

The algorithm is efficient. For example, for numbers N, $10^7 < N < 10^8$, the algorithm will proceed between 16 and 19 necessary tests for the bisection algorithm, at the worst (see figure 1.3).

For all programs we have considered $ORIGIN := 1$. By means of the algorithm 1.4 (The Sieve of Eratosthenes, Pritchard's improved version)
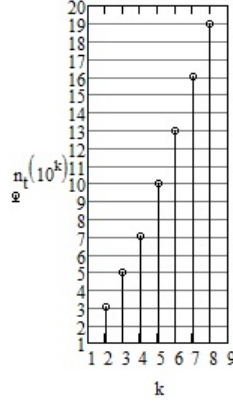
Figure 1.3: The graph of function $n_t(10^n)$ for $n = 2, 3, \ldots, 8$

and of command

$$p := CEPb(2 \cdot 10^7)$$

all prime numbers up to $2 \cdot 10^7$ are generated in vector p.

Program 1.11. The program is an efficient primality test for N. A binary search is used (the bisection algorithm), i.e., if N, which finds itself between the prime numbers p_ℓ and p_r, is in the list of prime numbers p.

$$Cb(N, \ell, r) := \begin{vmatrix} while \ \ell < r \\ \quad \begin{vmatrix} M \leftarrow \dfrac{\ell + r}{2} \\ m \leftarrow ceil\,(M) \\ return \ 1 \ \ if \ N{=}p_m \\ \ell \leftarrow m \ \ if \ N > p_m \\ r \leftarrow floor\,(M) \ \ if \ N < p_m \end{vmatrix} \\ return \ 0 \end{vmatrix}$$

The subprogram 1.11 calls the components p_k of the vector that contains the prime numbers. If N is prime, the subprogram returns 1, if N is not prime, it returns 0. The necessary time to test the primality of all odd numbers up to 10^6 is $8.283sec$ on a 2.2 GHz processor.

Other deterministic tests:

1. Pepin's test or the $p-1$ test. If we study attentively a list that contains the greatest known prim numbers, p, we will remark that most of them has a particular form, namely, $p-1$ or $p+1$ and can be decomposed very fast. This result is not unexpected as there exist deterministic primality tests for such numbers. In 1891, Lucas, [Williams, 1998], has converted the Fermat's Little Theorem into a practical primality test, improved afterwards by Kraitchik and Lehmer [Brillhart et al., 1975], [Dan, 2005].

2. n+1 tests or Lucas-Lehmer test for Mersenne numbers. Approximately half of the prime numbers in the list that contains the greatest known prim numbers are of the form $N-1$, where N can be easily factorized.

   Program 1.12. The program for Lucas-Lehmer algorithm is:

$$LL(n) := \left| \begin{array}{l} return \,"Error \, n < 3 \, or \, n > 53" \, if \, n \leq 2 \vee n \geq 54 \\ M \leftarrow 2^n - 1 \\ f \leftarrow Fa(n) \\ return \, (M \,"is \, not \, prime") \, if \, (f_{1,1})^2 < n \\ s \leftarrow 4 \\ for \, k \in 1..n-2 \\ \quad \left| \begin{array}{l} S \leftarrow s^2 - 2 \\ s \leftarrow mod(S, M) \\ return \,"Error" \, if \, floor\left(\frac{S}{M}\right) \cdot M + s \neq S \end{array} \right. \\ return \, (M \,"is \, prime") \, if \, s{=}0 \\ return \, (M \,"is \, prime") \, otherwise \end{array} \right.$$

Run examples:

$$LL(11) = (2047 \,"is \, not \, prime") \quad LL(13) = (8191 \,"is \, prime")$$

$$LL(19) = (524287 \,"is \, prime") \quad LL(23) = (8388607 \,"is \, not \, prime") \,.$$

3. The Miller-Rabin test. If we apply the Miller's test for numbers lesser than $2.5 \cdot 10^{10}$ but different from 3215031751, and they pass the test for basis 2, 3, 5 and 7, they are prime. Similarly, if we apply a test in seven steps, the previously obtained results allow to verify the primality of all prime numbers up to $3.4 \cdot 10^{14}$. If we choose 25 iterations for Miller's algorithm applied to a number, the probability that this is not composite is lesser than 2^{-50}. Hence, the Miller-Rabin test becomes a deterministic test for numbers lesser than $3.4 \cdot 10^{10}$,[Dan, 2005].

Program 1.13. The program for Miller-Rabin test is:

$$
MR(n) := \left| \begin{array}{l} return \; "Error \; n < 2 \; or \; n \; even" \; if \; n < 2 \vee mod(n,2) = 1 \\ s \leftarrow 0 \\ t \leftarrow n - 1 \\ while \; mod(t,2) = 0 \\ \quad \left| \begin{array}{l} s \leftarrow s + 1 \\ t \leftarrow \frac{t}{2} \end{array} \right. \\ \lambda \leftarrow \frac{\sqrt{n}}{2} \\ for \; k \in 1..25 \\ \quad \left| \begin{array}{l} b \leftarrow 2 + 2 \cdot floor(rnd(\lambda)) + 1 \\ y \leftarrow RRP(b,t,n) \\ if \; y \neq 1 \wedge y \neq n - 1 \\ \quad \left| \begin{array}{l} j \leftarrow 1 \\ while \; j \leq s - 1 \wedge j \neq n - 1 \\ \quad \left| \begin{array}{l} y \leftarrow mod(y^2,n) \\ return \; 0 \; if \; y{=}1 \\ j \leftarrow j + 1 \end{array} \right. \\ return \; 0 \; if \; y \neq n - 1 \end{array} \right. \\ return \; 1 \end{array} \right.
$$

The test of the program ha been made for $n = 2^{47} - 1 > 3.4 \cdot 10^{10}$ and cu $n = 2^{19} - 1$.

$$MR(2^{47} - 1) = 0 \quad MR(2^{19} - 1) = 1$$

$n = 2^{47} - 1$ is indeed a composite number

$$Fa(2^{47} - 1) = \begin{pmatrix} 2351 & 1 \\ 4513 & 1 \\ 13264529 & 1 \end{pmatrix} ,$$

and $2^{19} - 1 = 524287$ is a prime number. For factorization of a natural numbers has been done with the programs Fa, 1.29, emphasized in Section 1.3.1 .

The program MR calls the program RRP for repeatedly squaring modulo m, i.e. it calculates $mod(b^n, m)$ for great numbers.

$$RRP(b, n, m) := \left| \begin{array}{l} N \leftarrow 1 \\ return\ N\ if\ n\text{=}0 \\ A \leftarrow b \\ a \leftarrow Cb2(n) \\ N \leftarrow b\ if\ a_0\text{=}1 \\ for\ k \in 1..last(a) \\ \quad \left| \begin{array}{l} A \leftarrow mod(A^2, m) \\ N \leftarrow mod(A \cdot N, m)\ if\ a_k\text{=}1 \end{array} \right. \\ return\ N \end{array} \right.$$

The test of this program has been made on following example:

$$RRP(5, 596, 1234) = 1013 ,$$

provided in the paper [Dan, 2005, p. 60]. Concerning this program, it calls a program for finding the digits of basis 2 for a decimal number.

$$Cb2(n) := \left| \begin{array}{l} j \leftarrow 0 \\ c_0 \leftarrow n \\ while\ trunc\left(\dfrac{c_j}{2}\right) \text{=}0 \\ \quad \left| \begin{array}{l} r_j \leftarrow mod(c_j, 2) \\ j \leftarrow j + 1 \end{array} \right. \end{array} \right.$$

$$\left| \; \left| c_j \leftarrow trunc\left(\frac{c_{j-1}}{2}\right) \right. \right.$$
$$r_j \leftarrow c_j$$
$$return \; r$$

The test of this program is made by following example:

$$Cb2(107) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

4. AKS test. Agrawal, Kayal and Saxena, [Agrawal et al., 2004], have found a deterministic algorithm, relative easy, that isn't based on any unproved statement. The idea of AKS test results form a simple version of the Fermat's Little Theorem . The AKS algorithm is:

INPUT a natural number > 2;

OUTPUT 0 if n is not prime, 1 if n is prime;

1. If n is of the form a^b, with $b > 1$, then return: n is not prime and stop the algorithm.

2. Let $r \leftarrow 2$.

3. As long as $r < n$; execute:

   3.1. If $(n, r) \neq 1$, return: n is not prime and stop the algorithm.

   3.2. If $r \geq 2$ and it is prime, then execute: let q be the greatest factor of $r - 1$, then, if $q > 4\sqrt{r}\lg(n)$ and $n^{(r-1)/q} \neq 1$ $(mod \; r)$, then go to item 4.

   3.3. Let $r \leftarrow r + 1$.

4. For a from 1 to $2\sqrt{r}\lg(n)$, execute:

   4.1. If $(x - a)^n \neq x^n - a \; (mod \; x^r - 1, n)$, then return: n is not prime and stop the algorithm.

5. Return: n is prime and stop the algorithm.

1.2.3   Smarandache's criteria of primality

In this section we present four necessary and sufficient conditions for a natural number to be prime, [Smarandache, 1981b].

Definition 1.14. We say that integers a are b congruent modulo m (denoted $a \equiv b \,(mod\, m)$) if and only if $m \mid a - b$ (i.e. m divides $a - b$) or $a - b = k \cdot m$, where $k \in \mathbb{Z}$, $k \neq 1$ and $k \neq m$ (i.e. m is a proper factor of $a - b$). Therefore, we have

$$a \equiv b \,(mod\, m) \Leftrightarrow mod(a - b, m) = 0 \,, \qquad (1.12)$$

where $mod(x, y)$ is the function that returns the rest of the division of x by y, with $x, y \in \mathbb{Z}$.

In 1640 Fermat shows without demonstrate the following theorem:

Theorem 1.15 (Fermat)**.** *If $a \in \mathbb{N}$ and p is prime and $p \nmid a$, then*

$$a^{p-1} \equiv 1 \,(mod\, p) \ .$$

The first proof of the this theorem was given in 1736 by Euler.

Theorem 1.16 (Wilson)**.** *If p is prime, then*

$$(p - 1)! + 1 \equiv 0 \,(mod\, p) \ .$$

The theorem Wilson 1.16 was published by Waring [1770], but it was known long before even Leibniz.

Theorem 1.17. *Let $p \in \mathbb{N}^*$, $p \geq 3$, then p is prime if and only if*

$$(p - 3)! \equiv \frac{p - 1}{2} \,(mod\, p) \ . \qquad (1.13)$$

Proof.
  Necessity: p is prime $\Rightarrow (p - 1)! \equiv -1 \,(mod\, p)$ conform to Wilson's theorem 1.16. It results that $(p-1)(p-2)(p-3)! \equiv -1 \,(mod\, p)$, or $2(p-3)! \equiv p - 1 \,(mod\, p)$. But p being a prime number ≥ 3 it results that $(2, p) = 1$

and $(p-1)/2 \in \mathbb{Z}$. It has sense the division of the congruence by 2, and therefore we obtain the conclusion.

Sufficiency: We multiply the congruence $(p-3)! \equiv (p-1)/2 \,(mod\,p)$ with $(p-1)(p-2) \equiv 2 \,(mod\,p)$, [Popovici, 1973, pp. 10-16], and it results that $(p-1)! \equiv -1 \,(mod\,p)$ from Wilson's theorem 1.16, which makes that p is prime. $\square$

Program 1.18. The primality criterion (1.13), given by Theorem 1.17 can be implemented in Mathcad as follows:

$$CSP1(p) := \left| \begin{array}{l} return\ -1\ if\ p < 3 \vee p \neq trunc(p) \\ return\ 1\ if\ mod\left[(p-3)! - \dfrac{p-1}{2}, p\right] = 0 \\ return\ 0\ otherwise \end{array} \right.$$

The call of this criterion using the symbolic computation is:

$$
\begin{array}{lcr}
CSP1(2) & \rightarrow & -1\,, \\
CSP1(3.5) & \rightarrow & -1\,, \\
CSP1(61) & \rightarrow & 1\,, \\
CSP1(87) & \rightarrow & 0\,, \\
CSP1(127) & \rightarrow & 1\,, \\
CSP1(1057) & \rightarrow & 0\,,
\end{array}
$$

where 1 indicates that the number is prime, 0 the contrary and -1 error, i.e. $p < 3$ or p is not integer.

Lemma 1.19. *Let m be a natural number > 4. Then m is a composite number if and only if $(m-1)! \equiv 0 \,(mod\,m)$.*

Proof.

The *sufficiency* is evident conform to Wilson's theorem 1.16.

Necessity: m can be written as $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ where p_i prime numbers, two by two distinct and $\alpha_i \in \mathbb{N}^*$, for any $i \in I_s = \{1, 2, \ldots, s\}$.

If $s \neq 1$ then $p_i^{\alpha_i} < m$, for any $i \in I_s$. Therefore $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ are distinct factors in the product $(m-1)!$, thus $(m-1)! \equiv 0 \,(mod\,m)$.

If $s = 1$ then $m = p^\alpha$ with $\alpha \geq 2$ (because non-prime). When $\alpha = 2$ we have $p < m$ and $2p < m$ because $m > 4$. It results that p and $2p$ are different factors in $(m - 1)!$ and therefore $(m - 1)! \equiv 0 \, (mod \, m)$. When $\alpha > 2$, we have $p < m$ and $p^{\alpha-1} < m$, and p and $p^{\alpha-1}$ are different factors in product $(m - 1)!$.

Therefore $(m - 1)! \equiv 0 \, (mod \, m)$ and the lemma is proved for all cases.

$\square$

Theorem 1.20. *Let p be a natural number $p > 4$. Then p is prime if and only if*

$$(p - 4)! \equiv (-1)^{\left[\frac{p}{3}\right]+1} \cdot \left[\frac{p+1}{6}\right] (mod \, p) \,, \qquad (1.14)$$

where $[x]$ is the integer part of x, i.e. the largest integer less than or equal to x.

Proof.

Necessity: $(p - 4)!(p - 3)(p - 2)(p - 1) \equiv -1 \, (mod \, p)$ from Wilson's theorem 1.16, or $6(p - 4)! \equiv 1 \, (mod \, p)$; p being prime and greater than 4, it results that $(6, p) = 1$. It results that $p = 6k \pm 1$, with $k \in \mathbb{N}^*$.

1. If $p = 6k - 1$, then $6 \mid (p + 1)$ and $(6, p) = 1$, and dividing the congruence $6(p - 4)! \equiv p + 1 \, (mod \, p)$, which is equivalent with the initial one, by 6 we obtain:

$$(p - 4)! \equiv \frac{p + 1}{6} \equiv (-1)^{\left[\frac{p}{3}\right]+1} \cdot \left[\frac{p+1}{6}\right] (mod \, p) \,.$$

2. If $p = 6k + 1$, then $6 \mid (1 - p)$ and $(6, p) = 1$, and dividing the congruence $6(p - 4)! \equiv 1 - p \, (mod \, p)$, which is equivalent to the initial one, by 6 it results:

$$(p - 4)! \equiv \frac{1 - p}{6} \equiv -k \equiv (-1)^{\left[\frac{p}{3}\right]+1} \cdot \left[\frac{p+1}{6}\right] (mod \, p) \,.$$

Sufficiency: We must prove that p is prime. First of all we'll show that $p \neq \mathcal{M}6$. Let's suppose by absurd that $p = 6k$, $k \in \mathbb{N}^*$. By substituting

in the congruence from hypothesis, it results $(6k - 4)! \equiv -k \,(mod\,6k)$. From the inequality $6k - 5 \geq k$ for $k \in \mathbb{N}^*$, it results that $k \mid (6k - 5)!$. From $22 \mid (6k - 4)$, it results that $2k \mid (6k - 5)!(6k - 4)$. Therefore $2k \mid (6k - 4)!$ and $2k \mid 6k$, it results (conform with the congruencies' property), [Popovici, 1973, pp. 9-26], that $2k \mid (-k)$, which is not true; and therefore $p \neq \mathcal{M}6$.

From $(p - 1)(p - 2)(p - 3) \equiv -6 \,(mod\,p)$ by multiplying it with the initial congruence it results that:

$$(p - 1)! \equiv (-1)^{\left[\frac{p}{3}\right]} \cdot 6 \cdot \left[\frac{p + 1}{6}\right] (mod\,p) \ .$$

Let's consider lemma 1.19, for $p > 4$ we have:

$$(p - 1)! \equiv \begin{cases} 0 & (mod\,p) \quad \text{if } p \text{ is not prime;} \\ -1 & (mod\,p) \quad \text{if } p \text{ is prime;} \end{cases}$$

1. If $p = 6k + 2 \Rightarrow (p - 1)! \equiv 6k \not\equiv 0 \,(mod\,p)$.

2. If $p = 6k + 3 \Rightarrow (p - 1)! \equiv -6k \not\equiv 0 \,(mod\,p)$.

3. If $p = 6k + 4 \Rightarrow (p - 1)! \equiv -6k \not\equiv 0 \,(mod\,p)$.

Thus $p \neq \mathcal{M}6 + r$ with $r \in \{0, 2, 3, 4\}$. It results that p is of the form: $p = 6k \pm 1$, $k \in \mathbb{N}^*$ and then we have: $(p - 1)! \equiv -1 \,(mod\,p)$, which means that p is prime. $\qquad\square$

Program 1.21. The primality criterion (1.14), given by Theorem 1.20 can be implemented in Mathcad as follows:

$$CSP2(p) := \left| \begin{array}{l} return \ -1 \ if \ p < 5 \lor p \neq trunc(p) \\ m \leftarrow trunc\left(\dfrac{p}{3}\right) + 1 \\ n \leftarrow trunc\left(\dfrac{p + 1}{6}\right) \\ return \ 1 \ if \ mod\,[(p - 4)! - (-1)^m \cdot n, p] = 0 \\ return \ 0 \ otherwise \end{array} \right.$$

The call of this criterion using the symbolic computation is:

$$
\begin{aligned}
CSP2(4)   &\rightarrow -1\,, \\
CSP2(5.5)   &\rightarrow -1\,, \\
CSP2(61)   &\rightarrow 1\,, \\
CSP2(87)   &\rightarrow 0\,, \\
CSP2(127)   &\rightarrow 1\,, \\
CSP2(1057)   &\rightarrow 0\,,
\end{aligned}
$$

where 1 indicates that the number is prime, 0 the contrary and -1 error, i.e. $p < 5$ or p is not integer.

Theorem 1.22. *If p is a natural number $p \geq 5$, then p is prime if and only if*

$$
(p - 5)! \equiv r \cdot h + \frac{r^2 - 1}{24} \, (mod \ p) \,, \tag{1.15}
$$

where

$$
h = \left[\frac{p}{24} \right] \quad \text{and} \quad r = p - 24h \,.
$$

Proof.

   Necessity: if p is prime, it results that:

$$
(p - 5)!(p - 4)(p - 3)(p - 2)(p - 1) \equiv -1 \, (mod \ p)
$$

or

$$
24(p - 5)! \equiv -1 \, (mod \ p) \,.
$$

But p could be written as $p = 24h + r$, with $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$, because it is prime. It can be easily verified that

$$
\frac{r^2 - 1}{24} \in \{0, 1, 2, 5, 7, 12, 15, 22\} \subset \mathbb{Z} \,.
$$

$$
24(p - 5)! \equiv -1 + r(24h + r) \equiv 24rh + r^2 - 1 \, (mod \ p)
$$

Because $(24, p) = 1$ and $24 \mid (r^2 - 1)$ we can divide the congruence by 24, obtaining:

$$
(p - 5)! \equiv rh + \frac{r^2 - 1}{24} \, (mod \ p) \,.
$$

Sufficiency: p can be written $p = 24h + r$, $h, r \in \mathbb{N}$, $0 \le r < 24$. Multiplying the congruence $(p-4)(p-3)(p-2)(p-1) \equiv 24 \, (mod \, p)$ with the initial one, we obtain: $(p-1)! \equiv r(24h + r) - 1 \equiv -1 \, (mod \, p)$. $\qquad \square$

Program 1.23. The implementation of the primality criterion (1.15) given by Theorem 1.22 is:

$$CSP3(p) := \left| \begin{array}{l} return \; -1 \; if \; p < 5 \lor p \ne trunc(p) \\ h \leftarrow trunc\left(\dfrac{p}{24}\right) \\ r \leftarrow p - 24 \cdot h \\ return \; 1 \; if \; mod\left[(p-5)! - \left(r \cdot h + \dfrac{r^2 - 1}{24}\right), p\right] = 0 \\ return \; 0 \; otherwise \end{array} \right.$$

The call of this criterion using the symbolic computation is:

$$\begin{array}{rcl} CSP3(4) & \to & -1 \, , \\ CSP3(5.5) & \to & -1 \, , \\ CSP3(61) & \to & 1 \, , \\ CSP3(87) & \to & 0 \, , \\ CSP3(127) & \to & 1 \, , \\ CSP3(1057) & \to & 0 \, , \end{array}$$

where 1 indicates that the number is prime, 0 the contrary and -1 error, i.e. $p < 5$ or p is not integer.

Theorem 1.24. *Let's consider* $p = (k-1)! \cdot h \pm 1$, *with* $k > 2$ *a natural number. Then* p *is prime if and only if*

$$(p-k)! \equiv (-1)^{k + \left[\frac{p}{h}\right] + 1} \cdot h \, (mod \, p) \; . \qquad (1.16)$$

Proof.

Necessity: If p is prime then, according to Wilson's theorem 1.16, results that $(p-1)! \equiv -1 \, (mod \, p) \Leftrightarrow (-1)^{k-1}(p-k)!(k-1)! \equiv -1 \, (mod \, p) \Leftrightarrow (p-k)!(k-1)! \equiv (-1)^k \, (mod \, p)$. We have:

$$((k-1)!, p) = 1 \; . \qquad (1.17)$$

(A) $p = (k-1)! \cdot h - 1$.

  (a) k is an even number $\Rightarrow (p-k)!(k-1)! \equiv 1 + p \, (mod \, p)$, and because of the relation (1.17) and $(k-1)! \mid (1+p)$, by dividing with $(k-1)!$ we have: $(p-k)! \equiv h \, (mod \, p)$.

  (b) k is an odd number $\Rightarrow (p-k)!(k-1)! \equiv -1 - p \, (mod \, p)$, and because of the relation (1.17) and $(k-1)! \mid (-1-p)$, by dividing with $(k-1)!$ we have: $(p-k)! \equiv -h \, (mod \, p)$.

(B) $p = (k-1)! \cdot h + 1$.

  (a) k is an even number $\Rightarrow (p-k)!(k-1)! \equiv 1 - p \, (mod \, p)$, and because $(k-1)! \mid (1-p)$ and of the relation (1.17), by dividing with $(k-1)!$ we have: $(p-k)! \equiv -h \, (mod \, p)$.

  (b) k is an odd number $\Rightarrow (p-k)!(k-1)! \equiv -1 + p \, (mod \, p)$, and because $(k-1)! \mid (-1+p)$ and of the relation (1.17), by dividing with $(k-1)!$ we have $(p-k)! \equiv h \, (mod \, p)$.

Putting together all these cases, we obtain: if p is prime, $p = (k-1)! \cdot h \pm 1$, with $k > 2$ and $h \in \mathbb{N}^*$, then the relation (1.16) is true.

Sufficiency: Multiplying the relation (1.16) by $(k-1)!$ it results that:

$$(p-k)!(k-1)! \equiv (k-1)! \cdot h \cdot (-1)^{\left[\frac{p}{h}\right]+1} \cdot (-1)^k \, (mod \, p) \ .$$

Analyzing separately each of these cases:

(A) $p = (k-1)! \cdot h - 1$ and

(B) $p = (k-1)! \cdot h + 1$, we obtain for both, the congruence:

$$(p-k)!(k-1)! \equiv (-1)^k \, (mod \, p)$$

which is equivalent (as we showed it at the beginning of this proof) with $(p-1)! \equiv -1 \, (mod \, p)$ and it results that p is prime. $\square$

Program 1.25. The implementation of the primality criterion given by (1.16) using the symbolic computation is:

$$CSP4(p) := \begin{vmatrix} return \ -1 \ if \ p < 2 \vee p \neq trunc(p) \\ return \ 1 \ if \ p{=}2 \\ h \leftarrow 0 \\ j \leftarrow 3 \\ while \ (j-1)! \leq p+1 \\ \quad \begin{vmatrix} if \ mod[p+1,(j-1)!]{=}0 \\ \quad \begin{vmatrix} h \leftarrow \dfrac{p+1}{(j-1)!} \\ k \leftarrow j \end{vmatrix} \\ j \leftarrow j+1 \end{vmatrix} \\ return \ 0 \ if \ h{=}0 \\ return \ 1 \ if \ mod\left[(p-k)! - (-1)^{k+trunc\left(\frac{p}{h}\right)+1} \cdot h, p\right]{=}0 \\ return \ 0 \ otherwise \end{vmatrix}$$

The test of the program 1.25 has been done as follows. We know that we have 24 odd prime numbers up to 99. Vector I of odd numbers from 3 to 99 was generated with the sequence:

$$ORIGIN := 2 \quad j := 2..50 \quad I_j := 2 \cdot j - 1 \ .$$

For each component of vector I program 1.25 was called and the result was assigned to vector v. As the values of vector v are 1 for prime numbers and 0 for non-prime numbers, it follows that the sum of the components of vector v will give the number of prime numbers. If this sum is 24, it follows that criterion 1.16 and program 1.25 are correct for all odd numbers up to 99.

$$v_j := CSP4(I_j) \quad \sum v = 24 \ .$$

The call of this criterion using the symbolic computation is:

$$\begin{aligned} CSP4(1) \quad &\rightarrow \quad -1 \ , \\ CSP4(2) \quad &\rightarrow \quad 1 \ , \\ CSP4(3.5) \quad &\rightarrow \quad -1 \ , \\ CSP4(47) \quad &\rightarrow \quad 1 \ , \\ CSP4(147) \quad &\rightarrow \quad 0 \ , \\ CSP4(149) \quad &\rightarrow \quad 1 \ , \\ CSP4(150) \quad &\rightarrow \quad 0 \ . \end{aligned}$$

where 1 indicates that the number is prime, 0 the contrary and -1 error, i.e. $p < 2$ or p is not integer.

1.3   Decomposition product of prime factors

The factorization problem of integers is: given a positive integer n let find its prime factors, which means the pairs (p_i, α_i), p_i are distinct prime numbers and α_i are positive integers, such that $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

In the Number Theory, the factorization of integers is the process of finding the divisors of a given composite number. This seems to be a trivial problem, but for huge numbers there doesn't exist any efficient factorization algorithm, the most efficient algorithm has an exponential complexity, relative to the numbers of digits. Hence, a factorization experiment of a number containing 200 decimal digits was successfully ended only after several months. In this experiment were used 80 computers Opteron processor of 2.2 GHz, connected in a network of Gigabit type.

Many algorithms were conceived to determine the prime factors of a given number. They can vary very little in sophistication and complexity. It is very difficult to build a general algorithm for this "complex" computing problem, such that any additional information about the number or its factors can be often useful to save an important amount of time.

The algorithms for factorizing an integer n can be divided into two types:

1. General algorithms. Algorithm trial division is:

INPUT $n \in \mathbb{N}$, $n \geq 3$, n is neither prime nor perfect square and $b \in \mathbb{N}^*$.

OUTPUT Smallest prime factor n if it is $< b$, otherwise failure.

    1. for $q \in \{2, 3, 5, 7, 11, \ldots, p\}$, $p \leq b$.

      1.1. Return q if $mod(n, q) = 0$.

      1.2. Otherwise continue.

    2. Return failure.

The number of steps for trial division is $O \sim (\sqrt[3]{n})$ most of the time, [Myasnikov and Backes, 2008].

2. Special algorithms. Their execution time depends on the special properties of number n, as, for example, the size of the greatest prime factor. This category includes:

  (a) The *rho* algorithm of Pollard, [Pollard, 1975, Brent, 1980, Weisstein, 2014d];

  (b) The $p-1$ algorithm of Pollard [Cormen et al., 2001];

  (c) The algorithm based on elliptic curves [Galbraith, 2012];

  (d) The Pollard-Strassen algorithm [Pomerance, 1982, Hardy et al., 1990, Weisstein, 2014f], which was proved to be the fastest factorization algorithm. For $a \in \mathbb{N}$ we denote $\bar{a} = mod(a, n)$. Let c, $1 \le c \le \sqrt{n}$

$$F(x) = (x+1)(x+2)\cdots(x+c) \in \mathbb{Z}[x]$$

  and

$$f(x) = \overline{F}(x) \in \mathbb{Z}_N[x]$$

  then

$$\overline{c^2!} = \prod_{k=0}^{c} f(\overline{k \cdot c}) \,.$$

This algorithm has the following steps:

INPUT $n \in \mathbb{N}$, $n \ge 3$, n is neither prime, nor perfect square, $b \in \mathbb{N}^*$.

OUTPUT If the smallest prime factor of n is $< b$, otherwise failure.

  1. Compute $c \leftarrow \left\lceil \sqrt{b} \right\rceil$.

  2. Determine the coefficients of polynomial $f \in \mathbb{Z}_N[x]$:

$$f(x) = \prod_{k=1}^{c} (x + \overline{k}) \,.$$

3. Compute $g_k \in \{0, 1, \ldots, n-1\}$ such that

$$g_k = mod\big(f(\overline{k \cdot c}), n\big) \ \text{ for } \ 0 \leq k < c \,.$$

   4.1. If $\gcd(g_k, n) = 1$ for $\forall k \in \{0, 1, \ldots c-1\}$ then return failure.

   4.2. On the contrary, let

$$k = \min\{0 \leq k < c; \ \gcd(g_k, n) > 1\} \,.$$

5. Return $\min\{d \,; \ mod(n, d) = 0, \ k \cdot c + 1 \leq d \leq k \cdot c + c\}$.

Pollard's and Strassen's integer factoring algorithm works correctly and uses $O(M(\sqrt{b})M(log(n))(log(b) + log(log(n)))$ word operations, where M is the time for multiplication, and space for $O(\sqrt{b} \cdot log(n))$ words, [Myasnikov and Backes, 2008, von zur Gathen and Gerhard, 2013].

Program 1.26. This program uses the Schema of Horner [1819], the fastest algorithm to compute the value of a polynomial, [Cira, 2005]. The input variables are the vector a which defines the polynomial $a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$ and x.

$$Horner(a, x) := \begin{vmatrix} m \leftarrow last(a) \\ f \leftarrow a_m \\ for \ k \in m-1..0 \\ \quad f \leftarrow f \cdot x + a_k \\ return \ f \end{vmatrix}$$

Program 1.27. Computation program for the coefficients of the polynomial $(x + 1)(x + 2) \cdots (x + c)$.

$$Prod(c) := \begin{vmatrix} v \leftarrow (1 \ 1)^{\mathrm{T}} \\ return \ v \ if \ c{=}1 \\ for \ k \in 2..c \\ \quad v \leftarrow stack(0, v) + stack(k \cdot v, 0) \\ return \ v \end{vmatrix}$$

Program 1.28. This program applies the Pollard-Strassen algorithm for finding the smallest prime factor, not greater than b, of number n.

$$PS(n,b) := \begin{vmatrix} c \leftarrow ceil\left(\sqrt{b}\right) \\ C \leftarrow Prod(c) \\ for \; k \in 0..c \\ \quad a_k \leftarrow mod(C_k, n) \\ for \; k \in 0..c - 1 \\ \quad \begin{vmatrix} g_k \leftarrow mod(Horner(a, mod(k \cdot c, n)), n) \\ d_k \leftarrow \gcd(g_k, n) \\ return \; d_k \; if \; d_k > 1 \end{vmatrix} \\ return \; "Fail" \end{vmatrix}$$

This program calls programs 1.27 and 1.26. The program was tested by means of following examples:

$$n := 143 \quad b := floor(\sqrt{n}) = 11 \quad PS(n,b) = 11$$

$$n := 667 \quad b := floor(\sqrt{n}) = 25 \quad PS(n,b) = 23$$

$$n := 4009 \quad b := floor(\sqrt{n}) = 63 \quad PS(n,b) = 19$$

$$n := 10097 \quad b := floor(\sqrt{n}) = 100 \quad PS(n,b) = 23$$

1.3.1 Direct factorization

The most easy method to find factors is the so-called "direct search". In this method, all possible factors are systematically tested using a division of testings to see if they really divide the given number. This algorithm is useful only for small numbers ($< 10^6$).

Program 1.29. The program of factorization of a natural number. This program uss the vector of prime numbers p generated by the Sieve of Eratosthenes, the fastest program that generates prime numbers up to a given

limit. The Call of the Sieve of Eratosthenes, the program 1.4, is made using the sequence:

$$L := 2 \cdot 10^7 \quad t_0 = time(0) \quad p := CEPb(L) \quad t_1 = time(1)$$

$$(t_1 - t_0)s = 5.064s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999$$

$$Fa(m) := \left| \begin{array}{l} return \ ("m = " \ m \ " \ > that \ the \ last \ p^{2}") \ if \ m > (p_{last(p)})^2 \\ j \leftarrow 1 \\ k \leftarrow 0 \\ f \leftarrow (1 \ 1) \\ while \ m \geq p_j \\ \quad \left| \begin{array}{l} if \quad \mod \ (m, p_j) = 0 \\ \quad \left| \begin{array}{l} k \leftarrow k + 1 \\ m \leftarrow \dfrac{m}{p_j} \end{array} \right. \\ otherwise \\ \quad \left| \begin{array}{l} f \leftarrow stack[f, (p_j, k)] \ if \ k > 0 \\ j \leftarrow j + 1 \\ k \leftarrow 0 \end{array} \right. \end{array} \right. \\ f \leftarrow stack[f, (p_j, k)] \ if \ k > 0 \\ return \ submatrix(f, 2, rows(f), 1, 2) \end{array} \right.$$

We give a remark that can simplify the primality test in some cases.

Observation 1.30. If p is the first prime factor of n and $p^2 > q = \frac{n}{p}$, then q is a prime number. Hence, the decomposition in prime factors of number n is $p \cdot q$.

Proof. Let us suppose that q is a composite number, which means $q = a \cdot b$. As p is the first prime factor of n, it follows that $a, b > p$. Hence, a contradiction is obtained, namely $n = p \cdot q = p \cdot a \cdot b > p^3 > n$. Therefore, q is a prime number. Hence, the decomposition in prime factors of n is $p \cdot q$. $\qquad\qquad\square$

Examples of factorization:

$$Fa(2^{36} - 1) = \begin{pmatrix} 3 & 3 \\ 5 & 1 \\ 7 & 1 \\ 13 & 1 \\ 19 & 1 \\ 37 & 1 \\ 73 & 1 \\ 109 & 1 \end{pmatrix}, \quad Fa(3^{20} - 1) = \begin{pmatrix} 2 & 4 \\ 5 & 2 \\ 11 & 2 \\ 61 & 1 \\ 1181 & 1 \end{pmatrix},$$

$$Fa(11^{7} - 1) = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 43 & 1 \\ 45319 & 1 \end{pmatrix}, \quad Fa(7^{11} - 1) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 1123 & 1 \\ 293459 & 1 \end{pmatrix}.$$

1.3.2 Other methods of factorization

1. The method of Fermat and the generalized method of Fermat are recommended for the case where n has two factors of similar extension. For a natural number n, two integers are searched, x and y such that $n = x^2 - y^2$. Then $n = (x - y)(x + y)$ and we obtain a first decomposition of n, where one factor is very small. This factorization may be inefficient if the factors a and b do not have close values, it is possible to be necessary $\frac{n+1}{2} - \sqrt{n}$ verifications for testing if the generated numbers are squares. In this situation we can use a generalized Fermat ethod which applies better in such cases, [Dan, 2005].

2. The method of Euler of factorization can be applied for odd numbers n that can be written as the sum of two squares in two different ways

$$n = a^2 + b^2 = c^2 + d^2$$

where a, c are even and c, d odd.

3. The method of Pollard-rho or the Monte Carlo method. We suppose that a great number n is composite. The simplest test, much more faster than the method of divisions, is due to Pollard [1975]. It is also called the rho method, or the Monte Carlo method. This test has a special purpose, used to find the small prime factors for a composite number.

   For the Pollard-rho algorithm, a certain function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ is chosen, such that, for example, its values to be determined easily. Hence, f is usually a polynomial function; for example $f(x) = x^2 + a$, where $a \neq \{0, 2\}$.

   Pollard-rho algorithm with the chosen function $f(x) = x^2 + 1$, is:

INPUT: A composite number $n > 2$, which is not the power of a prime number.

OUTPUT: A proper divisor of n.

    1. Let $a \leftarrow 2$ and $b \leftarrow 2$.

    2. For $k = 1, 2, \ldots$, run:

      2.1 Compute $a \leftarrow mod(f(a), n)$ and $b \leftarrow mod(f(b), n)$.

      2.2 Compute $d = (a - b, n)$.

      2.3 If $1 < d < n$, then return d proper divisor of n and stop the algorithm.

      2.4 If $d = n$, then return the message *"Failure, another function must be chosen"*.

4. Pollard $p - 1$ method. This method has a special purpose, being used for the factorization of numbers n which have a prime factor p with the property that $p - 1$ is a product of prime factors smaller than a relative small number. Pollard $p - 1$ algorithm is:

INPUT: A composite number $n > 2$, which is not the power of a prime number.

OUTPUT: A proper divisor of n.

1. Choose a margin B.

2. Choose, randomly, an a, $2 \leq a \leq n - 1$ and compute $d = (a, n)$. If $d \geq 2$, return d proper divisor of n and stop the algorithm.

3. For every prim $q \leq B$, run:

   3.1. Compute $\ell = [\ln(n)/\ln(q)]$.

   3.2. Compute $a \leftarrow mod\left(a^{q^\ell}, n\right)$.

4. Compute $d = (n - 1, a)$.

5. If $d = 1$ or $d = n$, then return the message "*E'sec*", else, return d proper divisor of n and stop the algorithm.

1.4 Counting of the prime numbers

1.4.1 Program of counting of the prime numbers

If we have the list of prime numbers, we can, obviously, write a program to count them up to a given number $x \in \mathbb{N}^*$. We read the file of prime numbers available on the site [Caldwell, 2014b] and we assign it to vector p with the sequence:

$$p := READPRN(" \ldots \backslash Prime.prn")$$

$$last(p) = 6 \cdot 10^6 \quad p_{last(p)} = 104395301 \,.$$

Command $last(p)$ states that vector p contains the first $6 \cdot 10^6$ prime numbers, and the last prime number of vector p is 104395301.

Program 1.31. Program for counting the prime numbers up to a natural number x.

$$\pi(x) := \begin{vmatrix} for\ n \in 1..last(p) \\ \quad if\ p_n \geq x \\ \quad \quad \begin{vmatrix} return\ n - 1\ if\ p_n > x \\ return\ n\ otherwise \end{vmatrix} \end{vmatrix}$$

For example, let us count the prime numbers up to 10^n, for $n = 1, 2, \ldots, 8$. This counting can be made by using following commands:

$$n := 1..8 \quad \pi(10^n) = \begin{pmatrix} 4 \\ 25 \\ 168 \\ 1229 \\ 9592 \\ 78498 \\ 664579 \\ 5761455 \end{pmatrix}.$$

1.4.2   Formula of counting of the prime numbers

By means of Smarandache's function we obtain a formula for counting the prime numbers less or equal to n, [Seagull, 1995].

Theorem 1.32. *If n is an integer ≥ 4, then*

$$\pi(n) = -1 + \sum_{k=2}^{n} \left\lfloor \frac{\eta(k)}{k} \right\rfloor \tag{1.18}$$

Proof. Knowing the $\eta(n)$ has the property that if $p > 4$ then $\eta(p) = p$ if only if p is prime, and $\eta(n) < n$ for any n, and $\eta(4) = 4$ (the only exception from the first rule), then

$$\left\lfloor \frac{\eta(k)}{k} \right\rfloor = \begin{cases} 1 \,, & \text{if } k \text{ is prime} \\ 0 \,, & \text{if } k \text{ is not prime} \end{cases} .$$

We easily find an exact formula for the number of primes less than or equal to n. $\square$

If we read the file $\eta.prn$ and attribute to the values of the vector η the sequence

$$ORIGIN := 1 \quad \eta := READPRN(" \ldots \backslash \eta.prn")$$

then formula (1.18) becomes:

$$\pi(n) := \begin{cases} return \ "Error \ n < 1 \ or \ n \notin \mathbb{Z}" \ if \ n < 1 \vee n \neq trunc(n) \\\\ return \ -1 + \sum_{k=2}^{n} \left\lfloor \frac{\eta_k}{k} \right\rfloor \ if \ n \geq 4 \\\\ return \ 2 \ if \ n = 3 \\\\ return \ 1 \ if \ n = 2 \\\\ return \ 0 \ if \ n = 1 \end{cases}$$

(1.19)

Using this formula, the number of primes up to $n = 10$, $n = 10^2$, ..., $n = 10^6$ has been determined and the obtained results are:

$$\pi(10) = 4 \quad \pi(10^2) = 25 \quad \pi(10^3) = 168 \quad \pi(10^4) = 1229$$

$$\pi(10^5) = 9592 \quad \pi(10^6) = 78498 \ .$$

Chapter 2

Smarandache's function η

The function that associates to each natural number n the smallest natural number m which has the property that $m!$ is a multiple of n was considered for the first time by Lucas [1883]. Other authors who have considered this function in their works are: Neuberg [1887], Kempner [1918]. This function was rediscovered by Smarandache [1980a]. The function is denoted by Smarandache with S or η, and on the site *Wolfram MathWorld*, [Sondow and Weisstein, 2014], it is denoted μ. In this volume we have adopted the notation η found in the paper [Smarandache, 1999b].

Therefore, function $\eta : \mathbb{N}^* \to \mathbb{N}^*$, $\eta(n) = m$, where m is the smallest natural that has the property that n divides $m!$, (or $m!$ is a multiple of n) is known in the literature as *Smarandache's function*. The values of the function, for $n = 1, 2, \ldots, 18$, are: 1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, 6 obtained by means of an algorithm that results from the definition of function η, as follows:

Program 2.1.

$$\eta(n) = \left| \begin{array}{l} for\ m = 1..n \\ \quad return\ m\ if\ mod(m!, n) \text{=} 0 \end{array} \right.$$

The program 2.1 can not be used for $n \geq 19$ as the numbers 19!, 20!, …has much more than 17 decimal digits and in the classic computation
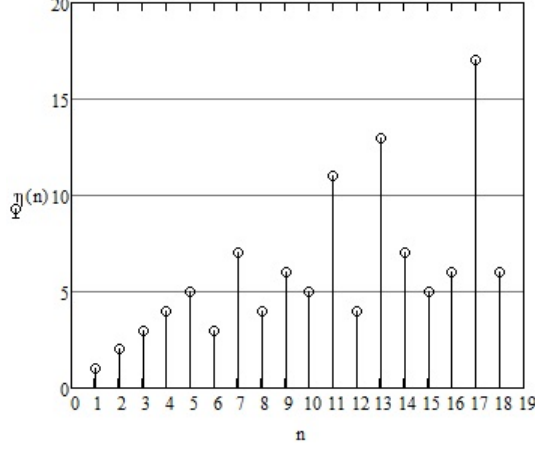
Figure 2.1: η function

approach (without an arithmetics of random precisions [Uznanski, 2014]) will be generated errors due to the classic representation in the memory of computers.

Kempner [1918], gave an algorithm to compute $\eta(n)$ using the classic factorization $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with prime numbers of n, and the generalized base $(\alpha_i)_{[p_i]}$, for $i = \overline{1, s}$. Partial solutions for algorithms that compute $\eta(n)$ were given previously by Lucas and Neuberg, [Sondow and Weisstein, 2014] .

We give Kempner's algorithm, that computes Smarandache's function η. At the beginning, let us define the recursive sequence

$$a_{j+1} = p \cdot a_j + 1 \quad \text{with} \quad j = 1, 2, \ldots \quad \text{and} \quad a_1 = 1 \,,$$

where p is a prime number. This sequence represents the generalized base of p. As $a_2 = p + 1$, $a_3 = p^2 + p + 1$, $\ldots$ we can prove by induction that

$$a_j = 1 + p + \ldots + p^{j-1} = \frac{p^j - 1}{p - 1} \quad \text{for} \ \ \forall j \geq 2 \,.$$

The value of ν, such that $a_\nu \leq \alpha < a_{\nu+1}$, is given by the formula

$$\nu = \lfloor \log_p \left(1 + \alpha(p-1)\right) \rfloor , \tag{2.1}$$

where $\lfloor \cdot \rfloor$ is the function *lower integer part*. With the help Euclid's algorithm we can determine the unique sequences κ_i and r_i, as follows

$$\alpha = \kappa_\nu \cdot a_\nu + r_\nu , \tag{2.2}$$

$$r_\nu = \kappa_{\nu-1} \cdot a_{\nu-1} + r_{\nu-1} , \tag{2.3}$$

$$\vdots$$

$$r_{\nu-(\lambda-2)} = \kappa_{\nu-(\lambda-1)} \cdot a_{\nu-(\lambda-1)} + r_{\nu-(\lambda-1)} , \tag{2.4}$$

$$r_{\nu-(\lambda-1)} = \kappa_{\nu-\lambda} \cdot a_{\nu-\lambda} . \tag{2.5}$$

which means, until the rest $r_{\nu-\lambda} = 0$. At each step κ_i is the integer part of the ratio r_i/a_i and r_i is the rest of the division. For example, for the first step we have $\kappa_\nu = \lfloor \alpha/a_\nu \rfloor$ and $r_\nu = \alpha - \kappa_\nu \cdot a_\nu$. Then, we have

$$\eta(p^\alpha) = (p-1)\alpha + \sum_{i=\nu}^{\lambda} \kappa_i . \tag{2.6}$$

In general, for

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s} , \tag{2.7}$$

the value of function η is given by the formula:

$$\eta(n) = \max \left\{ \eta(p_1^{\alpha_1}), \eta(p_2^{\alpha_2}), \ldots, \eta(p_s^{\alpha_s}) \right\} , \tag{2.8}$$

formula due to Kempner [1918].

Remark 2.2. If $n \in \mathbb{N}^*$ has the decomposition in product of prime numbers (2.7), where p_i are prime numbers such that $p_1 < p_2 < \ldots < p_s$, and $s \geq 1$, then the Kempner's algorithm of computing function η is

$$\eta(n) = \max \left\{ p_1 \cdot \left(\alpha_{1_{[p_1]}} \right)_{(p_1)} , \; p_2 \cdot \left(\alpha_{2_{[p_2]}} \right)_{(p_2)} , \right.$$

$$\left. \ldots, \; p_s \cdot \left(\alpha_{s_{[p_s]}} \right)_{(p_s)} \right\} , \tag{2.9}$$

where $\left(\alpha_{[p]}\right)_{(p)}$ means that α is *"written"* in the generalized base p (denoted $\alpha_{[p]}$) and is *"read"* in base p (denoted $\beta_{(p)}$, where $\beta = \alpha_{[p]}$), [Smarandache, 1999a, p.39].

On the site *The On-Line Encyclopedia of Integer Sequences*, [Sloane, 2014, A002034], is given a list of 1000 values of function η, due to T. D. Noe. We remark that on the site *The On-Line Encyclopedia of Integer Sequences* it is defined $\eta(1) = 1$, while Ashbacher [1995] and Russo [2000] consider that $\eta(1) = 0$.

2.1   The properties of function η

The greater values for function η are obtained for 4 and for the prime numbers and are $\eta(p) = p$, [Sloane, 2014, A046022].

The smallest values for n are, [Sloane, 2014, A094371]:

$$\frac{\eta(n)}{n} = 1, \; \frac{1}{2}, \; \frac{1}{3}, \; \frac{1}{4}, \; \frac{1}{6}, \; \frac{1}{8}, \; \frac{1}{12}, \; \frac{3}{40}, \; \frac{1}{15}, \; \frac{1}{16}, \; \frac{1}{24}, \; \frac{1}{30}, \; \ldots$$

for the values [Sloane, 2014, A002034]:

$$n = 1, \; 6, \; 12, \; 20, \; 24, \; 40, \; 60, \; 80, \; 90, \; 112, \; 120, \; 180, \; \ldots$$

This function is important because it characterize the prime numbers – by the following fundamental property.

Theorem 2.3. *Let p be an integer > 4. Then p is prime if and only if $\eta(p) = p$.*

Proof. See [Smarandache, 1999a, p. 31]. $\qquad\qquad\qquad\qquad\qquad$ □

Hence, the fixed points of this function are prime numbers (to which 4 is added). Due to this property, function η is used as a primality test.

The formula (2.9) used to compute Smarandache's function η allows us to give several values of the function for particular numbers n

$$
\begin{aligned}
\eta(1) &= 1 \, , \\
\eta(n!) &= n \, , \\
\eta(p) &= p \, , \\
\eta(p_1 \cdot p_2 \cdots p_s) &= p_1 \cdot p_2 \cdots p_s \, , \\
\eta(p^\alpha) &= p \cdot \alpha \, ,
\end{aligned}
\tag{2.10}
$$

where p and p_i are distinct prime numbers with $p_1 < p_2 < \ldots < p_s$ and $\alpha \leq p$, [Kempner, 1918].

Other special numbers for which we can give the values of function η are:

$$
\eta(P_p) = M_p \, ,
\tag{2.11}
$$

where $P_2 = 6$, $P_3 = 28$, $P_5 = 496$, $P_7 = 8128$, ..., [Sloane, 2014, A000396], are the perfect numbers corresponding to the prime numbers $2, 3, 5, 7, \ldots$, and $M_2 = 2^2 - 1 = 3$, $M_3 = 2^3 - 1 = 7$, $M_5 = 2^5 - 1 = 31$, $M_7 = 2^7 - 1 = 127$, ..., [Sloane, 2014, A000668], are Mersenne numbers corresponding to the prime numbers prime $2, 3, 5, 7, \ldots$, see the papers [Ashbacher, 1997, Ruiz, 1999a].

Function η has following properties:

$$
\eta(n_1 \cdot n_2) \leq \eta(n_1) + \eta(n_2) \, ,
\tag{2.12}
$$

$$
\max \left\{ \eta(n_1), \eta(n_2) \right\} \leq \eta(n_1 \cdot n_2) \leq \eta(n_1) \cdot \eta(n_2) \, ,
\tag{2.13}
$$

where $n_1, n_2 \in \mathbb{N}^*$.

If p is a prime number and $\alpha \geq 2$ an integer, then

$$
\eta\left(p^{p^\alpha}\right) = p^{\alpha+1} - p^\alpha + p \, .
\tag{2.14}
$$

This result is due to Ruiz [1999b].

The case p^α with $\alpha > p$ is more complicated to which applies the Kempner's algorithm.

According to formula (2.8), it results that for all $n \in \mathbb{N}^*$ we have

$$
\eta(n) \geq g_{pf}(n) \, ,
\tag{2.15}
$$

where $g_{pf}(n)$ is the function *the greatest prime factor* of n. Therefore, $\eta(n)$ can be computed by determining $g_{pf}(n)$ and testing if $n \mid g_{pf}(n)!$. If $n \mid g_{pf}(n)!$ then $\eta(n) = g_{pf}(n)$, if $n \nmid g_{pf}(n)!$ then $\eta(n) > g_{pf}(n)$ and we call Kempner's algorithm.

Let $A \subset \mathbb{N}^*$ a set of strictly nondecreasing positive integers. We denote by $A(n)$ the number of numbers of the set A up to n. In what follows we give the definition of the density of a set of natural numbers, [Guy, 1994, p. 199].

Definition 2.4. We name *density of a set* $A \subset \mathbb{N}^*$, the number

$$\lim_{n \to \infty} \frac{A(n)}{n},$$

if it exists.

For example, *the density of the set of the even natural numbers* is $1/2$ because

$$\lim_{n \to \infty} \frac{\lfloor \frac{n}{2} \rfloor}{n} = \frac{1}{2} .$$

The set of numbers $n \in \mathbb{N}^*$ with the property that $n \nmid g_{pf}(n)!$ has *zero density*, such as Erdös [1991] supposed and Kastanas [1994] proved.

The first numbers with the property that $n \nmid g_{pf}(n)!$ are: 4, 8, 12, 16, 18, 24, 25, 27, 32, 36, 45, 48, 49, 50, ... [Sloane, 2014, A057109].

If we denote by $N(x)$ the number of numbers $n \in \mathbb{N}^*$ which have the properties $2 \leq n \leq x$ and $n \nmid g_{pf}(n)!$, then we obtain the estimation

$$N(x) \ll x \cdot e^{-\frac{1}{4}\sqrt{\ln(x)}} , \tag{2.16}$$

due to Akbik [1999], where the notation $f(x) \ll g(x)$ means that there exists $c \in \mathbb{R}_+$ such that $|f(x)| < c \cdot |g(x)|, \forall x$. As

$$\frac{N(x)}{x} \ll e^{-\frac{1}{4}\sqrt{\ln(x)}} , \quad \text{and} \quad \lim_{x \to \infty} e^{-\frac{1}{4}\sqrt{\ln(x)}} = 0 ,$$

we may say that the set has *zero density*.

This result was later improved by Ford [1999] and by the authors De Koninck and Doyon [2003] . Ford proposed following asymptotic formula:

$$N(x) \approx \frac{\sqrt{\pi}\left(1 + \ln(2)\right)}{\sqrt[4]{2^3}} \sqrt[4]{\ln(x)^3 \ln(\ln(x))^3} \cdot x^{1 - \frac{1}{u_0}} \cdot \rho(u_0) \,, \qquad (2.17)$$

where $\rho(u)$ is the Dickman's function, [Dickman, 1930, Weisstein, 2014a], and u_0 is defined implicitly by equation

$$\ln(x) = u_0 \left(x^{\frac{1}{u_0^2}} - 1 \right) \,. \qquad (2.18)$$

The estimation made in formula (2.17) was rectified by Ivić [2003], in two consecutive postings,

$$N(x) = x \left(2 + O\left(\sqrt{\frac{\ln(\ln(x))}{\ln(x)}} \right) \right) \int_2^x \rho\left(\frac{\ln(x)}{\ln(t)} \right) \frac{\ln(t)}{t^2} dt \,, \qquad (2.19)$$

or, by means of elementary functions

$$N(x) = x \cdot \exp\left[-\sqrt{2\ln(x)\ln(\ln(x))} \left(1 + O\left(\frac{\ln(\ln(\ln(x)))}{\ln(\ln(x))} \right) \right) \right] \,. \qquad (2.20)$$

Tutescu [1996] assumed that function η does not have the same value for two consecutive values of the argument, which means

$$\forall\, n \in \mathbb{N}^* \,, \quad \eta(n) \neq \eta(n+1) \,.$$

Weisstein published, on the 3rd of March 2004, [Sondow and Weisstein, 2014], the fact that he has verified this result, by means of a program, up to 10^9.

Several numbers $n \in \mathbb{N}^*$ may have the same value for η function, i.e. function η is not injective. In table 2.1 we emphasize numbers n for which $\eta(n) = k$.

k	n for which we have $\eta(n) = k$
1	1
2	2
3	3, 6
4	4, 8, 12, 24
5	5, 10, 15, 20, 30, 40, 60, 120
6	6, 16, 18, 36, 45, 48, 72, 80, 90, 144, 240, 360, 720

Table 2.1: Values n for which $\eta(n) = k$

Let $a(k)$ be the smallest inverse of $\eta(n)$, i.e. the smallest n for which $\eta(n) = k$. Then $a(k)$ is given by

$$a(k) = g_{pf}(n)^\omega ,$$

$$\text{where } \omega = \sum_{i=1}^{L} \left\lfloor \frac{n-1}{g_{pf}(k)^i} \right\rfloor , \text{ and } L = \left\lfloor \log_{g_{pf}(k)}(n-1) \right\rfloor . \quad (2.21)$$

This result was published by Sondow [2005]. For $k = 1, 2, \ldots$, function $a(k)$ is equal to 1, 2, 3, 4, 5, 9, 7, 32, 27, 25, 11, 243, $\ldots$ as seen in [Sloane, 2014, A046021].

Some values of $\eta(n)$ function are obtained for huge values of n. An increasing sequence of great values of $a(k)$ is 1, 2, 3, 4, 5, 9, 32, 243, 4096, 59049, 177147, 134217728, 31381059609, $\ldots$, (see [Sloane, 2014, A092233]), the sequence that corresponds to $n = 1, 2, 3, 4, 5, 6, 8, 12, 24, 27, 32, 48, 54,$ $\ldots$ (see [Sloane, 2014, A092232]).

In the process of finding number n for which $\eta(n) = k$, we remark that n is a divisor of $\eta(n)!$ but not of $\eta(n-1)!$. Therefore, in order to find all the numbers n for each $\eta(n)$ has a value, we consider all n with $\eta(n) = k$, where n is in the set of all divisors of $k!$ minus the divisors of $(k-1)!$. In particular, $b(k)$ of n for which $\eta(n) = k$, for $k > 1$ is

$$b(k) = \sigma_0(k!) - \sigma_0\big((k-1)!\big) , \quad (2.22)$$

where $\sigma_0(m)$ is the *divisors counting* function of m. Hence, the number of

integers n with $\eta(n) = 1, 2, \ldots$ are given by the sequence 1, 1, 2, 4, 8, 14, 30, 36, 64, 110, ... (see [Sloane, 2014, A038024]).

Particularly, equation (2.22) shows that the inverse of Smarandache's function, $a(n)$, exists always, as for each n there exist an m such that $\eta(n) = m$ (i.e. the smallest $a(n)$), because

$$\sigma_0(n!) - \sigma_0\big((n-1)!\big) > 0 \, ,$$

for $n > 1$.

Sondow [2004] showed that $\eta(n)$ appears unexpectedly in an irrational limit for e and it suppose that the inequality $n^2 < \eta(n)!$ holds for *"almost every n"*, where *"almost every n"* means the set of integers minus an exception set of *zero density*. The exception set is 2, 3, 6, 8, 12, 15, 20, 24, 30, 36, 40, 45, 48, 60, 72, 80, ..., (see [Sloane, 2014, A122378]).

As equation $g_{pf}(n) = \eta(n)$, considered by Erdös [1991], Kastanas [1994] for *"almost every n"*, is equivalent with the inequality $n^2 < g_{pf}(n)!$ for *"almost every n"* of Sondow's conjecture, it results that the conjecture of Erdös and Kastanas is equivalent with Sondow's conjectures. The exception set, in this case, of *zero density* is: 2, 3, 4, 6, 8, 9, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, ..., (see [Sloane, 2014, A122380]).

D. Wilson, underlines, in the case where

$$I(n,p) = \frac{n - \Sigma(n,p)}{p-1} \, , \tag{2.23}$$

is a power of p prime in $n!$, where $\Sigma(n,p)$ is the function *sum in base p of n*, then following relation

$$a(n) = \min_{p|n} p^{I(n-1,p)+1} \, , \tag{2.24}$$

hold, where the minimum is reached for every prime number p that divides n. This minimum seems to be always attainable when $p = g_{pf}(n)$.

2.2   Programs for Kempner's algorithm

In this section we emphasize Kempner's algorithm by means of the Mathcad programs necessary to the algorithm.

Program 2.5. The function that counts the digits in base p of n

$$ncb(n,p) := \left| \begin{array}{l} return\ ceil(\log(n,p))\ if\ n > 1 \\ return\ 1\ otherwise \end{array} \right.$$

where the utility function Mathcad $ceil(\cdot)$ is the upper integer part function.

Program 2.6. The program that generates the generalized base p (denoted by Smarandache $[p]$) for a number with m digits

$$a(p,m) := \left| \begin{array}{l} for\ i \in 1..m \\ \quad a_i \leftarrow \dfrac{p^i - 1}{p - 1} \\ return\ a \end{array} \right.$$

Program 2.7. The program that generates the base p (denoted by Smarandache (p)) to write number α

$$b(\alpha,p) := \left| \begin{array}{l} return\ (1)\ if\ p = 1 \\ for\ i \in 1..ncb(\alpha,p) \\ \quad b_i \leftarrow p^{i-1} \\ return\ b \end{array} \right.$$

Program 2.8. The program of finding the digits of the generalized base $[p]$ for number n

$$Nbg(n,p) := \left| \begin{array}{l} m \leftarrow ncb(n,p) \\ a \leftarrow a(p,m) \\ return\ (1)\ if\ m{=}0 \\ for\ i \in m..1 \\ \quad \left| \begin{array}{l} c_i \leftarrow trunc\left(\dfrac{n}{a_i}\right) \\ n \leftarrow \ \ mod\ (n,a_i) \end{array} \right. \\ return\ c \end{array} \right.$$

Program 2.9. The program for Smarandache's function

$$\eta(n) := \begin{vmatrix} return\ "Error\ n\ is\ not\ integer"\ if\ n \neq trunc(n) \\ return\ "Error\ n < 1"\ if\ n < 1 \\ return\ (1)\ if\ n{=}1 \\ f \leftarrow Fa(n) \\ p \leftarrow f^{\langle 1 \rangle} \\ \alpha \leftarrow f^{\langle 2 \rangle} \\ for\ k = 1..rows(p) \\ \quad \eta_k \leftarrow p_k \cdot Nbg(\alpha_k, p_k) \cdot b(\alpha_k, p_k) \\ return\ \max(\eta) \end{vmatrix}$$

This program calls the program $Fa(n)$ of factorization by prime numbers. The program uses Smarandache's remark 2.2 relative to Kempner's algorithm.

If we introduce number n as a product of prime numbers p_i raised at power α_i (α_i integer ≥ 0) it will result a variant of the program 2.9 which can compute the values of η function for huge numbers.

Program 2.10. The program for computing the values of η function for huge numbers.

$$\eta_s(f) := \begin{vmatrix} Prop \leftarrow "Matrix\ f\ is\ not\ at\ least\ one\ row\ with\ two\ columns" \\ return\ Prop\ if\ \neg(IsArray(f) \wedge rows(f) \geq 1 \wedge cols(f){=}2) \\ p \leftarrow f^{\langle 1 \rangle} \\ \alpha \leftarrow f^{\langle 2 \rangle} \\ for\ k = 1..rows(p) \\ \quad \eta_k \leftarrow p_k \cdot Nbg(\alpha_k, p_k) \cdot b(\alpha_k, p_k) \\ return\ \max(\eta) \end{vmatrix}$$

Program 2.11. Program that generates the matrix that contains al values n for which $\eta(n) = k$.

$$EK(N) := \begin{vmatrix} for\ n \in 2..N \\ \quad K_n \leftarrow \eta(n) \\ for\ q \in 2..max(K) \\ \quad \begin{vmatrix} j \leftarrow 1 \\ for\ k \in 2..N \end{vmatrix} \end{vmatrix}$$

$$
\begin{array}{l}
\qquad\quad if\ K_k\text{=}q \\
\qquad\qquad\quad EK_{q,j} \leftarrow k \\
\qquad\qquad\quad j \leftarrow j+1 \\
\quad return\ EK
\end{array}
$$

2.2.1 Applications

Several applications for the given programs are given in what follows:

1. *Compute the values of η function for numbers n_1, n_2 given as products of prime numbers raised at a positive integer power.*

   (a) Let $n_1 = 2^{12} \cdot 7^{13} \cdot 11^{23} = 89543024325533426 1261034$, then

   $$
   n_1 := \begin{pmatrix} 2 & 12 \\ 7 & 13 \\ 11 & 23 \end{pmatrix} \quad \eta_s(n_1) = 242 \ .
   $$

   (b) Let $n_2 = 3^{33} \cdot 5^{55} \cdot 7^{51} \cdot 11^{11} =$

   $$
   12589532854288041315477068297463914028063002 \ ,
   $$

   then

   $$
   n_2 := \begin{pmatrix} 3 & 33 \\ 5 & 55 \\ 7 & 51 \\ 11 & 11 \end{pmatrix} \quad \eta_s(n_2) = 315 \ ,
   $$

2. *Find the number whose factorial ends in 1000 zeros.*

   To answer this question we remark that for $n = 10^{1000}$ we have $\eta(n)! = M \cdot 10^{1000}$ and this $\eta(n)$ is the smallest natural number whose factorial ends in 1000 zeros. We have $\eta(n) = \eta(2^{1000} \cdot 5^{1000})$, then

   $$
   n := \begin{pmatrix} 2 & 1000 \\ 5 & 1000 \end{pmatrix} \quad \eta_s(n) = 4005
   $$

   and, hence, the number whose factorial ends in 1000 zeros is 4005. The numbers 4006, 4007, 4008, 4009 have also the required property, but 4010 has the property that its factorial has 1001 zeros.

3. *Determine all values n for which $\eta(n) = 7$.*

   With the help of program 2.11 we can generate the matrix that con-
   tains all values n for which $\eta(n) = k$. Line 7 of the matrix is the
   answer to the problem:

   $$7 = \eta(n), \text{for } n = 7, 14, 21, 28, 35, 42, 56, 63, 70, 84, 105,$$
   $$112, 126, 140, 168, 210, 252, 280, 315, 336, 420, 504, 560, 630,$$
   $$840, 1008, 1260, 1680, 2520, 5040 \ .$$

2.2.2 Calculation the of values η function

Generating the file $\eta.prn$ once and reading the generated file in Math-
cad documents that determine solutions of the Diophantine equations lead
to an important saving of the execution time for the program that searches
the solutions.

Program 2.12. The program by means of which the file $\eta.prn$ is generated
is:

$$ValFS(N) := \begin{vmatrix} \eta_1 \leftarrow 1 \\ for \ n \in 2..N \\ \quad \eta_n \leftarrow \eta(n) \\ return \ \eta \end{vmatrix}$$

This program calls the program 2.9 which calculates the values of η func-
tion. The generating sequence of the file $\eta.prn$ is:

$$t_0 := time(0) \ WRITEPRN("\eta.prn") := ValFS(10^6) \ t_1 := time(1)$$

$$(t_1 - t_0)sec = "1 : 7 : 32.625" hhmmss$$

The execution time of generating the values of η function up to 10^6 exceeds
one hour on a computer with an Intel processor of 2.20GHz with RAM of
4.00GB (3.46GB usable).
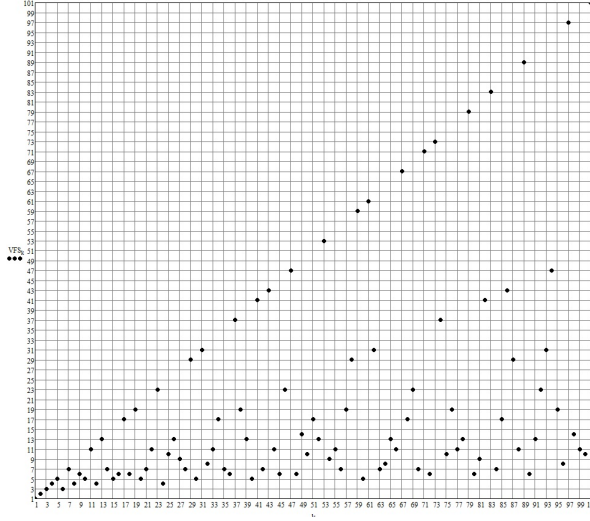
Figure 2.2: The graph of η function on the set $\{1, 2, \ldots, 101\}$

We give the list of the first 400 and the last 256 values of η function:

1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, 6, 19, 5, 7, 11, 23, 4, 10, 13, 9, 7, 29, 5, 31, 8, 11, 17, 7, 6, 37, 19, 13, 5, 41, 7, 43, 11, 6, 23, 47, 6, 14, 10, 17, 13, 53, 9, 11, 7, 19, 29, 59, 5, 61, 31, 7, 8, 13, 11, 67, 17, 23, 7, 71, 6, 73, 37, 10, 19, 11, 13, 79, 6, 9, 41, 83, 7, 17, 43, 29, 11, 89, 6, 13, 23, 31, 47, 19, 8, 97, 14, 11, 10, 101, 17, 103, 13, 7, 53, 107, 9, 109, 11, 37, 7, 113, 19, 23, 29, 13, 59, 17, 5, 22, 61, 41, 31, 15, 7, 127, 8, 43, 13, 131, 11, 19, 67, 9, 17, 137, 23, 139, 7, 47, 71, 13, 6, 29, 73, 14, 37, 149, 10, 151, 19, 17, 11, 31, 13, 157, 79, 53, 8, 23, 9, 163, 41, 11, 83, 167, 7, 26, 17, 19, 43, 173, 29, 10, 11, 59, 89, 179, 6, 181, 13, 61, 23, 37, 31, 17, 47, 9, 19, 191, 8, 193, 97, 13, 14, 197, 11, 199, 10, 67, 101, 29, 17, 41, 103, 23, 13, 19, 7, 211, 53, 71, 107, 43, 9, 31, 109, 73, 11, 17, 37, 223, 8, 10, 113, 227, 19, 229, 23, 11, 29, 233, 13, 47, 59, 79, 17, 239, 6, 241, 22, 12, 61, 14, 41, 19, 31, 83, 15, 251, 7, 23, 127, 17, 10, 257, 43, 37, 13, 29, 131, 263, 11, 53, 19, 89, 67, 269, 9, 271, 17, 13, 137, 11, 23, 277, 139, 31, 7, 281, 47, 283, 71, 19, 13, 41, 8, 34, 29, 97, 73, 293, 14, 59, 37, 11, 149, 23, 10, 43, 151, 101, 19, 61, 17, 307, 11, 103, 31, 311, 13, 313, 157, 7, 79, 317, 53, 29, 8, 107, 23, 19, 9, 13, 163, 109, 41, 47, 11,

331, 83, 37, 167, 67, 7, 337, 26, 113, 17, 31, 19, 21, 43, 23, 173, 347, 29, 349, 10, 13, 11, 353, 59, 71, 89, 17, 179, 359, 6, 38, 181, 22, 13, 73, 61, 367, 23, 41, 37, 53, 31, 373, 17, 15, 47, 29, 9, 379, 19, 127, 191, 383, 8, 11, 193, 43, 97, 389, 13, 23, 14, 131, 197, 79, 11, 397, 199, 19, 10,

$\vdots$

607, 389, 1669, 83311, 569, 1193, 83, 7351, 239, 55541, 1451, 193, 14489,
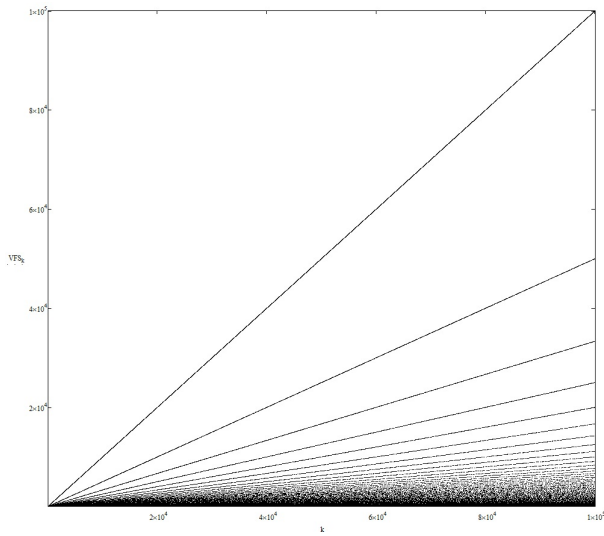
Figure 2.3: The graph of η function on the set $\left\{1, 2, \ldots, 10^5\right\}$

26309, 1531, 127, 2531, 1567, 2267, 2293, 999749, 43, 14081, 9613, 19603, 71411, 3389, 9257, 90887, 499879, 333253, 12497, 7517, 166627, 999763, 10867, 1709, 499883, 5779, 541, 999769, 5881, 2207, 249943, 999773, 829, 197, 199, 9007, 38453, 937, 877, 32251, 71413, 12343, 2659, 4877, 166631, 2557, 249947, 47609, 149, 76907, 131, 23251, 499897, 66653, 5101, 2309, 593, 521, 4999, 10099, 1319, 613, 29, 199961, 499903, 333269, 107, 999809, 46, 2053, 733, 333271, 229, 557, 41659, 10987, 317, 111091, 49991, 571, 2347, 8263, 113, 13331, 137, 7193, 27773, 5987, 7691, 1013, 124979, 1499, 15149, 199967, 5813, 1949, 4201, 3533, 2083, 14923, 3067, 827, 1381, 53, 55547, 2141,

124981, 333283, 19997, 443, 11903, 999853, 499927, 1307, 23, 9173, 166643, 142837, 49993, 333287, 17239, 999863, 1543, 643, 71419, 739, 249967, 76913, 33329, 7873, 919, 269, 16127, 421, 859, 1447, 967, 337, 3571, 13697, 4273, 999883, 249971, 349, 499943, 142841, 563, 5347, 99989, 1277, 249973, 14083, 179, 15383, 124987, 333299, 9433, 3257, 101, 1721, 21737, 4219, 31247, 6451, 9803, 999907, 67, 461, 99991, 90901, 683, 52627, 499957, 107, 547, 999917, 18517, 1009, 431, 25639, 151, 449, 809, 47, 1901, 111103, 124991, 677, 33331, 999931, 223, 193, 38459, 881, 31, 8849, 499969, 2237, 173, 1733, 166657, 20407, 1033, 823, 499973, 76919, 3623, 87, 2857, 331, 62497, 999953, 761, 18181, 249989, 2801, 499979, 999959, 641, 999961, 13513, 811, 503, 4651, 139, 32257, 31249, 333323, 277, 6211, 197, 97, 29411, 199, 523, 90907, 821, 999979, 49999, 111109, 6329, 999983, 251, 28571, 38461, 1297, 22727, 52631, 271, 997, 2551, 333331, 21739, 199999, 499, 1321, 254, 37, 25 .

Chapter 3

Divisor functions σ

3.1 The divisor function σ

The divisor function of order k is given by the formula:

$$\sigma_k(n) = \sum_{d|n} d^k .$$ (3.1)

For $k = 0$, we have function $\sigma_0(n)$ (see figure 3.1) which counts the number of divisors of n. For example, 12 has 1, 2, 3, 4, 6, 12 as divisors and, hence, their number is 6.
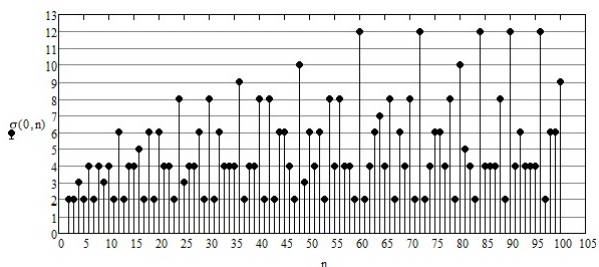
Figure 3.1: Function $\sigma_0(n)$

For $k = 1$ we have function $\sigma_1(n)$, (see figure 3.2) the function sum of the divisors of n. For example, $\sigma_1(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

Function $\sigma_1(n)$, which gives the sum of the divisors of n, is usually written without index, i.e. $\sigma(n)$.

The function sum of the proper divisors of s, [Madachy, 1979], and is given by the formula:

$$s(n) = \sigma(n) - n . \tag{3.2}$$

For example, $s(12) = 1 + 2 + 3 + 4 + 6 = 16$.

For $k = 2$ function $\sigma_2(n)$ is the sum of the squares of the divisors. Fo examples, $\sigma_2(12) = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2 = 210$.

Let n be a natural number whose decomposition into prime factors is

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s} , \tag{3.3}$$

where $p_1 < p_2 < \ldots p_s$ are prime numbers, and $\alpha_j \in \mathbb{N}$ for $j = 1, 2, \ldots, s$.

Theorem 3.1. *For two positive natural numbers n and m, relative prime, $(n, m) = 1$, then*

$$\sigma(n \cdot m) = \sigma(n) \cdot \sigma(m) . \tag{3.4}$$

Proof. For each divisor d_j of $n \cdot m$ we have $d_j = n_{j_1} \cdot m_{j_2}$, where $n_{j_1} | n$ and $m_{j_2} | m$. The numbers 1, n_1, n_2, ..., n are the divisors of n and 1, m_1, m_2, ..., m are the divisors of m. Then we have

$$\sigma(n) = 1 + n_1 + n_2 + \ldots + n$$

and

$$\sigma(m) = 1 + m_1 + m_2 + \ldots + m .$$

According to the previous relations we can write $n_{j_1}(1 + m_1 + m_2 + \ldots + m) = n_{j_1} \cdot \sigma(m)$. If we sum relative to n_{j_1} it follows that $(1 + n_1 + n_2 + \ldots + n)\sigma(m) = \sigma(n) \cdot \sigma(m)$, i.e. relation (3.4) holds. $\square$

We owe to Berndt, [Berndt, 1985, p. 94], [Weisstein, 2014c], next result.
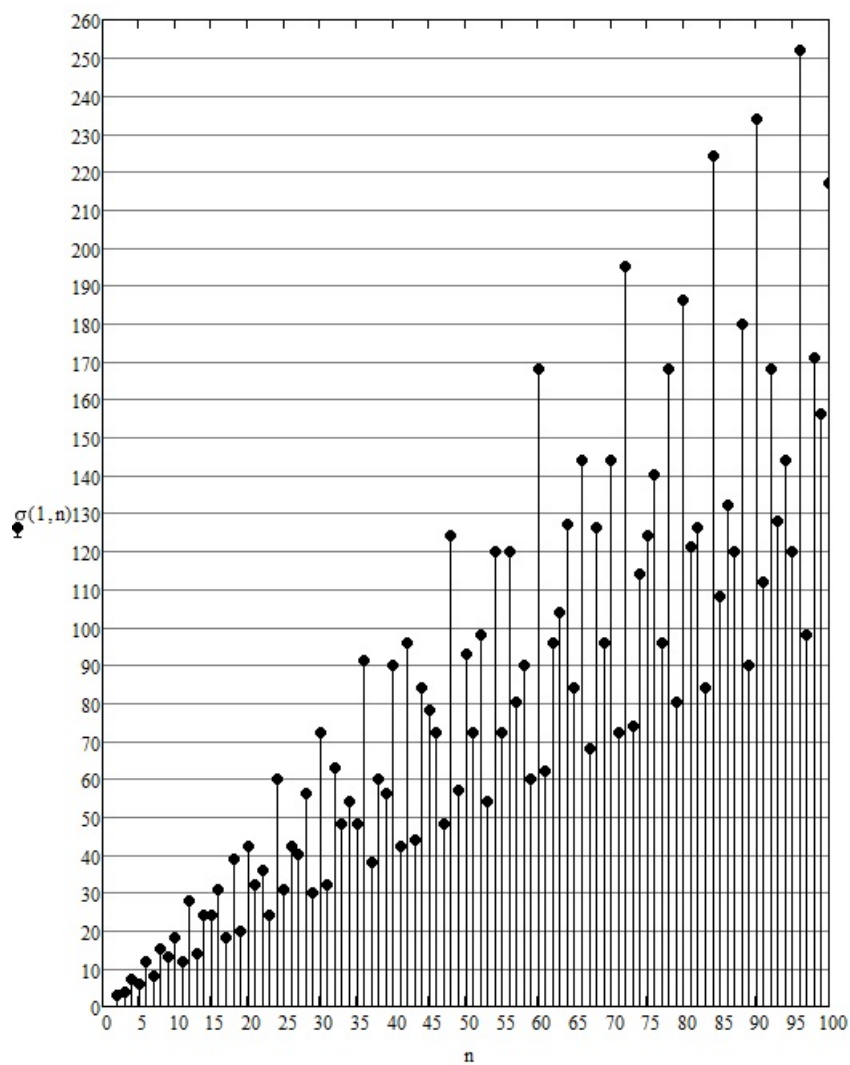
Figure 3.2: Function $\sigma(n)$

Theorem 3.2. *For every natural number n, whose decomposition into prime factors is (3.3), we have that*

$$\sigma(n) = \prod_{j=1}^{s} \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} . \tag{3.5}$$

Proof. According to relations (3.3) and (3.4) it follows that

$$\sigma(n) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2}) \cdots \sigma(p_s^{\alpha_s}) .$$

The divisors of $p_j^{\alpha_j}$ are $1, p_j, p_j^2, \ldots, p_j^{\alpha_j}$, therefore, the sum of the divisors of $p_j^{\alpha_j}$ is

$$\sigma(p_j^{\alpha_j}) = 1 + p_j + p_j^2 + \ldots + p_j^{\alpha_j} = \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} .$$

Hence, according to Proposition 3.1 we can state that

$$\sigma(n) = \sigma(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_s}) = \prod_{j=1}^{s} \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} .$$

$$\square$$

By generalizing formula (3.5) it results a relation for function σ_k. Function $\sigma_k : \mathbb{N}^* \to \mathbb{N}$, [Weisstein, 2014c], is given by the relations:

$$\sigma_0(n) = \prod_{j=1}^{s} (\alpha_j + 1) \tag{3.6}$$

and

$$\sigma_k(n) = \prod_{j=1}^{s} \frac{p_j^{(\alpha_j+1)k} - 1}{p_j^k - 1} . \tag{3.7}$$

3.1.1 Computing the values of σ_k functions

Program 3.3. The program for computing the values of function σ_k, for $k = 0, 1, \dots$.

$$\sigma(k, n) := \begin{vmatrix} f \leftarrow Fa(n) \\[2mm] return \ \displaystyle\prod_{j=1}^{rows(f)} (f_{j,2} + 1) \ \ if \ k{=}0 \\[4mm] return \ \displaystyle\prod_{j=1}^{rows(f)} \frac{(f_{j,1})^{(f_{j,2}+1)k} - 1}{(f_{j,1})^{k} - 1} \ \ if \ k > 0 \end{vmatrix}$$

The program 3.3 calls the program 1.29 of factorization in product of prime factors.

Program 3.4. The program by means of which the files $\sigma k.prn$ are generated is:

$$G\sigma(k, N) := \begin{vmatrix} f\varphi_1 \leftarrow 1 \\ for \ n \in 2..N \\ \quad f\sigma_n \leftarrow \sigma(k, n) \\ return \ f\sigma \end{vmatrix}$$

Obviously this program calls the program 3.3 for computing the values of function σ_k. The sequence for generating the file $\sigma 0.prn$ is:

$$t_0 := time(0) \ \ WRITEPRN("\sigma 0.prn") := G\sigma(0, 10^6) \ \ t_1 := time(1)$$

$$(t_1 - t_0)sec = "0 : 0 : 2.833" \, hhmmss$$

The sequences for generating the files $\sigma 1.prn$ and $\sigma 2.prn$ are similar.

3.2 k–**hyperperfect numbers**

A number $n \in \mathbb{N}^*$ is called k–*hyperperfect* if following identity

$$n = 1 + k \sum_j d_j$$

holds, or

$$n = 1 + k(\sigma(n) - n - 1)\,, \qquad n = 1 + k(s(n) - 1)\,,$$

where $\sigma(n) = \sigma_1(n)$ is the function that represents sum of the divisors d_j of n and $s(n)$ the sum of the proper divisors of n, where $1 < d_j < n$. After rearranging, we obtain relation

$$k\sigma(n) = (k + 1)n + k - 1$$

which, if it is verified, means that n is k–*hyperperfect* number. If $k = 1$ we say that n is a *perfect* number.

The conjecture of McCranie [2000] states: *the number* $n = p^2q$ *is a k–hyperperfect number if* $k \in 2\mathbb{N}^* + 1$, $p = \frac{3k+1}{2}$, $q = 3k + 4$, p *and* q *prime numbers.*

If p and q are distinct odd prime numbers such that $k(p + q) = pq - 1$ for a $k \in \mathbb{N}^*$, then $n = pq$ is k–*hyperperfect*.

If $k \in \mathbb{N}^*$ and $p = k + 1$ is prime, then, if there exists a $j \in \mathbb{N}^*$ such that $q = p^j - p + 1$ prime, then $n = p^{j-1}q$ is k–*hyperperfect*.

The first k-hyperperfect numbers are: 6, 21, 28, 301, 325, 496, 697, 1333, ... [Sloane, 2014, A034897], which correspond to the values of k: 1, 2, 1, 6, 3, 1, 12, 18, ... McCranie [2000] gave the list of all *hyperperfect* numbers up to 10^{11}.

Chapter 4

Euler's totient function φ

Euler's totient function, denoted φ, counts the number of factors relative prime to n, where 1 is considered relative prime to every natural number. For example, factors relative prime to 36 are 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35 and, therefore, it results that $\varphi(36) = 12$. By convention, we have $\varphi(0) = 1$.

Program 4.1. The program for computing the values of Euler's totient function which applies the definition of the function is

$$\varphi(n) := \left| \begin{array}{l} return\ 1\ if\ n\texttt{=}0 \\ j \leftarrow 0 \\ for\ k \in 1..n \\ \quad j \leftarrow j + 1\ if\ \gcd(k, n)\texttt{=}1 \\ return\ j \end{array} \right.$$

This program can not be used for computing the values of Euler's totient function for great numbers.

Function $n - \varphi(n)$ is called cototient function.
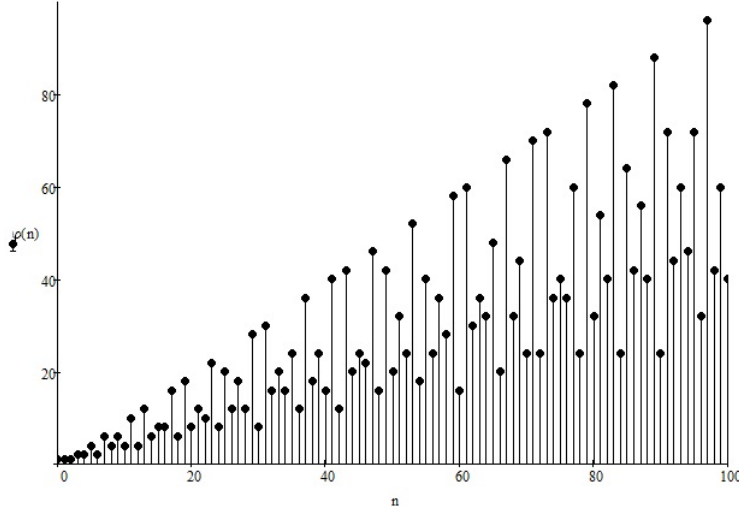
Figure 4.1: Euler's totient function

4.1 The properties of function φ

For p prime number we have $\varphi(p) = p - 1$, and

$$\varphi(p^\alpha) = p^{\alpha-1}(p - 1) = p^\alpha \left(1 - \frac{1}{p} \right) .$$

Let m be a prime multiple of p. We define function $\varphi_p(m)$ which counts the positive integers $\leq m$ which are not divisible by p. As p, $2p$, ..., $\frac{m}{p}p$ have common factor p, it follows that

$$\varphi_p(m) = m - \frac{m}{p} = m \left(1 - \frac{1}{p} \right) . \tag{4.1}$$

Let q be another prime number that divides m, or let m be a multiple of q. Then q, $2q$, ..., $\frac{m}{q}q$ have q common factor, but there exist also duplicate common factors pq, $2pq$, ..., $\frac{m}{pq}pq$. Therefore, the number of terms that

have to be subtracted from $\varphi_p(m)$ to obtain $\varphi_{pq}(m)$ is

$$\frac{m}{q} - \frac{m}{pq} = \frac{m}{q}\left(1 - \frac{1}{p}\right) . \tag{4.2}$$

Then, from (4.1) and (4.2) it results that

$$\varphi_{pq}(m) = m\left(1 - \frac{1}{p}\right) - \frac{m}{q}\left(1 - \frac{1}{p}\right) = m\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) . \tag{4.3}$$

Similarly, by mathematical induction it can be proved that if n is divisible by p_1, p_2, ..., p_s, prime numbers (or n is a multiple of p_1, p_2, ..., p_s, prime numbers), then we have

$$\varphi(n) = n\prod_{k=1}^{s}\left(1 - \frac{1}{p_k}\right) . \tag{4.4}$$

We have an interesting identity, due to Olofsson [2004], regarding $\varphi(n^\ell)$ and $\varphi(n)$, given by relation

$$\varphi(n^\ell) = n^{\ell-1}\varphi(n) . \tag{4.5}$$

Euler's totient function satisfies the inequality $\varphi(n) > \sqrt{n}$ for all $n \in \mathbb{N}^*$ excepting 2 and 6, [Kendall and Osborn, 1965], [Mitrinović and Sándor, 1995, p. 9]. Consequently, $\varphi(n) = 2$ only for $n = 3$, $n = 4$ and $n = 6$. Also, in the monograph [Sierpiński, 1988], was proved that $\varphi(n) \leq n - \sqrt{n}$.

The solutions of the φ–Diophantine equation $\varphi(n) = \varphi(n+1)$ are: 1, 3, 15, 104, 164, 194, 255, 495, 584, 975, ... [Sloane, 2014, A003275].

In the search domain $D_c = \left\{1, 2, \ldots, 10^{10}\right\}$ there exists only one solution $n = 5186 = 2^5 \cdot 3^4$ for which the double identity $\varphi(n) = \varphi(n+1) = \varphi(n+2)$ holds, [Guy, 2004, p. 139].

The smallest three close numbers (the difference between them is ≤ 6), for which the double equality $\varphi(n_1) = \varphi(n_2) = \varphi(n_3)$ holds, are: 404471, 404473 and 404477. These numbers verify the equalities:

$$\varphi(404471) = \varphi(404473) = \varphi(404477) = 403200 .$$

The smallest four close numbers (the difference between them is ≤ 12), for which the triple equality $\varphi(n_1) = \varphi(n_2) = \varphi(n_3) = \varphi(n_4)$ hold, are: 25930, 25935, 24940 and 25942. They verify the equalities:

$$\varphi(25930) = \varphi(25935) = \varphi(25940) = \varphi(25942) = 10368 .$$

These results were published in [Guy, 2004, p. 139].

McCranie [2000] found the arithmetic progression $a_k = a_0 + k \cdot r$, where the first term is $a_0 = 583200$ and $r = 30$ is the ratio, for which we have

$$\varphi(a_k) = 155520 \text{ for all } k = 0, 1, \ldots, 5 .$$

Other arithmetic progressions with six consecutive terms, with $a_0 = 1166400$ and $r = 583200$, which have the same property, are also known [Sloane, 2014, A050518].

An interesting conjecture due to Guy [2004] has following predication. *If Goldach's conjecture holds, then, for every $m \in \mathbb{N}^*$, there exist the prime numbers p and q such that $\varphi(p) + \varphi(q) = 2m$.* Erdös wondered if this statement also holds for p and q not necessarily primes, but this "relaxed" conjecture remains unproved.

Guy [2004] considered the φ–σ–Diophantine equation $\varphi(\sigma(n)) = n$. F. Helenius found 365 solutions, of which the first are: 2, 8, 12, 128, 240, 720, 6912, 32768, 142560, 712800, ..., [Sloane, 2014, A001229].

4.1.1 Computing the values of φ function

Program 4.2. Considering formula (4.5), an efficient program for computing the values of function φ can be written.

$$\varphi(n) := \begin{vmatrix} return \ 1 \ if \ n\text{=}0 \vee n\text{=}1 \\ f \leftarrow Fa(n) \\ \phi \leftarrow n \\ for \ k \in 1..rows(f) \\ \quad \phi \leftarrow \phi \cdot \dfrac{f_{k,1} - 1}{f_{k,1}} \\ return \ \phi \end{vmatrix}$$

This program calls the program 1.29 for factorization of a number.

Program 4.3. The program by means of which the file $\varphi.prn$ is generated is:

$$G\varphi(N) := \left| \begin{array}{l} f\varphi_1 \leftarrow 1 \\ for\ n \in 2..N \\ \quad f\varphi_n \leftarrow \varphi(n) \\ return\ f\varphi \end{array} \right.$$

This program calls the program 4.2 for computing the values of Euler's totient function. The sequence for generating the file $\varphi.prn$ is:

$$t_0 := time(0)\ \ WRITEPRN("\varphi.prn") := G\varphi(10^6)\ \ t_1 := time(1)$$

$$(t_1 - t_0)sec = "5:30:33.558"hhmmss$$

The execution time for generating the values of function φ up to 10^6 is of 5 hours and 30 minutes on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

4.2   A generalization of Euler's theorem

In the sections which follow we will prove a result which replaces the theorem of Euler: "If $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1\,(mod\ m)$", for the case when a and m are not relatively primes.

One supposes that $m > 0$. This assumption will not affect the generalization, because Euler's indicator satisfies the equality: $\varphi(m) = \varphi(-m)$, [Popovici, 1973], and that the congruencies verify the following property: $a \equiv b\,(mod\ m) \Leftrightarrow a \equiv b\,(mod\ -m)$, [Popovici, 1973, pp. 12–13].

In the case of congruence modulo 0, there is the relation of equality. One denotes $(a, b) = gcf(a, b)$ greatest common factor of the two integers a and b, and one chooses $(a, b) > 0$. Note gcf is the same as gcd for numbers, so $(a, b) = gcf(a, b) = \gcd(a, b)$.

Lemma 4.4. *Let be a an integer and m a natural number > 0. The exist $d_0, m_0 \in \mathbb{N}$ such that $a = a_0 d_0$, $m = m_0 d_0$ and $(a_0, m_0) = 1$.*

Proof. It is sufficient to choose $d_0 = (a, m)$. In accordance with the definition of the gcf (greatest common factor), the quotients of a_0 and m_0 of a and m by their gcf are relatively primes, [Creangă et al., 1965, pp. 25–26]. □

Lemma 4.5. *With the notations of lemma 4.4, if $d_0 \neq 1$ and if: $d_0 = d_0^1 d_1$, $m_0 = m_1 d_1$, $(d_0^1, m_1) = 1$ and $d_1 \neq 1$, then $d_0 > d_1$ and $m_0 > m_1$, and if $d_0 = d_1$, then after a limited number of steps i one has $d_0 > d_{i+1} = (d_i, m_i)$.*

Proof.

$$(0) \begin{cases} a = a_0 d_0 & ; \quad (a_0, m_0) = 1 \\ m = m_0 d_0 & ; \quad d_0 \neq 1 \end{cases},$$

$$(1) \begin{cases} d_0 = d_0^1 d_1 & ; \quad (d_0^1, m_1) = 1 \\ m_0 = m_1 d_1 & ; \quad d_1 \neq 1 \end{cases}.$$

From (0) and from (1) it results that $a = a_0 d_0 = a_0 d_0^1 d_1$ therefore $d_0 = d_0^1 d_1$ thus $d > d_1$ if $d_0^1 \neq 1$.

From $m_0 = m_1 d_1$ we deduct that $m_0 > m_1$. If $d_0 = d_1$ then $m_0 = m_1 d_0 = k \cdot d_0^z$, where $z \in \mathbb{N}^*$ and $d_0 \nmid k$. Therefore $m_1 = k \cdot d_0^{k-1}$; $d_2 = (d_1, m_1) = (d_0, k \cdot d_0^{z-1})$. After $i = z$ steps, it results $d_{i+1} = (d_0, k) < d_0$. □

Lemma 4.6. *For each integer a and for each natural number $m > 0$ one can build the following sequence of relations:*

$$(0) \begin{cases} a = a_0 d_0 & ; \quad (a_0, m_0) = 1 \\ m = m_0 d_0 & ; \quad d_0 \neq 1 \end{cases},$$

$$(1) \begin{cases} d_0 = d_0^1 d_1 & ; \quad (d_0^1, m_1) = 1 \\ m_0 = m_1 d_1 & ; \quad d_1 \neq 1 \end{cases},$$

$$\cdots\cdots\cdots$$

$$(s-1) \begin{cases} d_{s-2} = d_{s-2}^1 d_{s-1} & ; \quad (d_{s-2}^1, m_{s-1}) = 1 \\ m_{s-2} = m_{s-1} d_{s-1} & ; \quad d_{s-1} \neq 1 \end{cases},$$

$$(s) \begin{cases} d_{s-1} = d_{s-1}^1 d_s & ; \quad \left(d_{s-1}^1, m_s\right) = 1 \\ m_{s-1} = m_s d_s & ; \quad d_s \neq 1 \end{cases} .$$

Proof. One can build this sequence by applying lemma 4.4. The sequence is limited, according to lemma 4.5, because after r_1 steps, one has $d_0 > d_{r_1}$, and $m_0 > m_{r_1}$, and after r_2 steps, one has $d_{r_1} > d_{r_1 + r_2}$ and $m_{r_1} > m_{r_1 + r_2}$, etc., and the m_i are natural numbers. One arrives at $d_s = 1$ because if $d_s \neq 1$ one will construct again a limited number of relations $(s+1)$, ..., $(s+r)$ with $d_{s+r} < d_s$. □

Theorem 4.7. *Let us have* $a, m \in \mathbb{Z}$, *and* $m \neq 0$. *Then*

$$a^{\varphi(m_s)+s} \equiv a^s \ (mod \ m) \ ,$$

where s *and* m_s, *are the same ones as in the lemmas above.*

Proof. Similar with the method followed previously, one can suppose $m > 0$ without reducing the generality. From the sequence of relations from lemma 4.6, it results that:

$$\begin{array}{cccccc} (0) & (1) & (2) & (3) & & (s) \\ a & = a_0 d_0 & = a_0 d_0^1 d_1 & = a_0 d_0^1 d_1^1 d_2 & \cdots & = a_0 d_0^1 d_1^1 \cdots d_{s-1}^1 d_s \end{array}$$

and

$$\begin{array}{cccccc} (0) & (1) & (2) & (3) & & (s) \\ m & = m_0 d_0 & = m_1 d_1 d_0 & = m_2 d_2 d_1 d_0 & \cdots & = m_s d_s d_{s-1} \cdots d_1^1 d_0 \end{array}$$

and

$$m_s d_s d_{s-1} \cdots d_1 d_0 = d_0 d_1 \cdots d_{s-1} d_s m_s \ .$$

From (0) it results that $d_0 = (a, m)$, and from (i) that $d_i = (d_{i-1}, m_{i-1})$, for all $i \in \{1, 2, \ldots, s\}$.

$$\begin{aligned} d_0 &= d_0^1 d_1^1 d_2^1 \cdots d_{s-1}^1 d_s \ , \\ d_1 &= d_1^1 d_2^1 d_3^1 \cdots d_{s-1}^1 d_s \ , \\ &\vdots \\ d_{s-1} &= d_{s-1}^1 d_s \ , \\ d_s &= d_s \ . \end{aligned}$$

Therefore

$$d_0 d_1 d_2 \cdots d_{s-1} d_s = (d_0^1)^1 (d_1^1)^2 (d_2^1)^3 \cdots (d_{s-1}^1)^s (d_s^1)^{s+1}$$
$$= (d_0^1)^1 (d_1^1)^2 (d_2^1)^3 \cdots (d_{s-1}^1)^s,$$

because $d_s = 1$.

Thus $m = (d_0^1)^1 (d_1^1)^2 (d_2^1)^3 \cdots (d_{s-1}^1)^s \cdot m_s$; therefore $m_s \mid m$.

$$(d_s, m_s) \overset{(s)}{=} (1, m_s)$$

and $\left(d_{s-1}^1, m_s\right) = 1$

$$1 \overset{(s-1)}{=} \left(d_{s-2}^1, m_{s-1}\right) = \left(d_{s-2}^1, m_s d_s\right)$$

therefore $\left(d_{s-2}^1, m_s\right) = 1$,

$$1 \overset{(s-2)}{=} \left(d_{s-3}^1, m_{s-2}\right) = \left(d_{s-3}^1, m_{s-1} d_{s-1}\right) = \left(d_{s-3}^1, m_s d_s d_{s-1}\right)$$

therefore $\left(d_{s-3}, m_s\right) = 1$,

$$\vdots$$

$$1 \overset{(i+1)}{=} \left(d_i^1, m_{i+1}\right) = \left(d_i^1, m_{i+2} d_{i+2}\right) = \left(d_i^1, m_{i+3} d_{i+3} d_{i+2}\right) = \ldots$$
$$= \left(d_i^1, m_s d_s d_{s-1} \cdots d_{i+2}\right),$$

thus $\left(d_i^1, m_s\right) = 1$, and this is for all $i \in \{1, 2, \ldots, s-2\}$,

$$\vdots$$

$$1 \overset{(0)}{=} \left(a_0, m_0\right) = \left(a_0, d_1 \cdots d_{s-1} d_s m_s\right)$$

,

thus $\left(a_0, m_s\right) = 1$.

From the Euler's theorem results that: $(d_i^1)^{\varphi(m_s)} \equiv 1 \, (mod \, m_s)$ for all $i \in \{0, 1, \ldots, s\}$, $a_0^{\varphi(m_s)} \equiv 1 \, (mod \, m_s)$, but

$$a_0^{\varphi(m_s)} = a_0^{\varphi(m_s)}(d_0^1)^{\varphi(m_s)}(d_1^1)^{\varphi(m_s)} \cdots (d_{s-1}^1)^{\varphi(m_s)} \, ,$$

therefore $a^{\varphi(m_s)} \equiv \underbrace{1 \cdots 1}_{(s+1) \; times} \, (mod \, m_s)$, then $a^{\varphi(m_s)} \equiv 1 \, (mod \, m_s)$. We

equivalence

$$a_0(d_0^1)^{s-1}(d_1^1)^{s-2}(d_2^1)^{s-3} \cdots (d_{s-2}^1)^1 \cdot a^{\varphi(m_s)}$$
$$\equiv a_0^s(d_0^1)^{s-1}(d_1^1)^{s-2} \cdots (d_{s-2}^1)^1 \cdot 1 \, (mod \, m_s) \, .$$

If you multiply the $(d_0^1)^1(d_1^1)^2(d_2^1)^3 \cdots (d_{s-2}^1)^{s-1}(d_{s-1}^1)^s$ we obtain:

$$a_0^s(d_0^1)^s(d_1^1)^s \cdots (d_{s-2}^1)^s(d_{s-1}^1)^s \cdot a^{\varphi(m_s)}$$
$$\equiv a_0^s(d_0^1)^s(d_1^1)^s \cdots (d_{s-2}^1)^s(d_{s-1}^1)^s \, \left(mod \, (d_0^1)^1 \cdots (d_{s-1}^1)^s \cdot m_s\right) \, ,$$

but $a_0^s(d_0^1)^s(d_1^1)^s \cdots (d_{s-1}^1)^s \cdot a^{\varphi(m_s)}$ and $a_0^s(d_0^1)^s(d_1^1)^s \cdots (d_{s-1}^1)^s = a^s$ there-fore $a^{\varphi(m_s)+s} \equiv a^s \, (mod \, m)$, for all $a, m \in \mathbb{Z}, m \neq 0$. $\square$

Observation 4.8. If $\left(a, m\right) = 1$ then $d = 1$. Thus $s = 0$, and according the theorem 4.7 one has $a^{\varphi(m_0)+0} \equiv a^0 \, (mod \, m)$ therefore $a^{\varphi(m_0)} \equiv 1 \, (mod \, m)$. But $m = m_0 d_0 = m_0 \cdot 1 = m_0$. Thus $a^{\varphi(m)} \equiv 1 \, (mod \, m)$, and one obtains Euler's theorem.

Observation 4.9. Let us have a and m two integers, $m \neq 0$ and $\left(a, m\right) = d_0 \neq 1$, and $m = m_0 d_0$. If $\left(d_0, m_0\right) = 1$, then $a^{\varphi(m_0)+1} \equiv a \, (mod \, m)$. Which, in fact, it results from the theorem 4.7 with $s = 1$ and $m_1 = m_0$. This relation has a similar to Fermat's theorem: $a^{\varphi(p)+1} \equiv a \, (mod \, p)$.

4.2.1   An algorithm to solve congruences

One will construct an algorithm to calculate s and m_s of the theorem 4.7.

Program 4.10. The program is:

$$S(a,m) := \begin{vmatrix} s \leftarrow 0 \\ m_s \leftarrow m \\ d \leftarrow \gcd(a, m_s) \\ while \ d \neq 1 \\ \quad \begin{vmatrix} s \leftarrow s + 1 \\ m_s \leftarrow \dfrac{m_s}{d} \\ d \leftarrow \gcd(d, m_s) \end{vmatrix} \\ return \ \begin{pmatrix} s \\ m_s \end{pmatrix} \end{vmatrix}$$

The program calls the function Mathcad gcd computation of the greatest common divisor.

4.2.2 Applications

In the resolution of the exercises one uses the theorem 4.7 and the algorithm to calculate s and m_s.

Example 1: $6^{\varphi(m_s)} \equiv 6^s \ (mod \ 105765)$. One thus applies the algorithm to calculate s and m_s and then the theorem 4.7:

$$a := 6 \quad m := 105765$$

$$\begin{pmatrix} s \\ m_s \end{pmatrix} := S(a, m) = \begin{pmatrix} 1 \\ 35255 \end{pmatrix}$$

$$\phi := \varphi(m_s) + s = 25601$$

$$mod(a^{\phi}, m) = 6 \quad mod(a^s, m) = 6 \,,$$

where we used the programs 4.10 and 4.2.

Example 2: $847^{\varphi(m_s)} \equiv a^s \ (mod \ 283618125)$. One thus applies the algorithm to calculate s and m_s and then the theorem 4.7:

$$a := 847 \quad m := 283618125$$

$$\begin{pmatrix} s \\ m_s \end{pmatrix} := S(a, m) = \begin{pmatrix} 5 \\ 16875 \end{pmatrix}$$

$$\phi := \varphi(m_s) + s = 9005$$

$$mod(a^\phi, m) \rightarrow 7^5 \cdot 9601 \quad mod(a^s, m) \rightarrow 7^5 \cdot 9601 \ ,$$

where we used the programs 4.10 and 4.2.

Example 2: $847^{\varphi(m_s)} \equiv a^s \ (mod \ 283618125)$. One thus applies the algorithm to calculate s and m_s and then the theorem 4.7:

$$a := 847 \quad m := 283618125$$

$$\begin{pmatrix} s \\ m_s \end{pmatrix} := S(a, m) = \begin{pmatrix} 5 \\ 16875 \end{pmatrix}$$

$$\phi := \varphi(m_s) + s = 9005$$

$$mod(a^\phi, m) \rightarrow 7^5 \cdot 9601 \quad mod(a^s, m) \rightarrow 7^5 \cdot 9601 \ ,$$

where we used the programs 4.10 and 4.2.

Example 3: $437^{\varphi(m_s)} \equiv a^s \ (mod \ 2058557375)$. One thus applies the algorithm to calculate s and m_s and then the theorem 4.7:

$$a := 437 \quad m := 2058557375$$

$$\begin{pmatrix} s \\ m_s \end{pmatrix} := S(a, m) = \begin{pmatrix} 3 \\ 300125 \end{pmatrix}$$

$$\phi := \varphi(m_s) + s = 205803$$

$$mod(a^\phi, m) \rightarrow 19^3 \cdot 23^3 \quad mod(a^s, m) \rightarrow 19^3 \cdot 23^3 \ ,$$

where we used the programs 4.10 and 4.2.

Example 4: $4433^{\varphi(m_s)} \equiv a^s \ (mod \ 789310951)$. One thus applies the algorithm to calculate s and m_s and then the theorem 4.7:

$$a := 4433 \quad m := 789310951$$

$$\begin{pmatrix} s \\ m_s \end{pmatrix} := S(a, m) = \begin{pmatrix} 5 \\ 29 \end{pmatrix}$$

$$\phi := \varphi(m_s) + s = 33$$

$$mod(a^\phi, m) \rightarrow 2^3 \cdot 11^5 \cdot 13^2 \quad mod(a^s, m) \rightarrow 2^3 \cdot 11^5 \cdot 13^2 \ ,$$

where we have used the programs 4.10 and 4.2.

Chapter 5

Generalization of congruence theorems

5.1 Notions introductory

Let us consider a positive integer, which we will call *modulus*. With its help we introduce in the set $\mathbb{Z}$ of integers a binary relation, called *of congruence* and denoted $\equiv$, such that:

Definition 5.1. The integers a and b are congruent relative to modulus m is and only if m divides the difference $a - b$.

Hence, we have

$$a \equiv b \ (mod \ m) \ \Leftrightarrow \ a - b = k \cdot m, \ \text{where } k \in \mathbb{Z}. \tag{5.1}$$

Consequence 5.2. $a \equiv b \ (mod \ m) \Leftrightarrow a$ and b give, by division trough m, the same residue.

It is known that the congruence relation given by (5.1) is an equivalence relation (is reflexive, symmetric and transitive). It also has following remarkable properties:

$$a_1 \equiv b_1 \ (mod \ m) \text{ and } a_2 \equiv b_2 \ (mod \ m) \Rightarrow,$$

(i) $a_1 + a_2 \equiv b_1 + b_2 \pmod{m}$,

(ii) $a_1 - a_2 \equiv b_1 - b_2 \pmod{m}$,

(iii) $a_1 a_2 \equiv b_1 b_2 \pmod{m}$.

More generally, if $a_i \equiv b_i \pmod{m}$, for $i = 1, 2, \ldots, n$, and $f(x_1, x_2, \ldots, x_n)$ is a polynomial with integer coefficients, then

(iv) $f(a_1, a_2, \ldots, a_n) \equiv f(b_1, b_2, \ldots, b_n) \pmod{m}$.

One can also prove following properties of the congruence relations:

(v) $a \equiv b \pmod{m}$ and $c \in \mathbb{N}^* \Rightarrow ac \equiv bc \pmod{m}$,

(vi) $a \equiv b \pmod{m}$ and $n \in \mathbb{N}^*$, n divide $m \Rightarrow a \equiv b \pmod{n}$,

(vii) $a \equiv b \pmod{m_i}$, $i = \overline{1, s}$, $\Rightarrow a \equiv b \pmod{m}$,

where $m = [m_1, m_2, \ldots, m_s] = lcm(m_1, m_2, \ldots, m_s)$ is the smallest common multiple of numbers m_i.

(viii) $a \equiv b \pmod{m} \Rightarrow (a, m) = (b, m)$,

where by $(x, y) = gcd(x, y)$ we denote the greatest common divisor of numbers x and y.

As the relation *congruence mod m* is an equivalence relation, it divides the set $\mathbb{Z}$ of integers into equivalence classes (classes of *congruence mod m*). Two such classes either are disjoint or coincide.

As every integer provides by division through m one of the residues 0, $1, 2, \ldots, m - 1$, it follows that

$$C_0, \ C_1, \ \ldots, \ C_{m-1}$$

are the m classes of residues *mod m*, where C_i is the set of all integers congruent with $i \pmod{m}$.

Sometimes it is useful to consider, instead of the classes, representatives that satisfy certain conditions. Hereby, following terminology is established.

Definition 5.3. The integers a_1, a_2, ..., a_m compose *a complete system of mod m residues* if any two of them are not congruent *mod m*.

It results that a complete system of *mod m* residues contains a representative of each class.

If φ is Euler's totient function ($\varphi(n)$, denoted also φ_n, is the number of natural numbers smaller than n and prime to n), then we also have:

Definition 5.4. The integers a_1, a_2, ..., $a_{\varphi(m)}$ build *a reduced system of mod m residues* if each is prime with the modulus and if any two of them are not congruent *mod m*.

Following result is known:

Theorem 5.5.

1. *If a_1, a_2, ..., a_m is a complete system of mod m residues and a is an integer, prime to m, then the sequence $a \cdot a_1$, $a \cdot a_2$, ..., $a \cdot a_m$ is also a complete system of mod m residues.*

2. *If a_1, a_2, ..., $a_{\varphi(m)}$ is a reduced system of mod m residues and a is an integer, prime to m, then the sequence $a \cdot a_1$, $a \cdot a_2$, ..., $a \cdot a_{\varphi(m)}$ is also a reduced system of mod m residues.*

If we denote by Z_m the set of the classes of residues mod m:

$$Z_m = \{C_0, C_1, \ldots, C_{m-1}\}$$

and we define the relations

$$+ : Z_m \times Z_m \to Z_m , \quad \cdot : Z_m \times Z_m \to Z_m ,$$

by

$$C_i + C_j = C_k, \text{ where } k \equiv i + j \ (mod \ m) ,$$

$$C_i \cdot C_j = C_h, \text{ where } h \equiv i \cdot j \ (mod \ m) ,$$

then following result holds:

Theorem 5.6.

1. $(Z_m, +)$ *is a commutative group,*

2. $(Z_m, +, \cdot)$ *is a commutative ring,*

3. $(G_m, \cdot)$ *is a commutative group,*

where $G_m = \left\{ C_{r_1}, C_{r_2}, \ldots, C_{r_{\varphi(m)}} \right\}$ the set of the classes of residues prime to the modulus.

Consequence 5.7. The set Z_p of the classes of residues relative to a prime modulus p builds a commutative field relative to the previously defined operations of addition and multiplication.

5.2   Theorems of congruence of the Number Theory

In this section we will recall some congruence theorems of the Number Theory (Theorems of Fermat, Euler, Wilson, Gauss, Lagrange, Leibniz, Moser and Sierpinski) which we will generalize in the next section. Equally, we will emphasize a unifying point of view.

In 1640 Fermat states, without proof, the next result:

Theorem 5.8 (Fermat). *If integer a is not divisible by the prime number p, then*

$$a^{p-1} \equiv 1 \ (mod \ p) \ . \tag{5.2}$$

The first proof of this theorem was given in 1736 by Euler.

As it is known, the reciprocal of Fermat's Theorem is not true. In another words, the fact that $a^{m-1} \equiv 1 \ (mod \ m)$ and m is not divisible by a, does not necessarily imply that m is a prime number.

It is not even true that, if (5.2) holds for all numbers a prime relative to m, then m is prime, as one can remark in the following example.

Example 5.9. Let $m = 561 = 3 \cdot 11 \cdot 17$. If a is an integer that is not divisible by 3, by 11 or by 17, we surely have:

$$a^2 \equiv 1 \ (mod \ 3), \ a^{10} \equiv 1 \ (mod \ 11), \ a^{16} \equiv 1 \ (mod \ 17) \ ,$$

according to the direct Theorem of Fermat. But, as 560 is divisible by 2 and 10, as well as by 16, we deduce that:

$$a^{560} \equiv 1 \ (mod \ m_i) \, , \ i = 1, 2, 3 \, ,$$

where $m_1 = 3$, $m_2 = 11$ and $m_3 = 17$.

According to property (vii) of the previous section, it follows that

$$a^{560} \equiv 1 \ (mod \ m) \, , \ \text{for } m = 561.$$

Actually, it is known that 561 is the smallest composite number that satisfies (5.2). Next numbers follow: $1105, 1729, 2465, 2821, \dots$.

Consequently, the congruence (5.2) can be true for a certain integer a and a composite number m.

Definition 5.10. If relation (5.2) is satisfied for a composite number m and an integer a, it is said that m is *pseudoprime relative to a*. If m is pseudoprime relative to every integer a, prime to m, it is said that m is a *Carmichael number*.

The American mathematician Robert Carmichael was the first who, in 1910 has emphasized such numbers, called *fake prime numbers*.

Until recently, it was not known if there exists or not an infinity of Carmichael numbers. In the very first issue of the journal "What's Happening in the Mathematical Sciences", where, yearly, the most important recent results in mathematics are emphasized, it is that three mathematicians: Alford, Granville and Pomerance, have proved that there exists an infinity of Carmichael numbers.

The proof of the trio of American mathematicians is based on an heuristic remark from 1956 of the internationally known Hungarian mathematician P. Erdös. The main idea is to chose a number L for which there exist a lot of prime numbers p that do not divide L, but having the property that $p - 1$ divides L. Afterwards it is shown that these prime numbers can be multiplied among themselves in several ways such that each product is congruent with $1 \ (mod \ L)$. It results that every such product is a Carmichael number.

For example, for $L = 120$, the prime numbers that satisfy the previous condition are: $p_1 = 7$, $p_2 = 11$, $p_3 = 13$, $p_4 = 31$, $p_5 = 41$, $p_6 = 61$. It follows that $41041 = 7 \cdot 11 \cdot 13 \cdot 41$, $172081 = 7 \cdot 13 \cdot 31 \cdot 61$ and $852841 = 11 \cdot 31 \cdot 41 \cdot 61$ are congruent with $1 \pmod{120}$, and, hence, they are Carmichael numbers.

We mention that the heuristic remark of P. Erdös is based on the following theorem that characterizes the Carmichael numbers, proved in 1899.

Theorem 5.11 (A. Korselt). *The number n is a Carmichael number if and only if following conditions hold:*

(C_1) *n is squares free,*

(C_2) *$p - 1$ divides $n - 1$ as long as p is a prime divisor of n.*

The three American mathematicians have proved the following result:

Theorem 5.12 (Alford, Granville, Pomerance). *There exist at least $x^{2/7}$ Carmichael numbers, not greater than x, for x sufficiently big.*

By means of the heuristic argument due to P. Erdös it can be proved that the exponent of Theorem 5.12 can be replaced by any other sub unitary exponent.

Theorem 5.13 (Euler). *If $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 \pmod{m}$.*

The notation $(a, m) = 1$ means that the greatest common divisor of a and m is 1, which means that the numbers are relatively prime.

Theorem 5.13 generalizes Theorem 5.8 and was proved by Euler in 1760.

Theorem 5.14 (Wilson). *If p is a prime number, then $(p - 1)! + 1 \equiv 0 \pmod{p}$.*

It is known that the reciprocal of Theorem 5.14 is true, which means that following result holds

Theorem 5.15. *If $n > 1$ is an integer and $(n - 1)! + 1 \equiv 0 \pmod{n}$ then n is prime.*

Theorem 5.14, of Wilson, was published in 1770 by mathematician Waring (*Meditationes Algebraicae*), but it was known long before, by Leibniz.

Lagrange generalizes Theorem 5.14, of Wilson, as follows:

Theorem 5.16 (Lagrange). *If p is a prime number, then $a^{p-1} - 1 \equiv (a+1)(a+2)\dots(a+p-1) \pmod{p}$.*

Leibniz states following theorem:

Theorem 5.17 (Leibniz). *If p is a prime number, then $(p-2)! \equiv 1 \pmod{p}$.*

The reciprocal of Theorem 5.17, of Leibniz, is also true, i.e. a natural number $n > 1$ is prime if and only if $(n-2)! \equiv 1 \pmod{p}$.

Another result concerning congruences with prime numbers is the next theorem:

Theorem 5.18 (L. Moser). *If p is a prime number, then $(p-1)!a^p + a \equiv 0 \pmod{p}$.*

Sierpinski proves that following result holds:

Theorem 5.19 (Sierpinski). *If p is a prime number, then $a^p + (p-1)! \equiv 0 \pmod{p}$.*

We remark that this statement unify the Theorems 5.8 of Fermat and 5.14 of Wilson.

In the next section we will define a function $L : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, by means of which we will be able to prove several results that unify all previous theorems.

5.3   A unifying point of convergence theorems

Let A be the set $\{m \in \mathbb{Z}/m = \pm p^\beta, \pm 2p^\beta\}$ with p an odd prime, $\beta \in \mathbb{N}^*$, or $m = \pm 2^\alpha$, with $\alpha = 0, 1, 2$, or $m = 0$.

Let $m = \varepsilon p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, with $\varepsilon = \pm 1$, all $\alpha_i \in \mathbb{N}^*$ and $p_1, p_2, \dots, p_r$ are distinct positive primes.

We construct the function $L : \mathbb{Z} \times \mathbb{Z}$,

$$L(x, m) = (x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) , \qquad (5.3)$$

where C_1, C_2, $\ldots C_{\varphi(m)}$ are all modulo m rests relatively prime to m, and φ is Euler's function.

If all distinct primes which divide x and m simultaneously are p_{i_1}, p_{i_2}, $\ldots p_{i_r}$, then:

$$L(x, m) \equiv \pm 1 \left(mod \ p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}} \right) \text{ when } m \in A \qquad (5.4)$$

respectively $m \notin A$, and

$$L(x, m) \equiv 0 \left(mod \ m / \left(p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}} \right) \right) . \qquad (5.5)$$

For $d = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$, and $m' = m/d$ we find

$$L(x, m) \equiv \pm 1 + k_1^0 d \equiv k_2^0 m' \pmod{m} , \qquad (5.6)$$

where k_1^0 and k_2^0 constitute a particular integer solution of the Diophantine equation $k_2 m' - k_1 d = \pm 1$ (the signs are chosen in accordance with the affiliation of m to A).

This result generalizes Gauss' theorem, ($C_1 \cdot C_2 \cdots C_{\varphi(m)} \equiv \pm 1 \pmod{m}$ when $m \in A$ respectively $m \notin A$), see [Dirichlet, 1894], which generalized in its turn the Wilson's theorem (if p is prime then $(p - 1)! \equiv -1 \pmod{m}$).

Lemma 5.20. *If C_1, C_2, $\ldots$, $C_{\varphi(p^\alpha)}$ are all modulo p^α rests, relatively prime to p^α, with p an integer and $\alpha \in \mathbb{N}^*$, then for $k \in \mathbb{Z}$ and $\beta \in \mathbb{N}^*$ we have also that $kp^\beta + C_1$, $kp^\beta + C_2$, $\ldots$, $kp^\beta + C_{\varphi(p^\alpha)}$ constitute all modulo p^α rests relatively prime to p^α.*

Proof. It is sufficient to prove that for $1 \leq i \leq \varphi(p^\alpha)$ we have $kp^\beta + C_i$ relatively prime to p^α, but this is obvious. $\square$

Lemma 5.21. *If C_1, C_2, $\ldots$, $C_{\varphi(m)}$ are all modulo m rests relatively prime to m, $p_i^{\alpha_i}$ divides m and $p_i^{\alpha_i+1}$ does not divide m, then C_1, C_2, $\ldots$, $C_{\varphi(m)}$ constitute $\varphi(m/p_i^{\alpha_i})$ systems of all modulo $p_i^{\alpha_i}$ rests relatively prime to $p_i^{\alpha_i}$.*

Proof. Proof is obvious. □

Lemma 5.22. *If C_1, C_2, ..., $C_{\varphi(m)}$ are all modulo q rests relatively prime to b and $(b, q) = 1$ then $b + C_1$, $b + C_2$, ..., $b + C_{\varphi(q)}$ contain a representative of the class $\widehat{0}$ modulo q.*

Proof. Of course, because $(b, q - b) = 1$ there will be a $C_{i_0} = q - b$, whence $b + C_{i_0} = \mathcal{M}q$ (multiple of q). □

From this we have:

Theorem 5.23. *If $\left(x, m/(p_1^{\alpha_{i_1}} \cdot p_2^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}})\right) = 1$ then*

$$L(x, m) = (x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)})$$
$$\equiv 0 \left(mod\ m/\left(p_1^{\alpha_{i_1}} \cdot p_2^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}\right)\right)\ .$$

Proof. Proof is obvious. □

Lemma 5.24. *Because $C_1 \cdot C_2 \cdots C_{\varphi(m)} \equiv \pm 1\ (mod\ m)$ it results that*

$$C_1 \cdot C_2 \cdots C_{\varphi(m)} \equiv \pm 1\ (mod\ p_i^{\alpha_i})\ ,$$

for all i, when $m \in A$, respectively $m \notin A$.

Proof. Proof is obvious. □

Lemma 5.25. *If p_i divides x and m simultaneously, then*

$$(x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) \equiv \pm 1\ (mod\ p_i^{\alpha_i})\ ,$$

when $m \in A$ respectively $m \notin A$.

Proof. Of course, from the lemmas 5.21 and 5.20, respectively 5.24, we have

$$(x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) \equiv C_1 \cdot C_2 \cdots C_{\varphi(m)} \equiv \pm 1\ (mod\ p_i^{\alpha_i})\ .$$

□

From the lemma 5.25 we obtain:

Theorem 5.26. *If p_{i_1}, p_{i_2}, ..., p_{i_r} are all primes which divide x and m simultaneously then*

$$(x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) \equiv \pm 1 \left(mod\ p_1^{\alpha_{i_1}} \cdot p_2^{\alpha_{i_2}} \cdots p_i^{\alpha_{i_r}} \right) ,$$

when $m \in A$ respectively $m \notin A$.

From the theorems 5.23 and 5.26 it results $L(x, m) = \pm 1 + k_1 \cdot d = k_2 \cdot m'$, where $k_1, k_2 \in \mathbb{Z}$. Because $(d, m') = 1$ the Diophantine equation $k_2 \cdot m' - k_1 \cdot d = \pm 1$ admits integer solutions (the unknowns being k_1 and k_2). Hence $k_1 = m' \cdot t + k_1^0$ and $k_2 = d \cdot t + k_2^0$, with $t \in \mathbb{Z}$, and k_1^0, k_2^0 constitute a particular integer solution of our equation. Thus

$$L(x, m) \equiv \pm 1 + m' \cdot d \cdot t + k_1^0 \cdot d \equiv \pm 1 + k_1^0 \ (mod\ m)$$

or

$$L(x, m) \equiv k_2^0 \cdot m' \ (mod\ m) \ .$$

5.4   Applications

1. The theorem Lagrange was extended of Wilson as follows: "if p is prime, then $x^{p-1} - 1 \equiv (x + 1)(x + 2) \cdots (x + p - 1) \ (mod\ p)$"; we shall extend this result in the following way: For any $m \neq 0, \pm 4$ we have for $x^2 + s^2 \neq 0$ that

$$x^{\varphi(m_s)+s} - x^s \equiv (x + 1)(x + 2) \cdots (x + |m| - 1) \ (mod\ m) ,$$

where m_s and s are obtained from the algorithm:

Algorthm 5.27.

$$(0) \begin{pmatrix} x = x_0 d_0 \ ; & (x_0, m_0) = 1 \\ m = m_0 d_0 \ ; & d_0 \neq 1 \end{pmatrix} ,$$

$$\text{(1)} \quad \left(\begin{array}{ll} d_0 = d_0^1 d_1 \; ; & (d_0^1, m_1) = 1 \\ m_0 = m_1 d_1 \; ; & d_1 \neq 1 \end{array} \right) ,$$

$$\ldots\ldots\ldots$$

$$(s-1) \quad \left(\begin{array}{ll} d_{s-2} = d_{s-2}^1 d_{s-1} \; ; & (d_{s-2}^1, m_{s-1}) = 1 \\ m_{s-2} = m_{s-1} d_{s-1} \; ; & d_{s-1} \neq 1 \end{array} \right) ,$$

$$(s) \quad \left(\begin{array}{ll} d_{s-1} = d_{s-1}^1 d_s \; ; & (d_{s-1}^1, m_s) = 1 \\ m_{s-1} = m_s d_s \; ; & d_s \neq 1 \end{array} \right) ,$$

[Smarandache, 1981a, 1984].

For m positive prime we have $m_s = m$, $s = 0$ and $\varphi(m) = m - 1$, that is Lagrange's theorem.

2. L. Moser enunciated the following theorem: "*if p is prime, then $(p-1)!a^p + a = \mathcal{M}p$*", and Sierpinski [1966]: "*if p is prime then $a^p + (p-1)!a = \mathcal{M}p$* which merges Wilson's and Fermat's theorems in a single one.

The function L and the algorithm 5.27 will help us to generalize them too, so: if a and m are integers, $m \neq 0$, and $C_1, C_2, \ldots, C_{\varphi(m)}$ are all modulo rests relatively prime to m then

$$C_1 \cdot C_2 \cdots C_{\varphi(m)} a^{\varphi(m_s)+s} - L(0, m) \cdot a^s = \mathcal{M}m ,$$

respectively

$$-L(0, m) a^{\varphi(m_s)+s} + C_1 \cdot C_2 \cdots C_{\varphi(m)} \cdot a^s = \mathcal{M}m ,$$

or more,

$$(x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) a^{\varphi(m_s)+s} - L(x, m) \cdot a^s = \mathcal{M}m$$

respectively

$$-L(x, m) a^{\varphi(m_s)+s} + (x + C_1)(x + C_2) \cdots (x + C_{\varphi(m)}) \cdot a^s = \mathcal{M}m ,$$

which reunites Fermat, Euler, Wilson, Lagrange and Moser (respectively Sierpinski).

3. The author also obtained a partial extension of Moser's and Sier-pinski's results, [Smarandache, 1983], so: if m is positive integer, $m \neq 0, 4$, and a is an integer, then $(a^m - a)(m - 1)! = \mathcal{M}m$, reuniting Fermat's and Wilson's theorem in another way.

4. Leibniz enunciated that: *"if p is prime then $(p - 2)! \equiv 1 \pmod{p}$"*; we consider *"$C_i < C_{i+1} \pmod{m}$"* if $C_i < C_{i+1}$ where $0 \leq C_i < |m|$, $0 \leq C_{i+1} < |m|$ and $C_i \equiv C_i \pmod{m}$, $C_{i+1} \equiv C_{i+1} \pmod{m}$; one simply gives that if C_1, C_2, ..., $C_{\varphi(m)}$ are all modulo m rests relatively prime to m ($C_i < C_{i+1} \pmod{m}$ for all i, $m \neq 0$) then $C_1 \cdot C_2 \cdots C_{\varphi(m)-1} \equiv \pm 1 \pmod{m}$ if $m \in A$ respectively $m \notin A$, because $C_{\varphi(m)} \equiv -1 \pmod{m}$.

Chapter 6

Analytical solving of Diophantine equations

6.1 General Diophantine equations

A Diophantine equation is an equation in which only integer solutions are allowed.

Hilbert's 10th problem asked if an algorithm existed for determining whether an arbitrary Diophantine equation has a solution. Such an algorithm does exist for the solution of first-order Diophantine equations. However, the impossibility of obtaining a general solution was proven by Matiyasevich [1970], Davis [1973], Davis and Hersh [1973], Davis [1982], Matiyasevich [1993] by showing that the relation $n = F_{2m}$ (where F_2 is the $2m$-th Fibonacci number) is Diophantine. More specifically, Matiyasevich showed that there is a polynomial P in n, m, and a number of other variables x, y, z, ... having the property that $n = F_{2m}$ if there exist integers x, y, z, ... such that $P(n, m, x, y, z, \ldots) = 0$.

Matiyasevich's result filled a crucial gap in previous work by Martin Davis, Hilary Putnam, and Julia Robinson. Subsequent work by Matiyasevich and Robinson proved that even for equations in thirteen variables, no algorithm can exist to determine whether there is a solution. Matiya-

sevich then improved this result to equations in only nine variables Jones and Matiyasevich [1981].

Ogilvy and Anderson [1988] give a number of Diophantine equations with known and unknown solutions.

A linear Diophantine equation (in two variables) is an equation of the general form

$$m \cdot x + n \cdot y = \ell \, ,$$

where solutions are sought with m, n, and ℓ integers. Such equations can be solved completely, and the first known solution was constructed by Brahmagupta, [Weisstein, 2014b]. Consider the equation

$$m \cdot x + n \cdot y = 1 \, .$$

Now use a variation of the Euclidean algorithm, letting $m = r_1$ and $n = r_2$

$$
\begin{aligned}
r_1 &= q_1 \cdot r_2 + r_3 \, , \\
r_2 &= q_2 \cdot r_3 + r_4 \, , \\
&\vdots \qquad \vdots \\
r_{n-3} &= q_{n-3} \cdot r_{n-2} + r_{n-1} \, , \\
r_{n-2} &= q_{n-2} \cdot r_{n-1} + 1.
\end{aligned}
$$

Starting from the bottom gives

$$
\begin{aligned}
1 &= r_{n-2} - q_{n-2} \cdot r_{n-1} \\
r_{n-1} &= r_{n-3} - q_{n-3} \cdot r_{n-2}, \\
r_{n-2} &= r_{n-4} - q_{n-4} \cdot r_{n-3} \, , \\
&\vdots \qquad \vdots \\
n = r_2 &= r_4 - q_4 \cdot r_3 \, , \\
m = r_1 &= r_3 - q_3 \cdot r_2
\end{aligned}
$$

so

$$
\begin{aligned}
1 &= r_{n-2} - q_{n-2} \cdot r_{n-1} \\
&= r_{n-2} - q_{n-2}(r_{n-3} - q_{n-3} \cdot r_{n-2}) \\
&= -q_{n-2} \cdot r_{n-3} + (1 + q_{n-2} \cdot q_{n-3})r_{n-2} \\
&= -q_{n-2} \cdot r_{n-3} + (1 + q_{n-2} \cdot q_{n-3})(r_{n-4} - q_{n-4} \cdot r_{n-3}) \\
&= (1 + q_{n-2} \cdot q_{n-3})r_{n-4} - (q_{n-2} + q_{n-4} + q_{n-2} \cdot q_{n-3} \cdot q_{n-4})r_{n-3} \\
&= \ldots .
\end{aligned}
$$

Continue this procedure all the way back to the top.

6.2 General linear Diophantine equation

The utility of this section is that it establishes if the number of natural solutions of a general linear equation is limited or not. We will show also a method of solving, using integer numbers, the equation $ax - by = c$ (which represents a generalization of lemmas 1 and 2 of [Andrica and Andreescu, 1981]), an example of solving a linear equation with 3 unknowns in $\mathbb{N}$, and some considerations on solving, using natural numbers, equations with n unknowns.

Let's consider the equation:

$$
a \cdot x = b \,, \tag{6.1}
$$

where $a \in \mathbb{Z}^n$, $b \in \mathbb{Z}$ or in explicit form

$$
\sum_{i=1}^{n} a_i x_i = b \,, \tag{6.2}
$$

with all $a_i, b \in \mathbb{Z}$, $a_i \neq 0$ and the greatest common factor

$$
d = gcf(a_1, a_2, \ldots, a_n) \,. \tag{6.3}
$$

Observation 6.1. The notion of gcd (greatest common divisor) is the same with the notion of gcf (greatest common factor) for numbers, gcf being used for algebraic expressions.

1. the notion of gcf refers to numbers and algebraic expressions, for example: $gcf(2abc, 8a^2b, 10abc) = 2ab$.

2. gcd refers only to numbers, for example: $gcd(2, 8, 10) = 2$.

Analogously, the notion of lcm (least common multiple) is the same with the notion of lcf (least common factory) for numbers, lcf being used for algebraic expressions.

1. the notion of lcf refers to numbers and algebraic expressions, for example: $lcf(2abc, 8a^2b, 10abc) = 40a^2b$.

2. lcm refers only to numbers, for example: $lcm(2, 8, 10) = 40$.

Lemma 6.2. *The equation* (6.2) *admits at least a solution in the set of integer, if d,* (6.3), *divides b.*

Proof. This result is classic.   □

In (6.2), one does not diminish the generality by considering

$$gcf(a_1, a_2, \ldots, a_n) = 1 \; ,$$

because in the case when $d \neq 1$, one divides the equation by this number; if the division is not an integer, then the equation does not admit natural solutions.

It is obvious that each homogeneous linear equation admits solutions in $\mathbb{N}$: at least the banal solution!

6.2.1   The number of solutions of equation
$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n = b$

We will introduce the following definition:

Definition 6.3. The equation (6.2) has variations of sign if there are at least two coefficients a_i, a_j, with $1 \leq i, j \leq n$, such that $sign(a_i, a_j) = -1$.

Lemma 6.4. *An equation* (6.2) *which has sign variations admits an infinity of natural solutions.*

Observation 6.5. Lemma 6.4 generalization of lemma 1 of [Andrica and Andreescu, 1981].

Proof. From the hypothesis of the lemma it results that the equation has h no null positive terms, $1 \leq h \leq n$, and $k = n - h$ non null negative terms. We have $1 \leq k \leq n$; it is supposed that the first h terms are positive and the following k terms are negative (if not, we rearrange the terms).

We can then write:

$$\sum_{i=1}^{h} a_i x_i - \sum_{j=h+1}^{n} a'_j x_j = b \text{ where } a'_j = -a_j > 0 \, .$$

Let's consider $0 < M = lcm(a_1, a_2, \ldots, a_n)$, the least common multiple, and $c_i = |M/a_i|, i \in I_n = \{1, 2, \ldots, n\}$.

Let's also consider $0 < P = lcm(h, k)$, the least common multiple, and $h_1 = P/h$, and $k_1 = P/k$. Taking

$$\begin{cases} x_t = h_1 c_t \cdot z + x_t^0 \, , & 1 \leq t \leq h \\ x_j = k_1 c_j \cdot z + x_j^0 \, , & h+1 \leq j \leq n \end{cases}$$

where $z \in \mathbb{N}$,

$$z \geq \max_{1 \leq t \leq h < j \leq n} \left\{ \left[\frac{-x_t^0}{h_1 c_t} \right] , \left[\frac{x_j^0}{k_1 c_j} \right] \right\} + 1 \, ,$$

where $[\gamma]$ meaning integer part of γ, i.e. the greatest integer less than or equal to γ, and $x_i^0, i \in I_n$, a particular integer solution (which exists according to lemma 6.2), we obtain an infinity of solutions in the set of natural numbers for the equation (6.2). □

Lemma 6.6.

1. *An equation (6.2) which does not have variations of sign has at maximum a limited number of natural solutions.*

2. *In this case, for $b \neq 0$, constant, the equation has the maximum number of solutions if and only if all $a_i = 1$ for $i \in I_n$.*

Proof.

1. One considers all $a_i > 0$ (otherwise, multiply the equation by -1).

   If $b < 0$, it is obvious that the equation does not have any solution in $\mathbb{N}$.

   If $b = 0$, the equation admits only the trivial solution.

   If $b > 0$, then each unknown x_i, takes positive integer values between 0 and $d_i = b/a_i$ (finite), and not necessarily all these values. Thus the maximum number of solutions is lower or equal to $\prod_{i=1}^{n}(1 + d_i)$, which is finite.

2. For $b \neq 0$, constant, $\prod_{i=1}^{n}(1 + d_i)$ is maximum if and only if d_i are maximum, i.e. *if* $a_i = 1$ for all $i \in I_n$.

$\square$

Theorem 6.7. *The equation (6.2) admits an infinity of natural solutions if and only if it has variations of sign.*

Proof. This naturally follows from the previous results. $\square$

6.2.2 Diophantine equation of first order with two unknown

Theorem 6.8. *Let's consider the equation $ax - by = c$, with integer coefficients, where $a > 0$ and $b > 0$ and $(a, b) = gcd(a, b) = 1$. Then the general solution in natural numbers of this equation is:*

$$\begin{cases} x = bk + x_0 \\ y = ak + y_0 \end{cases} \tag{6.4}$$

where (x_0, y_0) is a particular integer solution of the equation, and

$$k \geq \max \left\{ \left\lceil \frac{-x_0}{b} \right\rceil, \left\lceil \frac{-y_0}{a} \right\rceil \right\}$$

is an integer parameter.

Observation 6.9. The theorem 6.8 generalization of lemma 2 of [Andrica and Andreescu, 1981].

Proof. It results from [Creangă et al., 1965] that the general integer solution of the equation is (6.4). Since x and y are natural integers, it is necessary for us to impose conditions for k such that $x \geq 0$ and $y \geq 0$, from which it results the theorem 6.8. □

The solve in the set of natural numbers a linear equation with n unknowns we will use the previous results in the following way:

(a) If equation does not have variations of sign, because it has a limited number of natural solutions, the solving is made by tests.

(b) If it has variations of sign and if b is divisible by d, then it admits an infinity of natural solutions. One finds its general integer solution, see [Ion and Niță, 1978];

$$x_i = \sum_{j=1}^{n-1} \alpha_{ij} k_j + \beta_j, \quad i \in I_n,$$

where all the $\alpha_{ij}, \beta_j \in \mathbb{Z}$ and the k_j are integer parameters.

By applying the restriction $x_i \geq 0$ for $i \in I_n$, one finds the conditions which must be satisfied by the parameters k_j:

$$k_j \in \mathbb{Z}, \quad \text{for all } j \in I_{n-1}. \tag{6.5}$$

The case $n = 2$ and $n = 3$ can be done by this method, but when n is bigger, the conditions (6.5) becomes more and more difficult to find.

Example 6.10. Solve in $\mathbb{N}$ the equation $3x - 7y + 2z = -18$.

Solution: In $\mathbb{Z}$ one obtains the general integer solution:

$$\begin{cases} x = k_1 \\ y = k_1 + 2k_2 \\ z = 2k_1 + 7k_2 - 9 \end{cases} ,$$

with k_1 and k_2 in $\mathbb{Z}$.

From the conditions (6.5) result the inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$. It results that $k_1 \geq 0$ and also:

$$k_2 \geq -\frac{k_1}{2} \text{ if } -\frac{k_1}{2} \notin \mathbb{Z},$$

or

$$k_2 \geq -\frac{k_1}{2} \text{ if } -\frac{k_1}{2} \in \mathbb{Z};$$

and

$$k_2 \geq \frac{9 - 2k_1}{7} + 1 \text{ if } \frac{9 - 2k_1}{7} \notin \mathbb{Z},$$

or

$$k_2 \geq \frac{9 - 2k_1}{7} \text{ if } \frac{9 - 2k_1}{7} \in \mathbb{Z};$$

that is

$$k_2 \geq \frac{2 - 2k_1}{7} + 2 \text{ if } \frac{2 - 2k_1}{7} \notin \mathbb{Z},$$

or

$$k_2 \geq \frac{2 - 2k_1}{7} + 1 \text{ if } \frac{2 - 2k_1}{7} \in \mathbb{Z}.$$

With these conditions on k_1 and k_2 we have the general solution in natural numbers of the equation.

Procedure for solving Diophantine equations of first order with two unknowns

For automatically solving the Diophantine equations of order 1 with 2 unknowns $ax - by = c$ we need the following program.

Program 6.11. Program for finding a solution.

$$
S12(a, b, c) := \left| \begin{array}{l} return \; "Error(a, b) \neq 1" \; if \; \gcd(a, b) \neq 1 \\ m \leftarrow 10^6 \\ for \; x \in 1..m \\ \quad for \; y \in floor(\frac{ax-c-1}{b})..ceil(\frac{ax-c+1}{b}) \\ \qquad return \; \left(\begin{array}{cc} x & y \end{array} \right)^{\mathrm{T}} \; if \; a \cdot x - b \cdot y - c{=}0 \\ return \; "Not \; found \; a \; solution" \end{array} \right.
$$

Example 6.12. We consider the Diophantine equation on the set of natural numbers $1245x - 365y = 4567$. This case is solvable, as $gcd(a, b) = 1$.

$$
a := 124 \quad b := 365 \quad c := 4567 \quad gcd(a, b) = 1
$$

$$
\left(\begin{array}{c} x_0 \\ y_0 \end{array} \right) := S12(a, b, c) = \left(\begin{array}{c} 28 \\ -3 \end{array} \right)
$$

$$
k_0 := \max \left(\left(\begin{array}{c} ceil \left(\frac{-x_0}{b} \right) \\ ceil \left(\frac{-y_0}{a} \right) \end{array} \right) \right) = 1
$$

$$
x(k) := b \cdot k + x_0
$$
$$
y(k) := a \cdot k + y_0
$$

for $k := k_0..10$ we obtain the solutions

$$x(k) \to \begin{pmatrix} 393 \\ 758 \\ 1123 \\ 1488 \\ 1853 \\ 2218 \\ 2583 \\ 2948 \\ 3313 \\ 3678 \end{pmatrix} \quad y(k) \to \begin{pmatrix} 121 \\ 245 \\ 369 \\ 493 \\ 617 \\ 741 \\ 865 \\ 989 \\ 1113 \\ 1237 \end{pmatrix}.$$

Solving of Diophantine equation $a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n = b$

In this section we will present the problem of solving for integers equations of the form:

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n = b \tag{6.6}$$

where $a_k, b \in \mathbb{Z}$ for $k \in I_n$.

We suppose that not all the numbers a_k, for $k \in I_n$, are null. Obviously, to exist an integer solution of the equation (6.6) it is necessary that

$$d = (a_1, a_2, \ldots, a_n) = \gcd(a_1, a_2, \ldots, a_n) \mid b .$$

We will prove that this condition is also sufficient.

Let $a'_k = a_k/d$, $k \in I_n$ and $b' = b/d$. We consider following equation equivalent with (6.6)

$$a'_1 \cdot x_1 + a'_2 \cdot x_2 + \ldots + a'_n \cdot x_n = b' \tag{6.7}$$

then $(a'_1, a'_2, \ldots, a'_n) = 1$. Let a'_k, a'_j be non-null numbers with $k < j$ and $|a'_k| > |a'_j|$. According to the Theorem of division with remainder, [Burton, 2010], there exist the numbers q and r such that

$$a'_k = a'_j \cdot q + r$$

and, by substituting a'_k in (6.7) equation

$$a'_1 \cdot x_1 + \ldots + r \cdot x_k + \ldots + a'_j(x_j + q \cdot x_k) + \ldots + a'_n \cdot x_n = b' \qquad (6.8)$$

is obtained.

Equation (6.8) can be written as:

$$a''_1 \cdot x''_1 + \ldots + a''_n \cdot x''_n = b' \qquad (6.9)$$

where

$$a''_i = \left\{ \begin{array}{ll} a'_i, & i \neq k \\ r, & i = k \end{array} \right. , \quad x''_i = \left\{ \begin{array}{ll} x_i, & i \neq k \\ x_j + q \cdot x_k, & i = k \end{array} \right. .$$

It can be easily observed that there exists a one-to-one correspondence between the solutions of equations (6.7) and (6.9). Furthermore, knowing the solutions of equation (6.9) and taking into account the previous transformations, the solutions of equation (6.7) can also be given.

We mention that, for every $i, k \in I_n$, $i \neq k$ following relations hold:

$$a''_i = a'_i \text{ and } |a''_k| < |a'_k| .$$

Additionally, we have

$$(a''_1, \ldots, a''_n) = (a'_1, \ldots, a'_k - a'_j \cdot q, \ldots, a'_n) = (a'_1, \ldots, a'_n) = 1.$$

After summing all the previous relations, we conclude that equation (6.7) can be reduced to the form

$$\widetilde{a}_1 \cdot \widetilde{x}_1 + \ldots + \widetilde{a}_n \cdot \widetilde{x}_n = b' \qquad (6.10)$$

after a finite number of steps, where $\widetilde{a}_i$ with $i \in I_n$ are non-null numbers, whose absolute values are pairwise distinct.

Hence, we deduce that the numbers $\widetilde{a}_i$, $i \in I_n$ have only the values 0 or ± 1 and are not all null. Without any loss of generality of the problem, we suppose that $\widetilde{a}_1 = 1$. Then equation (6.10) has following solutions

$$\left\{ \begin{array}{l} \widetilde{x}_1 = b' - \widetilde{a}_2 \cdot t_2 - \ldots - \widetilde{a}_n \cdot t_n \\ \widetilde{x}_2 = t_2 \\ \ldots \\ \widetilde{x}_n = t_n \end{array} \right.$$

where $t_2, t_3, \ldots, t_n$ are arbitrary integers. Using the transformations done along the previous reasonings, the solutions of equation (6.7) are also obtained.

We insist on mentioning that in solving equation (6.10) the fact that $\widetilde{a_1} = 1$ was used, and, therefore, if at a certain step of the indicated algorithm an equation with at least one coefficient equal to ± 1 is obtained, the solution of this equation can be written similarly with the solution of the equation (6.10).

6.3   Solving the Diophantine linear systems

More generally, every system of linear Diophantine equations may be solved by computing the *Smith normal form* of its matrix, in a way that is similar to the use of the *reduced row echelon form* to solve a system of linear equations over a field.

ABS algorithm for solving linear Diophantine equations, Gao and Dong [2008] introduce an algorithm for solving a system of m linear integer inequalities in n variables, $m \leq n$, with full rank coefficient matrix.

6.3.1   Procedure of solving with row–reduced echelon form

Echelon form (or row echelon form) is:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry (i.e. leftmost nonzero entry) of a row is in a column to the right of leading entry of the row above it.

3. All entries in a column below a leading entry are zero.

Example 6.13. Echelon forms:

$$
\begin{pmatrix}
\blacksquare & * & * & * & * \\
0 & \blacksquare & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\blacksquare & * & * \\
0 & \blacksquare & * \\
0 & 0 & \blacksquare \\
0 & 0 & 0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 & \blacksquare & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & *
\end{pmatrix}.
$$

where we noted with $\blacksquare$ any nonzero integer and with $*$ any integer.

Reduced echelon form: Add the following conditions to conditions 1, 2 and 3 above.

4. The leading entry in each nonzero row is1.

5. Each leading 1 is the only nonzero entry in its column.

A matrix is in reduced row echelon form, also called *row canonical form*, if it satisfies the following conditions, [Meyer, 2000].

Example 6.14. Reduced echelon form:

$$
\begin{pmatrix}
0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & *
\end{pmatrix}.
$$

The theorem uniqueness of the reduced echelon form, [Nakos and Joyner, 1998, Meyer, 2000]:

Theorem 6.15. *Each matrix is row-equivalent to one and only one reduced echelon matrix.*

Definition 6.16. Pivot position is a position of a leading entry in an echelon form of the matrix.

Definition 6.17. Pivot is a nonzero number that either is used in a pivot position to create $0's$ or is changed into a leading 1, which in turn is used to create $0's$.

Definition 6.18. Pivot column is a column that contains a pivot position.

The theorem existence and uniqueness, [Nakos and Joyner, 1998, Meyer, 2000].

Theorem 6.19.

1. *A linear system is consistent if and only if the rightmost column of augmented matrix is not a pivot column (i.e. if and only if an echelon form of the augmented matrix has no row of the form* $[0\ 0\ \ldots\ 0\ b]$, *where* $b \neq 0$).

2. *If a linear system is consistent, then the solution contains either*

    (a) *a unique solution (when there are no free variables) or*

    (b) *infinitely many solutions (when there is at least one free variable).*

Algorthm 6.20. The algorithm using reduced row echelon form to solve linear system:

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not stop; otherwise go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. State the solution by expressing each basic variable in terms of free variables and declare the free variables.

Example 6.21. Solving by means of symbolic computation of a Diophantine linear system using the method *row reduced echelon form*. We consider the origin of the vectors and matrices equal to 1.

$$ORIGIN := 1$$

We consider the system $A \cdot x = b$, where

$$A := \begin{pmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{pmatrix} \quad b := \begin{pmatrix} -5 \\ 9 \\ 15 \end{pmatrix} .$$

We concatenate matrix A to the free term b and we determine the number of lines and columns of matrix E

$$E := augment(A, b) \ \ n := rows(E) \to 3 \ \ cols(E) \to 6 .$$

By means of the Mathcad function $rref$ we determine the *row reduced echelon form* of matrix E.

$$R := rref(E) \to \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & 24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} .$$

According to this matrix, it follows that the main unknowns are x_1, x_2 and x_5, while x_3 and x_4 are the secondary unknowns.

$$S(x_1, x_2, x_3, x_4, x_5) :=$$

$$submatrix(R, 1, n, 1, m - 1) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} - R^{\langle m \rangle} \to$$

$$\begin{pmatrix} x_1 - 2x_3 + 3x_4 + 24 \\ x_2 - 2x_3 + 2x_4 + 7 \\ x_5 - 4 \end{pmatrix}$$

We determine x_1, x_2 and x_5 relative to x_3 and x_4.

$$\begin{pmatrix} x_1 & x_2 & x_5 \end{pmatrix} :=$$
$$S(x_1, x_2, x_3, x_4, x_5) solve \begin{pmatrix} x_1 & x_2 & x_5 \end{pmatrix} \to$$
$$\begin{pmatrix} 2x_3 - 3x_4 - 24 & 2x_3 - 2x_4 - 7 & 4 \end{pmatrix} .$$

We verify if the obtained solution satisfies the equation

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} - b = 0 \, ,$$

$$A \cdot \begin{pmatrix} 2x_3 - 3x_4 - 24 \\ 2x_3 - 2x_4 - 7 \\ x_3 \\ x_4 \\ 4 \end{pmatrix} - b \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \, .$$

It is obvious that for different integer values of x_3 and x_4 there result integer solutions for the linear system.

Example 6.22. Linear Diophantine system that has a unique solution. We consider the linear system with

$$A := \begin{pmatrix} 3 & 4 \\ 2 & 5 \\ -2 & -3 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \, .$$

We consider matrix E and we count the number of lines and columns of matrix E.

$$E := augment(A, b) \ \ n := rows(E) \to 3 \ \ cols(E) \to 3 \, .$$

The matrix *row reduced echelon form* is

$$R := rref(E) \to \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \, .$$

We compute

$$S(x_1, x_2) :=$$

$$submatrix(R, 1, n, 1, m - 1) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - R^{\langle m \rangle} \rightarrow \begin{pmatrix} x_1 + 5 \\ x_2 - 3 \\ 0 \end{pmatrix}.$$

From this result follows that the main unknowns are x_1 and x_2 which do not depend on any other variable and we have $x_1 = -5$ and $x_2 = 3$.

Indeed, it is verified that

$$A \cdot \begin{pmatrix} -5 \\ 3 \end{pmatrix} - b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example 6.23. Linear Diophantine system that has no solutions. We consider matrix A and vector b

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 3 \\ 5 \end{pmatrix}.$$

Let be matrix E

$$E := augment(A, b) \;\; n := rows(E) \rightarrow 4 \;\; m := cols(E) \rightarrow 4.$$

We compute the matrix *row-reduced echelon form* by means of function *rref*

$$R := rref(E) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Following calculation is done

$$S(x_1, x_2, x_3) :=$$

$$submatrix(R, 1, n, 1, m-1) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - R^{\langle m \rangle} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -1 \end{pmatrix}.$$

We try to solve by means of symbolic computation, using function *solve*, the equation $S(x_1, x_2, x_3)=0$:

$$S(x_1, x_2, x_3)=0 \ solve(x_1, x_2, x_3) \quad \rightarrow$$
$$\boxed{No \ solution \ was \ found} \quad .$$

The Mathcad's answer is: *No solution was found*.

Example 6.24. Linear Diophantine system with matrix A and free term b

$$A := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 9 & 16 & 25 & -36 \\ 1 & 8 & 27 & 64 & 125 & 216 \\ 1 & 16 & 81 & 256 & 625 & -1296 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 \end{pmatrix} \quad b := \begin{pmatrix} 104 \\ -140 \\ 2750 \\ -7952 \\ 87374 \end{pmatrix}.$$

Matrix E is obtained by concatenating vector b to matrix A.

$$E := augment(A, b) \ \ n := rows(E) \rightarrow 5 \ \ m := cols(E) \rightarrow 7 \ .$$

We compute the matrix *row-reduced echelon form* by means of function *rref*

$$R := rref(E) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1980 & 17833 \\ 0 & 1 & 0 & 0 & 0 & -3465 & -31185 \\ 0 & 0 & 1 & 0 & 0 & 3080 & 27719 \\ 0 & 0 & 0 & 1 & 0 & -1386 & -12469 \\ 0 & 0 & 0 & 0 & 1 & 252 & 2272 \end{pmatrix}.$$

We compute

$$S(y_1, y_2, y_3, y_4, y_5, y_6) :=$$

$$submatrix(R, 1, n, 1, m-1) \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} - R^{\langle m \rangle} \rightarrow$$

$$\begin{pmatrix} y_1 + 1980 \cdot y_6 - 17833 \\ y_2 - 3465 \cdot y_6 + 31185 \\ y_3 + 3080 \cdot y_6 - 27719 \\ y_4 - 1386 \cdot y_6 + 12469 \\ y_5 + 252 \cdot y_6 - 2272 \end{pmatrix} .$$

We solve by means the symbolic computation, using function *solve*, equation $S(y_1, y_2, y_3, y_4, y_5, y_6)=0$:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} := S(y_1, y_2, y_3, y_4, y_5, y_6)=0 \; solve \; \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 17833 - 1980 \cdot y_6 \\ -31185 + 3465 \cdot y_6 \\ 27719 - 3080 \cdot y_6 \\ -12469 + 1386 \cdot y_6 \\ 2272 - 252 \cdot y_6 \end{pmatrix} .$$

6.3.2 Solving with Smith normal form

Using matrix notation every system of linear Diophantine equations may be written

$$A \cdot X = C$$

where A is a $m \times n$ matrix of integers, X is a $n \times 1$ column matrix of unknowns and C is a $m \times 1$ column matrix of integers.

The computation of the Smith normal form of A provides two unimodular matrices (that is matrices that are invertible over the integers, which have ± 1 as determinant) U and V of respective dimensions mm and $n \times n$, such that the matrix

$$B = [b_{i,j}] = UAV$$

is such that $b_{i,i}$ is not zero for i not greater than some integer k, and all the other entries are zero. The system to be solved may thus be rewritten as

$$B(V^{-1} \cdot X) = U \cdot C.$$

Calling y_i the entries of $V^{-1} \cdot X$ and d_i those of $D = U \cdot C$, this leads to the system

$$\begin{aligned} b_{i,i} \cdot y_i &= d_i \quad \text{for } 1 \le i \le k \\ 0 \cdot y_i &= d_i \quad \text{for } k < i \le n \end{aligned}$$

This system is equivalent to the given one in the following sense: A column matrix of integers x is a solution of the given system if and only if $x = V \cdot y$ for some column matrix of integers y such that $By = D$.

It follows that the system has a solution if and only if $b_{i,i}$ divides d_i for $i \le k$ and $d_i = 0$ for $i > k$. If this condition is fulfilled, the solutions of the given system are

$$V \cdot \begin{pmatrix} \frac{d_1}{b_{1,1}} \\ \vdots \\ \frac{d_k}{b_{k,k}} \\ h_{k+1} \\ \vdots \\ h_n \end{pmatrix}$$

where $h_{k+1}, \dots, h_n$ are arbitrary integers, [Schmidt, 1991, Lazebnik, 1996, Smart, 1998].

6.4   Solving the Diophantine equation of order n with an unknown

The Diophantine equation of order n with a single unknown is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \,, \tag{6.11}$$

where $a_k \in \mathbb{Z}$, $a_n \neq 0$. The problem that arises is to find the solutions that are rational numbers ($\mathbb{Q}$) or integers ($\mathbb{Z}$) or natural numbers ($\mathbb{N}$). As it is well known, the Fundamental Theorem of Algebra, [Krantz, 1999], assures the existence of n complex solutions for the algebraic equation of order n. Therefore, the Diophantine equation (6.11) can not have more than n rational, integer or natural solutions.

Leaving from "Vieta's formula", [Viète, 1646, Girard, 1884]:

$$s_1 \cdot s_2 \cdots s_n = (-1)^n \frac{a_0}{a_n} \,,$$

where s_k are the roots of polynomial P, which means $P(s_k) = 0$, for $k \in I_n$, it results a classic method. This method supposes finding all the divisors of a_n and of a_0. Afterwards, the set of numbers that divide a_n/a_0 is generated (finite set, $\leq \sigma_0(a_n)\sigma_0(a_0)$) and it is tested which of those divisors are roots of polynomial P.

We give an automatic procedure for finding the rational, integer or natural solutions which uses following 3 programs.

Program 6.25. Program to find the divisors of a natural number.

$$Div(m) := \begin{vmatrix} j \leftarrow 0 \\ d_j \leftarrow 1 \\ for \ k \in 2..floor(\frac{m}{2}) \\ \quad if \ mod(m,k)\text{=}0 \\ \qquad \begin{vmatrix} j \leftarrow j+1 \\ d_j \leftarrow k \end{vmatrix} \\ d \leftarrow stack(d,m) \\ return \ d \end{vmatrix}$$

Program 6.26. Program to find the factors (repetition excluded) of the number a_0/a_n. This program calls the program 6.25.

$$Factori(a) := \left| \begin{array}{l} d0 \leftarrow Div(|a_0|) \\ dn \leftarrow Div(|a_{last(a)}|) \\ f \leftarrow d0 \\ i \leftarrow last(f) + 1 \\ for\ k \in 0..last(d0) \\ \quad \left| \begin{array}{l} for\ j \in 1..last(dn) \\ \quad \left| \begin{array}{l} f_i \leftarrow \frac{d0_k}{dn_j} \\ i \leftarrow i + 1 \end{array} \right. \end{array} \right. \\ f \leftarrow sort(f) \\ j \leftarrow 0 \\ w_j \leftarrow f_0 \\ for\ k \in 1..last(f) \\ \quad \left| \begin{array}{l} if\ w_j \neq f_k \\ \quad \left| \begin{array}{l} j \leftarrow j + 1 \\ w_j \leftarrow f_k \end{array} \right. \end{array} \right. \\ return\ w \end{array} \right.$$

Program 6.27. Program to find the rational solutions for the input parameter $t = 0$, the integer solutions for the input parameter $t = 1$ and the natural solutions for the input parameter $t = 2$. This program calls the programs 6.26 and 1.26.

$$Sqzn(a,t) := \left| \begin{array}{l} retur\ "Error\ t \neq 0 \land 1 \land 2"\ if\ t \neq 0 \land t \neq 1 \land t \neq 2 \\ f \leftarrow Factori(a) \\ fs \leftarrow sort(f) \\ f \leftarrow stack(-reverse(fs), fs) \\ w \leftarrow f\ if\ t{=}0 \\ if\ t{=}1 \\ \quad \left| \begin{array}{l} j \leftarrow 0 \\ for\ k \in 0..last(f) \\ \quad \left| \begin{array}{l} if\ f_k{=}trunc(f_k) \\ \quad \left| w_j \leftarrow f_k \right. \end{array} \right. \end{array} \right. \end{array} \right.$$

$$
\begin{array}{|l}
\quad\;\; \big|\;\; \big|\; j \leftarrow j + 1 \\
if\; t{=}2 \\
\quad \big|\; j \leftarrow 0 \\
\quad \big|\; for\; k \in 0..last(f) \\
\quad\;\; \big|\; if\; f_k{=}trunc(f_k) \wedge f_k \geq 0 \\
\quad\;\;\;\; \big|\; w_j \leftarrow f_k \\
\quad\;\;\;\; \big|\; j \leftarrow j + 1 \\
i \leftarrow 0 \\
for\; k \in 0..last(w) \\
\quad \big|\; if\; Horner(a, w_k){=}0 \\
\quad\;\; \big|\; s_i \leftarrow w_k \\
\quad\;\; \big|\; i \leftarrow i + 1 \\
return\; s^{\mathrm{T}}
\end{array}
$$

Example 6.28. We consider the polynomial defined by the vector

$$
a := \begin{pmatrix} -96 & 776 & -1568 & 134 & 1620 & -359 & -466 & 49 & 30 \end{pmatrix}^{\mathrm{T}},
$$

afterwards we consider the calls

$$
S(a, 2) \to \begin{pmatrix} 3 \end{pmatrix},
$$

$$
S(a, 1) \to \begin{pmatrix} -4 & -2 & 3 \end{pmatrix},
$$

$$
S(a, 0) \to \begin{pmatrix} -4 & -2 & \dfrac{1}{5} & \dfrac{1}{2} & \dfrac{2}{3} & 3 \end{pmatrix}.
$$

The first call states that the polynomial defined by vector a has a unique natural solution; the second call tells us that the polynomial has 3 integer solutions; while the third call shows hat the polynomial has 6 integer solutions.

The second method supposes finding all the roots of the polynomial by means of formula for polynomials of order $n \leq 4$ and emphasizing the rational, integer or natural roots.

As the polynomials of degree $n > 4$ can not be solved with square roots (Impossibility Theorem, Abel [1826, 1881, 1988] and Galois 1832, [Artin, 1944], (this was also shown by Ruffini in 1813 [Wells, 1986]), the roots of the polynomial will be approximated by a numerical method, usually Laguerre's method, [Cira, 2005](probably the best numerical method for approximating the solutions of a algebraic equation. This method is implemented in most of the mathematics softwares, such as Maple, Mathematica, Matlab, Mathcad.

The approximative roots which are "close" to rational, integer or natural numbers are verified by direct computation (Schema of Horner [1819], the fastest algorithm for computing the values of an algebraic polynomial) if those numbers are the solutions of the equation $P(x) = 0$. We give an example of how this method is applied.

Example 6.29. Let be the algebraic equation $P(x) = 0$, where polynomial P is defined by the vector

$$
\begin{aligned}
a := (\,&2074506308666643852,\ -4170138555243755952, \\
&3708600060698625999,\ -2371615921694294428, \\
&1144052588009550927,\ -392768652155202268, \\
&93951730922422481,\ -15744238825971732, \\
&1864646677195241,\ -156394532149220, \\
&9205044609900,\ -370727876000, \\
&9701590000,\ -148200000,\ 1000000)^{\mathrm{T}}
\end{aligned}
$$

The call of Mathcad function *polyroots* will give approximations for the

solutions of the algebraic equation $P(x) = 0$

$$s := polyroots(a)$$

$$= \begin{pmatrix} -0.00000000595668852 - 2.0000000743130992i \\ -0.00000000540100408 + 2.00000007346564i \\ 1.0999993750452355 \\ 5.097397296153272 \\ 5.911537626129313 \\ 6.877530938768753 \\ 8.128108099781006 \\ 9.08388413343586 \\ 10.99946985840438 \\ 13.00329871837791 \\ 16.99735261469869 \\ 19.00160713253058 \\ 22.99980560963407 \\ 29.00000860839862 \end{pmatrix} .$$

Aside the fist two solution which are complex numbers,the other solutions are "close" to natural numbers. By direct computation it can be established which of these "close" integer are solutions natural numbers.

$$P(1) \rightarrow 72560416394188800 ,$$
$$P(5) \rightarrow -856324819703808 ,$$
$$P(6) \rightarrow -202256700445800 ,$$
$$P(7) \rightarrow 153045758638080 ,$$
$$P(8) \rightarrow 161732639306700 ,$$
$$P(9) \rightarrow -274961651328000$$

$$P(11) \rightarrow 0 ,$$
$$P(13) \rightarrow 0 ,$$
$$P(17) \rightarrow 0 ,$$
$$P(19) \rightarrow 0 ,$$
$$P(23) \rightarrow 0 ,$$
$$P(29) \rightarrow 0 .$$

The conclusion is that equation $P(x) = 0$ has following solutions natural numbers: 11, 13, 17, 19, 23 and 29.

In the case of algebraic equations of order 1, 2, 3 and 4 the solutions are obtained by means of symbolic calculus, by applying well known formulas.

Equation $29x^2 - 490x + 1469 = 0$ has the solutions

$$29x^2 - 490x + 1469 \; solve \begin{pmatrix} 13 \\ \dfrac{113}{29} \end{pmatrix}$$

which implies that there exists a rational solution and a natural solution.

Equation $127x^3 - 28829x^2 - 12767x + 2898109 = 0$ has the solutions

$$127x^3 - 28829x^2 - 12767x + 2898109 \; solve \rightarrow \begin{pmatrix} 227 \\ \dfrac{\sqrt{1621409}}{127} \\ -\dfrac{\sqrt{1621409}}{127} \end{pmatrix}$$

hence, we have a unique solution natural number.

Equation $3x^4 - 7x^3 + 17x^2 - 35x + 10 =$ has the solutions

$$3x^4 - 7x^3 + 17x^2 - 35x + 10 \; solve \rightarrow \begin{pmatrix} 2 \\ \dfrac{1}{3} \\ \sqrt{5}i \\ -\sqrt{5}i \end{pmatrix}$$

therefore we have a solution natural number and a solution rational number.

6.5 The Diophantine equation of second order and with two unknowns

We consider the equation

$$ax^2 - by^2 + c = 0 , \tag{6.12}$$

with $a, b \in \mathbb{N}^*$ and $c \in \mathbb{Z}^*$. It is a generalization of Pell's equation $x^2 - Dy^2 = 1$, [Dickson, 2005]. Here, we show that: if the equation has an integer solution and $a \cdot b$ is not a perfect square, then (6.12) has an infinitude of integer solutions; in this case we find a closed expression (x_n, y_n), the general positive integer solution, by an original method. More, we generalize it for any Diophantine equation of second degree and with two unknowns.

6.5.1 Existence and number of solutions of Diophantine quadratic equations with two unknowns in $\mathbb{Z}$ and $\mathbb{N}$

We study the existence and number of solutions in the set of integers, $\mathbb{Z}$ and the set of natural numbers, $\mathbb{N}$ of Diophantine equations of second degree with two unknowns of the general form (6.12).

Theorem 6.30. *The equation $x^2 - y^2 = c$ admits integer solutions if and only if c belongs to $4\mathbb{Z}$ or is odd.*

Proof. The equation $(x - y)(x + y) = c$ admits solutions in $\mathbb{Z}$ if there exist c_1 and c_2 in $\mathbb{Z}$ such that $x - y = c_1$, $x + y = c_2$ and $c_1 c_2 = c$. Therefore

$$x = \frac{c_1 + c_2}{2} \text{ and } y = \frac{c_2 - c_1}{2} .$$

But x and y are integers if and only if $c_1 + c_2 \in 2\mathbb{Z}$, i.e.:

1. or c_1 and c_2 are odd, then c is odd (and reciprocally),

2. or c_1 and c_2 are even, then $c \in 4\mathbb{Z}$.

Reciprocally, if $c \in 4\mathbb{Z}$, then we can decompose up c into two even factors c_1 and c_2, such that $c_1 c_2 = c$. □

Remark 6.31. The theorem 6.30 is true also for solving in $\mathbb{N}$, because we can suppose $c \geq 0$ (in the contrary case, we can multiply the equation by -1), and we can suppose $c_2 \geq c_1 \geq 0$, from which $x \geq 0$ and $y \geq 0$.

Theorem 6.32. *The equation $x^2 - dy^2 = c^2$ (where d is not a perfect square) admits infinity of solutions in $\mathbb{N}$.*

Proof. Let's consider $x = ck_1$, $k_1 \in \mathbb{N}$ and $y = ck_2$, $k_2 \in \mathbb{N}$, $c \in \mathbb{N}$. It results that $k_1^2 - dk_2^2 = 1$, which we can recognize as being the Pell-Fermat's equation, which admits an infinity of solutions in $\mathbb{N}$, (u_n, v_n). Therefore $x_n = cu_n$, $y_n = cv_n$ constitute an infinity of natural solutions for our equation. □

Theorem 6.33. *The equation (6.12), $c \neq 0$, where $ab = k^2$, $k \in \mathbb{Z}$, admits a finite number of natural solutions.*

Proof. We can consider a, b, c as positive numbers, otherwise, we can multiply the equation by -1 and we can rename the variables.

Let us multiply the equation by a, then we will have:

$$z^2 - t^2 = d \text{ with } z = ax \in \mathbb{N}, \quad t = ky \in \mathbb{N} \text{ and } d = ac > 0. \quad (6.13)$$

We will solve it as in theorem 6.30, which gives z and t. But in (6.13) there is a finite number of natural solutions, because there is a finite number of integer divisors for a number in $\mathbb{N}^*$. Because the pairs (z, t) are in a limited number, it results that the pairs $(z/a, t/k)$ also are in limited number, and the same for the pairs (x, y). □

Theorem 6.34. *If the equation (6.2), where $ab \neq k^2$, $k \in \mathbb{Z}$, admits a particular nontrivial solution in $\mathbb{N}$, then it admits an infinity of solutions in $\mathbb{N}$.*

Proof. Let's consider:

$$\begin{cases} x_n = x_0 \cdot u_n + b \cdot y_0 \cdot v_n, \\ y_n = y_0 \cdot u_n + a \cdot x_0 \cdot v_n, \end{cases} \quad (6.14)$$

for $n \in \mathbb{N}$, where (x_0, y_0) is the particular natural solution for the equation (6.12), and (u_n, v_n) is the general natural solution for the equation $u^2 - abv^2 = 1$, called the solution Pell, which admits an infinity of solutions. Then $ax_n^2 - by_n^2 = (ax_0^2 - by_0^2)(u_n^2 - abv_n^2) = c$. Therefore (6.14) verifies the equation (6.12). $\qquad\square$

6.5.2 Method of solving the Diophantine equation of second order

Suppose (6.12) has many integer solutions. Let (x_0, y_0), (x_1, y_1) be the smallest positive integer solutions for (6.12), with $0 \le x_0 < x_1$. We construct the recurrent sequences:

$$\begin{cases} x_{n+1} = \alpha x_n + \beta y_n \\ y_{n+1} = \gamma x_n + \delta y_n \end{cases} \tag{6.15}$$

putting the condition (6.15) verify (6.12). It results:

$$a\alpha\beta = b\gamma\delta \tag{6.16}$$
$$a\alpha^2 - b\gamma^2 = a \tag{6.17}$$
$$a\beta^2 - b\delta^2 = -b \tag{6.18}$$

having the unknowns $\alpha, \beta, \gamma, \delta$.

We pull out $a\alpha^2$ and $a\beta^2$ from (6.17), respectively (6.18), and replace them in (6.16) at the square; it obtains

$$a\delta^2 - b\gamma^2 = a . \tag{6.19}$$

We subtract (6.19) from (6.17) and find

$$\alpha = \pm\beta . \tag{6.20}$$

Replacing (6.20) in (6.16) it obtains

$$\beta = \pm\frac{b}{a}\gamma . \tag{6.21}$$

Afterwards, replacing (6.20) in (6.17), and (6.21) in (6.18) it finds the same equation:

$$a\alpha^2 - b\gamma^2 = a . \tag{6.22}$$

Because we work with positive solutions only, we take

$$x_{n+1} = \alpha_0 x_n + \frac{b}{a}\gamma_0 y_n \tag{6.23}$$

$$y_{n+1} = \gamma_0 x_n + \alpha_0 y_n \tag{6.24}$$

where (α_0, γ_0) is the smallest, positive integer solution of (6.22) such that $\alpha_0\gamma_0 \neq 0$. Let

$$A = \begin{pmatrix} \alpha_0 & \frac{b}{a}\gamma_0 \\ \gamma_0 & \alpha_0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}) . \tag{6.25}$$

Of course, if (x', y') is an integer solution for (6.12), then

$$A \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} , \quad A^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

are another ones, where

$$A^{-1} = \frac{1}{\gamma^2 b - a\alpha^2} \begin{pmatrix} -a\alpha & \gamma b \\ \gamma b & -a\alpha \end{pmatrix}$$

is the inverse matrix of A, i.e. $A^{-1} \cdot A = A \cdot A^{-1} = I$ (unit matrix). Hence, if (6.12) has an integer solution it has an infinite ones. Clearly $A^{-1} \in \mathcal{M}_2(\mathbb{Z})$.

The general positive integer solution of the equation (6.12) is $(x'_n, y'_n) = (|x_n|, |y_n|)$.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{for all } n \in \mathbb{Z} , \tag{6.26}$$

where by conversion $A^0 = I$ and

$$A^{-k} = \underbrace{A^{-1} \cdots A^{-1}}_{k \text{ times}} .$$

In problems it is better to write general solution as

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = A^n \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad n \in \mathbb{N} \tag{6.27}$$

and

$$\begin{pmatrix} x''_n \\ y''_n \end{pmatrix} = A^n \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad n \in \mathbb{N}^* . \tag{6.28}$$

We proof, by *reduction ad absurdum*, (6.28) is a general positive integer solution for (6.12).

Let (u, v) be a positive integer particular solution for (6.12). If

$$\exists \, k_0 \in \mathbb{N} \; : \; (u, v) = A^{k_0} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or

$$\exists \, k_1 \in \mathbb{N}^* \; : \; (u, v) = A^{k_1} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} ,$$

then $(u, v) \in$(6.28). Contrary to this, we calculate

$$(u_{i+1}, v_{i+1}) = A^{-1} \cdot \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

for $i = 0, 1, 2, \ldots$, where $u_0 = u$, $v_0 = v$. Clearly $u_{i+1} < u_i$ for all i. After a certain rank $x_0 < u_{i_0} < x_1$ it finds either $0 u_{i_0} < x_0$ but that is absurd.

It is clear we can put

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \cdot \begin{pmatrix} x_0 \\ \varepsilon \cdot y_0 \end{pmatrix} \quad n \in \mathbb{N} , \;\; \text{where} \;\; \varepsilon = \pm 1 . \tag{6.29}$$

We shall now transform the general solution (6.29) in closed expression.

Let λ be real number, then $det(A - \lambda I) = 0$ involves the solutions $\lambda_{1,2}$ and the proper vectors $v_{1,2}$ i.e.

$$A \cdot v_i = \lambda_i \cdot v_i , \;\; \text{for} \;\; i \in I_2 .$$

Note

$$P = \left(\begin{array}{cc} v_1 & v_2 \end{array} \right) \in \mathcal{M}_2(\mathbb{R}) \ .$$

Then

$$P^{-1} \cdot A \cdot P = \left(\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right) \ ,$$

whence

$$A^n = P \cdot \left(\begin{array}{cc} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{array} \right) \cdot P^{-1}$$

and replacing it in (6.29) and doing the calculus we find a closed expression for (6.29).

6.5.3   Procedure for solving of Diophantine equation of second order with two unknowns

We will present an automatic procedure for solving Diophantine equations (6.12). We have two programs that establish the basis matrix (6.25) and the particular minimal solution. The input variables are the integer constants a, b and c. Finding the basis matrix and the minimal solution is done up to a given limit (in our case up to $m = 10^6$, obviously this limit can be augmented).

Program 6.35. Program for finding the basis matrix.

$$M(a,b) := \left| \begin{array}{l} m \leftarrow 10^6 \\ for\ \alpha \in 2..m \\ \quad \left| \begin{array}{l} q \leftarrow \sqrt{\dfrac{b}{a}(\alpha - 1)(\alpha + 1)} \\ break\ if\ q{=}trunc(q) \wedge \dfrac{a}{b}q{=}trunc\left(\dfrac{a}{b}q\right) \end{array} \right. \\ return\ \left(\begin{array}{cc} \alpha & q \\ \dfrac{a}{b}q & \alpha \end{array} \right) if\ \alpha < m \\ return\ "Error\ Matrix\ A\ was\ not\ found"\ otherwise \end{array} \right.$$

Program 6.36. Program for finding the minimal solutions.

$$
SM(a,b,c) := \left| \begin{array}{l} m \leftarrow 10^6 \\ for\ y \in 1..m \\ \quad \left| \begin{array}{l} d \leftarrow \dfrac{b \cdot y^2 - c}{a} \\ if\ d \geq 0 \\ \quad \left| \begin{array}{l} x \leftarrow \sqrt{d} \\ break\ if\ x{=}trunc(x) \end{array} \right. \end{array} \right. \\ return\ \begin{pmatrix} x & x \\ y & -y \end{pmatrix}\ if\ y < m \\ return\ "Error\ S\ was\ not\ found"\ otherwise \end{array} \right.
$$

Example 6.37. For the Diophantine equation $2x^2 - 3y^2 = 5$, we give the sequence of symbolic Mathcad commands which completely solve analytically the Diophantine equation. Origin indices are considered 1 by using the command $ORIGIN := 1$.

We initialize the constants a, b and c

$$a := 2\ \ b := 3\ \ c := -5$$

The determine the basis matrix A and the eigenvalues of the matrix by means of the Mathcad function $eigenvals$

$$A := M(a,b) = \begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}\ \ \lambda := eigenvals(A) \rightarrow \begin{pmatrix} 5 + 2\sqrt{6} \\ 5 - 2\sqrt{6} \end{pmatrix}$$

We determine the eigenvectors of matrix A with the aid of the Mathcad function $eigenvec$ and we build matrix V

$$V := augment(eigenvec(A, \lambda_1), eigenvec(A, \lambda_2)) \rightarrow \begin{pmatrix} \dfrac{\sqrt{6}}{2} & -\dfrac{\sqrt{6}}{2} \\ 1 & 1 \end{pmatrix}$$

We have matrix $P(n)$ given by formula (where $P(n) = A^n$):

$$P(n) := V \cdot \begin{pmatrix} (\lambda_1)^n & 0 \\ 0 & (\lambda_2)^n \end{pmatrix} \cdot V^{-1}$$

We determine the minimal solutions

$$SM(a, b, c) \rightarrow \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$S_0 := SM(a, b, c)^{\langle 1 \rangle} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad S_1 := SM(a, b, c)^{\langle 2 \rangle} \rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The general solutions of the Diophantine equation are $S0(n)$ and $S1(n)$

$$S0(n) := (A^n \cdot S_0)^{\mathrm{T}} \quad S1(n) := (A^n \cdot S_1)^{\mathrm{T}}$$

The explicit formulas for the general solutions are $T0(n)$ and $T1(n)$

$$T0(n) := P(n) \cdot S_0 \; factor \rightarrow$$
$$\begin{pmatrix} \dfrac{(\sqrt{6} + 4)(5 + 2\sqrt{6})^n - (\sqrt{6} - 4)(5 - 2\sqrt{6})^n}{4} \\ \dfrac{(2\sqrt{6} - 3)(5 - 2\sqrt{6})^n - (2\sqrt{6} + 3)(5 + 2\sqrt{6})^n}{6} \end{pmatrix}$$

$$T1(n) := P(n) \cdot S_1 \; factor \rightarrow$$
$$\begin{pmatrix} \dfrac{(\sqrt{6} + 4)(5 - 2\sqrt{6})^n - (\sqrt{6} - 4)(5 + 2\sqrt{6})^n}{4} \\ \dfrac{(2\sqrt{6} + 3)(5 - 2\sqrt{6})^n - (2\sqrt{6} - 3)(5 + 2\sqrt{6})^n}{6} \end{pmatrix}$$

Let $n = 0, 1, 2, \ldots, 10$

$$n := 0..10$$

We display the solutions for n

$$S0(n) \rightarrow \begin{pmatrix} 2 & 1 \\ 16 & 13 \\ 158 & 129 \\ 1564 & 1277 \\ 15482 & 12641 \\ 153256 & 125133 \\ 1517078 & 1238689 \\ 15017524 & 12261757 \\ 148658162 & 121378881 \\ 1471564096 & 1201527053 \\ 14566982798 & 11893891649 \end{pmatrix}$$

$$S1(n) \rightarrow \begin{pmatrix} 2 & -1 \\ 4 & 3 \\ 38 & 31 \\ 376 & 307 \\ 3722 & 3039 \\ 36844 & 30083 \\ 364718 & 297791 \\ 3610336 & 2947827 \\ 35738642 & 29180479 \\ 353776084 & 288856963 \\ 3502022198 & 2859389151 \end{pmatrix}$$

The displayed solutions can be tested if we verify the Diophantine equation by the aid of the sequences:

$$a \cdot (S0(n)_1)^2 - b \cdot (S0(n)_2)^2 + c \rightarrow$$

$$(\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix})^{\mathrm{T}}$$

$$a \cdot (S1(n)_1)^2 - b \cdot (S1(n)_2)^2 + c \rightarrow$$

$$(\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix})^{\mathrm{T}}$$

The solutions given by the expressions $T0(n)$ and $T1(n)$ can also be displayed, as follows:

$$
\left(\begin{array}{cc} T0(n)_1 & T0(n)_2 \end{array} \right) \rightarrow
\begin{pmatrix}
2 & 1 \\
16 & 13 \\
158 & 129 \\
1564 & 1277 \\
15482 & 12641 \\
153256 & 125133 \\
1517078 & 1238689 \\
15017524 & 12261757 \\
148658162 & 121378881 \\
1471564096 & 1201527053 \\
14566982798 & 11893891649
\end{pmatrix} ,
$$

$$
\left(\begin{array}{cc} T1(n)_1 & T1(n)_2 \end{array} \right) \rightarrow
\begin{pmatrix}
2 & -1 \\
4 & 3 \\
38 & 31 \\
376 & 307 \\
3722 & 3039 \\
36844 & 30083 \\
364718 & 297791 \\
3610336 & 2947827 \\
35738642 & 29180479 \\
353776084 & 288856963 \\
3502022198 & 2859389151
\end{pmatrix} .
$$

Obviously these solutions are identical with those given by $S0(n)$ and $S1(n)$.

Basically, any Diophantine equation of the type (6.12) can be completely solved with this set of commands.

Example 6.38. Let us consider the equation $13x^2 - 17y^2 + 2636 = 0$. The basis matrix is

$$
A := M(13, 17) = \begin{pmatrix} 1665 & 1904 \\ 1456 & 1665 \end{pmatrix} .
$$

The minimal solutions are:

$$S_0 \rightarrow \begin{pmatrix} 19 \\ 11 \end{pmatrix} \quad S_1 \rightarrow \begin{pmatrix} 19 \\ -11 \end{pmatrix} .$$

the solutions are given by the formulas:

$$S0(n) := (A^n \cdot S_0)^{\mathrm{T}} \quad S1(n) := (A^n \cdot S_1)^{\mathrm{T}} .$$

The values given by $S0$ for $n = 0, 1, 2$ and 10 are:

$$\boxed{19, \ 11}, \boxed{52579, \ 45979}, \boxed{175088051, \ 153110059}$$

and

$$\boxed{264734208132703398942304179191\4721331,}$$
$$\boxed{231503349286334972644202580334291\9339},$$

and the values provided by $S1$ for $n = 0, 1, 2$ and 10 are:

$$\boxed{19, -11}, \boxed{10691, \ 9349}, \boxed{35601011, \ 31132181}$$

and

$$\boxed{5382894721815312111185495966880\06131,}$$
$$\boxed{470720488200496189286367630993\971861} .$$

The explicit solutions are:

$$T0(n) := P(n) \cdot S_0 \ factor \rightarrow$$

$$\begin{pmatrix} \dfrac{(11\sqrt{221} + 247)\theta_1^n - (11\sqrt{221} - 247)\theta_2^n}{26} \\ \dfrac{(19\sqrt{221} + 187)\theta_1^n - (19\sqrt{221} - 187)\theta_2^n}{34} \end{pmatrix}$$

where

$$\theta_1 = 1665 + 112\sqrt{221} , \quad \theta_2 = 1665 - 112\sqrt{221}$$

and

$$T1(n) := P(n) \cdot S_1 \; factor \; \rightarrow$$

$$\left(\begin{array}{c} \dfrac{(11\sqrt{221} + 247)\theta_2^n - (11\sqrt{221} - 247)\theta_1^n}{26} \\[2mm] \dfrac{(19\sqrt{221} + 187)\theta_2^n - (19\sqrt{221} - 187)\theta_1^n}{34} \end{array} \right) \, ,$$

for $n \in \mathbb{N}$.

6.5.4   Generalizations

If $f(x, y) = 0$ is a Diophantine equation of second degree and with two unknowns, by linear transformations it becomes (6.12).

If $a \cdot b \geq 0$ the equation has at most a finite number of integer solutions which can be found attempts. It is easier to present an example.

The Diophantine equation

$$18x^2 + 12xy - 26y^2 - 12x - 32y + 40 = 0 \qquad (6.30)$$

becomes

$$2u^2 - 7v^2 + 45 = 0 \, , \qquad (6.31)$$

where (unfortunately, finding these substitutions is a difficult problem)

$$\begin{cases} u &=& 3x + y - 1 \, , \\ v &=& 2y + 1 \, . \end{cases} \qquad (6.32)$$

The basis matrix for the Diophantine equation (6.31) is

$$A := M(2, 7) = \left(\begin{array}{cc} 15 & 28 \\ 8 & 15 \end{array} \right)$$

and the minimal solutions are $S_0 = (3 \; 3)^{\mathrm{T}}$ and $S_1 = (3 \; -3)^{\mathrm{T}}$. In this conditions we obtain the solutions $S0(n) = A^n \cdot S_0$ and $S1(n) = A^n \cdot S_1$. Formula $S1(n)$ produces as solutions negative integers. Back from the

solutions obtain with formula $S0(n)$ to variables x and y by means of the substitutions

$$\begin{cases} x = \dfrac{2u - v - 3}{6} \\[2mm] y = \dfrac{v - 1}{2} \end{cases}$$

we obtain the solution of the Diophantine equation (6.30). The first 11 positive integer solutions are:

$$\begin{pmatrix} 1 & 1 \\ 32 & 34 \\ 945 & 1033 \\ 28304 & 30970 \\ 848161 & 928081 \\ 25416512 & 27811474 \\ 761647185 & 833416153 \\ 22823999024 & 24974673130 \\ 683958323521 & 748406777761 \\ 20495925706592 & 22427228659714 \\ 614193812874225 & 672068453013673 \end{pmatrix}.$$

We solve (6.31). Thus:

$$\begin{cases} u_{n+1} & = & 15u_n + 28v_n \,, \\ v_{n+1} & = & 8u_n + 15v_n \,, \end{cases} \tag{6.33}$$

$n \in \mathbb{N}$, with $(u_0,\ v_0) = (3,\ 3\varepsilon)$.

First solution

By induction we proof that: for all $n \in \mathbb{N}$ we have v_n is odd, and u_n as well as v_n are multiple of 3. Clearly $v_0 = 3\varepsilon \cdot u_0$. For $n + 1$ we have $v_{n+1} = 8u_n + 15v_n = even + odd = odd$, and of course u_{n+1}, v_{n+1} are multiples of 3 because u_n, v_n are multiple 3, too.

Hence, there exist x_n, y_n, in positive integers for all $n \in \mathbb{N}$:

$$\begin{cases} x_n = \dfrac{2u_n - v_n + 3}{6} \,, \\\\ y_n = \dfrac{v_n - 1}{2} \,, \end{cases} \tag{6.34}$$

(from (6.32)). Now we find the (6.29) for (6.31) as closed expression, and by means of (6.34 it results the general integer solution of the equation (6.30).

Second solution

Another expression of the (6.29) for (6.30) we obtain if we transform (6.32) as: $u_n = 3x_n + y_n - 1$ and $v_n = 2y_n + 1$, for all $n \in \mathbb{N}$. Whence, using (6.33) and doing the calculus, it finds

$$\begin{cases} x_{n+1} = 11x_n + \dfrac{52}{3}y_n + \dfrac{11}{3} \,, \\\\ y_{n+1} = 12x_n + 19y_n + 3 \,, \end{cases} \tag{6.35}$$

for $n \in \mathbb{N}$, with $(x_0, y_0) = (1,\ 1)$ or $(2,\ -2)$ (two infinitude of integer solutions). Let

$$A = \begin{pmatrix} 11 & \dfrac{52}{3} & \dfrac{11}{3} \\\\ 12 & 9 & 3 \\\\ 0 & 0 & 1 \end{pmatrix} .$$

Then

$$\begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix} = A^n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix} = A^n \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} , \tag{6.36}$$

always $n \in \mathbb{N}$.

From (6.35) we have always $y_{n+1} \equiv y_n \equiv \ldots \equiv y_0 \equiv 1 \ (mod\ 3)$, hence always $x_n \in \mathbb{Z}$. Of course (6.36) and (6.34) are equivalent as general integer solution (6.30).

This method can be generalized for Diophantine equations

$$\sum_{i=1}^{n} a_i \cdot x_i^2 = b \,, \tag{6.37}$$

will all $a_i, b \in \mathbb{Z}$.

It always $a_i \cdot a_j \geq 0 \ 1 \leq i \leq j < n$, the equation (6.37) has most finite number of integer solution.

Now, we suppose $\exists i_0, j_0 \in I_n$ for which $a_{i_0} \cdot a_{j_0} < 0$ (the equation presents at least a variation of sign). Analogously, for $n \in \mathbb{N}$. We define the recurrent sequence:

$$x_h^{(n+1)} = \sum_{i=1}^{n} a_{ih} \cdot x_i^{(n)} \quad 1h \in I_n \tag{6.38}$$

considering $(x_1^0, x_2^0, \ldots, x_n^0)$ the smallest positive integer solution of (6.37). It replaces (6.38) in (6.37), it identifies the coefficients and it look for the n^2 unknowns a_{ih}, where $i, h \in I_n$. This calculus is very intricate, but it can done by means of a computer. The method goes on similarly, but the calculus becomes more and more intricate – for example to calculate A^n. It must computer may be.

Other results referring to Diophantine equations can be found in the papers [Landau, 1955, Long, 1965, Ogibvy and Anderson, 1966, Mordell, 1969, Hardy and Wright, 1984, Bencze, 1985, Borevich and Shafarevich, 1985, Pérez et al., 2013].

6.6 The Diophantine equation $x^2 - 2y^4 + 1 = 0$

In this section we present a method of solving this Diophantine equation, method which is different from Ljunggren's, Mordell's and Guy's.

In the book [Guy, 1981, pp. 84-85] to shows that equation $x^2 = 2y^4 - 1$ has, in the set of positive integers, only solutions $\boxed{1, 1}$ and $\boxed{239, 13}$; Ljunggren [1966] has proved it in a complicated way. But Mordell [1964] gave an easier proof.

We'll note $t = y^2$. The general integer solution for $x^2 - 2t^2 + 1$ is

$$\begin{cases} x_{n+1} = 3x_n + 4t_n \ , \\ t_{n+1} = 2x_n + 3t_n \end{cases}$$

for all $n \in \mathbb{N}$, where $(x_0, y_0) = (1, \varepsilon)$, with $\varepsilon = \pm 1$ or

$$\begin{pmatrix} x_n \\ t_n \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \ ,$$

for all $n \in \mathbb{N}$, where a matrix to the power zero is equal to the unit matrix I.

Let's consider

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \ ,$$

and $\lambda \in \mathbb{R}$. Then $det(A - \lambda \cdot I) = 0$ implies $\lambda_{1,2} = 3 \pm \sqrt{2}$, whence if v is a vector of dimension two, then $Av = \lambda_{1,2} \cdot v$.

Let's consider

$$P = \begin{pmatrix} 2 & 2 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

and

$$D = \begin{pmatrix} 3 + \sqrt{2} & 0 \\ 0 & 3 - \sqrt{2} \end{pmatrix} \ .$$

We have $P^{-1} \cdot A \cdot P = D$, or

$$A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix} \dfrac{a_n + b_n}{2} & \dfrac{\sqrt{2}(a_n - b_n)}{2} \\ \dfrac{\sqrt{2}(a_n - b_n)}{4} & \dfrac{a_n + b_n}{2} \end{pmatrix} \ .$$

where $a_n = (3 + 2\sqrt{2})^n$ and $b_n = (3 - 2\sqrt{2})^n$. Hence, we find

$$
\begin{pmatrix} x_n \\ t_n \end{pmatrix} = \begin{pmatrix} \dfrac{1 + \varepsilon\sqrt{2}}{2} a_n + \dfrac{1 - \varepsilon\sqrt{2}}{2} b_n \\ \dfrac{2\varepsilon + \sqrt{2}}{4} a_n + \dfrac{2\varepsilon - \sqrt{2}}{4} b_n \end{pmatrix} .
$$

for all $n \in \mathbb{N}$.

Or $y_n^2 = t_n$, for all $n \in \mathbb{N}$. For $n = 0$, $\varepsilon = 1$ we obtain $y_0^2 = 1$ (whence $x_0 = 1$), and for $n = 3$, $\varepsilon = 1$ we obtain $y_3^2 = 169$ (whence $x_3 = 239$).

$$
y_n^2 = \varepsilon \sum_{k=0}^{\left[\frac{n}{2}\right]} C_n^{2k} \cdot 3^{n-2k} \cdot 2^k + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_n^{2k+1} 3^{n-2k-1} \cdot 2^{3k+1} . \tag{6.39}
$$

We still prove that y_n^2 is perfect square if and only if $n = 0, 3$. We can use a similar method the Diophantine equation $x^2 = Dy^4 \pm 1$, or more generally: $C \cdot X^{2a} = DY^{2b} + E$, with $a, b \in \mathbb{N}^*$ and $C, D, E \in \mathbb{Z}^*$; denoting $x^a = U$, $y^b = V$, and applying the results from [Smarandache, 1988], the relation (6.39) becomes very complicated.

May be found following works [Mordell, 1964, Ljunggren, 1966, Cohn, 1978] and [Guy, 1981, pp 84-85].

Chapter 7

Partial empirical solving of η–Diophantine equations

7.1 Empirical determination of solutions

A method often used to find some solutions of Diophantine equations is the empirical search, Alanen [1972], of certain numbers that satisfy the Diophantine equation, [Abraham et al., 2010], [Cohen, 2007, Niven et al., 1991, Rossen, 1987].

The empirical search or exhaustive search, also known as generating and testing, is a very general technique of problem solving, which consists in systematically enumerating all possible candidates as solutions and testing if they verify the problem.

An algorithm of empirical search for finding the divisors of a natural number n would enumerate all integers from 1 to $\lfloor \sqrt{n} \rfloor$, and verify each number if it divides n.

An empirical search is easy to implement, and it will always find a solution in the case that those solutions exist, its cost being proportional with the number of candidate solutions – which, in may practical problems, tends to grow very fast along with the problem's dimension. Therefore, the empirical search is used when the dimension of the problem is lim-

ited, or when, for specific heuristic causes, the problem can be reduced to a more manageable dimension. The method is also used when the simplicity of the implementation is more important than the speed of the problem solving.

For example, this is the case of critical applications, when any error in the algorithm would have serious consequences; or when a computer is used to prove a mathematical theorem. The empirical search is also useful as a basic method when benchmarks or other meta-heuristic algorithm are used. Indeed, the empirical search can be viewed as the simplest meta-heuristic algorithm. The empirical search should not be confounded with the backtracking, where a great number of solutions can be avoided without being explicitly enumerated. The empirical search method is useful for finding an element in a table – namely, it verifies sequentially all inputs – that's why it is a linear search.

A possibility to accelerate the empirical algorithm is to reduce the search space, that is the set of candidate solutions, by using heuristic techniques that are specific to the problem.

By means of a brief analysis we can often bring dramatic reductions to the number of candidate solutions, and solving the problem can turn from a difficult issue into a trivial one.

In some cases, the analysis can reduce the candidate solutions toe a set of viable solutions. This can be obtained with an algorithm that enumerates directly all candidates, without losing time on testing, and generates also invalid candidates. For example, for the problem "*find all integers between* 1 *and* 10^9*, divisible by* 571", a naive solution would generate all integers, testing afterwards each of them for divisibility by 571. However, this problem can be solved more efficiently by beginning with la 571 and, repeatedly, adding 571 up to number 10^9 – which would necessitate only 1751314 ($= 10^9/571$) steps and tests.

7.1.1 Partial empirical solving of Diophantine equations

Examples of problems solved by means of the empirical search:

1. D. Wilson, [Sloane, 2014, A030052], has compiled a list of the smallest

nth powers of positive integers that are the sums of the nth powers of distinct smaller positive integers. The first few are:

$$
\begin{aligned}
3^1 &= 1^1 + 2^1 \,, \\
5^2 &= 3^2 + 4^2 \,, \\
6^3 &= 3^3 + 4^3 + 5^3 \,, \\
15^4 &= 4^4 + 6^4 + 8^4 + 9^4 + 14^4 \,, \\
12^5 &= 4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 \,,
\end{aligned}
$$

$$
\begin{aligned}
25^6 = 1^6 + 2^6 + 3^6 + 5^6 + 6^6 + 7^6 + 8^6 + 9^6 + 10^6 + 12^6 \\
+ 13^6 + 15^6 + 16^6 + 17^6 + 18^6 + 23^6 \,,
\end{aligned}
$$

$$
\begin{aligned}
40^7 = 1^7 + 3^7 + 5^7 + 9^7 + 12^7 + 14^7 + 16^7 + 17^7 + 18^7 \\
+ 20^7 + 21^7 + 22^7 + 25^7 + 28^7 + 39^7 \,,
\end{aligned}
$$

$$
\begin{aligned}
84^8 = 1^8 + 2^8 + 3^8 + 5^8 + 7^8 + 9^8 + 10^8 + 11^8 + 12^8 + 13^8 \\
+ 14^8 + 15^8 + 16^8 + 17^8 + 18^8 + 19^8 + 21^8 + 23^8 \\
+ 24^8 + 25^8 + 26^8 + 27^8 + 29^8 + 32^8 + 33^8 + 35^8 \\
+ 37^8 + 38^8 + 39^8 + 41^8 42^8 + 43^8 + 45^8 + 46^8 \\
+ 47^8 + 48^8 + 49^8 + 51^8 + 52^8 + 53^8 + 57^8 + 58^8 \\
+ 59^8 + 61^8 + 63^8 + 69^8 + 73^8 \,,
\end{aligned}
$$

$$
\begin{aligned}
47^9 = 1^9 + 2^9 + 4^9 + 7^9 + 11^9 + 14^9 + 15^9 + 18^9 + +26^9 + 27^9 \\
+ 30^9 + 31^9 + 32^9 + 33^9 + 36^9 + 38^9 + 39^9 + 43^9 \,,
\end{aligned}
$$

$$
\begin{aligned}
63^{10} = 1^{10} + 2^{10} + 4^{10} + 5^{10} + 6^{10} + 8^{10} + 12^{10} + 15^{10} + 16^{10} \\
+ 17^{10} + 20^{10} + 21^{10} + 25^{10} + 26^{10} + 27^{10} + 28^{10} + 30^{10} \\
+ 36^{10} + 37^{10} + 38^{10} + 40^{10} + 51^{10} + 62^{10} \,.
\end{aligned}
$$

2. The first prime number with the special property that the result of the addition to its reverse is also a prime number is 229. We will call the prime numbers with this property numbers having *the 229 property*. By means of an empirical search algorithm were found all 50598 prime numbers having *the 229 property*, for p prime, $p < 10^7$, in approximately 25 seconds on a computer with Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable) [Cira, 2014a].

   The list of solutions begins with the prime numbers:
   229, 239, 241, 257, 269, 271, 277, 281, 439, 443, 463, 467, 479, 499, 613, 641, 653, 661, 673, 677, 683, 691, 811, 823, 839, 863, 881 ...
   and ends with the prime numbers:
   8998709, 8998813, 8998919, 8999099, 8999161, 8999183, 8999219, 8999311, 8999323, 8999339, 8999383, 8999651, 8999671, 8999761, 8999899, 8999981 .

3. The natural numbers that satisfy the Diophantine relation $\overline{c_{n-1}c_{n-2}\dots c_0} = c_{n-1}^n + c_{n-2}^n + \dots + c_0^n$, are called *narcissistic numbers*, [Cira and Cira, 2010].

   (a) Solutions in base 3 numeral system:
      i. For $n = 1$ we have the trivial solutions: $1 = 1^1$, $2 = 2^1$, out of 2 possible cases, and solution $0 = 0^1$.
      ii. For $n = 2$ we have the solutions:
      $$12 = 1^2 + 2^2 = 1 + 11 ,$$
      $$22 = 2^2 + 2^2 = 11 + 11 ,$$
      out of 7 possible cases.
      iii. For $n = 3 = 10_3$ we have a sole solution:
      $$122 = 1^{10} + 2^{10} + 2^{10} = 1 + 22 + 22 ,$$
      out of 19 possible cases.
      (4-7) for $n = 4 = 11_3$, $n = 5 = 12_3$ $n = 6 = 20_3$ and $n = 7 = 21_3$ there do not exist solutions, out of, respectively 55, 163, 487 and 1459 possible cases.

(b) Solutions in base 4 numeral system:

    i. For $n = 1$ we have the trivial solutions: $1 = 1^1$, $2 = 2^1$, $3 = 3^1$, out of 3 possible cases, and solution $0 = 0^1$.

    ii. For $n = 2$ we do not have solutions, out of 13 possible cases.

    iii. For $n = 3$ we have 6 solutions:

$$130 = 1^3 + 3^3 + 0^3 = 1 + 123 + 0 \ ,$$
$$131 = 1^3 + 3^3 + 1^3 = 1 + 123 + 1 \ ,$$
$$203 = 2^3 + 0^3 + 3^3 = 20 + 0 + 123 \ ,$$
$$223 = 2^3 + 2^3 + 3^3 = 20 + 20 + 123 \ ,$$
$$313 = 3^3 + 1^3 + 3^3 = 123 + 1 + 123 \ ,$$
$$332 = 3^3 + 3^3 + 2^3 = 123 + 123 + 20 \ ,$$

out of 49 possible cases.

    iv. For $n = 4 = 10_4$ we 2 solutions:

$$1103 = 1^{10} + 1^{10} + 0^{10} + 3^{10} = 1 + 1 + 0 + 1101 \ ,$$
$$3303 = 3^{10} + 3^{10} + 0^{10} + 3^{10} = 1101 + 1101 + 0 + 1101 \ ,$$

out of 193 possible cases.

(5-13) For $n = 5 = 11_4$, $n = 6 = 12_4$, $\ldots$, $n = 13 = 31_4$ we do not have solutions, out of 769, 3073, $\ldots$, 503316493 possible cases.

(c) etc.

4. Conjecture of Erdös-Straus: *for n natural number $n \geq 2$ the equation*

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

admits at least a solution $(x, y, z) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^*$. Important theoretical results were obtained by Tao [2011] and Elsholtz and Tao [2012], but the previous statement was not yet proved. Swett [2006] announced that he has verified the statement for $n \leq 10^{14}$. We give solutions of this equation in the form $\boxed{x, y, z}$:

n=2 $\boxed{1,2,2}$;

n=3 $\boxed{1,4,12}$, $\boxed{1,6,6}$, $\boxed{2,2,3}$;

n=4 $\boxed{2,3,6}$, $\boxed{2,4,4}$, $\boxed{3,3,3}$;

n=5 $\boxed{2,4,20}$, $\boxed{2,5,10}$;

n=6 $\boxed{2,7,42}$, $\boxed{2,8,24}$, $\boxed{2,9,18}$, $\boxed{2,10,15}$, $\boxed{2,12,12}$, $\boxed{3,4,12}$,
  $\boxed{3,6,6}$, $\boxed{4,4,6}$;

n=7 $\boxed{2,18,63}$, $\boxed{2,21,42}$, $\boxed{2,28,28}$, $\boxed{3,6,14}$, $\boxed{4,4,14}$;

n=8 $\boxed{3,7,42}$, $\boxed{3,8,24}$, $\boxed{3,9,18}$, $\boxed{3,10,15}$, $\boxed{3,12,12}$, $\boxed{4,5,20}$,
  $\boxed{4,6,12}$, $\boxed{4,8,8}$, $\boxed{5,5,10}$, $\boxed{6,6,6}$;

n=9 $\boxed{3,10,90}$, $\boxed{3,12,36}$, $\boxed{3,18,18}$, $\boxed{4,6,36}$, $\boxed{4,9,12}$, $\boxed{6,6,9}$;

n=10 $\boxed{3,18,90}$, $\boxed{3,20,60}$, $\boxed{3,24,40}$, $\boxed{3,30,30}$, $\boxed{4,8,40}$, $\boxed{4,10,20}$,
  $\boxed{4,12,15}$, $\boxed{5,6,30}$, $\boxed{5,10,10}$, $\boxed{6,6,15}$;

n=11 $\boxed{3,66,66}$, $\boxed{4,11,44}$, $\boxed{4,12,33}$, $\boxed{6,6,33}$;

n=12 $\boxed{4,14,84}$, $\boxed{4,15,60}$, $\boxed{4,16,48,}$, $\boxed{4,18,36}$, $\boxed{4,20,30}$, $\boxed{4,21,28}$,
  $\boxed{4,24,24}$, $\boxed{5,9,45}$, $\boxed{5,10,30}$, $\boxed{5,12,20}$, $\boxed{5,15,15}$, $\boxed{6,7,42}$,
  $\boxed{6,8,24}$, $\boxed{6,9,18}$, $\boxed{6,10,15}$, $\boxed{6,12,12}$, $\boxed{7,7,21}$, $\boxed{8,8,12}$,
  $\boxed{9,9,9}$;

n=13 $\boxed{4,26,52}$.

Obviously, these solutions verify the equation, as the solution for
$n = 13$

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{26} + \frac{1}{52} \ .$$

7.2 The η–Diophantine equations

Let $m, n \in \mathbb{N}^*$ fixed and x and y unknown positive integers. The
Diophantine equations in which function η is involved are called η–
Diophantine. The list of η–Diophantine equations, considered from

[Smarandache, 1999b], which we have into consideration to solve empirically are:

(2069) $\eta(m \cdot x + n) = x$,

(2070) $\eta(m \cdot x + n) = m + n \cdot x$,

(2071) $\eta(m \cdot x + n) = x!$,

(2072) $\eta(x^m) = x^n$,

(2073) $\eta(x)^m = \eta(x^n)$,

(2074) $\eta(m \cdot x + n) = \eta(x)^y$,

(2075) $\eta(x) + y = x + \eta(y)$, where $x \neq y$, x and y are not prime,

(2076) $\eta(x) + \eta(y) = \eta(x + y)$, where x and y are not siblings prime,

(2077) $\eta(x + y) = \eta(x) \cdot \eta(y)$,

(2078) $\eta(x \cdot y) = \eta(x) \cdot \eta(y)$,

(2079) $\eta(m \cdot x + n) = x^y$,

(2080) $\eta(x) \cdot y = x \cdot \eta(y)$, where x and y are not prime,

(2081) $\eta(x) \cdot \eta(y) = x \cdot y$, where x and y are not prime,

(2082) $\eta(x)^y = x^{\eta(y)}$, where x and y are not prime,

(2083) $\eta(x)^{\eta(y)} = \eta(x^y)$,

(2084) $\eta(x^y) - \eta(z^w) = 1$, with $y \neq 1 \neq w$,

(2085) $\eta(x^y) = y$, with $y \geq 2$,

(2086) $\eta(x^x) = y^y$,

(2087) $\eta(x^y) = y^x$,

(2088) $\eta(x) = y!$,

(2089) $\eta(m \cdot x) = m \cdot \eta(x)$, with $m \geq 2$,

(2090) $m^{\eta(x)} + \eta(x)^n = m^n$.

(2091) $n \cdot \eta(x^2) \pm m \cdot \eta(y^2) = m \cdot n$,

(2092) $\eta(x_1^{y_1} + x_2^{y_2} + \ldots + x_r^{y_r}) = \eta(x_1)^{y_1} + \eta(x_2)^{y_2} + \ldots + \eta(x_r)^{y_r}$,

(2093) $\eta(x_1! + x_2! + \ldots + x_r!) = \eta(x_1)! + \eta(x_2)! + \ldots + \eta(x_r)!$,

(2094) $(x, y) = (\eta(x), \eta(y))$, where by $(\cdot, \cdot)$ we understand is the greatest common divisor and x and y are not prime,

(2095) $[x, y] = [\eta(x), \eta(y)]$, where by $[\cdot, \cdot]$ we understand is the smallest common multiple and x and y are not prime.

7.2.1 Partial empirical solving of η–Diophantine equations

For all Diophantine equations solved in this section the file $\eta.prn$ is read, generated by the program 2.9, by means of Mathcad function $READPRN$

$$\eta := READPRN(\text{"...}\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6$$

where the command $last(\eta)$ indicates the last index of vector η.

7.2.2 The equation 2069

Program 7.1. Given vector η, the equation $\eta(mx + n) = x$ is equivalent with the relation $\eta_{mx+n} = x$. The program to find the solutions of equation (2069) is:

$$Ed2069(a_m, b_m, a_n, b_n, a_x, b_x) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{" "}n\text{" "}x\text{"}) \\ u \leftarrow last(\eta) \\ for \ m \in a_m..b_m \\ \quad for \ n \in a_n..b_n \end{array} \right.$$

$$\begin{array}{l} for\ x \in a_x..b_x \\ \quad \left|\begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta{=}x \\ S \leftarrow stack[S, (m\ n\ x)]\ if\ q \end{array}\right. \\ return\ S \end{array}$$

The call of the program is done by the sequence:

$$a_m := 2\ b_m := 10 \quad a_n := 1\ b_n := 10 \quad a_x := 1\ b_x := 16 \ ,$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 10\} \times \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 16\} \ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 1440 \ .$$

The call of the program Ed2069:

$$t_0 : time(0) \ \ Sol := Ed2069(a_m, b_m, a_n, b_n, a_x, b_x) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.011 \cdot s \quad rows(Sol) - 1 = 36$$

For $m \in \{2, 3, \ldots, 10\}$, $n \in \{1, 2, \ldots, 10\}$ and $x \in \{1, 2, \ldots, 16\}$ the 36 solutions of the Diophantine equation $\eta(m \cdot x + n) = x$, given as $\boxed{m, n, x}$, are:

$\boxed{2, 4, 4}$ $\boxed{2, 4, 6}$ $\boxed{2, 5, 5}$ $\boxed{2, 5, 10}$ $\boxed{2, 6, 6}$ $\boxed{2, 7, 7}$ $\boxed{2, 9, 9}$ $\boxed{2, 10, 5}$;

$\boxed{3, 5, 5}$ $\boxed{3, 7, 7}$ $\boxed{3, 7, 14}$ $\boxed{3, 8, 8}$;

$\boxed{4, 7, 7}$ $\boxed{4, 8, 4}$ $\boxed{4, 10, 5}$ $\boxed{4, 10, 10}$;

$\boxed{5, 4, 4}$ $\boxed{5, 5, 5}$ $\boxed{5, 6, 6}$ $\boxed{5, 7, 7}$ $\boxed{5, 9, 9}$;

$\boxed{6,9,6}$ $\boxed{6,10,5}$;

$\boxed{7,3,6}$ $\boxed{7,5,5}$ $\boxed{7,5,10}$ $\boxed{7,6,6}$ $\boxed{7,7,7}$ $\boxed{7,8,8}$;

$\boxed{8,5,15}$ $\boxed{8,7,7}$ $\boxed{8,9,9}$;

$\boxed{9,7,7}$ $\boxed{9,10,10}$;

$\boxed{10,7,14}$ $\boxed{10,10,5}$.

The maximum value of solutions x is 15.

By a similar call, the program $Ed2069$ provides the 40 solutions of the Diophantine equation $\eta(m \cdot x + n) = x$ of the search domain

$$D_c = \{97, 98, \ldots, 100\} \times \{11, 12, \ldots, 99\} \times \{43, 44, \ldots, 89\}$$

in the form $\boxed{m, n, x}$:

$\boxed{97, 43, 43}$ $\boxed{97, 47, 47}$ $\boxed{97, 53, 53}$ $\boxed{97, 59, 59}$ $\boxed{97, 61, 61}$ $\boxed{97, 67, 67}$
$\boxed{97, 71, 71}$ $\boxed{97, 73, 73}$ $\boxed{97, 79, 79}$ $\boxed{97, 83, 83}$ $\boxed{97, 86, 43}$ $\boxed{97, 89, 89}$
$\boxed{97, 94, 47}$;

$\boxed{98, 43, 43}$ $\boxed{98, 47, 47}$ $\boxed{98, 53, 53}$ $\boxed{98, 59, 59}$ $\boxed{98, 61, 61}$ $\boxed{98, 67, 67}$
$\boxed{98, 71, 71}$ $\boxed{98, 73, 73}$ $\boxed{98, 79, 79}$ $\boxed{98, 83, 83}$ $\boxed{98, 86, 43}$ $\boxed{98, 89, 89}$
$\boxed{98, 94, 47}$;

$\boxed{99, 43, 43}$ $\boxed{99, 47, 47}$ $\boxed{99, 53, 53}$ $\boxed{99, 59, 59}$ $\boxed{99, 61, 61}$ $\boxed{99, 67, 67}$
$\boxed{99, 71, 71}$ $\boxed{99, 73, 73}$ $\boxed{99, 79, 79}$ $\boxed{99, 83, 83}$ $\boxed{99, 89, 89}$;

$\boxed{100, 86, 43}$ $\boxed{100, 87, 58}$ $\boxed{100, 94, 47}$.

The maximum value of solutions x is 89.

7.2.3   The equation 2070

Program 7.2. Given vector η, the equation $\eta(mx+n) = m+nx$ is equivalent with the relation $\eta_{mx+n} = m + nx$. The program for finding the solutions of the equation (2070) is:

$$Ed2070(a_m, b_m, a_n, b_n, a_x, b_x) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{"}\ \text{"}n\text{"}\ \text{"}x\text{"}) \\ u \leftarrow last(\eta) \\ for\ m \in a_m..b_m \\ \quad for\ n \in a_n..b_n \\ \qquad for\ x \in a_x..b_x \\ \qquad \left| \begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta = m + n \cdot x \\ S \leftarrow stack[S, (m\ n\ x)]\ if\ q \end{array} \right. \\ return\ S \end{array} \right.$$

The call of the program is done by the sequence:

$$a_m := 2\ \ b_m := 20\ \ \ a_n := 1\ \ b_n := 20\ \ \ a_x := 1\ \ b_x := 16\ ,$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 20\} \times \{1, 2, \ldots, 20\} \times \{1, 2, \ldots, 16\}\ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 7220\ .$$

The call of the program Ed2070:

$$t_0 : time(0)\ \ Sol := Ed2070(a_m, b_m, a_n, b_n, a_x, b_x)\ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.853 \cdot s\ \ \ \ rows(Sol) - 1 = 14$$

For $m \in \{2, 3, \ldots, 20\}$, $n \in \{1, 2, \ldots, 20\}$ and $x \in \{2, 3, \ldots, 20\}$ the 14 solutions of the Diophantine equation $\eta(m \cdot x + n) = m + n \cdot x$, given as $\boxed{m, n, x}$, are:

$\boxed{2,1,4}$;

$\boxed{4,1,2}$ $\boxed{4,1,6}$;

$\boxed{6,1,4}$ $\boxed{6,1,8}$;

$\boxed{8,1,6}$ $\boxed{8,1,6}$;

$\boxed{10,1,12}$;

$\boxed{12,1,10}$ $\boxed{12,1,14}$;

$\boxed{14,1,12}$;

$\boxed{16,1,18}$;

$\boxed{18,1,16}$ $\boxed{18,1,20}$;

$\boxed{20,1,18}$.

The maximum value of solutions x is 20.

7.2.4 The equation 2071

Program 7.3. Given vector η, the equation $\eta(mx + n) = x!$ is equivalent with the relation $\eta_{mx+n} = x!$. The program for finding the solutions of the equation (2071) is:

$$Ed2071(a_m, b_m, a_n, b_n, a_x, b_x) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{"} \ \text{"}n\text{"} \ \text{"}x\text{"}) \\ u \leftarrow last(\eta) \\ for \ m \in a_m..b_m \\ \quad for \ n \in a_n..b_n \\ \quad \quad for \ x \in a_x..b_x \\ \quad \quad \quad \left| \begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta = x! \\ S \leftarrow stack[S, (m \ n \ x)] \ if \ q \end{array} \right. \end{array} \right.$$

$$\left| return\ S \right.$$

The call of the program is done by the sequence:

$$a_m := 2\ b_m := 15\ \ a_n := 1\ b_n := 15\ \ a_x := 1\ b_x := 19\ ,$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 15\} \times \{1, 2, \ldots, 15\} \times \{1, 2, \ldots, 19\}\ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 3990\ .$$

The call of the program Ed2071:

$$t_0 : time(0)\ \ Sol := Ed2071(a_m, b_m, a_n, b_n, a_x, b_x)\ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.02 \cdot s \quad rows(Sol) - 1 = 24$$

For $m \in \{2, 3, \ldots, 15\}$, $n \in \{1, 2, \ldots, 15\}$ and $x \in \{1, 2, \ldots, 19\}$ the 24 solutions of the Diophantine equation $\eta(m \cdot x + n) = x!$, given as $\boxed{m, n, x}$, are:

$\boxed{2, 3, 3}\ \boxed{2, 10, 3}\ \boxed{2, 12, 3}$;

$\boxed{3, 7, 3}\ \boxed{3, 9, 3}$;

$\boxed{4, 4, 3}\ \boxed{4, 6, 3}$;

$\boxed{5, 1, 3}\ \boxed{5, 3, 3}$;

$\boxed{7, 15, 3}$;

$\boxed{8, 12, 3}$;

$\boxed{9,9,3}$;

$\boxed{10,6,3}$ $\boxed{10,15,3}$;

$\boxed{11,3,3}$ $\boxed{11,12,3}$ $\boxed{11,15,3}$;

$\boxed{12,9,3}$ $\boxed{12,12,3}$;

$\boxed{13,6,3}$ $\boxed{13,9,3}$;

$\boxed{14,3,3}$ $\boxed{14,6,3}$;

$\boxed{15,3,3}$.

The maximum value of solutions x is 3.

7.2.5   The equation 2072

Program 7.4. Given vector η, the equation $\eta(x^m) = x^n$ is equivalent with the relation $\eta_{x^m} = x^n$. The program for finding the solutions of the equation (2072) is:

$$Ed2072(a_m, b_m, a_n, b_n, a_x, b_x) := \begin{array}{|l} S \leftarrow (\text{''}m\text{''} \ \text{''}n\text{''} \ \text{''}x\text{''}) \\ u \leftarrow last(\eta) \\ for \ m \in a_m..b_m \\ \quad for \ n \in a_n..b_n \\ \quad\quad for \ x \in a_x..b_x \\ \quad\quad\quad \begin{array}{|l} \eta \leftarrow x^m \\ q \leftarrow \eta \leq u \wedge \eta_\eta = x^n \\ S \leftarrow stack[S, (m \ n \ x)] \ if \ q \end{array} \\ return \ S \end{array}$$

The call of the program is done by the sequence:

$$a_m := 2 \ b_m := 9 \quad a_n := 2 \ b_n := 9 \quad a_x := 2 \ b_x := 10 \ ,$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 9\} \times \{2, 3, \ldots, 9\} \times \{2, 3, \ldots, 10\} \ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 576 \ .$$

The call of the program Ed2072:

$$t_0 : time(0) \ \ Sol := Ed2072(a_m, b_m, a_n, b_n, a_x, b_x) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.11 \cdot s \quad rows(Sol) - 1 = 12 \ .$$

For $m \in \{2, 3, \ldots, 9\}$, $n \in \{2, 3, \ldots, 9\}$ and $x \in \{1, 2, \ldots, 10\}$ the 12 solutions of the Diophantine equation $\eta(x^m) = x^n$, given as $\boxed{m, n, x}$, are:

$\boxed{2, 2, 2}$;

$\boxed{3, 2, 2}$ $\boxed{3, 2, 3}$

$\boxed{4, 2, 3}$;

$\boxed{5, 2, 5}$ $\boxed{5, 3, 2}$;

$\boxed{6, 2, 4}$ $\boxed{6, 2, 5}$ $\boxed{6, 3, 2}$;

$\boxed{7, 2, 4}$ $\boxed{7, 2, 7}$ $\boxed{7, 3, 2}$.

The maximum value of solutions x is 7.

7.2.6 The equation 2073

Program 7.5. Given vector η, the equation $\eta(x)^m = \eta(x^n)$ is equivalent with the relation $(\eta_x)^m = \eta_{x^n}$. The program for finding the solutions of the equation (2073) is:

$$Ed2073(a_m, b_m, a_n, b_n, a_x, b_x) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{"} \ \text{"}n\text{"} \ \text{"}x\text{"}) \\ u \leftarrow last(\eta) \\ for \ m \in a_m..b_m \\ \quad for \ n \in a_n..b_n \\ \qquad for \ x \in a_x..b_x \\ \qquad \quad \left| \begin{array}{l} q \leftarrow x^n \leq u \wedge (\eta_x)^m = \eta_{x^n} \\ S \leftarrow stack[S, (m \ n \ x)] \ if \ q \end{array} \right. \\ return \ S \end{array} \right.$$

The call of the program is done by the sequence:

$$a_m := 2 \ b_m := 9 \quad a_n := 2 \ b_n := 9 \quad a_x := 2 \ b_x := 25 \ ,$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 9\} \times \{2, 3, \ldots, 9\} \times \{2, 3, \ldots, 25\} \ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 1536 \ .$$

The call of the program Ed2073:

$$t_0 : time(0) \ \ Sol := Ed2073(a_m, b_m, a_n, b_n, a_x, b_x) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.014 \cdot s \quad rows(Sol) - 1 = 20$$

For $m \in \{2, 3, \ldots, 9\}$, $n \in \{2, 3, \ldots, 9\}$ and $x \in \{2, 3, \ldots, 25\}$ the 20 solutions the Diophantine equation $\eta(x)^m = \eta(x^n)$, in the form $\boxed{m, n, x}$ are:

$\boxed{2,2,2}$;

$\boxed{2,3,2}$ $\boxed{2,3,3}$ $\boxed{2,3,6}$;

$\boxed{2,4,3}$ $\boxed{2,4,6}$ $\boxed{2,4,8}$ $\boxed{2,4,24}$;

$\boxed{2,5,5}$ $\boxed{2,5,8}$ $\boxed{2,5,10}$ $\boxed{2,5,15}$;

$\boxed{2,6,4}$ $\boxed{2,6,5}$ $\boxed{2,6,10}$;

$\boxed{2,7,4}$ $\boxed{2,7,7}$;

$\boxed{3,5,2}$ $\boxed{3,6,2}$ $\boxed{3,7,2}$.

The maximum value of solutions x is 24.

7.2.7   The equation 2074

Program 7.6. Given vector η, the equation $\eta(mx+n) = \eta(x)^m$ is equivalent with the relation $\eta_{mx+n} = (\eta_x)^m$. The program for finding the solutions of the equation (2074) is:

$$Ed2074(a_m, b_m, a_n, b_n, a_x, b_x) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{" "}n\text{" "}x\text{"}) \\ u \leftarrow last(\eta) \\ for\ m \in a_m..b_m \\ \quad for\ n \in a_n..b_n \\ \quad\quad for\ x \in a_x..b_x \\ \quad\quad\quad \left| \begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta = (\eta_x)^m \\ S \leftarrow stack[S, (m\ n\ x)]\ if\ q \end{array} \right. \\ return\ S \end{array} \right.$$

The call of the program is done by the sequence:

$$a_m := 1\ b_m := 6 \quad a_n := 1\ b_n := 9 \quad a_x := 1\ b_x := 10^5 ,$$

then the search domain is

$$D_c = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 9\} \times \{1, 2, \ldots, 10^5\} \ .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 5400000 \ .$$

The call of the program Ed2074:

$$t_0 : time(0) \ \ Sol := Ed2074(a_m, b_m, a_n, b_n, a_x, b_x) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.017 \cdot s \quad rows(Sol) - 1 = 24$$

For $m \in \{1, 2, \ldots, 6\}$, $n \in \{1, 2, \ldots, 9\}$ and $x \in \{1, 2, \ldots, 10^5\}$ the 24 solutions of the Diophantine equation $\eta(mx+n) = \eta(x)^m$, in the form $\boxed{m, n, x}$, are:

$\boxed{1, 2, 16}$;

$\boxed{1, 3, 3}$ $\boxed{1, 3, 45}$;

$\boxed{1, 4, 4}$ $\boxed{1, 4, 8}$;

$\boxed{1, 5, 5}$ $\boxed{1, 5, 10}$ $\boxed{1, 5, 15}$;

$\boxed{1, 7, 7}$ $\boxed{1, 7, 9}$ $\boxed{1, 7, 14}$ $\boxed{1, 7, 21}$ $\boxed{1, 7, 28}$ $\boxed{1, 7, 35}$ $\boxed{1, 7, 56}$ $\boxed{1, 7, 63}$ $\boxed{1, 7, 105}$;

$\boxed{1, 8, 4}$ $\boxed{1, 8, 72}$ $\boxed{1, 8, 1792}$;

$\boxed{1, 9, 9}$ $\boxed{1, 9, 36}$;

$\boxed{2, 4, 2}$ $\boxed{2, 8, 2}$.

The maximum value of solutions x is 1792.

The equation (2074) has a similar version in the form of the equation $(2074')$ $\eta(mx + n) = \eta(x)^n$.

Program 7.7. Given vector η, the equation $\eta(mx + n) = \eta(x)^n$ is equivalent with the relation $\eta_{mx+n} = (\eta_x)^n$. The program for finding the solutions of the equation $(2074')$ is:

$$Ed20741(a_m, b_m, a_n, b_n, a_x, b_x) := \left|\begin{array}{l} S \leftarrow (\text{"}m\text{"} \; \text{"}n\text{"} \; \text{"}x\text{"}) \\ u \leftarrow last(\eta) \\ for \; m \in a_m..b_m \\ \quad for \; n \in a_n..b_n \\ \qquad for \; x \in a_x..b_x \\ \qquad \left|\begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta = (\eta_x)^n \\ S \leftarrow stack[S, (m \; n \; x)] \; if \; q \end{array}\right. \\ return \; S \end{array}\right.$$

The call of the program is done by the sequence:

$$a_m := 1 \; b_m := 9 \quad a_n := 1 \; b_n := 9 \quad a_x := 1 \; b_x := 10^5 \; ,$$

hence, the search domain is

$$D_c = \{1, 2, \ldots, 9\} \times \{1, 2, \ldots, 9\} \times \{1, 2, \ldots, 10^5\} \; .$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 8100000 \; .$$

The call of the program Ed20741:

$$t_0 : time(0) \; \; Sol := Ed20741(a_m, b_m, a_n, b_n, a_x, b_x) \; \; t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 7.566 \cdot s \quad \; rows(Sol) - 1 = 3$$

For $m \in \{1, 2, \ldots, 6\}$, $n \in \{1, 2, \ldots, 9\}$ and $x \in \{1, 2, \ldots, 10^5\}$ the 3 solutions of the Diophantine equation $\eta(mx+n) = \eta(x)^m$, in the form $\boxed{m, n, x}$, are:

$$\boxed{1, 2, 2}\ \boxed{3, 2, 2}\ \boxed{5, 2, 2}\ .$$

The maximum value of solutions x is 2.

7.2.8 The equation 2075

Program 7.8. Given vector η, the equation $\eta(x) + y = x + \eta(y)$ cu $x \neq y$, where x and y are not prime, is equivalent with the relation $\eta_x + y = x + \eta_y$, with $x < y$ (we consider condition $x < y$ instead of $x \neq y$ for symmetry reasons of the equation relative to x and y), where x and y are not prime. The program for finding the solutions of the equation (2075) is:

$$Ed2075(a_{xy}, b_{xy}) := \left|
\begin{array}{l}
S \leftarrow (\text{"}x\text{" "}y\text{"}) \\
for\ x \in a_{xy}..b_{xy} - 1 \\
\quad for\ y \in x + 1..b_{xy} \\
\qquad \left|
\begin{array}{l}
u \leftarrow Tp\eta(x) = 0 \\
v \leftarrow Tp\eta(y) = 0 \\
notprime \leftarrow u \wedge v \\
q \leftarrow \eta_x + y = x + \eta_y \\
S \leftarrow stack[S, (x\ y)]\ if\ notprime \wedge q
\end{array}
\right. \\
return\ S
\end{array}
\right.$$

The program $Ed2075$ calls the program 1.8 to run the primality test for x and y.The call of the program is done by the sequence:

$$a_{xy} := 2\ b_{xy} := 1000$$

hence, the search domain is

$$D_c = \{2, 3, \ldots, 999\} \times \{3, 4, \ldots, 1000\}, \quad \text{with } x < y\ .$$

The total number of verified cases is:

$$\sum_{x=2}^{999} \sum_{y=x+1}^{1000} 1 = 498501 \ .$$

The call of the program Ed2075:

$$t_0 : time(0) \ \ Sol := Ed2075(a_{xy}, b_{xy}) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 5.765 \cdot s \quad rows(Sol) - 1 = 157$$

For $x \in \{2, 3, \ldots, 999\}$ and $y \in \{3, 4, \ldots, 1000\}$ the 157 solutions of the Diophantine equation $\eta(x) + y = x + \eta(y)$ in the form of pairs $\boxed{x, y}$ are:

$6, 9$	$15, 16$	$20, 25$	$40, 42$	$40, 49$	$42, 49$	$45, 52$	$60, 66$	$63, 64$
$72, 77$	$75, 78$	$80, 111$	$84, 88$	$90, 98$	$96, 99$;		

$108, 110$	$108, 121$	$110, 121$	$120, 138$	$126, 136$	$140, 147$	$140, 152$
$144, 161$	$147, 152$	$154, 156$	$154, 169$	$156, 169$	$160, 171$	$162, 170$
$168, 184$	$175, 176$	$180, 203$	$192, 207$	$195, 196$	$198, 204$;	

$200, 209$	$210, 232$	$216, 230$	$220, 228$	$225, 258$	$231, 242$	$234, 238$
$243, 245$	$250, 282$	$256, 287$	$260, 266$	$264, 276$	$270, 290$	$272, 289$
$280, 286$	$288, 294$;					

$300, 319$	$312, 322$	$320, 325$	$320, 338$	$325, 338$	$330, 348$	$336, 376$
$340, 342$	$340, 361$	$342, 361$	$343, 345$	$350, 357$	$352, 363$	$352, 372$
$360, 413$	$363, 372$	$378, 410$	$384, 423$	$390, 406$;		

$408, 414$	$416, 434$	$420, 472$	$432, 470$	$441, 488$	$448, 450$	$455, 459$
$456, 460$	$462, 492$	$480, 531$	$486, 553$;			

$500, 582$	$504, 568$	$506, 529$	$507, 518$	$510, 522$	$525, 618$	$528, 564$
$540, 590$	$544, 558$	$546, 574$	$560, 632$	$561, 578$	$567, 589$	$570, 580$
$572, 602$	$576, 639$	$588, 615$	$594, 605$	$594, 636$;		

$600, 649$	$605, 636$	$608, 620$	$616, 625$	$624, 658$	$630, 637$	$630, 712$
$637, 712$	$640, 711$	$648, 710$	$660, 708$	$663, 665$	$672, 747$	$675, 684$
$675, 686$	$684, 686$	$690, 696$	$693, 713$;			

$702, 742$	$714, 738$	$720, 729$	$735, 824$	$736, 744$	$748, 774$	$756, 830$
$770, 782$	$780, 826$	$792, 852$	$798, 820$;			

$800, 869$	$810, 890$	$812, 841$	$816, 846$	$819, 837$	$825, 851$	$832, 845$
$836, 860$	$840, 850$	$840, 867$	$850, 867$	$874, 888$	$875, 903$	$880, 948$
$882, 899$	$896, 925$;					

$900, 979$	$910, 920$	$912, 940$	$918, 954$	$924, 996$	$928, 930$	$928, 961$
$930, 961$	$936, 994$	$966, 984$	$968, 989$	$975, 999$.		

The maximum value of solutions x is 975 and of y is 999.

7.2.9 The equation 2076

Program 7.9. Given vector η, the equation $\eta(x) + \eta(y) = \eta(x + y)$ cu $x \neq y$, where x and y are not prime, is equivalent with the relation $\eta_x + \eta_y = \eta_{x+y}$, with $x < y$ (we consider condition $x < y$ instead of $x \neq y$ for symmetry reasons of the equation relative to x and y), where x and y are not prime. The program for finding the solutions of the equation (2076) is:

$$Ed2076(a_{xy}, b_{xy}) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\ for \ x \in a_{xy}..b_{xy} - 1 \\ \quad for \ y \in x + 1..b_{xy} \\ \qquad \left| \begin{array}{l} u \leftarrow Tp\eta(x)\text{=}0 \\ v \leftarrow Tp\eta(y)\text{=}0 \end{array} \right. \end{array} \right.$$

$$\left|\begin{array}{l} \left|\begin{array}{l} notprime \leftarrow u \wedge v \\ q \leftarrow \eta_x + \eta_y {=} \eta_{x+y} \\ S \leftarrow stack[S, (x\ y)]\ if\ notprime \wedge q \end{array}\right. \\ return\ S \end{array}\right.$$

The program $Ed2076$ calls the program 1.8 to run the primality test for x and y. The call of the program is done by the sequence:

$$a_{xy} := 4\ b_{xy} := 1000$$

hence, the search domain is

$$D_c = \{4, 5, \ldots, 999\} \times \{5, 6, \ldots, 1000\}, \ \ \text{with} \ x < y\,.$$

The total number of verified cases is:

$$\sum_{x=4}^{999} \sum_{y=x+1}^{1000} 1 = 496506\,.$$

The call of the program Ed2076:

$$t_0 : time(0) \ \ Sol := Ed2076(a_{xy}, b_{xy}) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 3.877 \cdot s \ \ \ \ rows(Sol) - 1 = 1277$$

For $x \in \{4, 5, \ldots, 999\}$ and $y \in \{5, 6, \ldots, 1000\}$ the 1277 solutions of the Diophantine equation $\eta(x) + \eta(y) = \eta(x+y)$, with $y > x$, x and y non-prime numbers in the form of pairs $\boxed{x, y}$ (the first 96 and the last 75 solutions) are:

$\boxed{4, 84}$ $\boxed{4, 234}$ $\boxed{4, 455}$ $\boxed{4, 456}$;

$\boxed{6, 8}$ $\boxed{6, 48}$ $\boxed{6, 147}$ $\boxed{6, 150}$ $\boxed{6, 192}$;

$8,14$ $8,24$ $8,26$ $8,38$ $8,74$ $8,86$ $8,125$ $8,134$ $8,135$ $8,158$
$8,168$ $8,194$ $8,206$ $8,218$ $8,254$ $8,326$ $8,386$ $8,446$ $8,458$
$8,468$ $8,475$ $8,554$ $8,614$ $8,626$ $8,698$ $8,758$ $8,794$ $8,878$
$8,910$ $8,912$ $8,914$ $8,926$ $8,974$ $8,998$;

$9,56$ $9,110$ $9,143$ $9,221$ $9,297$ $9,368$ $9,390$ $9,420$ $9,612$
$9,620$ $9,851$;

$10,15$ $10,40$ $10,45$ $10,144$ $10,224$ $10,294$ $10,595$ $10,640$;

$12,15$ $12,21$ $12,27$ $12,39$ $12,57$ $12,111$ $12,129$ $12,201$
$12,237$ $12,252$ $12,260$ $12,291$ $12,309$ $12,327$; $12,378$
$12,381$ $12,489$ $12,494$ $12,579$ $12,669$ $12,687$ $12,702$
$12,729$ $12,831$ $12,921$ $12,939$ $12,960$;

$14,35$ $14,84$ $14,90$ $14,280$ $14,363$ $14,700$;

$\vdots$

$630,840$ $630,900$ $636,637$ $637,696$ $637,705$ $640,767$ $640,880$
$646,798$ $650,702$ $650,841$ $658,720$ $660,792$ $666,703$ $672,756$
$672,924$ $675,864$ $680,765$ $684,760$ $690,897$ $693,880$ $696,986$
$697,984$;

$700,731$ $700,851$ $700,899$ $700,910$ $702,819$ $704,990$ $715,975$
$720,749$ $720,759$ $720,819$ $720,840$ $722,779$ $725,957$ $726,840$
$729,792$ $735,944$ $738,943$ $750,837$ $754,833$ $754,928$ $756,765$
$759,828$ $768,798$ $768,927$ $770,924$ $777,810$ $779,902$ $780,910$
$782,805$ $783,980$ $784,952$;

$805, 1000$	$810, 867$	$810, 900$	$812, 870$	$816, 918$	$819, 896$
$820, 861$	$825, 990$	$832, 858$	$836, 968$	$836, 969$	$845, 918$
$845, 1000$	$850, 884$	$855, 950$	$860, 989$	$867, 988$	$875, 896$
$884, 972$	$891, 924$	$891, 972$	;

| $903, 946$ | $930, 992$ | .

The maximum value of solutions x is 930 and of y is 1000.

7.2.10   The equation 2077

Let us consider the Diophantine equation $\eta(x + y) = \eta(x) \cdot \eta(y)$ with $x \neq y$ on the set $\{1, 2, \ldots, 5 \cdot 10^4 - 1\} \times \{2, 3, \ldots, 5 \cdot 10^4\}$. Given vector η, the Diophantine equation (2077) is equivalent with relation $\eta_{x+y} = \eta_x \cdot \eta_y$, with $x < y$ (we use condition $x < y$ instead of $x \neq y$ for symmetry reasons of the equation relative to x and y). The search of the solutions was done by means of the program:

$$Ed2077(a_{xy}, b_{xy}) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{" "}y\text{"}) \\ for\ x \in a_{xy}..b_{xy} - 1 \\ \quad \left| for\ y \in x + 1..b_{xy} \\ \quad \quad \left| q \leftarrow \eta_{x+y} = \eta_x \cdot \eta_y \\ \quad \quad\ S \leftarrow stack[S, (x\ y)]\ if\ q \right. \\ return\ S \end{array} \right.$$

The search time was of $853.816s$ of 1249975000 possible cases. There was no solution found on the search domain $\{1, 2, \ldots, 5 \cdot 10^4 - 1\} \times \{2, 3, \ldots, 5 \cdot 10^4\}$.

Instead of the Diophantine equation (2077), which seems not to have any solutions, we propose equation $\eta(x \cdot y) = \eta(x) + \eta(y)$ for $x \neq y$, where x and y are not prime. Vector η allows us to write the equivalent relation to the Diophantine equation $\eta_{x \cdot y} = \eta_x + \eta_y$ for $x < y$ (we use condition $x < y$ instead of $x \neq y$ for symmetry reasons of the equation relative to x

and y). The program for finding the solutions of the Diophantine equation $\eta(x \cdot y) = \eta(x) + \eta(y)$ is:

$$Ed20771(a_{xy}, b_{xy}) := \begin{vmatrix} S \leftarrow (\text{"}x\text{" "}y\text{"}) \\ for\ x \in a_{xy}..b_{xy} - 1 \\ \quad for\ y \in x + 1..b_{xy} \\ \qquad \begin{vmatrix} u \leftarrow Tp\eta(x)\text{=}0 \\ v \leftarrow Tp\eta(y)\text{=}0 \\ notprime \leftarrow u \wedge v \\ q \leftarrow \eta_{x\cdot y}\text{=}\eta_x + \eta_y \\ S \leftarrow stack[S, (x\ y)]\ if\ notprime \wedge q \end{vmatrix} \\ return\ S \end{vmatrix}$$

The program $Ed20771$ calls the program 1.8 to run the primality test for x and y. The call of the program is done by the sequence:

$$a_{xy} := 4\ b_{xy} := 10^3$$

hence, the search domain is

$$D_c = \{4, 5, \ldots, 999\} \times \{5, 6, \ldots, 1000\}, \quad \text{with } x < y.$$

The total number of verified cases is:

$$\sum_{x=4}^{999} \sum_{y=x+1}^{1000} 1 = 496506.$$

The call of the program Ed20772:

$$t_0 : time(0)\ \ Sol := Ed20771(a_{xy}, b_{xy})\ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 4.979 \cdot s \quad rows(Sol) - 1 = 9659$$

For $x \in \{4, 5, \ldots, 999\}$ and $y \in \{5, 6, \ldots, 1000\}$ the 9659 solutions of the Diophantine equation $\eta(x \cdot y) = \eta(x) + \eta(y)$, with $x < y$, x and y being non-prime in the form of pairs $\boxed{x, y}$ (the first 131 and the last 55 solutions) are:

$4,8$ | $4,24$ | $4,128$ | $4,384$ | $4,640$ | $4,896$;

$6,9$ | $6,18$ | $6,36$ | $6,45$ | $6,72$ | $6,81$ | $6,90$ | $6,144$ | $6,162$ | $6,180$
$6,243$ | $6,324$ | $6,360$ | $6,405$ | $6,486$ | $6,567$ | $6,648$ | $6,720$ | $6,729$
$6,810$ | $6,972$;

$8,12$ | $8,24$ | $8,64$ | $8,128$ | $8,192$ | $8,256$ | $8,320$ | $8,384$ | $8,448$
$8,512$ | $8,576$ | $8,640$ | $8,768$ | $8,896$ | $8,960$;

$9,81$ | $9,162$ | $9,243$ | $9,324$ | $9,405$ | $9,486$ | $9,567$ | $9,648$ | $9,810$
$9,972$;

$10,15$ | $10,20$ | $10,25$ | $10,30$ | $10,40$ | $10,50$ | $10,60$ | $10,75$ | $10,100$
$10,120$ | $10,125$ | $10,150$ | $10,175$ | $10,200$ | $10,225$ | $10,250$
$10,300$ | $10,350$ | $10,375$ | $10,400$ | $10,450$ | $10,500$ | $10,525$
$10,600$ | $10,625$ | $10,675$ | $10,700$ | $10,750$ | $10,800$ | $10,875$
$10,900$ | $10,1000$;

$12,24$ | $12,128$ | $12,384$ | $12,640$ | $12,896$;

$14,21$ | $14,28$ | $14,35$ | $14,42$ | $14,49$ | $14,56$ | $14,63$ | $14,70$ | $14,84$
$14,98$ | $14,105$ | $14,112$ | $14,126$ | $14,140$ | $14,147$ | $14,168$ | $14,196$
$14,210$ | $14,245$ | $14,252$ | $14,280$ | $14,294$ | $14,315$ | $14,336$
$14,343$ | $14,392$ | $14,420$ | $14,441$ | $14,490$ | $14,504$ | $14,539$
$14,560$ | $14,588$ | $14,630$ | $14,637$ | $14,686$ | $14,735$ | $14,784$
$14,840$ | $14,882$ | $14,980$;

$\vdots$

$888,925$ | $888,962$ | $888,999$ | $890,979$ | $891,924$ | $891,968$ | $891,990$
$893,940$ | $893,987$ | $896,960$ | $897,920$ | $897,966$ | $899,930$ | $899,961$
$899,992$;

$900, 1000$	$901, 954$	$902, 943$	$902, 984$	$903, 946$	$903, 989$
$910, 936$	$910, 975$	$912, 931$	$912, 950$	$912, 969$	$912, 988$	$913, 996$
$915, 976$	$918, 935$	$918, 952$	$920, 966$	$923, 994$	$924, 968$	$924, 990$
$925, 962$	$925, 999$	$928, 957$	$928, 986$	$930, 961$	$930, 992$	$931, 950$
$931, 969$	$931, 988$	$935, 952$	$936, 975$	$940, 987$	$943, 984$	$946, 989$
$950, 969$	$950, 988$	$957, 986$	$961, 992$	$962, 999$	$968, 990$	;

7.2.11 The equation 2078

The Diophantine equation $\eta(x \cdot y) = \eta(x) \cdot \eta(y)$ for $x \in \{1, 2, \ldots, 10^3 - 1\}$ and $y \in \{2, 3, \ldots, 10^3\}$ with $x \neq y$ has the only the 999 trivial solutions, in the form $\boxed{x, y}$: $\boxed{1, 2}$ $\boxed{1, 3}$ ... $\boxed{1, 1000}$.

7.2.12 The equation 2079

Program 7.10. Given vector η, the equation $\eta(mx + n) = x^y$ is equivalent with the relation $\eta_{mx+n} = x^y$. The program for finding the solutions of the equation (2079) is:

$$Ed2079(a_m, b_m, a_n, b_n, a_x, b_x, y) := \left| \begin{array}{l} S \leftarrow (\text{"}m\text{" "}n\text{" "}x\text{"}) \\ u \leftarrow last(\eta) \\ for\ m \in a_m..b_m \\ \quad for\ n \in a_n..b_n \\ \quad\quad for\ x \in a_x..b_x \\ \quad\quad\quad \left| \begin{array}{l} \eta \leftarrow m \cdot x + n \\ q \leftarrow \eta \leq u \wedge \eta_\eta = x^y \\ S \leftarrow stack[S, (m\ n\ x)]\ if\ q \end{array} \right. \\ return\ S \end{array} \right.$$

The call of the program is done by the sequence:

$$y := 2 \ a_m := 1 \ b_m := 10 \quad a_n := 1 \ b_n := 10 \quad a_x := 1 \ b_x := 10^4 \ ,$$

hence, the search domain is

$$D_c = \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 10^4\} . \qquad (7.1)$$

The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 10^6 .$$

The call of the program Ed2079:

$$t_0 : time(0) \;\; Sol := Ed2079(a_m, b_m, a_n, b_n, a_x, b_x, y) \;\; t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.834 \cdot s \quad rows(Sol) - 1 = 16$$

For $m \in \{1, 1, \ldots, 10\}$, $n \in \{1, 2, \ldots, 10\}$ and $x \in \{1, 2, \ldots, 10^4\}$ the 16 solutions of the Diophantine equation $\eta(m \cdot x + n) = x^2$, in the form $\boxed{m, n, x}$, are:

$\boxed{1, 2, 2}$ $\boxed{1, 6, 2}$ $\boxed{1, 10, 2}$;

$\boxed{2, 4, 2}$ $\boxed{2, 8, 2}$;

$\boxed{3, 2, 2}$ $\boxed{3, 6, 2}$;

$\boxed{4, 4, 2}$;

$\boxed{5, 2, 2}$;

$\boxed{6, 9, 3}$;

$\boxed{7, 6, 3}$ $\boxed{7, 10, 2}$;

$\boxed{8, 3, 3}$ $\boxed{8, 8, 2}$;

$\boxed{9, 6, 2}$;

$\boxed{10, 4, 2}$.

The maximum value of solutions m, n and x is, respectively, 10, 10 and 2.

For $y = 3$ the Diophantine equation 2079 does not have solutions in the search domain (7.1).

It should be noted that for $y = \frac{1}{2}$ (y is not an integer!) the η–Diophantine equation $\eta(mx + n) = x^{1/2}$ has 8 solutions in the search domain (7.1). The η–Diophantine equation $\eta(mx + n) = x^{1/2}$ is equivalent with the relation $\eta_{mx+n} = \sqrt{x}$. The call of the program is done by the sequence:

$$y := \frac{1}{2} \ \ a_m := 1 \ b_m := 10 \ \ a_n := 1 \ b_n := 10 \ \ a_x := 1 \ b_x := 10^4 \ ,$$

hence we have the same search domain D_c given by (7.1). The total number of verified cases is:

$$(b_m - a_m + 1)(b_n - a_n + 1)(b_x - a_x + 1) = 10^6 \ .$$

The call of the program Ed2079:

$$t_0 : time(0) \ \ Sol := Ed2079(a_m, b_m, a_n, b_n, a_x, b_x, y) \ \ t_1 := time(1)$$

The execution time in seconds and the number of solutions follow from:

$$(t_1 - t_0) \cdot s = 0.928 \cdot s \ \ \ rows(Sol) - 1 = 16$$

For $m \in \{1, 2, \ldots, 10\}$, $n \in \{1, 2, \ldots, 10\}$ and $x \in \{1, 2, \ldots, 10^4\}$ the 8 solutions of the Diophantine equation $\eta(m \cdot x + n) = x^{1/2}$, in the form $\boxed{m, n, x}$, are:

$\boxed{1, 5, 25}$ $\boxed{1, 7, 49}$ $\boxed{1, 8, 16}$ $\boxed{1, 9, 36}$;

$\boxed{2, 7, 49}$ $\boxed{2, 8, 36}$ $\boxed{2, 10, 25}$;

$\boxed{5, 7, 49}$.

The maximum value of solutions m, n and x is, respectively 5, 10 and 49.

7.2.13   The equation 2080

Program 7.11. Given vector η, the equation $\eta(x) \cdot y = x \cdot \eta(y)$ is equivalent with the relation $\eta_x \cdot y = x \cdot \eta_y$. The program for finding the solutions of the equation (2080) is:

$$Ed2080(a_{xy}, b_{xy}) := \begin{vmatrix} S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\ for \ x \in a_{xy}..b_{xy} - 1 \\ \quad \begin{vmatrix} u \leftarrow Tp\eta(x) = 0 \\ for \ y \in x + 1..b_{xy} \\ \quad \begin{vmatrix} q \leftarrow u \wedge Tp\eta(y) = 0 \\ S \leftarrow stack[S, (x \ y)] \ if \ q \wedge \eta_x \cdot y = x \cdot \eta_y \end{vmatrix} \\ return \ S \end{vmatrix}$$

The program calls the subprogram 1.8 of establishing the primality of x and y. The call of the program is done by $a_{xy} := 2$, $b_{xy} := 10^3$ and hereby it results that we have the search domain

$$D_c = \{2, 3, \ldots, 999\} \times \{3, 4, \ldots, 1000\} \ ,$$

with $x < y$, where x and y are not prime. Hence, we obtain a number of 498501 possible cases. These cases have been ran through by the program $Ed2080$ in 5.444 seconds. A number of 13200 solutions was obtained, of which we present the first 95 and the last 81:

6, 8	6, 10	6, 14	6, 22	6, 26	6, 34	6, 38	6, 46	6, 58	6, 62

6, 74	6, 82	6, 86	6, 94	;

6, 106	6, 118	6, 122	6, 134	6, 142	6, 146	6, 158	6, 166	6, 178

6, 194	;

6, 202	6, 206	6, 214	6, 218	6, 226	6, 254	6, 262	6, 274	6, 278

6, 298	;

6, 302	6, 314	6, 326	6, 334	6, 346	6, 358	6, 362	6, 382	6, 386

6, 394	6, 398	;

$\boxed{6,422}$ $\boxed{6,446}$ $\boxed{6,454}$ $\boxed{6,458}$ $\boxed{6,466}$ $\boxed{6,478}$ $\boxed{6,482}$;

$\boxed{6,502}$ $\boxed{6,514}$ $\boxed{6,526}$ $\boxed{6,538}$ $\boxed{6,542}$ $\boxed{6,554}$ $\boxed{6,562}$ $\boxed{6,566}$ $\boxed{6,586}$;

$\boxed{6,614}$ $\boxed{6,622}$ $\boxed{6,626}$ $\boxed{6,634}$ $\boxed{6,662}$ $\boxed{6,674}$ $\boxed{6,694}$ $\boxed{6,698}$;

$\boxed{6,706}$ $\boxed{6,718}$ $\boxed{6,734}$ $\boxed{6,746}$ $\boxed{6,758}$ $\boxed{6,766}$ $\boxed{6,778}$ $\boxed{6,794}$;

$\boxed{6,802}$ $\boxed{6,818}$ $\boxed{6,838}$ $\boxed{6,842}$ $\boxed{6,862}$ $\boxed{6,866}$ $\boxed{6,878}$ $\boxed{6,886}$ $\boxed{6,898}$;

$\boxed{6,914}$ $\boxed{6,922}$ $\boxed{6,926}$ $\boxed{6,934}$ $\boxed{6,958}$ $\boxed{6,974}$ $\boxed{6,982}$ $\boxed{6,998}$;

$\vdots$

$\boxed{900,990}$ $\boxed{902,946}$ $\boxed{903,987}$ $\boxed{905,955}$ $\boxed{905,965}$ $\boxed{905,985}$ $\boxed{905,995}$
$\boxed{906,942}$ $\boxed{906,978}$ $\boxed{908,916}$ $\boxed{908,932}$ $\boxed{908,956}$ $\boxed{908,964}$ $\boxed{909,927}$
$\boxed{909,963}$ $\boxed{909,981}$;

$\boxed{910,980}$ $\boxed{913,979}$ $\boxed{914,922}$ $\boxed{914,926}$ $\boxed{914,934}$ $\boxed{914,958}$ $\boxed{914,974}$
$\boxed{914,982}$ $\boxed{914,998}$ $\boxed{916,932}$ $\boxed{916,956}$ $\boxed{916,964}$ $\boxed{917,959}$
$\boxed{917,973}$;

$\boxed{921,933}$ $\boxed{921,939}$ $\boxed{921,951}$ $\boxed{921,993}$ $\boxed{922,926}$ $\boxed{922,934}$ $\boxed{922,958}$
$\boxed{922,974}$ $\boxed{922,982}$ $\boxed{922,998}$ $\boxed{923,949}$ $\boxed{926,934}$ $\boxed{926,958}$ $\boxed{926,974}$
$\boxed{926,982}$ $\boxed{926,998}$ $\boxed{927,963}$ $\boxed{927,981}$ $\boxed{928,992}$;

$\boxed{932,956}$ $\boxed{932,964}$ $\boxed{933,939}$ $\boxed{933,951}$ $\boxed{933,993}$ $\boxed{934,958}$ $\boxed{934,974}$
$\boxed{934,982}$ $\boxed{934,998}$ $\boxed{938,994}$ $\boxed{939,951}$ $\boxed{939,993}$;

$\boxed{942,978}$ $\boxed{943,989}$ $\boxed{944,976}$ $\boxed{948,996}$;

$\boxed{951,993}$ $\boxed{955,965}$ $\boxed{955,985}$ $\boxed{955,995}$ $\boxed{956,964}$ $\boxed{958,974}$ $\boxed{958,982}$
$\boxed{958,998}$ $\boxed{959,973}$;

$$\boxed{963, 981} \; \boxed{965, 985} \; \boxed{965, 995} \; ;$$

$$\boxed{974, 982} \; \boxed{974, 998} \; ;$$

$$\boxed{982, 998} \; \boxed{985, 995} \; .$$

Taking into consideration the great number of solutions of equation (2080), we propose the same equation for $x \in \{a_x, a_x + m_x, a_x + 2m_x, \ldots, b_x\}$, x non-prime and $y \in \{a_y, a_y + m_y, a_y + 2m_y, \ldots, b_y\}$, y non-prime. The program for finding the solutions of the equation (2080) becomes:

$$Ed20801(a_x, m_x, b_x, a_y, m_y, b_y) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{"} \; \text{"}y\text{"}) \\ for \; x \in a_x, a_x + m_x..b_x \\ \quad \left| \begin{array}{l} u \leftarrow Tp\eta(x) = 0 \\ for \; y \in a_y, a_y + m_y..b_y \\ \quad \left| \begin{array}{l} q \leftarrow u \wedge Tp\eta(y){=}0 \\ if \; q \wedge \eta_x \cdot y{=}x \cdot \eta_y \\ \quad S \leftarrow stack[S, (x \; y)] \end{array} \right. \end{array} \right. \\ return \; S \end{array} \right.$$

The program calls the subprogram 1.8 for establishing the primality of x and y. For $a_x := 2$, $m_x := 111$, $b_x := 3 \cdot 10^3$, $a_y := 3$, $m_y := 203$, $b_y := 2 \cdot 10^4$, then the search set is

$$\{2, 113, 224, 335, 446, 557, 668, 779, 890, \ldots, 2999\}$$
$$\times \{3, 206, 409, 612, 815, 1018, 1221, 1424, 1627, \ldots, 19897\}$$

and the solutions, in the form $\boxed{x, y}$, are:

$$\boxed{335, 815} \; \boxed{335, 2845} \; \boxed{335, 6905} \; \boxed{335, 8935} \; ;$$

$$\boxed{446, 206} \; \boxed{446, 1018} \; \boxed{446, 2642} \; \boxed{446, 5078} \; \boxed{446, 10762} \; \boxed{446, 13198}$$
$$\boxed{446, 14822} \; \boxed{446, 15634} \; \boxed{446, 17258} \; ;$$

$$\boxed{668, 7108} \; ;$$

$\boxed{779, 2033}$ $\boxed{779, 17461}$;

$\boxed{1112, 6296}$ $\boxed{1112, 9544}$ $\boxed{1112, 19288}$;

$\boxed{1334, 8326}$;

$\boxed{1556, 7108}$;

$\boxed{2222, 3454}$ $\boxed{2222, 12386}$;

$\boxed{2444, 2236}$;

$\boxed{2888, 13604}$.

7.2.14   The equation 2081

Program 7.12. Given vector η, the equation $\eta(x) \cdot \eta(y) = x \cdot y$, with $x \neq y$ and x, y non-prime, is equivalent to the relation $\eta_x \cdot \eta_y = x \cdot y$, cu $x < y$ (for symmetry reasons of the equation relative to x and y, in order to elapse the equivalent solutions, condition $x < y$ was considered). The program for finding the solutions of the equation (2081), where $x < y$, with x and y non-prime, is:

$$Ed2081(a_{xy}, b_{xy}) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\ for \ x \in a_{xy}..b_{xy} - 1 \\ \quad \left| \begin{array}{l} u \leftarrow Tp\eta(x)\text{=}0 \\ for \ y \in x + 1..b_{xy} \\ \quad \left| \begin{array}{l} q \leftarrow u \wedge Tp\eta(y)\text{=}0 \\ S \leftarrow stack(S, (x \ y)) \ if \ q \wedge \eta_x \cdot \eta_y\text{=}x \cdot y \end{array} \right. \\ \end{array} \right. \\ return \ S \end{array} \right.$$

The program calls the subprogram 1.8 to establish the primality of x and y. The call of the program is done by the sequence:

$$a_{xy} := 2 \quad b_{xy} := 10^4 \ ,$$

hence, the search domain is

$$D_c = \{2, 3 \ldots, 9999\} \times \{3, 4, \ldots, 10000\}$$

cu $x < y$. The total number of verified cases is:

$$\sum_{x=2}^{999} \sum_{y=x+1}^{10000} = 49985001 \; ,$$

and no solution has been found.

7.2.15   The equation 2082

The η–Diophantine equation $\eta(x)^y = x^{\eta(y)}$, with x and y non-prime, is equivalent with the relation $(\eta_x)^y = x^{\eta_y}$, with x and y non-prime, taking into consideration the meaning of vector η.

The search program is

$$Ed2082(a_x, b_x, a_y, b_y) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{" "}y\text{"}) \\ for\ x \in a_x..b_x \\ \quad \left| \begin{array}{l} u \leftarrow Tp\eta(x){=}0 \\ for\ y \in a_y..b_y \\ \quad \left| \begin{array}{l} q \leftarrow u \wedge Tp\eta(y){=}0 \\ S \leftarrow stack(S, (x\ y))\ if\ q \wedge (\eta_x)^y{=}x^{\eta_y} \end{array} \right. \\ return\ S \end{array} \right.$$

The program calls the subprogram 1.8 to establish the primality of x and y.

The solutions of the problem for the search domain defined by $a_x := 2$, $b_x := 64$, x non-prime and $a_y := 2$, $b_y := 80$, y non-prime in the form $\boxed{x, y}$, are:

$\boxed{4, 4}$ $\boxed{8, 9}$ $\boxed{27, 9}$;

$\boxed{36, 6}$ $\boxed{36, 8}$ $\boxed{36, 10}$ $\boxed{36, 14}$ $\boxed{36, 22}$ $\boxed{36, 26}$ $\boxed{36, 34}$ $\boxed{36, 38}$ $\boxed{36, 46}$
$\boxed{36, 58}$ $\boxed{36, 62}$ $\boxed{36, 74}$;

$$\boxed{64,6}\ \boxed{64,8}\ \boxed{64,10}\ \boxed{64,14}\ \boxed{64,22}\ \boxed{64,26}\ \boxed{64,34}\ \boxed{64,38}\ \boxed{64,46}$$
$$\boxed{64,58}\ \boxed{64,62}\ \boxed{64,74}\ ;$$

The number of ran through cases is 4977, and the execution time is less then one second.

7.2.16 The equation 2083

The equation $\eta(x)^{\eta(y)} = \eta(x^y)$ is equivalent with the relation $(\eta_x)^{\eta_y} = \eta_{x^y}$ taking into consideration the significance of vector η.

The search program will have to take into consideration the fact that x^y should not be greater then the last index of file η and $x < y$, as the role of x and y is symmetric.

$$Ed2083(a_x, b_x, a_y, b_y) := \left| \begin{array}{l} S \leftarrow (\text{"}x\text{" "}y\text{"}) \\ for\ x \in a_x..b_x - 1 \\ \quad \left| \begin{array}{l} for\ y \in x+1..b_y \\ \quad \left| \begin{array}{l} u \leftarrow x^y \le last(\eta) \\ S \leftarrow stack[S,(x\ y)]\ if\ u \wedge (\eta_x)^{\eta_y} = \eta_{x^y} \end{array} \right. \end{array} \right. \\ return\ S \end{array} \right.$$

for $a_x := 2$, $b_x := 100$, $a_y := 2$, $b_y : 120$, with $x < y$, the number of ran trough cases is 6831, with the remark that some cases are excluded, namely when $x^y \ge 10^6$. We have only 2 solutions: $\boxed{2,6}$ and $\boxed{2,12}$ that satisfy equation (2083)

$$\eta(2)^{\eta(6)} = \eta(2^6)\ \text{ and }\ \eta(2)^{\eta(12)} = \eta(2^{12})$$

or the equivalent relation $(\eta_x)^{\eta_y} = \eta_{x^y}$

$$(\eta_2)^{\eta_6} = \eta_{2^6}\ \text{ and }\ (\eta_2)^{\eta_{12}} = \eta_{2^{12}}\ .$$

7.2.17 The equation 2084

The η–Diophantine equation $\eta(x^y) = \eta(z^w)$, with $x \ne z$, is equivalent with the relation $\eta_{x^y} = \eta_{z^w}$ taking into consideration the significance of the

file η. The program $Ed20840(a_x, b_x, a_y, b_y, a_z, b_z, a_w, b_w)$ that run trough all 6561 situations of the search domain

$$D_c = \{2, 3, \ldots, 10\} \times \{2, 3, \ldots, 10\} \times \{2, 3, \ldots, 10\} \times \{2, 3, \ldots, 10\} \ , \ (7.2)$$

where $x \neq z$. The real number of cases that have been ran trough is 4033 for reason of restrictions $x^y \leq last(\eta) = 10^6$ and $z^w \leq last(\eta) = 10^6$.

$Ed20840(a_x, b_x, a_y, b_y, a_z, b_z, a_w, b_w) :=$

$$\begin{vmatrix} S \leftarrow (\text{"}x\text{" "}y\text{" "}z\text{" "}w\text{"}) \\ u \leftarrow last(\eta) \\ for \ x \in a_x..b_x \\ \quad \begin{vmatrix} for \ y \in a_y..b_y \\ X \leftarrow x^y \\ for \ z \in a_z..b_z \\ \quad \begin{vmatrix} for \ w \in a_w..b_w \\ Z \leftarrow z^w \\ q \leftarrow x \neq z \wedge X \leq u \wedge Z \leq u \\ S \leftarrow stack[S, (x \ y \ z \ w)] \ if \ q \wedge \eta_X = \eta_Z \end{vmatrix} \end{vmatrix} \\ return \ S \end{vmatrix}$$

The solution of the Diophantine equation $\eta(x^y) = \eta(z^w)$, in the form $\boxed{x, y, z, w}$, are:

$\boxed{2, 5, 4, 3}$ $\boxed{2, 6, 4, 3}$ $\boxed{2, 7, 4, 3}$ $\boxed{2, 9, 3, 5}$ $\boxed{2, 9, 4, 5}$ $\boxed{2, 9, 6, 5}$ $\boxed{2, 9, 8, 3}$ $\boxed{2, 10, 3, 5}$ $\boxed{2, 10, 4, 5}$ $\boxed{2, 10, 6, 5}$ $\boxed{2, 10, 8, 3}$;

$\boxed{3, 2, 2, 4}$ $\boxed{3, 3, 6, 4}$ $\boxed{3, 3, 9, 2}$ $\boxed{3, 4, 9, 2}$ $\boxed{3, 5, 2, 9}$ $\boxed{3, 5, 2, 10}$ $\boxed{3, 7, 4, 8}$ $\boxed{3, 7, 9, 4}$ $\boxed{3, 8, 6, 7}$ $\boxed{3, 8, 9, 4}$ $\boxed{3, 10, 9, 5}$;

$\boxed{4, 3, 2, 5}$ $\boxed{4, 3, 2, 6}$ $\boxed{4, 3, 2, 7}$ $\boxed{4, 3, 8, 2}$ $\boxed{4, 4, 2, 8}$ $\boxed{4, 4, 5, 2}$ $\boxed{4, 4, 10, 2}$ $\boxed{4, 5, 2, 9}$ $\boxed{4, 5, 2, 10}$ $\boxed{4, 5, 8, 3}$ $\boxed{4, 6, 8, 5}$ $\boxed{4, 7, 8, 5}$ $\boxed{4, 8, 3, 7}$ $\boxed{4, 8, 6, 7}$ $\boxed{4, 9, 8, 6}$;

$\boxed{5,2,2,8}\,\boxed{5,3,3,6}\,\boxed{5,4,4,9}\,\boxed{5,4,8,6}\,\boxed{5,5,10,6}$;

$\boxed{6,2,2,4}\,\boxed{6,3,3,4}\,\boxed{6,3,9,2}\,\boxed{6,4,9,2}\,\boxed{6,5,2,9}\,\boxed{6,5,2,10}\,\boxed{6,5,8,3}$
$\boxed{6,6,5,3}\,\boxed{6,6,9,3}\,\boxed{6,6,10,3}\,\boxed{6,7,3,8}\,\boxed{6,7,4,8}\,\boxed{6,7,9,4}$;

$\boxed{7,3,3,9}\,\boxed{7,5,5,8}$;

$\boxed{8,2,2,5}\,\boxed{8,2,2,6}\,\boxed{8,2,2,7}\,\boxed{8,2,4,3}\,\boxed{8,3,2,9}\,\boxed{8,3,2,10}\,\boxed{8,3,3,5}$
$\boxed{8,3,4,5}\,\boxed{8,3,6,5}\,\boxed{8,4,4,6}\,\boxed{8,4,4,7}\,\boxed{8,5,4,6}\,\boxed{8,5,4,7}\,\boxed{8,6,4,9}$
$\boxed{8,6,5,4}\,\boxed{8,6,10,4}$;

$\boxed{9,2,3,4}\,\boxed{9,2,6,3}\,\boxed{9,2,6,4}\,\boxed{9,3,3,6}\,\boxed{9,4,3,7}\,\boxed{9,4,3,8}\,\boxed{9,4,4,8}$
$\boxed{9,4,6,7}\,\boxed{9,5,3,10}$;

$\boxed{10,2,2,8}\,\boxed{10,3,3,6}\,\boxed{10,4,4,9}\,\boxed{10,4,8,6}\,\boxed{10,5,5,6}$.

We have 87 solutions of the Diophantine equation $\eta(x^y) = \eta(z^w)$ in the search domain D_c, given by (7.2), with $x \neq z$, $x^y \leq 10^6$ and $z^w \leq 10^6$.

Similarly, we can consider the Diophantine equations $\eta(x^y) - \eta(z^w) = k$, where $k = 1, 2, 3, 4, 5 \ldots$. The number of solutions for $k = 1$ is 61, for $k = 2$, 67, for $k = 3$, 67, for $k = 4$, 66 and for $k = 5$ we have 51 solutions, etc.

For example, the solutions of the equation $x^y - z^w = 23$ with $x \neq y \neq z \neq w$, $\boxed{x, y, z, w} \in \{2, 3, \ldots, 10\}^4$ are:

$\boxed{5,8,2,9}\,\boxed{5,8,2,10}\,\boxed{7,5,2,9}\,\boxed{7,5,2,10}\,\boxed{7,5,8,3}\,\boxed{9,6,2,3}$.

7.2.18 The equation 2085

Instead of the Diophantine equation (2085), $\eta(x^y) = y$, proposed in [Smarandache, 1999b], we solve the more general equation $\eta(x^y) - y = k$. The solutions of this equation with $\boxed{x, y, k} \in D_c \in \{2, 3, \ldots, 10^2\}^2 \times \{0, 1, \ldots, 10\}$ (where $\{2, 3, \ldots, 10^3\}^2 = \{2, 3, \ldots, 10^3\} \times \{2, 3, \ldots, 10^3\}$) are:

$\boxed{2, 3, 1}$ $\boxed{2, 7, 1}$ $\boxed{2, 15, 1}$;

$\boxed{2, 2, 2}$ $\boxed{2, 4, 2}$ $\boxed{2, 6, 2}$ $\boxed{2, 8, 2}$ $\boxed{2, 10, 2}$ $\boxed{2, 14, 2}$ $\boxed{2, 16, 2}$ $\boxed{2, 18, 2}$;

$\boxed{2, 5, 3}$ $\boxed{2, 9, 3}$ $\boxed{2, 11, 3}$ $\boxed{2, 13, 3}$ $\boxed{2, 17, 3}$ $\boxed{2, 19, 3}$;

$\boxed{2, 12, 4}$ $\boxed{3, 2, 4}$ $\boxed{4, 2, 4}$ $\boxed{6, 2, 4}$ $\boxed{12, 2, 4}$;

$\boxed{3, 4, 5}$ $\boxed{4, 3, 5}$ $\boxed{6, 4, 5}$;

$\boxed{3, 3, 6}$ $\boxed{4, 4, 6}$ $\boxed{6, 3, 6}$ $\boxed{8, 2, 6}$ $\boxed{12, 3, 6}$ $\boxed{12, 4, 6}$ $\boxed{24, 2, 6}$;

$\boxed{3, 5, 7}$ $\boxed{4, 5, 7}$ $\boxed{6, 5, 7}$ $\boxed{9, 2, 7}$ $\boxed{12, 5, 7}$ $\boxed{18, 2, 7}$ $\boxed{36, 2, 7}$ $\boxed{72, 2, 7}$;

$\boxed{5, 2, 8}$ $\boxed{10, 2, 8}$ $\boxed{15, 2, 8}$ $\boxed{16, 2, 8}$ $\boxed{20, 2, 8}$ $\boxed{30, 2, 8}$ $\boxed{40, 2, 8}$ $\boxed{45, 2, 8}$
$\boxed{48, 2, 8}$ $\boxed{60, 2, 8}$ $\boxed{80, 2, 8}$ $\boxed{90, 2, 8}$;

$\boxed{3, 6, 9}$ $\boxed{4, 7, 9}$ $\boxed{6, 6, 9}$ $\boxed{8, 3, 9}$ $\boxed{24, 3, 9}$;

$\boxed{3, 8, 10}$ $\boxed{4, 6, 10}$ $\boxed{4, 8, 10}$ $\boxed{32, 2, 10}$ $\boxed{96, 2, 10}$.

Let us remark that in D_c there exists no solution of equation $\eta(x^y) = y$.

The number of analyzed cases is 107811 of which only 3047 have satisfied the condition $x^y < last(\eta) = 10^6$.

Knowing the significance of vector η, the Diophantine equation $\eta(x^y) - y = k$ is equivalent with the relation $\eta_{x^y} - y = k$. In these conditions, the program for finding the solution is:

$Ed2085(a_k, b_k, a_x, b_x, a_y, b_y) :=$

$$
\begin{array}{|l}
j \leftarrow 0 \\
S \leftarrow (\text{"}x\text{" "}y\text{" "}k\text{"}) \\
for\ k \in a_k..b_k \\
\quad for\ x \in a_x..b_x \\
\qquad for\ y \in a_y..b_y
\end{array}
$$

$$\left|\begin{array}{l} \left|\begin{array}{l} \eta \leftarrow x^y \\ if\ \eta \leq last(\eta) \\ \quad\left|\begin{array}{l} j \leftarrow j + 1 \\ S \leftarrow stack[S, (x\ y\ k)]\ if\ \eta_\eta - y{=}k \end{array}\right. \end{array}\right. \\ return\ stack[S, (j\ j\ j)] \end{array}\right.$$

By means of variable j we count the number of concrete analyzed cases.

7.2.19 The equation 2086

The Diophantine equation $\eta(x^x) = y^y$, which is equivalent with the relation $\eta_{x^x} = y^y$, taking into consideration the significance of vector η, has only the trivial solutions $\boxed{x, y} = \boxed{1, 1}$ and $\boxed{x, y} = \boxed{2, 2}$ for $\boxed{x, y} \in D_c = \{1, 2, \ldots, 10^2\}^2$ of which there were in fact analyzed only 700 cases in which $x^x \leq last(\eta) = 10^6$.

7.2.20 The equation 2087

The Diophantine equation $\eta(x^y) = y^x$, equivalent with the relation $\eta_{x^y} = y^x$, taking into consideration the significance of vector η, has only the trivial solutions $\boxed{1, 1}$ and $\boxed{2, 2}$ for $\boxed{x, y} \in D_c = \{1, 2, \ldots, 10^2\}^2$ of which there were in fact analyzed only 476 cases in which $x^y \leq last(\eta) = 10^6$.

7.2.21 The equation 2088

The 26 solutions of the η–Diophantine equation $\eta(x) = y!$, in the form $\boxed{x, y}$, are:

$\boxed{2, 2}$;

$\boxed{9, 3}$ $\boxed{16, 3}$ $\boxed{18, 3}$ $\boxed{36, 3}$ $\boxed{45, 3}$ $\boxed{48, 3}$ $\boxed{72, 3}$ $\boxed{80, 3}$ $\boxed{90, 3}$ $\boxed{144, 3}$ $\boxed{180, 3}$ $\boxed{240, 3}$ $\boxed{360, 3}$ $\boxed{720, 3}$;

| 59049, 4 | 118098, 4 | 236196, 4 | 295245, 4 | 413343, 4 | 472392, 4 |

| 590490, 4 | 649539, 4 | 767637, 4 | 826686, 4 | .

The program for finding the solutions of the equation (2088) is

$$Ed2088(a_x, b_x, a_y, b_y) := \begin{vmatrix} S \leftarrow (\text{"}x\text{"} \text{ "}y\text{"}) \\ for \ x \in a_x..b_x \\ \qquad for \ y \in a_y..b_y \\ \qquad\qquad S \leftarrow stack[S, (x \ y)] \ if \ \eta_x = y! \\ return \ S \end{vmatrix}$$

The search domain is $D_c = \{1, 2, \ldots, 10^6\} \times \{1, 2, \ldots, 19\}$, where we have considered that 19! has 17 significant digits and 20! has 18 significant digits. Therefore, for $y > 19$ some errors will be produced, connected to the representation of the numbers in the intern memory of classic computers.

7.2.22   The equation 2089

The η–Diophantine equation $\eta(m \cdot x) = m \cdot \eta(x)$, if we take into consideration the significance of vector η, is equivalent with the relation

$$\eta_{m \cdot x} = m \cdot \eta_x \ .$$

In these conditions, the empirical search program for the solutions of the η–Diophantine equation is:

$$Ed2089(a_m, b_m, a_x, b_x) := \begin{vmatrix} j \leftarrow 0 \\ S \leftarrow (\text{"}m\text{"} \text{ "}x\text{"}) \\ for \ m \in a_m..b_m \\ \qquad for \ x \in a_x..b_x \\ \qquad\qquad if \ m \cdot x \leq last(\eta) \\ \qquad\qquad\qquad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack(S, (m \ x)) \ if \ \eta_{m \cdot x} = m \cdot \eta_x \end{vmatrix} \\ return \ stack(S, (j \ j)) \end{vmatrix}$$

The search domain is

$$D_c = \left\{2, 3, \ldots, 10^2\right\} \times \left\{2, 3, \ldots, 10^6\right\}$$

which imply that the number of cases is 98999901, but, due to condition $m \cdot x \leq last(\eta) = 10^6$, the number of real analyzed cases is 4187241, counted in the program by means of variable j.

On this search domain the equation has only a sole solution

$$\boxed{m, x} = \boxed{2, 2}.$$

Obviously, there exist also the trivial solutions where $m = 1$, $x \in \left\{1, 2, \ldots, 10^6\right\}$ and $m \in \left\{1, 2, \ldots, 10^6\right\}$ and $x = 1$, but those were avoided by choosing the search domain.

7.2.23 The equation 2090

The η–Diophantine equation $m^{\eta(x)} + \eta(x)^n = m^n$ is equivalent with the relation $m^{\eta_x} + (\eta_x)^n = m^n$ if we consider vector η which contains all the values of function $\eta(k)$ for $k \in \left\{1, 2, \ldots, 10^6\right\}$. Let us consider the next empirical search domain

$$D_c = \left\{2, 3, \ldots, 100\right\}^2 \times \left\{2, 3, \ldots 10^4\right\}$$

for the triplet (m, n, x). As we have operations to raise at power that can generate numbers greater than 10^{17}, a condition has been imposed to avoid the floating overflow errors and the numbers greater than 10^{17}. The search program is:

$Ed2090(a_m, b_m, a_n, b_n, a_x, b_x) :=$

$$\left|\begin{array}{l} j \leftarrow 0 \\ for\ m \in a_m..b_m \\ \quad for\ n \in a_n..b_n \\ \qquad for\ x \in a_x..b_x \\ \qquad \quad \left|\eta \leftarrow \eta_x \right. \end{array}\right.$$

$$\begin{vmatrix} & \begin{vmatrix} N \leftarrow \infty \text{ on error } N \leftarrow m^\eta + \eta^n \\ if\ N \leq 10^{17} \\ \quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack[S, (m\ n\ x)]\ if\ N{=}m^n \end{vmatrix} \end{vmatrix} \\ return\ stack[S, (j\ j\ j)] \end{vmatrix}$$

The search domain has 3799620 possible cases but in reality there were considered 236363 due to restriction $m^\eta + \eta^n \leq 10^{17}$. In these conditions, the η–Diophantine equation (2090) has no solutions on this search domain.

7.2.24   The equation 2091

The Diophantine equation (2091) is $\eta(x^2)/m \pm \eta(y^2)/n = 1$. The Diophantine equation (2091), variant with $+$, $\eta(x^2)/m + \eta(y^2)/n = 1$ has no solutions on this search domain $D_c = \{1, 2, \ldots, 10\}^2 \times \{2, 3, \ldots, 10^3\}^2$, with $m \neq n$ and $x \neq y$. The number of total possible cases is 80838081 of which, due to restriction $m \neq n$ and $x \neq y$, the number of real analyzed cases is 71784144. The search time was 37.397 seconds.

On the contrary, the Diophantine equation $\eta(x^2)/m - \eta(y^2)/n = 1$, which is equivalent with equation $n \cdot \eta(x^2) - m \cdot \eta(y^2) = m \cdot n$, are 54370 de solutions pe search domain $D_c = \{1, 2, \ldots, 10\}^2 \times \{2, 3, \ldots, 10^3\}^2$. The number of possible cases is 80838081 of which, due to restriction $m \neq n$ and $x \neq y$, the number of real analyzed cases is 71784144. The search time was 169.662 seconds. The first 49 and the last 28 solutions, in the form $\boxed{m, n, x, y}$, are:

$\boxed{2, 3, 3, 4}\ \boxed{2, 3, 3, 6}\ \boxed{2, 3, 3, 12}$;

$\boxed{2, 3, 4, 3}\ \boxed{2, 3, 4, 6}\ \boxed{2, 3, 4, 12}$;

$\boxed{2, 3, 5, 32}\ \boxed{2, 3, 5, 96}\ \boxed{2, 3, 5, 160}\ \boxed{2, 3, 5, 288}\ \boxed{2, 3, 5, 480}$;

$\boxed{2, 3, 6, 3}\ \boxed{2, 3, 6, 4}\ \boxed{2, 3, 6, 12}$;

$\boxed{2,3,7,81}$ $\boxed{2,3,7,162}$ $\boxed{2,3,7,256}$ $\boxed{2,3,7,324}$ $\boxed{2,3,7,405}$
$\boxed{2,3,7,567}$ $\boxed{2,3,7,648}$ $\boxed{2,3,7,768}$ $\boxed{2,3,7,810}$;

$\boxed{2,3,8,9}$ $\boxed{2,3,8,18}$ $\boxed{2,3,8,36}$ $\boxed{2,3,8,72}$;

$\boxed{2,3,10,32}$ $\boxed{2,3,10,96}$ $\boxed{2,3,10,160}$ $\boxed{2,3,10,288}$ $\boxed{2,3,10,480}$;

$\boxed{2,3,12,3}$ $\boxed{2,3,12,4}$ $\boxed{2,3,12,6}$;

$\boxed{2,3,14,81}$ $\boxed{2,3,14,162}$ $\boxed{2,3,14,256}$ $\boxed{2,3,14,324}$ $\boxed{2,3,14,405}$
$\boxed{2,3,14,567}$ $\boxed{2,3,14,648}$ $\boxed{2,3,14,768}$ $\boxed{2,3,14,810}$;

$\boxed{2,3,15,32}$ $\boxed{2,3,15,96}$ $\boxed{2,3,15,160}$ $\boxed{2,3,15,288}$ $\boxed{2,3,15,480}$;

$\vdots$

$\boxed{10,9,512,9}$ $\boxed{10,9,512,18}$ $\boxed{10,9,512,36}$ $\boxed{10,9,512,72}$;

$\boxed{10,9,525,9}$ $\boxed{10,9,525,18}$ $\boxed{10,9,525,36}$ $\boxed{10,9,525,72}$;

$\boxed{10,9,600,9}$ $\boxed{10,9,600,18}$ $\boxed{10,9,600,36}$ $\boxed{10,9,600,72}$;

$\boxed{10,9,675,9}$ $\boxed{10,9,675,18}$ $\boxed{10,9,675,36}$ $\boxed{10,9,675,72}$;

$\boxed{10,9,700,9}$ $\boxed{10,9,700,18}$ $\boxed{10,9,700,36}$ $\boxed{10,9,700,72}$;

$\boxed{10,9,800,9}$ $\boxed{10,9,800,18}$ $\boxed{10,9,800,36}$ $\boxed{10,9,800,72}$;

$\boxed{10,9,900,9}$ $\boxed{10,9,900,18}$ $\boxed{10,9,900,36}$ $\boxed{10,9,900,72}$.

7.2.25 The equation 2092

The η–Diophantine equation

$$\eta(x_1^{y_1} + x_2^{y_2} + \ldots + x_r^{y_r}) = \eta(x_1)^{y_1} + \eta(x_2)^{y_2} + \ldots + \eta(x_r)^{y_r} \,,$$

is equivalent with the relation

$$\eta_{x_1^{y_1}} + \eta_{x_2^{y_2}} + \ldots + \eta_{x_r^{y_r}} = (\eta_{x_1})^{y_1} + (\eta_{x_2})^{y_2} + \ldots + (\eta_{x_r})^{y_r} \,,$$

if we use vector η that contains all the values of function $\eta(k)$ for $k \in \{1, 2, \ldots, 10^6\}$. Unfortunately, equation (2092) surpasses slightly our possibilities of finding the solutions. It is sufficient to chose $r > 3$ and the powers $y_1, y_2, \ldots, y_r$ to be natural numbers > 2. In the case of this equation two cases were considered, that have no solutions (except the trivial solution $x_1 = 1, x_2 = 1, \ldots, x_r = 1$) on the given search domain:

1. Equation (2092), with $r = 2$, $y_1 = 2$ and $y_2 = 3$

$$\eta(x_1^2 + x_2^3) = \eta(x_1)^2 + \eta(x_2)^3$$

for $\{x_1, x_2\} \in D_c = \{2, 3, \ldots, 1000\} \times \{2, 3, \ldots, 100\}$.

2. Equation (2092), with $r = 3$, $y_1 = 1$, $y_2 = 2$ and $y_3 = 4$

$$\eta(x_1 + x_2^2 + x_3^4) = \eta(x_1) + \eta(x_2)^2 + \eta(x_3)^4 \,,$$

for

$$\{x_1, x_2, x_3\} \in D_c = \{2, 3, \ldots, 10^4\} \times \{2, 3, \ldots, 10^3\} \times \{2, 3, \ldots, 31\} \,.$$

7.2.26 The equation 2093

The η–Diophantine equation

$$\eta(x_1! + x_2! + \ldots + x_r!) = \eta(x_1)! + \eta(x_2)! + \ldots + \eta(x_r)! \,,$$

is equivalent with the relation

$$\eta_{x_1!} + \eta_{x_2!} + \ldots + \eta_{x_r!} = \eta_{x_1}! + \eta_{x_2}! + \ldots + \eta_{x_r}! \,,$$

if we use vector η that contains all the values of function $\eta(k)$ for $k \in \{1, 2, \ldots, 10^6\}$. This equation also surpasses slightly our possibilities of finding the solutions for $r > 2$. Let equation η–Diophantine for $r = 2$

$$\eta(x_1! + x_2!) = \eta(x_1)! + \eta(x_2)! \,, \qquad (7.3)$$

and equivalent relation

$$\eta_{x_1!} + \eta_{x_2!} = \eta_{x_1}! + \eta_{x_2}! \,.$$

As for $n = 9$, $n! = 362880$ and for $n = 10$, $n! = 3628800 > 10^6$, Then it follows that the biggest search domain for x_1 and x_2 is $D_c \{2, 3, \ldots, 9\}^2$. Equation (7.3) is symmetric relative to both unknowns. To avoid symmetric solutions we impose also the condition $x_1 < x_2$. We have avoided the values $x_1 = 1$ and $x_2 = 1$ because they lead to the trivial solution. The program for finding the solutions of the equation (7.3) is:

Program 7.13. Program $Ed20932$

$$Ed20932(a_{xy}, b_{xy}) := \left| \begin{array}{l} S \leftarrow (\text{''}x\text{''}\ \text{''}y\text{''}) \\ for\ x \in a_{xy}..b_{xy} - 1 \\ \quad \left| \begin{array}{l} for\ y \in x + 1..b_{xy} \\ \quad \left| \begin{array}{l} c \leftarrow \eta_{x!} + \eta_y = \eta_x! + \eta_y! \\ S \leftarrow stack[S, (x\ y)]\ if\ c \end{array} \right. \end{array} \right. \\ return\ S \end{array} \right.$$

With this program the solution $\boxed{2, 6}$ has be determined on the search domain $D_c \{2, 3, \ldots, 9\}^2$, solution for which we have $\eta(2!) + \eta(6!) = \eta(2)! + \eta(6)!$. Obviously, there exist also the trivial solutions $\boxed{1, 1}$ and $\boxed{2, 2}$ which verify:

$$\eta(1!) + \eta(1!) = \eta(1)! + \eta(1)! \ \text{ and } \ \eta(2!) + \eta(2!) = \eta(2)! + \eta(2)! \,.$$

7.2.27   The equation 2094

The η–Diophantine equation $(x, y) = \big(\eta(x), \eta(y)\big)$, (where by (x,y) was denoted the greatest common divisor of x and y), is equivalent

with the relation $\gcd(x, y) = \gcd(\eta_x, \eta_y)$, if we use vector η that contains all values of function $\eta(k)$ for $k \in \{1, 2, \ldots, 10^6\}$ and Mathcad function $\gcd(n_1, n_2, \ldots, n_\ell)$ for computing the greatest common divisor of $n_1, n_2, \ldots, n_\ell$ has 4799 solutions. The search domain ales is $D_c = \{2, 3, \ldots, 10^3\}^2$ (with $x < y$ as the role of x and y is symmetric) and $(x, y) \neq 1$, i.e. x and y not relative prime. The number of possible cases is 498501 of which the analyzed ones are 193351 due to restrictions $x < y$ and $(x, y) \neq 1$. We give all the solution for $x = 4, 6, 8$, in the form $\boxed{x, y}$:

$\boxed{4, 8}$ $\boxed{4, 12}$ $\boxed{4, 18}$ $\boxed{4, 24}$ $\boxed{4, 32}$ $\boxed{4, 50}$ $\boxed{4, 64}$ $\boxed{4, 90}$ $\boxed{4, 96}$ $\boxed{4, 98}$
$\boxed{4, 128}$ $\boxed{4, 150}$ $\boxed{4, 160}$ $\boxed{4, 192}$ $\boxed{4, 224}$ $\boxed{4, 242}$ $\boxed{4, 288}$ $\boxed{4, 294}$ $\boxed{4, 320}$
$\boxed{4, 338}$ $\boxed{4, 350}$ $\boxed{4, 384}$ $\boxed{4, 448}$ $\boxed{4, 450}$ $\boxed{4, 480}$ $\boxed{4, 490}$ $\boxed{4, 512}$ $\boxed{4, 576}$
$\boxed{4, 578}$ $\boxed{4, 640}$ $\boxed{4, 672}$ $\boxed{4, 722}$ $\boxed{4, 726}$ $\boxed{4, 882}$ $\boxed{4, 896}$ $\boxed{4, 960}$ $\boxed{4, 972}$;

$\boxed{6, 9}$ $\boxed{6, 27}$ $\boxed{6, 45}$ $\boxed{6, 81}$ $\boxed{6, 135}$ $\boxed{6, 189}$ $\boxed{6, 243}$ $\boxed{6, 375}$ $\boxed{6, 405}$
$\boxed{6, 567}$ $\boxed{6, 729}$ $\boxed{6, 945}$;

$\boxed{8, 12}$ $\boxed{8, 18}$ $\boxed{8, 50}$ $\boxed{8, 90}$ $\boxed{8, 98}$ $\boxed{8, 150}$ $\boxed{8, 242}$ $\boxed{8, 294}$ $\boxed{8, 338}$
$\boxed{8, 350}$ $\boxed{8, 450}$ $\boxed{8, 490}$ $\boxed{8, 578}$ $\boxed{8, 722}$ $\boxed{8, 726}$ $\boxed{8, 882}$ $\boxed{8, 972}$.

For $x \geq 860$ we give the solutions on $x = 86*, 87*, 88*, 89*, 90*, 91*, 92*, 93*, 94*, 95*, 96*$, in the same form $\boxed{x, y}$:

$\boxed{860, 903}$ $\boxed{860, 989}$ $\boxed{861, 902}$ $\boxed{861, 943}$ $\boxed{867, 884}$ $\boxed{867, 935}$ $\boxed{867, 952}$
$\boxed{868, 899}$ $\boxed{868, 961}$ $\boxed{869, 948}$;

$\boxed{871, 938}$ $\boxed{872, 981}$ $\boxed{873, 970}$ $\boxed{874, 897}$ $\boxed{876, 949}$;

$\boxed{880, 891}$ $\boxed{882, 968}$ $\boxed{884, 935}$ $\boxed{885, 944}$ $\boxed{888, 925}$;

$\boxed{890, 979}$ $\boxed{891, 968}$ $\boxed{893, 940}$ $\boxed{893, 987}$ $\boxed{896, 972}$ $\boxed{897, 920}$ $\boxed{899, 930}$
$\boxed{899, 961}$ $\boxed{899, 992}$;

$\boxed{901, 954}$ $\boxed{902, 943}$ $\boxed{903, 946}$ $\boxed{903, 989}$;

$\boxed{912, 931}$ $\boxed{913, 996}$ $\boxed{915, 976}$ $\boxed{918, 935}$;

$\boxed{923, 994}$ $\boxed{925, 962}$ $\boxed{925, 999}$ $\boxed{928, 957}$;

$\boxed{930, 961}$ $\boxed{931, 950}$ $\boxed{931, 969}$ $\boxed{931, 988}$ $\boxed{935, 952}$;

$\boxed{940, 987}$ $\boxed{943, 984}$ $\boxed{946, 989}$;

$\boxed{950, 969}$ $\boxed{957, 986}$;

$\boxed{961, 992}$ $\boxed{962, 999}$ $\boxed{969, 988}$.

The empirical search domain is:

$$Ed2094(a_{xy}, b_{xy}) := \begin{vmatrix} j \leftarrow 0 \\ S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\ for \ x \in a_{xy}..b_{xy} - 1 \\ \quad \begin{vmatrix} u \leftarrow Tp\eta(x) = 0 \\ for \ y \in x + 1..b_{xy} \\ \quad \begin{vmatrix} notprime \leftarrow u \wedge Tp\eta(y) = 0 \\ q \leftarrow \gcd(x, y) \\ if \ notprime \wedge q \neq 1 \\ \quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack(S, (x \ y)) \ if \ q = \gcd(\eta_x, \eta_y) \end{vmatrix} \end{vmatrix} \end{vmatrix} \\ return \ stack(S, (j \ j)) \end{vmatrix}$$

The program calls the subprogram 1.8 to establish the primality of x and y.

7.2.28 The equation 2095

We consider the search domain $D_c = \left\{2, 3, \ldots, 10^3\right\}^2$, x, y non-prime with $x < y$ and $(x, y) \neq 1$. The number of possible cases is 498501. Due to conditions $x < y$ and $(x, y) = 1$ the number of analyzed cases is in fact 193351. We have 145 solutions with $x = 4$ and solution $\boxed{x, y} = \boxed{6, 12}$

that verify the η–Diophantine equation $[x, y] = [\eta(x), \eta(y)]$, where by $[n_1, n_2]$ was denoted the smallest common multiple of numbers n_1 and n_2. This η–Diophantine equation is equivalent with the relation $lcm(x, y) = lcm(\eta_x, \eta_y)$, if we use vector η with the values of function η and Mathcad function $lcm(n_1, n_2, \ldots, n_\ell)$ to compute the the smallest common multiple of $n_1, n_2, \ldots n_\ell$. The solutions, in the form $\boxed{x, y}$, are:

$$\boxed{4, 6}\;\boxed{4, 10}\;\boxed{4, 14}\;\boxed{4, 20}\;\boxed{4, 22}\;\boxed{4, 26}\;\boxed{4, 28}\;\boxed{4, 34}\;\boxed{4, 38}\;\boxed{4, 44}\;\boxed{4, 46}$$
$$\boxed{4, 52}\;\boxed{4, 58}\;\boxed{4, 62}\;\boxed{4, 68}\;\boxed{4, 74}\;\boxed{4, 76}\;\boxed{4, 82}\;\boxed{4, 86}\;\boxed{4, 92}\;\boxed{4, 94}\;;$$

$$\boxed{4, 116}\;\boxed{4, 118}\;\boxed{4, 122}\;\boxed{4, 124}\;\boxed{4, 134}\;\boxed{4, 142}\;\boxed{4, 146}\;\boxed{4, 148}\;\boxed{4, 158}$$
$$\boxed{4, 164}\;\boxed{4, 166}\;\boxed{4, 172}\;\boxed{4, 178}\;\boxed{4, 188}\;\boxed{4, 194}\;;$$

$$\boxed{4, 202}\;\boxed{4, 206}\;\boxed{4, 212}\;\boxed{4, 214}\;\boxed{4, 218}\;\boxed{4, 226}\;\boxed{4, 236}\;\boxed{4, 244}\;\boxed{4, 254}$$
$$\boxed{4, 262}\;\boxed{4, 268}\;\boxed{4, 274}\;\boxed{4, 278}\;\boxed{4, 284}\;\boxed{4, 292}\;\boxed{4, 298}\;;$$

$$\boxed{4, 302}\;\boxed{4, 314}\;\boxed{4, 316}\;\boxed{4, 326}\;\boxed{4, 332}\;\boxed{4, 334}\;\boxed{4, 346}\;\boxed{4, 356}\;\boxed{4, 358}$$
$$\boxed{4, 362}\;\boxed{4, 382}\;\boxed{4, 386}\;\boxed{4, 388}\;\boxed{4, 394}\;\boxed{4, 398}\;;$$

$$\boxed{4, 404}\;\boxed{4, 412}\;\boxed{4, 422}\;\boxed{4, 428}\;\boxed{4, 436}\;\boxed{4, 446}\;\boxed{4, 452}\;\boxed{4, 454}\;\boxed{4, 458}$$
$$\boxed{4, 466}\;\boxed{4, 478}\;\boxed{4, 482}\;;$$

$$\boxed{4, 502}\;\boxed{4, 508}\;\boxed{4, 514}\;\boxed{4, 524}\;\boxed{4, 526}\;\boxed{4, 538}\;\boxed{4, 542}\;\boxed{4, 548}\;\boxed{4, 554}$$
$$\boxed{4, 556}\;\boxed{4, 562}\;\boxed{4, 566}\;\boxed{4, 586}\;\boxed{4, 596}\;;$$

$$\boxed{4, 604}\;\boxed{4, 614}\;\boxed{4, 622}\;\boxed{4, 626}\;\boxed{4, 628}\;\boxed{4, 634}\;\boxed{4, 652}\;\boxed{4, 662}\;\boxed{4, 668}$$
$$\boxed{4, 674}\;\boxed{4, 692}\;\boxed{4, 694}\;\boxed{4, 698}\;;$$

$$\boxed{4, 706}\;\boxed{4, 716}\;\boxed{4, 718}\;\boxed{4, 724}\;\boxed{4, 734}\;\boxed{4, 746}\;\boxed{4, 758}\;\boxed{4, 764}\;\boxed{4, 766}$$
$$\boxed{4, 772}\;\boxed{4, 778}\;\boxed{4, 788}\;\boxed{4, 794}\;\boxed{4, 796}\;;$$

$$\boxed{4, 802}\;\boxed{4, 818}\;\boxed{4, 838}\;\boxed{4, 842}\;\boxed{4, 844}\;\boxed{4, 862}\;\boxed{4, 866}\;\boxed{4, 878}\;\boxed{4, 886}$$
$$\boxed{4, 892}\;\boxed{4, 898}\;;$$

| $4,908$ | $4,914$ | $4,916$ | $4,922$ | $4,926$ | $4,932$ | $4,934$ | $4,956$ | $4,958$ |

| $4,964$ | $4,974$ | $4,982$ | $4,998$ | ;

| $6,12$ | .

The empirical search domain is:

$Ed2095(a_{xy}, b_{xy}) :=$

$$
\begin{vmatrix}
j \leftarrow 0 \\
S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\
for \ x \in a_{xy}..b_{xy} - 1 \\
\quad \begin{vmatrix}
u \leftarrow Tp\eta(x) = 0 \\
for \ y \in x + 1..b_{xy} \\
\quad \begin{vmatrix}
notprime \leftarrow u \wedge Tp\eta(y)\text{=}0 \\
if \ notprime \wedge \gcd(x, y) \neq 1 \\
\quad \begin{vmatrix}
j \leftarrow j + 1 \\
S \leftarrow stack(S, (x \ y)) \ if \ lcm(x, y)\text{=}lcm(\eta_x, \eta_y)
\end{vmatrix}
\end{vmatrix}
\end{vmatrix} \\
return \ stack(S, (j \ j))
\end{vmatrix}
$$

The program calls the subprogram 1.8 to establish the primality x and y.

7.3 The η–s–Diophantine equations

The function $s : \mathbb{N} \rightarrow \mathbb{N}$, where $s(n)$ is the sum of the divisors of n without n (the sum of the aliquot parts), for example $s(12) = 1 + 2 + 3 + 4 + 6 = 16$. The function $s(n)$ can be defined by means of function $\sigma(n) = \sigma_1(n)$ which is the sum of the divisors of n, $s(n) = \sigma(n) - n$ (see figure 7.1). Keeping the numbering from the paper [Smarandache, 1999b], the η–s–Diophantine equations are:

(2124) $\eta(x) = s(m \cdot x + n)$,

(2125) $\eta(x)^m = s(x^n)$,

(2126) $\eta(x) + y = x + s(y)$,
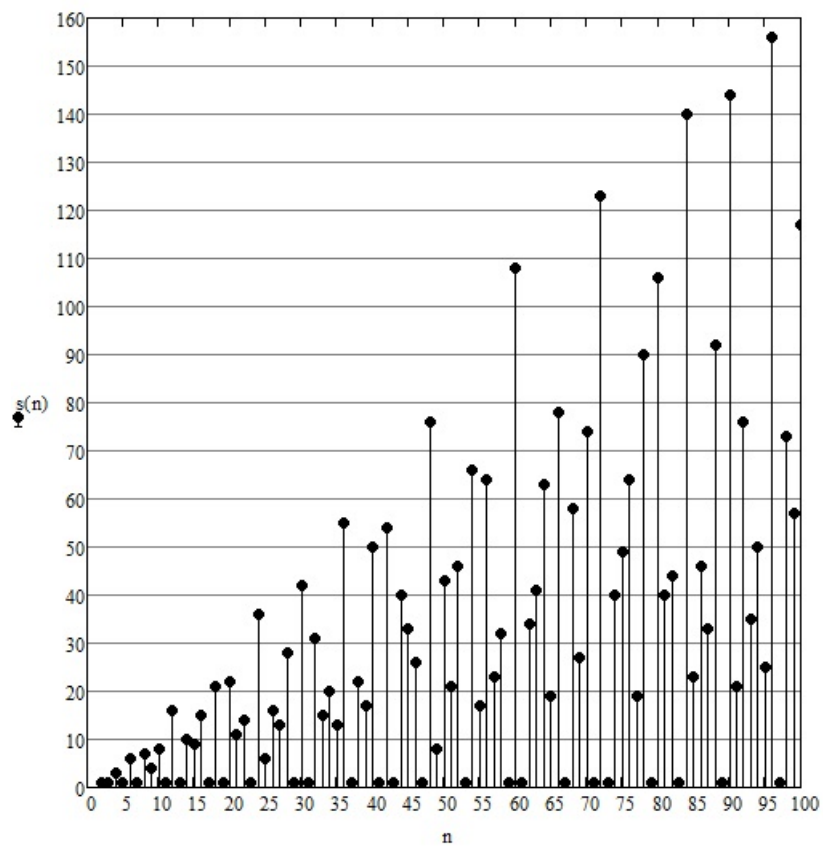
Figure 7.1: The function s

(2127) $\eta(x) \cdot y = x \cdot s(y)$,

(2128) $\eta(x)/y = x/s(y)$,

(2129) $\eta(x)^y = x^{s(y)}$,

(2130) $\eta(x)^y = s(y)^x$,

7.3.1 Partial empirical solving of η–s–Diophantine equations

To solve empirically the η–s–Diophantine equations (2124–2130), we will read the files $\eta.prn$ and $s.prn$ which contain the values of functions $\eta(n)$ and $s(n)$ for $n = 1, 2, \ldots, 10^6$, these files being generated previously. The files $\eta.prn$ and $s.prn$ will be assigned to vectors η and s with by means of the Mathcad sequences:

$$\eta := READPRN(\text{"}...\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6$$

$$s := READPRN(\text{"}...\backslash s.prn\text{"}) \ \ last(s) = 10^6 ,$$

where the commands $last(\eta)$ and $last(s)$ indicate the last index of vectors η and s. These manoeuvres are necessary to diminish the empirical search time of the solutions of the Diophantine equations.

7.3.2 The equation 2124

Equation η–s–Diophantine $\eta(x) = s(m \cdot x + n)$ is equivalent with the relation $\eta_x = s_{m \cdot x + n}$ if we take into consideration the significance of vectors η and s. Let be search domain

$$D_c = \{1, 2, \ldots, 10\}^2 \times \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor \cdot r_x \right\} \quad (7.4)$$

with $a_x = 1$, $r_x = 1$, $b_x = 10^6$ and additional conditions $m \cdot x + n \le last(s)$ and $\gcd(m, n) = 1$, i.e. m and n are relatively prime numbers. Then, the number of possible cases is 10^8. Due to the additional conditions, the number of analyzed cases is 21271688. The number of found solutions is 554, of which we present the first 80 and the last 85:

$\boxed{1,1,1}$ $\boxed{1,1,3}$ $\boxed{1,1,7}$ $\boxed{1,1,8}$ $\boxed{1,1,26}$ $\boxed{1,1,31}$ $\boxed{1,1,76}$ $\boxed{1,1,124}$
$\boxed{1,1,127}$ $\boxed{1,1,610}$ $\boxed{1,1,1072}$ $\boxed{1,1,2032}$ $\boxed{1,1,2186}$ $\boxed{1,1,4912}$
$\boxed{1,1,8191}$ $\boxed{1,1,16806}$ $\boxed{1,1,23488}$ $\boxed{1,1,25240}$ $\boxed{1,1,49900}$
$\boxed{1,1,50248}$ $\boxed{1,1,68920}$ $\boxed{1,1,78124}$ $\boxed{1,1,99100}$ $\boxed{1,1,131071}$
$\boxed{1,1,142090}$ $\boxed{1,1,205378}$ $\boxed{1,1,213052}$ $\boxed{1,1,357100}$ $\boxed{1,1,357910}$
$\boxed{1,1,371292}$ $\boxed{1,1,516496}$ $\boxed{1,1,524287}$ $\boxed{1,1,545146}$ $\boxed{1,1,704968}$
$\boxed{1,1,791716}$ $\boxed{1,1,929230}$;

$\boxed{1,2,1}$ $\boxed{1,2,2289}$ $\boxed{1,2,15947}$ $\boxed{1,2,19337}$ $\boxed{1,2,39447}$ $\boxed{1,2,52395}$
$\boxed{1,2,111045}$ $\boxed{1,2,135795}$ $\boxed{1,2,170255}$ $\boxed{1,2,186801}$ $\boxed{1,2,400449}$
$\boxed{1,2,485225}$ $\boxed{1,2,787461}$ $\boxed{1,2,996495}$;

$\boxed{1,3,296}$ $\boxed{1,3,382}$ $\boxed{1,3,1940}$ $\boxed{1,3,5174}$ $\boxed{1,3,7258}$ $\boxed{1,3,12824}$
$\boxed{1,3,101594}$ $\boxed{1,3,133748}$ $\boxed{1,3,210014}$ $\boxed{1,3,336374}$ $\boxed{1,3,441364}$
$\boxed{1,3,576884}$ $\boxed{1,3,855316}$;

$\boxed{1,4,1}$ $\boxed{1,4,51}$ $\boxed{1,4,217}$ $\boxed{1,4,22593}$ $\boxed{1,4,33657}$ $\boxed{1,4,34705}$
$\boxed{1,4,95109}$ $\boxed{1,4,205897}$ $\boxed{1,4,348609}$ $\boxed{1,4,355893}$ $\boxed{1,4,383443}$
$\boxed{1,4,433945}$ $\boxed{1,4,510147}$ $\boxed{1,4,578085}$ $\boxed{1,4,684697}$ $\boxed{1,4,926833}$
$\boxed{1,4,950137}$;

$\vdots$

$\boxed{7,6,1}$ $\boxed{7,6,34793}$ $\boxed{7,6,61993}$ $\boxed{7,8,4869}$ $\boxed{7,8,13403}$ $\boxed{7,8,123767}$
$\boxed{7,9,74050}$ $\boxed{7,9,82850}$ $\boxed{7,10,1}$ $\boxed{7,10,843}$ $\boxed{7,10,1477}$ $\boxed{7,10,89851}$
$\boxed{7,10,131679}$;

$\boxed{8,1,134}$ $\boxed{8,1,254}$ $\boxed{8,1,614}$ $\boxed{8,1,2936}$ $\boxed{8,1,3155}$ $\boxed{8,1,6281}$
$\boxed{8,1,8615}$ $\boxed{8,1,64562}$ $\boxed{8,1,88121}$ $\boxed{8,3,1}$ $\boxed{8,3,37}$ $\boxed{8,3,1603}$ $\boxed{8,5,1}$
$\boxed{8,5,622}$ $\boxed{8,5,2104}$ $\boxed{8,5,18372}$ $\boxed{8,5,48663}$ $\boxed{8,5,89193}$

$\boxed{8,5,114132}$ $\boxed{8,7,69023}$ $\boxed{8,7,80450}$ $\boxed{8,7,88187}$ $\boxed{8,9,1}$ $\boxed{8,9,604}$
$\boxed{8,9,4910}$ $\boxed{8,9,23386}$ $\boxed{8,9,36775}$;

$\boxed{9,2,1}$ $\boxed{9,2,4383}$ $\boxed{9,4,1}$ $\boxed{9,4,56683}$ $\boxed{9,5,1084}$ $\boxed{9,5,1374}$
$\boxed{9,5,3246}$ $\boxed{9,5,43256}$ $\boxed{9,8,1}$ $\boxed{9,8,1317}$ $\boxed{9,8,3787}$ $\boxed{9,8,13155}$
$\boxed{9,8,23955}$ $\boxed{9,8,46521}$ $\boxed{9,8,81827}$ $\boxed{9,10,1}$ $\boxed{9,10,71}$ $\boxed{9,10,16703}$
$\boxed{9,10,30769}$ $\boxed{9,10,31419}$ $\boxed{9,10,102417}$;

$\boxed{10,1,1}$ $\boxed{10,1,61}$ $\boxed{10,1,2524}$ $\boxed{10,1,4990}$ $\boxed{10,1,6892}$ $\boxed{10,1,9910}$
$\boxed{10,1,14209}$ $\boxed{10,1,35710}$ $\boxed{10,1,35791}$ $\boxed{10,1,92923}$ $\boxed{10,3,1}$
$\boxed{10,3,194}$ $\boxed{10,7,1}$ $\boxed{10,7,43}$ $\boxed{10,7,139}$ $\boxed{10,7,11947}$ $\boxed{10,7,25975}$
$\boxed{10,7,26581}$ $\boxed{10,7,64360}$ $\boxed{10,9,1}$ $\boxed{10,9,3928}$ $\boxed{10,9,29420}$
$\boxed{10,9,51835}$ $\boxed{10,9,57995}$.

7.3.3 The equation 2125

The η–s–Diophantine equation $\eta(x)^m = s(x^n)$, is equivalent with the relation $(\eta_x)^m = s_{x^n}$ if we take into consideration the significance of vectors η and s. On the search domain given by (7.4) with $a_x = 2$, $r_x = 1$, $b_x = 10^6$ and additional conditions $(\eta_x)^m < 10^{17}$ and $x^n \le last(s)$, the equation has a sole solution $\boxed{2,1,12}$, after having analyzed 4766682 cases of 99999900 possible situations. Obviously, this solution verifies equation (2125)

$$\eta(12)^2 = 4^2 = 16 \quad s(12^1) = 1 + 2 + 3 + 4 + 6 = 16 \ .$$

7.3.4 The equation 2126

The η–s–Diophantine equation $\eta(x) + y = x + s(y)$ is equivalent with the relation $\eta_x + y = x + s_y$ if we take into consideration the significance

of vectors η and s. Let

$$D_c = \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor \cdot r_x \right\}$$
$$\times \left\{ a_y, a_y + r_y, a_y + 2r_y, \ldots, \left\lfloor \frac{b_y - a_y}{r_y} \right\rfloor \cdot r_y \right\} . \quad (7.5)$$

be the search domain. As equation (2126) has a great number of solutions on the search domain D_c given by (7.5), with $a_x = 2$, $r_x = 1$, $b_x = 10^6$, $a_y = 3$, $r_y = 1$ and $b_y = 10^6$, we considered a restricted search domain with $a_x = 2$, $r_x = 113$, $b_x = 10^6$, $a_y = 3$, $r_y = 127$, $b_y = 10^6$ and the additional condition of x and y being relative prime. This domain has 69684900 possible cases and 42366956 cases to be analyzed as the cases when $(x, y) \neq 1$ are excluded. On this search domain equation (2126) has 44 de solutions given in the form $\boxed{x, y}$:

$3957, 3305$	$13449, 288928$	$21133, 99825$	$33111, 65662$
$38309, 76838$	$43733, 693296$	$67689, 480190$	$71757, 143386$
$93227, 790070$	$102041, 190503$	$103849, 653926$	$108369, 192662$
$116957, 303914$	$207357, 276482$	$239449, 239017$	$311091, 358397$
$320131, 832234$	$321487, 605158$	$323747, 411229$	$328267, 601983$
$332787, 336299$	$339567, 227333$	$349737, 237239$	$357873, 257051$
$407141, 910593$	$430871, 369319$	$431097, 348491$	$433131, 434089$
$459573, 310391$	$471325, 940438$	$505677, 370843$	$507259, 822963$
$508389, 509273$	$509745, 534673$	$516299, 696471$	$517881, 722125$
$520593, 433835$	$523983, 533657$	$567827, 849633$	$631785, 681739$
$635627, 637289$	$737779, 955551$	$765803, 739397$	$897561, 897893$

.

7.3.5   The equation 2127

The η–s–Diophantine equation $\eta(x) \cdot y = x \cdot s(y)$ is equivalent with the relation $\eta_x \cdot y = x \cdot s_y$ if we take into consideration the significance of

vectors η and s. On the search domain D_c given by (7.5) with $a_x = 100$, $r_x = 1$, $b_x = 200$, $a_y = 1$, $r_y = 1$, $b_x = 10^5$ and $(x, y) = 1$ (i.e. x and y are relative prime) equation has 83 solutions. The number of possible cases is 10099798 an the number of analyzed cases is 6151818. We give the solutions in the form $\boxed{x, y}$:

$$\boxed{101, 6} \; \boxed{101, 28} \; \boxed{101, 496} \; \boxed{101, 8128} \; ;$$

$$\boxed{103, 6} \; \boxed{103, 28} \; \boxed{103, 496} \; \boxed{103, 8128} \; ;$$

$$\boxed{107, 6} \; \boxed{107, 28} \; \boxed{107, 496} \; \boxed{107, 8128} \; ;$$

$$\boxed{109, 6} \; \boxed{109, 28} \; \boxed{109, 496} \; \boxed{109, 8128} \; ;$$

$$\vdots$$

As it is known, (2.10), for every prime number p we have $\eta(p) = p$. The numbers $6 = P_1$, $28 = P_2$, $496 = P_3$ and $8128 = P_4$ $33550336 = P_5 \ldots$, are perfect numbers, [Sloane, 2014, A000396], for which we have $s(P_k) = P_k$. As the solutions can also be given as pairs $\boxed{p, P_k}$, where p is a prime number and P_k is a perfect number, we can propose next theorem to be proved:

Theorem 7.14. *The solutions of the η–s–Diophantine equation $\eta(x) \cdot y = x \cdot s(y)$ are the pairs of numbers $\boxed{p, P_k}$, where p is a prime number and P_k, $k = 1, 2, \ldots$, is a perfect number.*

7.3.6 The equation 2128

The η–s–Diophantine equation $\eta(x)/y = x/s(y) \Leftrightarrow \eta(x) \cdot s(y) = x \cdot y$ is equivalent with the relation $\eta_x \cdot s_y = x \cdot y$ if we take into consideration the significance of vectors η and s. On the search domain D_c given by (7.5) with $a_x = 2$, $r_x = 1$, $b_x = 100$, $a_y = 6$, $r_y = 1$, $b_x = 8128$ and $(x, y) = 1$ (i.e. x and y are relative prime) the equation has 93 solutions. The search

domain has 804177 possible cases, but, due to the additional condition $(x, y) = 1$ the number of analyzed cases is 485989. We give the solutions in the form $\boxed{x, y}$:

$\boxed{3, 28}$ $\boxed{3, 496}$ $\boxed{3, 8128}$;

$\boxed{5, 6}$ $\boxed{5, 28}$ $\boxed{5, 496}$ $\boxed{5, 8128}$;

$\boxed{7, 6}$ $\boxed{7, 496}$ $\boxed{7, 8128}$;

$\boxed{11, 6}$ $\boxed{11, 28}$ $\boxed{11, 496}$ $\boxed{11, 8128}$;

$\boxed{13, 6}$ $\boxed{13, 28}$ $\boxed{13, 496}$ $\boxed{13, 8128}$;

$\boxed{17, 617, 28}$ $\boxed{17, 496}$ $\boxed{17, 8128}$;

$\boxed{19, 6}$ $\boxed{19, 28}$ $\boxed{19, 496}$ $\boxed{19, 8128}$;

$$\vdots$$

By a solutions' analysis, we remark that we have the same pairs $\boxed{p, P_k}$, where p is a prime and P_k perfect number, as in equation (2127). Also, it can be observed that there are missing pairs of solutions, due to the additional condition $(x, y) = 1$, for example, the pair $\boxed{3, 6}$ is missing, as $(3, 6) = 3$.

7.3.7   The equation 2129

The η–s–Diophantine equation $\eta(x)^y = x^{s(y)}$ is equivalent with the relation $(\eta_x)^y = x^{s_y}$, if we take into consideration the significance of vectors η and s. The solutions of the equation are pairs $\boxed{p, P_k}$, where p is a prime number and P_k a perfect number as in equation (2127).

7.3.8   The equation 2130

The η–s–Diophantine equation $\eta(x)^y = s(y)^y$ is equivalent with the relation $(\eta_x)^y = (s_y)^x$, if we take into consideration the significance of vectors η and s. This equation has no solutions on the search domain $D_c = \{2, 3, \ldots, 10^6\} \times \{1, 2, \ldots, 10^6\}$. which has 999999000000 possible cases of which only 2190820 where analyzed due to the restrictive conditions $(\eta_x)^y < 10^{307}$, $(s_y)^x < 10^{307}$ and $(x, y) = 1$.

7.4   The η–π–Diophantine equations

Let $m, n, k \in \mathbb{N}^*$ be fixed and x and y unknown positive integers. The Diophantine equation in which function η and π are involved, given by formula (1.19), are called η–π–Diophantine equations. The list of η–π–Diophantine equations, as given in [Smarandache, 1999b], which we intend to solve empirically are:

(2152) $\eta(x) = \pi(m \cdot x + n)$,

(2153) $\eta(x)^m = \pi(x^n)$,

(2154) $\eta(x) + y = x + \pi(y)$,

(2155) $\eta(x) \cdot y = x \cdot \pi(y)$,

(2156) $\eta(x)/y = x/\pi(y)$,

(2157) $\eta(x)^y = x^{\pi(y)}$,

(2158) $\eta(x)^y = \pi(y)^x$,

7.4.1   Partial empirical solving of η–π–Diophantine equations

To solve empirically the η–π–Diophantine equations (2152–2158) we will read the file $\eta.prn$ which contains the values of functions $\eta(n)$ for $n =$

$1, 2, \ldots, 10^6$, this file being generated previously. The file $\eta.prn$ will be attributed to vector η by means of the Mathcad sequence:

$$\eta := READPRN(\text{"}...\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6$$

where the command $last(\eta)$ indicates the last index of vectors η. This manoeuver is necessary to reduce a lot the time due to the empirical search of the solutions of the Diophantine equations.

7.4.2   The equation 2152

The η–π–Diophantine equation $\eta(x) = \pi(m \cdot x + n)$ is equivalent with the relation $\eta_x = \pi(m \cdot x + n)$ if we take into consideration the significance of vector η and formula (1.19). Let us consider the search domain

$$D_c = \{2, 3, \ldots, 20\}^2 \times \{1, 2, 3, \ldots, 10^3\} \qquad (7.6)$$

with $(m, n) = 1$, $m \cdot x + n \le last(\eta)$ and x non-prime. Then, the number of possible cases is 361000. Due to the additional conditions, the number of analyzed cases is 179712.

Program 7.15. The search program is:

$Ed2152(a_m, b_m, a_n, b_n, a_x, r_x, b_x) :=$

$$
\begin{vmatrix}
j \leftarrow 0 \\
S \leftarrow (\text{"}m\text{"} \ \text{"}n\text{"} \ \text{"}x\text{"}) \\
for \ m \in a_m..b_m \\
\quad for \ n \in a_n..b_n \\
\quad\quad for \ x \in a_x, a_x + r_x..b_x \\
\quad\quad\quad if \ Tp\eta(x){=}0 \wedge m \cdot x + n \le last(\eta) \wedge \gcd(m, n){=}1 \\
\quad\quad\quad\quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack[S, (m \ n \ x)] \ if \ \eta_x{=}\pi(m \cdot x + n) \end{vmatrix} \\
return \ stack[S, (j \ j \ j)]
\end{vmatrix}
$$

The program calls the subprogram 1.8 for establishing the primality of x and y. On the search domain D_c given by (7.7) we have 35 solutions, given in the form $\boxed{m, n, x}$:

$2, 3, 62$	$2, 3, 543$	$2, 5, 62$	$2, 7, 573$	$2, 9, 74$	$2, 9, 573$	$2, 11, 74$
$2, 11, 573$	$2, 13, 74$	$2, 13, 573$	$2, 13, 579$	$2, 15, 82$	$2, 15, 573$	
$2, 15, 579$	$2, 17, 579$	$2, 19, 86$	$2, 19, 579$	$2, 19, 591$;		

$3, 2, 362$	$3, 4, 362$	$3, 7, 382$	$3, 8, 382$	$3, 10, 382$	$3, 11, 382$
$3, 13, 382$	$3, 13, 386$	$3, 14, 382$	$3, 14, 386$	$3, 16, 382$	$3, 16, 386$
$3, 17, 386$	$3, 19, 386$	$3, 19, 394$	$3, 20, 386$	$3, 20, 394$.	

The necessary time for searching the solutions is of approximately 360 seconds on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

7.4.3 The equation 2153

The η–π–Diophantine equation $\eta(x)^m = \pi(x^n)$ is equivalent with the relation $(\eta_x)^m = \pi(x^n)$ if we take into consideration the significance of vector η and formula (1.19). Fie search domain

$$D_c = \{2, 3, \ldots, 10\}^2 \times \{1, 2, 3, \ldots, 10^3\} \tag{7.7}$$

with $x^n \le last(\eta)$. Then, the number of possible cases is 80919. Due to the additional condition, the number of analyzed cases is 10494.

Program 7.16. The search program is:

$Ed2153(a_m, b_m, a_n, b_n, a_x, r_x, b_x) :=$

$$
\begin{vmatrix}
j \leftarrow 0 \\
S \leftarrow (\text{"}m\text{" "}n\text{" "}x\text{"}) \\
for\ m \in a_m..b_m \\
\quad for\ n \in a_n..b_n \\
\qquad for\ x \in a_x, a_x + r_x..b_x \\
\qquad\quad if x^n \le last(\eta) \\
\qquad\qquad
\begin{vmatrix}
j \leftarrow j + 1 \\
S \leftarrow stack[S, (m\ n\ x)]\ if\ (\eta_x)^m = \pi(x^n)
\end{vmatrix} \\
return\ stack[S, (j\ j\ j)]
\end{vmatrix}
$$

on the search domain D_c given by (7.6) we have 3 de solutions, given in the form $\boxed{m, n, x}$: $\boxed{2, 2, 10}$, $\boxed{2, 3, 2}$ and $\boxed{2, 3, 3}$. The necessary time for searching the solutions is of approximately 1580 seconds on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

7.4.4   The equation 2154

The η–π–Diophantine equation $\eta(x) + y = x + \pi(y)$ is equivalent with the relation $\eta_x + y = x + \pi(y)$ if we take into consideration the significance of vector η and formula (1.19). Fie search domain

$$D_c = \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor \cdot r_x \right\}$$
$$\times \left\{ a_y, a_y + r_y, a_y + 2r_y, \ldots, \left\lfloor \frac{b_y - a_y}{r_y} \right\rfloor \cdot r_y \right\} . \quad (7.8)$$

with $a_x = 2$, $r_x = 3$, $b_x = 10^3$, $a_y = 2$, $r_y = 5$ and $b_y = 10^3$. Then, the number of possible and analyzed cases is 66600. In these conditions we have 52 solutions given in the form $\boxed{x, y}$:

$\boxed{14, 12}$ $\boxed{65, 72}$ $\boxed{74, 52}$ $\boxed{95, 102}$;

$\boxed{116, 117}$ $\boxed{128, 157}$ $\boxed{140, 172}$ $\boxed{152, 172}$ $\boxed{158, 107}$ $\boxed{176, 212}$;

$\boxed{230, 262}$ $\boxed{245, 292}$ $\boxed{290, 327}$;

$\boxed{305, 307}$ $\boxed{329, 352}$ $\boxed{350, 422}$ $\boxed{368, 427}$ $\boxed{374, 442}$ $\boxed{380, 447}$;

$\boxed{404, 377}$ $\boxed{410, 457}$ $\boxed{416, 497}$ $\boxed{422, 267}$ $\boxed{434, 497}$ $\boxed{455, 542}$;

$\boxed{500, 592}$ $\boxed{527, 607}$ $\boxed{533, 602}$ $\boxed{548, 507}$ $\boxed{551, 637}$ $\boxed{572, 682}$;

$\boxed{602, 682}$ $\boxed{614, 382}$ $\boxed{623, 652}$ $\boxed{635, 622}$ $\boxed{656, 747}$ $\boxed{668, 612}$ $\boxed{674, 417}$
$\boxed{680, 802}$ $\boxed{689, 772}$ $\boxed{698, 432}$;

$$\boxed{725,842}\;\boxed{746,462}\;\boxed{779,892}\;;$$

$$\boxed{803,882}\;\boxed{836,982}\;\boxed{848,957}\;\boxed{860,982}\;\boxed{866,532}\;;$$

$$\boxed{917,947}\;\boxed{956,867}\;.$$

The necessary time for searching the solutions is of approximately 112 seconds on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

7.4.5 The equation 2155

The η–π–Diophantine equation $\eta(x) \cdot y = x \cdot \pi(y)$ is equivalent with the relation $\eta_x \cdot y = x \cdot \pi(y)$ if we take into consideration the significance of vector η and formula (1.19). Fie search domain

$$D_c = \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor \cdot r_x \right\}$$

$$\times \left\{ a_y, a_y + r_y, a_y + 2r_y, \ldots, \left\lfloor \frac{b_y - a_y}{r_y} \right\rfloor \cdot r_y \right\} . \quad (7.9)$$

with $a_x = 2$, $r_x = 1$, $b_x = 10^3$, $a_y = 2$, $r_y = 1$ and $b_y = 10^3$. Then, the number of possible and analyzed cases is 998001. In these conditions we have 985 solutions of which we give the first 45 and the last 50 solutions in the form $\boxed{x, y}$:

$$\boxed{6,4}\;\boxed{6,8}\;\boxed{8,4}\;\boxed{8,6}\;;$$

$$\boxed{10,4}\;\boxed{10,6}\;\boxed{10,8}\;;$$

$$\boxed{12,3}\;\boxed{12,27}\;\boxed{12,30}\;\boxed{12,33}\;;$$

$$\boxed{14,4}\;\boxed{14,6}\;\boxed{14,8}\;;$$

$$\boxed{15,3}\;\boxed{15,27}\;\boxed{15,30}\;\boxed{15,33}\;;$$

$\boxed{16,24}$;

$\boxed{18,3}$ $\boxed{18,27}$ $\boxed{18,30}$ $\boxed{18,33}$;

$\boxed{20,96}$ $\boxed{20,100}$ $\boxed{20,120}$;

$\boxed{21,3}$ $\boxed{21,27}$ $\boxed{21,30}$ $\boxed{21,33}$;

$\boxed{22,4}$ $\boxed{22,6}$ $\boxed{22,8}$;

$\boxed{25,10}$ $\boxed{25,15}$ $\boxed{25,20}$;

$\boxed{26,4}$ $\boxed{26,6}$ $\boxed{26,8}$;

$\boxed{27,3}$ $\boxed{27,30}$ $\boxed{27,33}$;

$\boxed{28,96}$ $\boxed{28,100}$ $\boxed{28,120}$;

$\vdots$

$\boxed{951,3}$ $\boxed{951,27}$ $\boxed{951,30}$ $\boxed{951,33}$;

$\boxed{955,330}$ $\boxed{955,335}$ $\boxed{955,340}$ $\boxed{955,350}$ $\boxed{955,355}$ $\boxed{955,360}$;

$\boxed{956,96}$ $\boxed{956,100}$ $\boxed{956,120}$;

$\boxed{958,4}$ $\boxed{958,6}$ $\boxed{958,8}$;

$\boxed{964,96}$ $\boxed{964,100}$ $\boxed{964,120}$;

$\boxed{965,330}$ $\boxed{965,335}$ $\boxed{965,340}$ $\boxed{965,350}$ $\boxed{965,355}$ $\boxed{965,360}$;

$\boxed{974,4}$ $\boxed{974,6}$ $\boxed{974,8}$;

$\boxed{982,4}$ $\boxed{982,6}$ $\boxed{982,8}$;

$\boxed{985,330}$ $\boxed{985,335}$ $\boxed{985,340}$ $\boxed{985,350}$ $\boxed{985,355}$ $\boxed{985,360}$;

$\boxed{993,3}$ $\boxed{993,27}$ $\boxed{993,30}$ $\boxed{993,33}$;

$\boxed{995,330}$ $\boxed{995,335}$ $\boxed{995,340}$ $\boxed{995,350}$ $\boxed{995,355}$ $\boxed{995,360}$;

$\boxed{998,4}$ $\boxed{998,6}$ $\boxed{998,8}$.

The necessary time for searching the solutions is of approximately 116 seconds on a computer with an Intel processor of 2.20GHz with RAM of 4.00GB (3.46GB usable).

7.4.6 The equation 2156

The η–π–Diophantine equation $\eta(x)/y = x/\pi(y)$ is equivalent with the relation

$$\eta_x \cdot \pi(y) = x \cdot y , \qquad (7.10)$$

if we take into consideration the significance of vector η and formula (1.19). Fie search domain given by (7.9), with $a_x = 2$, $r_x = 1$, $b_x = 10^3$, $a_y = 2$, $r_y = 1$ and $b_y = 10^3$. Then, the number of possible and analyzed cases is 998001. In these conditions, equation (7.10) not has solutions.

7.4.7 The equation 2157

The η–π–Diophantine equation $\eta(x)^y = x^{\pi(y)}$ is equivalent with the relation

$$(\eta_x)^y = x^{\pi(y)} , \qquad (7.11)$$

if we take into consideration the significance of vector η and formula (1.19). Let us consider the search domain given by (7.9), with $a_x = 2$, $r_x = 1$, $b_x = 10^3$, $a_y = 2$, $r_y = 1$ and $b_y = 10^3$ with following restrictions $(x, y) = 1$, $(\eta_x)^y \leq 10^{307}$ and $x^{\pi(y)} \leq 10^{307}$. Then, the number of possible cases is 998001 and the number of analyzed cases is 112809. In these conditions, equation (7.11) has 3 solutions: $\boxed{32,5}$ $\boxed{81,4}$, $\boxed{81,8}$.

Program 7.17. The search program for the solutions of relation (7.11)

$Ed2157(a_x, r_x, b_x, a_y, r_y, b_y) :=$
$\quad \mid j \leftarrow 0$
$\quad \mid S \leftarrow ("x" \ "y")$
$\quad \mid for \ x \in a_x, a_x + r_x..b_x$
$\quad \quad \mid for \ y \in a_y, a_y + r_y..b_y$
$\quad \quad \quad \mid break \ on \ error \ (\eta_x)^y$
$\quad \quad \quad \mid break \ on \ error \ x^{\pi(y)}$
$\quad \quad \quad \mid if \ \gcd(x,y){=}1$
$\quad \quad \quad \quad \mid j \leftarrow j + 1$
$\quad \quad \quad \quad \mid S \leftarrow stack[S, (x \ y)] \ if \ (\eta_x)^y{=}x^{\pi(y)}$
$\quad \mid return \ stack[S, (j \ j)]$

7.4.8   The equation 2158

The η–π–Diophantine equation $\eta(x)^y = \pi(y)^x$ is equivalent with the relation

$$(\eta_x)^y = \pi(y)^x , \tag{7.12}$$

considering the meaning of vector η and formula (1.19). Let the search domain be given by (7.9), with $a_x = 2$, $r_x = 1$, $b_x = 10^3$, $a_y = 2$, $r_y = 1$ and $b_y = 10^3$ with restrictions $(x, y) = 1$, $(\eta_x)^y \leq 10^{307}$ and $\pi(y)^x \leq 10^{307}$. Then, the number of possible cases is 998001 and the number of analyzed cases are 35743. In these conditions, equation (7.12) has no solutions.

7.5   The η–σ_k–Diophantine equations

Let us consider $m, n, k \in \mathbb{N}^*$ fixed and x and y unknown positive integers. The Diophantine equations where functions η and σ_k are involved, ar called η–σ_k–Diophantine equations. The list of η–σ_k–Diophantine equations, as in [Smarandache, 1999b], which we intend to solve empirically, is:

(2166) $\eta(x) = \sigma_k(m \cdot x + n)$,

(2167) $\eta(x)^m = \sigma_k(x^n)$,

(2168) $\eta(x) + y = x + \sigma_k(y)$,

(2169) $\eta(x) \cdot y = x \cdot \sigma_k(y)$,

(2170) $\eta(x)/y = x/\sigma_k(y)$,

(2171) $\eta(x)^y = x^{\sigma_k(y)}$,

(2172) $\eta(x)^y = \sigma_k(y)^x$,

7.5.1 Partial empirical solving of η–σ_k–Diophantine equations

For every Diophantine equation solved in this section, the file $\eta.prn$ is read, generated by the program 2.9, by means of the Mathcad function $READPRN$

$$\eta := READPRN(\text{"...}\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6$$

where the command $last(\eta)$ indicates the last index of vector η. The reading time is of about 10 seconds, and, therefore, an important saving of the search time can be remarked.

The file $\sigma0.prn$ is read and the values will be assigned to vector $\sigma0$, by means of the Mathcad function $READPRN$, with the sequence:

$$\sigma0 := READPRN(\text{"...}\backslash\sigma0.prn\text{"}) \ \ last(\sigma0) = 10^6 ,$$

where each component $\sigma0_k$ contain the number of divisors of k, for $k = 1, 2, \ldots, 10^6$. The values were generated with the program 3.3. The time for generating the file $\sigma0.prn$ was *2:4:48.362hhmmss* . If we have a Diophantine equation in which function $\sigma_0(x)$ is involved, we will use vector $\sigma0$. The time saving that results is important.

If we have a Diophantine equation in which function $\sigma(x)$ is involved and we proceed to a systematic empirical search for many values x, we will read the file $\sigma1.prn$ in vector $\sigma1$ with the sequence:

$$\sigma1 := READPRN(\text{"...}\backslash\sigma1.prn\text{"}) \ \ last(\sigma1) = 10^6 ,$$

where each component $\sigma 1_k$ contains the sum of the divisors of k, for $k = 1, 2, \ldots, 10^6$. The necessary time for generating the file $\sigma 1.prn$ was *2:5:32.155hhmmss*.

By the same reasons as in reading files $\sigma 0.prn$ or $\sigma 1.prn$, we will read the file $\sigma 2.prn$ in vector $\sigma 2$ with the sequence:

$$\sigma 2 := READPRN(\text{"...\textbackslash}\sigma 2.prn\text{"})\ \ last(\sigma 2) = 10^6\ .$$

where each component $\sigma 2_k$ contains the sum of the squares of the divisors of k, for $k = 1, 2, \ldots, 10^6$. The necessary time for generating the file $\sigma 2.prn$ was *2:4:50.197hhmmss*. Thereby, an important time saving will be obtained in the empirical search.

7.5.2   The equation 2166

The Diophantine equation (2166) for $k = 0$ is $\eta(x) = \sigma_0(m \cdot x + n)$ and is equivalent with the relation $\eta_x = \sigma 0_{m \cdot x + n}$, considering the meaning of files η and $\sigma 0$. Let

$$D_c = \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 10^6\}\ , \qquad (7.13)$$

be the search domain. We have 10^8 possible cases. As the component $\sigma 0_{m \cdot x + n}$ with $m \cdot x + n > 10^6$ can not be addressed, for the search domain D_c given by (7.13), we consider the additional condition $m \cdot x + n \leq 10^6$. With this condition, the number of analyzed cases is 29289486. In these conditions we have 3869 solutions. We present the first 79 solutions in the form $\boxed{m, n, x}$:

$\boxed{1, 1, 2}$ $\boxed{1, 1, 3}$ $\boxed{1, 1, 15}$ $\boxed{1, 1, 63}$ $\boxed{1, 1, 175}$ $\boxed{1, 1, 384}$ $\boxed{1, 1, 896}$
$\boxed{1, 1, 2240}$ $\boxed{1, 1, 3024}$ $\boxed{1, 1, 4095}$ $\boxed{1, 1, 6720}$ $\boxed{1, 1, 13824}$ $\boxed{1, 1, 24255}$
$\boxed{1, 1, 25920}$ $\boxed{1, 1, 39375}$ $\boxed{1, 1, 39424}$ $\boxed{1, 1, 68607}$ $\boxed{1, 1, 81920}$
$\boxed{1, 1, 93555}$ $\boxed{1, 1, 281600}$ $\boxed{1, 1, 425984}$ $\boxed{1, 1, 552500}$ $\boxed{1, 1, 630784}$
$\boxed{1, 1, 649539}$;

$1, 2, 4$	$1, 2, 8$	$1, 2, 12$	$1, 2, 16$	$1, 2, 18$	$1, 2, 24$	$1, 2, 48$	$1, 2, 64$
$1, 2, 90$	$1, 2, 128$	$1, 2, 240$	$1, 2, 288$	$1, 2, 320$	$1, 2, 640$	$1, 2, 720$	
$1, 2, 960$	$1, 2, 1440$	$1, 2, 1470$	$1, 2, 2240$	$1, 2, 2688$	$1, 2, 2880$		
$1, 2, 3150$	$1, 2, 3402$	$1, 2, 4800$	$1, 2, 5346$	$1, 2, 5632$	$1, 2, 5760$		
$1, 2, 8064$	$1, 2, 20160$	$1, 2, 21384$	$1, 2, 26730$	$1, 2, 32768$			
$1, 2, 48750$	$1, 2, 73728$	$1, 2, 85050$	$1, 2, 90112$	$1, 2, 95550$			
$1, 2, 98304$	$1, 2, 114688$	$1, 2, 135168$	$1, 2, 138240$	$1, 2, 200704$			
$1, 2, 270336$	$1, 2, 308750$	$1, 2, 319488$	$1, 2, 401408$	$1, 2, 409600$			
$1, 2, 442368$	$1, 2, 630784$	$1, 2, 708750$	$1, 2, 716800$	$1, 2, 737280$			
$1, 2, 802816$	$1, 2, 901120$	$1, 2, 921600$;					

$$\vdots$$

The search time was 108.645 seconds.

As the number of solutions is big, we propose to restrict the search domain as follows: let m and n be relative prime and x to take the values of an arithmetic progression with the first term $a_x = 1$, the ratio $r_x = 113$ and the last term $\leq b_x = 10^6$. These restrictions are obtained by the composed condition, written in the Mathcad syntax, $\gcd(m, n) = 1$ and $x \in \{a_x, a_x + r_x..b_x\}$. In these conditions we have $D_c = \{1, 2, \ldots, 10\}^2 \times \{1, 14, 27, \ldots, 999938\}$ with $(m, n) = 1$. The number of analyzed cases is 188273 and the number of solutions is 16:

$1, 2, 1470$ $1, 9, 41472$;

$2, 9, 4860$ $2, 9, 41472$;

$3, 4, 41472$ $3, 8, 4860$;

$4, 3, 41472$;

$5, 3, 41472$ $5, 4, 41472$ $5, 8, 4860$ $5, 9, 4860$;

$\boxed{7, 8, 4860}$ $\boxed{7, 9, 4860}$;

$\boxed{8, 9, 4860}$;

$\boxed{9, 4, 4860}$;

$\boxed{10, 9, 4860}$.

The empirical search domain of the solutions of equation (2166) for $k = 0$ is:

$Ed2166(a_m, b_m, a_n, b_n, a_x, r_x, b_x) :=$

$$
\begin{vmatrix}
j \leftarrow 0 \\
S \leftarrow (\text{"}m\text{"} \ \text{"}n\text{"} \ \text{"}x\text{"}) \\
for \ m \in a_m..b_m \\
\quad for \ n \in a_n..b_n \\
\quad\quad for \ x \in a_x, a_x + r_x..b_x \\
\quad\quad\quad if \ m \cdot x + n \leq last(\sigma_0) \wedge \gcd(m, n){=}1 \\
\quad\quad\quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack[S, (m \ n \ x)] \ if \ \eta_x{=}\sigma 0_{m \cdot x + n} \end{vmatrix} \\
return \ stack[S, (j \ j \ j)]
\end{vmatrix}
$$

For $k = 1$ and $k = 2$ equation (2166) has no solutions on the search domain D_c given by (7.13).

As the values $\sigma 1_k$ overpass the values η_k, we have wondered if there exist solutions for the equation Diophantine

$$ \eta(m \cdot x + n) = \sigma(x) \ . $$

The equivalent relation is $\eta_{m \cdot x + n} = \sigma 1_x$, considering the meaning of vectors η and $\sigma 1$. On the search domain

$$ D_c = \{1, 2, \ldots, 10\}^2 \times \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor \cdot r_x \right\} \ , \quad (7.14) $$

with $a_x = 1$, $r_x = 1$ and $b_x = 10^6$ and $(m, n) = 1$ (i.e. m and n relative prime), there exist for this equation 29 solutions, given in the form $\boxed{m, n, x}$:

$$\boxed{1, 1, 2} \boxed{1, 1, 3} \boxed{1, 3, 4} \boxed{1, 4, 2} \boxed{1, 4, 5} \boxed{1, 4, 9} \boxed{1, 5, 3} \boxed{1, 6, 25} \boxed{1, 9, 3}$$
$$\boxed{1, 10, 4} ;$$

$$\boxed{3, 1, 5} \boxed{3, 2, 4} \boxed{3, 10, 13} ;$$

$$\boxed{4, 3, 9} \boxed{4, 5, 4} ;$$

$$\boxed{5, 1, 4} \boxed{5, 7, 9} \boxed{5, 8, 4} \boxed{5, 9, 3} ;$$

$$\boxed{6, 5, 25} ;$$

$$\boxed{7, 1, 5} \boxed{7, 2, 9} \boxed{7, 3, 3} \boxed{7, 10, 5} ;$$

$$\boxed{8, 3, 4} \boxed{8, 5, 5} ;$$

$$\boxed{9, 1, 7} \boxed{9, 10, 9} ;$$

$$\boxed{10, 1, 9} .$$

The search domain has 10^8 possible cases, but the analyzed are of only 21271688 due to restrictions $m \cdot x + n \leq last(\eta)$ and $\gcd(m, n) = 1$.

Analogously, the Diophantine equation

$$\eta(m \cdot x + n) = \sigma_2(x) .$$

was considered, which, on the search domain D_c given by (7.14) with $a_x = 1$, $r_x = 1$, $b_x = 10^6$ and additional conditions $m \cdot x + n \leq last(\eta)$ and $\gcd(m, n) = 1$ has 11 solutions:

$$\boxed{1, 3, 2} \boxed{1, 8, 2} \boxed{2, 1, 2} \boxed{3, 4, 2} \boxed{4, 7, 2} \boxed{6, 7, 3} \boxed{7, 1, 2} \boxed{7, 4, 3} \boxed{7, 6, 2}$$
$$\boxed{8, 1, 3} \boxed{9, 2, 2} .$$

7.5.3 The equation 2167

The Diophantine equation $\eta(x)^m = \sigma_k(x^n)$ with $k = 0, 1, 2$, can be equivalent with the relation $(\eta_x)^m = \sigma k_{x^n}$, with $k = 0, 1, 2$, if we consider the significance of vectors η and σk, with $k = 0, 1, 2$. If we impose the condition $m \neq n$ the number of solutions is 22, for $k = 0$, on the search domain

$$D_c = \{1, 2, \ldots, 10\}^2 \times \{2, 3, \ldots, 10^6\} . \qquad (7.15)$$

The number of possible cases is 99999900, as a consequence of conditions $m \neq n$ and $x^n \leq 10^6$, the number of analyzed cases is 9010449, and the search time is under one minute. The 22 solutions are presented, in the form $\boxed{m, n, x}$,

$\boxed{1, 2, 3}$ $\boxed{1, 2, 81}$ $\boxed{1, 2, 686}$ $\boxed{1, 4, 5}$ $\boxed{1, 6, 7}$;

$\boxed{2, 1, 13440}$ $\boxed{2, 1, 272160}$ $\boxed{2, 1, 340200}$ $\boxed{2, 1, 374220}$ $\boxed{2, 1, 427680}$
$\boxed{2, 1, 534600}$ $\boxed{2, 1, 598752}$ $\boxed{2, 1, 777600}$ $\boxed{2, 1, 935550}$ $\boxed{2, 3, 2}$ $\boxed{2, 3, 96}$
$\boxed{2, 4, 10}$ $\boxed{2, 4, 15}$ $\boxed{2, 5, 8}$ $\boxed{2, 8, 3}$;

$\boxed{3, 4, 30}$ $\boxed{3, 7, 2}$.

The Diophantine equations $\eta(x)^m = \sigma_k(x^n)$, for $k = 1, 2$ have 90 solutions on the search domain

$$D_c = \{1, 2, \ldots, 10\}^2 \times \{1, 2, \ldots, 10^6\}$$

and restriction $x^n \leq 10^6$. All solutions are trivial only with $x = 1$.

7.5.4 The equation 2168

The η–σ_0–Diophantine equation $\eta(x) + y = x + \sigma_0(y)$ has on thee search domain $D_c = \{2, 3, \ldots, 10^4\}^2$ 99980001 cases to be analyzed. This equation has 9893 solutions, obtained in approximately 50 seconds, of which we present the first 26 and the last 26 solutions in the form $\boxed{x, y}$:

| $2,2$ | $3,2$ | $4,2$ | $5,2$ | $6,5$ | $7,2$ | $8,8$ | $9,5$ |; |

| $10,7$ | $11,2$ | $13,2$ | $15,14$ | $16,14$ | $17,2$ | $18,18$ | $19,2$ |; |

| $20,17$ | $21,20$ | $22,13$ | $22,15$ | $22,16$ | $23,2$ | $25,17$ | $27,22$ | $28,23$ |
| $29,2$ |; |

$$\vdots$$

| $9971,9928$ | $9972,9697$ | $9973,2$ | $9976,9937$ | $9977,9076$ |
| $9977,9086$ | $9978,8317$ | $9978,8323$ |; |

$9981,8888$	$9981,8904$	$9982,9963$	$9982,9975$	$9983,9838$
$9983,9846$	$9984,9973$	$9986,4997$	$9987,6662$	$9989,8566$
$9989,8570$	;

| $9990,9957$ | $9991,9896$ | $9993,6668$ | $9994,9733$ | $9995,8016$ |
| $9996,9983$ | $9999,9902$ |. |

The search program uses the equivalent relation $\eta_x + y = x + \sigma 0_y$, taking into consideration the meaning of vectors η and $\sigma 0$. The simple search algorithm can be deduced from the source text of the program:

$Ed2168(a_x, r_x, b_x, a_y, r_y, b_y) :=$

$$
\begin{vmatrix}
S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \\
for \ x \in a_x, a_x + r_x..b_x \\
\quad for \ y \in a_y, a_y + r_y..b_y \\
\quad\quad S \leftarrow stack[S, (x \ y)] \ if \ \eta_x + y {=} x + \sigma 0_y \\
return \ S
\end{vmatrix}
$$

on the search domain

$$D_c = \{2, 505, 1008, \ldots, 999966\} \times \{3, 604, 1205, \ldots, 999466\} \ ,$$

the Diophantine equation $\eta(x) + y = x + \sigma_0(y)$ has two solutions $\boxed{262065, 244610}$ and $\boxed{741927, 494626}$.

The η–σ_k–Diophantine equation $\eta(x) + y = x + \sigma_k(y)$, with $k = 1, 2$ does not have solutions on the search domain $D_c = \left\{1, 2, \ldots, 10^6\right\}^2$, other than the 78500 trivial solutions when $y = 1$ and x is a prime number or $x = 4$.

7.5.5   The equation 2169

The η–σ_k–Diophantine equation $\eta(x) \cdot y = x \cdot \sigma_k(y)$ is equivalent with the relation $\eta_x \cdot y = x \cdot \sigma k_y$ if we consider the meaning of vectors η and σk for $k = 0, 1, 2$. We consider the search domain

$$D_c = \{a_x, a_x + r_x, a_x + 2r_x, \ldots, b_x\} \times \{a_y, a_y + r_y, a_y + 2r_y, \ldots, b_y\} ,$$
(7.16)

where $a_x = 2$, $r_x = 113$, $b_x = 10^6$, $a_y = 3$, $r_y = 127$ and $b_y = 10^6$. The number of possible cases is given by formula

$$\left(\left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor + 1\right) \left(\left\lfloor \frac{b_y - a_y}{r_y} \right\rfloor + 1\right) ,$$

and, therefore, for the previously given values we have 69684900 possible cases. The additional condition for the search domain is $x \neq y$. In this condition, the number of analyzed cases of the search domain (7.16) is 69684830. We have 81 solutions given in the form $\boxed{x, y}$:

$4296, 384$	$8816, 8512$	$15144, 384$	$20568, 384$	$22715, 36960$
$36162, 47628$	$47688, 384$	$53112, 384$	$64525, 52200$	$80232, 384$
$82944, 829440$	$87238, 1908$	$96504, 384$;		

$111194, 1908$	$145885, 29340$	$161592, 384$	$167016, 384$
$177864, 384$	$194136, 384$;		

$230974, 1908$	$251766, 61344$	$264648, 384$	$291768, 384$;

302842, 1908	307588, 12576	324312, 384	326798, 1908
356856, 384	369964, 65408	370755, 623700	372450, 280800
378552, 384	392225, 52200	394824, 384	398666, 1908 ;

405672, 384	421944, 384	427368, 384	459912, 384	476184, 384

478783, 5464 ;

514265, 29340	519576, 384	544436, 12576	573816, 384

590314, 1908 ;

617208, 384	622632, 384	627152, 8512	634158, 47628
649752, 384	653820, 97920	655176, 384	655854, 67440
660939, 494160	662182, 1908	666024, 384	682296, 384
686138, 1908	698455, 29340 ;		

703992, 384	718795, 36960	720264, 384	732242, 108080
736536, 384	758006, 1908	779928, 384	787499, 5464 ;

800042, 11052	805918, 1908	823546, 110112	829874, 1908
830552, 236096	833264, 8512	834168, 384	859254, 372240
882984, 384	893832, 384	898352, 122304 ;	

923890, 260480   948072, 384 .

The number of solutions for equation (2169) with $k = 0$ on the search domain (7.16) with $a_1 = 1$, $r_x = 1$, $b_x = 10^6$, $a_y = 1$, $r_y = 1$ and $b_y = 10^6$ is huge.

The search program is:

$Ed2169(a_x, r_x, b_x, a_y, r_y, b_y) :=$

$$\begin{vmatrix} j \leftarrow 0 \\ S \leftarrow (\text{"}x\text{"} \ \text{"}y\text{"}) \end{vmatrix}$$

$$\begin{vmatrix} for\ x \in a_x, a_x + r_x..b_x \\ \quad for\ y \in a_y, a_y + r_y..b_y \\ \quad\quad if\ x \neq y \\ \quad\quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack[S, (x\ y)]\ if\ \eta_x \cdot y{=}x \cdot \sigma 0_y \end{vmatrix} \\ return\ stack[S, (j\ j)] \end{vmatrix}$$

The Diophantine equation (2169) has, for $k = 1$ and $k = 2$, only trivial solutions in which $y = 1$.

7.5.6   The equation 2170

The η–σ–Diophantine equation (2170) can be written in the form $\eta(x) \cdot \sigma_k(y) = x \cdot y$, equation which is equivalent with the relation $\eta_x \cdot \sigma k_y = x \cdot y$ if we take into consideration the significance of vector η and σk for $k = 0, 1, 2$. The search program for equation $\eta(x) \cdot \sigma_0(y) = x \cdot y$ is

$$Ed2170(a_x, r_x, b_x, a_y, r_y, b_y) := \begin{vmatrix} S \leftarrow (\text{"}x\text{" "}y\text{"}) \\ for\ x \in a_x, a_x + r_x..b_x \\ \quad for\ y \in a_y, a_y + r_y..b_y \\ \quad\quad S \leftarrow stack[S, (x\ y)]\ if\ \eta_x \cdot \sigma 0_y{=}x \cdot y \\ return\ S \end{vmatrix}$$

The search domain is similar with the domain (7.16) in which, in order to cover all possible cases, we should have $a_x = 1$, $r_x = 1$, $b_x = 10^6$, $a_y = 1$, $r_y = 1$ and $b_y = 10^6$. Therefore we will have 10^{12} possible cases. The equation $\eta(x) \cdot \sigma_0(y) = x \cdot y$ does not have solutions on the search domain D_c given by (7.16), where $a_= 1$, $r_x = 1$, $b_x = 10^4$, $a_y = 1$, $r_y = 1$ and $b_y = 10^4$.

The Diophantine equation $\eta(x) \cdot \sigma(y) = x \cdot y$ on the search domain given by (7.16), where $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$ and $b_y = 10^4$, with 99970002 analyzed cases has 3625 solutions. As the number of solutions is that big, we intend that, on the same search domain, to consider the

additional condition $(x, y) = 1$ (i.e. x and y are relatively prime). Hence, the number of analyzed cases has decreased to 60769973 and, thereby, we have found 3 solutions, presented in the form $\boxed{x, y}$: $\boxed{25, 24}$, $\boxed{49, 4320}$ and $\boxed{49, 4680}$.

Program 7.18. The empirical search program is:

$Ed2170(a_x, r_x, b_x, a_y, r_y, b_y) :=$

$$
\begin{vmatrix}
j \leftarrow 0 \\
S \leftarrow (\text{"}x\text{" "}y\text{"}) \\
for\ x \in a_x, a_x + r_x..b_x \\
\quad for\ y \in a_y, a_y + r_y..b_y \\
\quad\quad if\ \gcd(x, y)\text{=}1 \\
\quad\quad\quad \begin{vmatrix} j \leftarrow j + 1 \\ S \leftarrow stack[S, (x\ y)]\ if\ \eta_x \cdot \sigma 1_y \text{=} x \cdot y \end{vmatrix} \\
return\ stack[S, (j\ j)]
\end{vmatrix}
$$

The Diophantine equation $\eta(x) \cdot \sigma_2(y) = x \cdot y$ on the given search domain (7.16), where $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$, $b_y = 10^4$ and $(x, y) = 1$, with 60769973 analyzed cases, has 211 solutions. We give in the form $\boxed{x, y}$ the first 60 and the last 57 solutions:

$\boxed{125, 6}$;

$221, 10$	$247, 10$	$299, 10$	$377, 10$	$403, 10$	$481, 10$	$533, 10$
$559, 10$	$611, 10$	$689, 10$	$767, 10$	$793, 10$	$871, 10$	$923, 10$
$949, 10$	$1027, 10$	$1079, 10$	$1157, 10$	$1261, 10$	$1313, 10$	$1339, 10$
$1391, 10$	$1417, 10$	$1469, 10$	$1651, 10$	$1703, 10$	$1781, 10$	$1807, 10$
$1937, 10$	$1963, 10$	$1972, 65$	$2041, 10$	$2119, 10$	$2171, 10$	$2327, 10$
$2353, 10$	$2249, 10$	$2483, 10$	$2509, 10$	$2561, 10$	$2587, 10$	$2743, 10$

$\boxed{2899, 10}$;

$\boxed{175, 12}$ $\boxed{245, 12}$;

$\boxed{1547, 60}$ $\boxed{1729, 60}$ $\boxed{2093, 60}$ $\boxed{2639, 60}$ $\boxed{2821, 60}$;

$\boxed{1292, 65}$ $\boxed{1564, 65}$ $\boxed{2108, 65}$ $\boxed{2312, 65}$ $\boxed{2516, 65}$ $\boxed{2788, 65}$;

$\boxed{2125, 84}$ $\boxed{2375, 84}$ $\boxed{2875, 84}$;

$\vdots$

$\boxed{7423, 10}$ $\boxed{7501, 10}$ $\boxed{7631, 10}$ $\boxed{7709, 10}$ $\boxed{7787, 10}$ $\boxed{7813, 10}$ $\boxed{7891, 10}$
$\boxed{7969, 10}$ $\boxed{8021, 10}$ $\boxed{8047, 10}$ $\boxed{8203, 10}$ $\boxed{8333, 10}$ $\boxed{8359, 10}$ $\boxed{8411, 10}$
$\boxed{8489, 10}$ $\boxed{8567, 10}$ $\boxed{8593, 10}$ $\boxed{8749, 10}$ $\boxed{8801, 10}$ $\boxed{8879, 10}$ $\boxed{8983, 10}$
$\boxed{9113, 10}$ $\boxed{9217, 10}$ $\boxed{9347, 10}$ $\boxed{9451, 10}$ $\boxed{9529, 10}$ $\boxed{9607, 10}$ $\boxed{9659, 10}$
$\boxed{9763, 10}$ $\boxed{9841, 10}$ $\boxed{9893, 10}$ $\boxed{9997, 10}$;

$\boxed{7553, 60}$ $\boxed{8099, 60}$ $\boxed{8827, 60}$ $\boxed{9191, 60}$ $\boxed{9373, 60}$ $\boxed{9737, 60}$
$\boxed{9919, 60}$;

$\boxed{7684, 65}$ $\boxed{8636, 65}$ $\boxed{8908, 65}$ $\boxed{9316, 65}$ $\boxed{9452, 65}$;

$\boxed{7625, 84}$ $\boxed{8375, 84}$ $\boxed{8875, 84}$ $\boxed{9125, 84}$ $\boxed{9875, 84}$;

$\boxed{8029, 150}$ $\boxed{8897, 150}$ $\boxed{9331, 150}$;

$\boxed{7626, 175}$ $\boxed{7998, 175}$ $\boxed{8742, 175}$ $\boxed{9858, 175}$;

$\boxed{8211, 260}$.

The empirical search program is similar with the program 7.18.

7.5.7 The equation 2171

The Diophantine equation $\eta(x)^y = x^{\sigma_k(y)}$ is equivalent with the relation $(\eta_x)^y = x^{\sigma k_y}$ for $k = 0, 1, 2$ if we take into consideration the meaning of vectors η and σk, with $k = 0, 1, 2$. For $k = 0$ let us consider the search domain (7.16) where $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$ and $b_y = 10^4$. Then, the number of considered cases is 99970002. Due to conditions $(\eta_x)^y > 10^{307}$ and $x^{\sigma k_y} > 10^{307}$ (upper floating overflow) the number of analyzed cases is of only 1460765. The number of found solutions is 45 and are presented in the form $\boxed{x, y}$:

$8, 3$	$8, 6$	$27, 3$	$27, 6$	$36, 8$	$36, 12$	$64, 8$	$64, 12$	$81, 8$	$81, 12$
$100, 8$	$100, 12$	$196, 8$	$196, 12$	$484, 8$	$484, 12$	$676, 8$	$676, 12$		
$1156, 8$	$1156, 12$	$1444, 8$	$1444, 12$	$2116, 8$	$2116, 12$	$3125, 5$			
$3125, 10$	$3364, 8$	$3364, 12$	$3375, 9$	$3375, 24$	$3844, 8$	$3844, 12$			
$4096, 9$	$4096, 18$	$4096, 24$	$5476, 8$	$5476, 12$	$6724, 8$	$6724, 12$			
$7396, 8$	$7396, 12$	$8836, 8$	$8836, 12$	$9261, 9$	$9261, 18$	.			

For the Diophantine equation $\eta(x)^y = x^{\sigma(y)}$ on the search domain (7.16), with $a_x = 2$, $r_x = 1$, $b_x = 10^5$, $a_y = 3$, $r_y = 1$ and $b_y = 10^5$, there are 9999700002 possible cases, of which only 2839629 cases were analyzed, by reasons of upper floating overflow generated by the raising to power $\eta(x)^y$ or $x^{\sigma(y)}$. The equation has no solutions.

The Diophantine equation $\eta(x)^y = x^{\sigma_2(y)}$, on the search domain (7.16) with $a_x = 2$, $r_x = 1$, $b_x = 10^5$, $a_y = 3$, $r_y = 1$ and $b_y = 10^5$ has 9999700002 possible cases, of which only 507015 cases were analyzed, by reasons of upper floating overflow generated by the raising to power $\eta(x)^y$ or $x^{\sigma_2(y)}$, has no solutions.

7.5.8 The equation 2172

The equation $\eta(x)^y = \sigma_k(y)^x$ is equivalent with the relation $(\eta_x)^y = (\sigma k_y)^x$ for $k = 0, 1, 2$ if we take into consideration the meaning of vectors

η and σk, with $k = 0, 1, 2$. For the case $k = 0$ fie search domain is (7.16), where $a_x = 2$, $r_x = 1$, $b_x = 10^6$, $a_y = 3$, $r_y = 1$ and $b_y = 10^6$. Then, the number of considered cases is 999997000002. Due to conditions $(\eta_x)^y > 10^{307}$ and $x^{\sigma k_y} > 10^{307}$ (upper floating overflow) the number of analyzed cases is 72776. The solutions found for $k = 0$ are: $\boxed{8, 8}$, $\boxed{18, 18}$, $\boxed{45, 45}$ and $\boxed{128, 128}$.

On the same search domain, for $k = 1$ the number of analyzed cases is 38794, for $k = 2$ the number of analyzed cases is only of 24736 and the Diophantine equations do not have solutions.

7.6   The η–φ–Diophantine equations

Let us consider $m, n \in \mathbb{N}^*$ fixed and x and y unknown positive integers. The Diophantine equations in which functions η and φ are involved are said to be η–φ–Diophantine equations. The list of η–φ–Diophantine equation, considered from [Smarandache, 1999b], and which we intend to solve empirically, is:

(2187) $\eta(x) = \varphi(m \cdot x + n)$,

(2188) $\eta(x)^m = \varphi(x^n)$,

(2189) $\eta(x) + y = x + \varphi(y)$,

(2190) $\eta(x) \cdot y = x \cdot \varphi(y)$,

(2191) $\eta(x)/y = x/\varphi(y)$,

(2192) $\eta(x)^y = x^{\varphi(y)}$,

(2193) $\eta(x)^y = \varphi(y)^x$.

7.6.1   Partial empirical solving of η–φ–Diophantine equations

For all Diophantine equations solved in this section the file $\eta.prn$ will be read, generated by the program 2.9, by means of the Mathcad function

READPRN

$$\eta := READPRN(\text{"...}\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6$$

where command $last(\eta)$ indicates the last index of vector η. The reading time is of about 10 seconds, therefore an important saving for the execution time of the search is obtained.

The file $\varphi.prn$ will be read and the values will be attributed to vector φ, by means of the Mathcad function $READPRN$, with the sequence:

$$\varphi := READPRN(\text{"...}\backslash\varphi.prn\text{"}) \ \ last(\varphi) = 10^6 \ ,$$

where each component φ contains the number of factors relatively prime to k, for $k = 1, 2, \ldots, 10^6$. The values were generated with the program 4.2. The generating time for the file $\varphi.prn$ was *5:30:33.558hhmmss* . If we have a Diophantine equation in which function $\varphi(x)$ is involved, we will use the vector φ. The time saving that results is important.

7.6.2 The equation 2187

The Diophantine equation $\eta(x) = \varphi(m \cdot x + n)$ is equivalent with the relation $\eta_x = \varphi_{m \cdot x + n}$ if we take into account the significance of vectors η and φ. Let the search domain be

$$D_c = \{1, 2, \ldots, 10\}^2 \times \{1, 2, \ldots, 10^6\} \tag{7.17}$$

and additional condition $(m, n) = 1$. The number of possible cases is 10^8 and we have only 21271688 analyzed cases. Under these conditions equation (2187) has 13 solutions given in the form $\boxed{m, n, x}$:

$$\boxed{1,1,1}\ \boxed{1,1,2}\ \boxed{1,1,4}\ \boxed{1,2,2}\ \boxed{1,2,8}\ \boxed{1,2,16}\ \boxed{1,4,2}\ \boxed{1,4,4}\ \boxed{1,4,8}$$
$$\boxed{1,5,9}\ \boxed{1,6,4}\ \boxed{1,8,4}\ \boxed{1,9,9}\ .$$

We can also consider the Diophantine equation $\eta(m \cdot x + n) = \varphi(x)$ which is equivalent with the relation $\eta_{m \cdot x + n} = \varphi_x$ if we take into account the significance of vectors η and φ. On the search domain (7.17) we have 26 solutions:

| $1,2,7$ | $1,2,10$ | $1,2,14$ | $1,2,30$ | $1,3,5$ | $1,3,22$ | $1,4,8$ | $1,4,14$ |

| $1,7,5$ | $1,7,9$ | $1,8,24$ | $1,9,7$ | $1,9,9$ | ;

| $2,3,11$ | $2,9,18$ | $3,4,20$ | $4,9,9$ | ;

| $5,1,7$ | $5,2,14$ | $5,3,9$ | $5,6,23$ | $5,8,24$ | ;

| $7,9,9$ | $9,1,11$ | $9,2,22$ | $9,8,24$ | .

7.6.3 The equation 2188

The η–φ–Diophantine equation $\eta(x)^m = \varphi(x^n)$ is equivalent with the relation $(\eta_x)^m = \varphi_{x^n}$ if we take into account the significance of vectors η and φ. Let the search domain be (7.17) with $a_m = 2$, $a_n = 2$, $a_x = 2$ and additional conditions $x^n \le last(\varphi)$, $(\eta_x)^n < 10^{17}$ and $(m,n) = 1$. Then we have 80999919 possible cases and 4362 analyzed cases. Under these conditions, the Diophantine equation (2188) has 12 solutions:

| $2,3,2$ | $3,2,32$ | $3,2,50$ | $3,4,2$ | $4,3,8$ | $4,5,2$ | $5,6,2$ | $6,7,2$ | $7,5,8$ |

| $7,8,2$ | $8,9,2$ | $9,10,2$ | .

7.6.4 The equation 2189

The η–φ–Diophantine equation $\eta(x) + y = x + \varphi(y)$ is equivalent with the relation $\eta_x + y = x + \varphi_y$ if we take into account the significance of vectors η and φ. Let the search domain be

$$
D_c = \left\{ a_x, a_x + r_x, a_x + 2r_x, \ldots, a_x + \left\lfloor \frac{b_x - a_x}{r_x} \right\rfloor r_x \right\}
$$
$$
\times \left\{ a_y, a_y + r_y, a_y + 2r_y, \ldots, a_y + \left\lfloor \frac{b_y - a_y}{r_y} \right\rfloor r_y \right\}, \quad (7.18)
$$

where $a_x = 2$, $r_x = 113$, $b_x = 10^6$, $a_y = 3$, $r_y = 127$ and $b_y = 10^6$. This search domain has 69684900 possible cases which are also analyzed cases.

In these conditions, equation (2189) has 60 solutions given in the form $\boxed{x, y}$:

$\boxed{2036, 518671}$ $\boxed{3618, 828551}$ $\boxed{7234, 907037}$ $\boxed{8816, 18291}$
$\boxed{20568, 610111}$ $\boxed{21585, 35182}$ $\boxed{25540, 993143}$ $\boxed{25653, 37722}$
$\boxed{32094, 575567}$ $\boxed{33676, 54105}$ $\boxed{34128, 743969}$ $\boxed{39552, 276101}$
$\boxed{47462, 304549}$ $\boxed{47914, 586997}$ $\boxed{66672, 695201}$ $\boxed{67915, 118748}$
$\boxed{71418, 416563}$ $\boxed{72435, 131956}$ $\boxed{81136, 403355}$ $\boxed{81362, 200409}$
$\boxed{85656, 217173}$ $\boxed{91758, 430025}$ $\boxed{92888, 276609}$;

$\boxed{123624, 864619}$ $\boxed{129274, 643385}$ $\boxed{130404, 597665}$ $\boxed{131421, 241684}$
$\boxed{135715, 199012}$ $\boxed{139218, 697487}$ $\boxed{143173, 271148}$ $\boxed{144755, 213744}$
$\boxed{146337, 193678}$ $\boxed{167355, 299596}$ $\boxed{184079, 212982}$ $\boxed{189051, 375542}$
$\boxed{194814, 828805}$ $\boxed{196735, 393322}$;

$\boxed{208035, 415928}$ $\boxed{208939, 273180}$ $\boxed{212216, 593601}$ $\boxed{229731, 381130}$
$\boxed{244647, 366906}$ $\boxed{246455, 347094}$ $\boxed{248376, 910085}$ $\boxed{259789, 426850}$
$\boxed{277078, 822201}$ $\boxed{277530, 443360}$ $\boxed{283745, 564518}$ $\boxed{293011, 405006}$;

$\boxed{352675, 503304}$ $\boxed{364427, 533784}$ $\boxed{370755, 627256}$ $\boxed{387592, 974855}$;

$\boxed{401491, 710314}$ $\boxed{430080, 851792}$ $\boxed{437425, 839854}$ $\boxed{440137, 702440}$
$\boxed{453019, 742318}$;

$\boxed{745915, 994540}$ $\boxed{754503, 838330}$.

7.6.5 The equation 2190

The η–φ–Diophantine equation $\eta(x) \cdot y = x \cdot \varphi(y)$ can be assimilated to relation $\eta_x \cdot y = x \cdot \varphi_y$, taking into account the significance of vectors η and φ. On the search domain $\{1, 2, \ldots, 10^6\}^2$ the equation has many solutions. Thus, we will consider a restricted search domain. Let us consider the

search domain (7.18) with $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$, $b_y = 10^4$ and additional condition $\gcd(x, y) = 1$ (i.e. x and y are relatively prime), then we have 99970002 possible cases, 60769973 analyzed cases and 0 solutions.

7.6.6   The equation 2191

The η–φ–Diophantine equation $\eta(x)/y = x/\varphi(y)$ is equivalent with relation $\eta_x \cdot \varphi_y = x \cdot y$ if we take into account the significance of vectors η and φ. The equation does not have solutions on the search domain D_c given by (7.18) with $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$ $b_y = 10^4$ and additional condition $(x, y) = 1$. The number of possible cases is 99970002 and the number of analyzed cases is 607699734, due to the additional condition.

7.6.7   The equation 2192

The η–φ–Diophantine equation $\eta(x)^y = x^{\varphi(y)}$ is equivalent with relation $(\eta_x)^y = x^{\varphi_y}$ if we take into account the significance of vectors η and φ. Let us consider the search domain D_c given by (7.18) with $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$ $b_y = 10^4$ and additional conditions $(x, y) = 1$, $(\eta_x)^y < 10^{17}$ and $x^{\varphi_y} < 10^{17}$. The number of possible cases is 99970002 and the number of analyzed cases is 534464, due to the additional conditions. The equation has 11 solutions given as $\boxed{x, y}$:

$$\boxed{8, 3}\ \boxed{8, 9}\ \boxed{8, 27}\ \boxed{8, 81}\ \boxed{8, 243}\ ,$$

$$\boxed{81, 4}\ \boxed{81, 8}\ \boxed{81, 16}\ \boxed{81, 32}\ \boxed{81, 64}\ \boxed{81, 128}\ .$$

7.6.8   The equation 2193

The η–φ–Diophantine equation $\eta(x)^y = \varphi(y)^x$ is equivalent with relation $(\eta_x)^y = (\varphi_y)^x$ if we take into account the significance of vectors η and φ. The equation does not have solutions on the search domain D_c given by (7.18) with $a_x = 2$, $r_x = 1$, $b_x = 10^4$, $a_y = 3$, $r_y = 1$ $b_y = 10^4$ and additional conditions $(x, y) = 1$, $(\eta_x)^y < 10^{17}$ and $(\varphi_y)^x < 10^{17}$. The number

of possible cases is 99970002 and the number of analyzed cases is of only 26936 cases, due to the additional conditions.

7.7 Guy type Diophantine equations

Guy [2004] considered the φ–σ–Diophantine equation $\varphi(\sigma(n)) = n$. F. Helenius determined 365 solutions. Similarly, the next Diophantine equations in which function η is involved were considered:

$$\eta(\varphi(x)) = x \tag{7.19}$$
$$\varphi(\eta(x)) = x \tag{7.20}$$
$$\eta(\varphi(x)) = \varphi(\eta(x)) \tag{7.21}$$
$$\eta(\sigma_0(x)) = x \tag{7.22}$$
$$\sigma_0(\eta(x)) = x \tag{7.23}$$
$$\eta(\sigma_0(x)) = \sigma_0(\eta(x)) \tag{7.24}$$
$$\eta(\sigma(x)) = x \tag{7.25}$$
$$\sigma(\eta(x)) = x \tag{7.26}$$
$$\eta(\sigma(x)) = \sigma(\eta(x)) \tag{7.27}$$
$$\eta(s(x)) = x \tag{7.28}$$
$$s(\eta(x)) = x \tag{7.29}$$
$$\eta(s(x)) = s(\eta(x)) \tag{7.30}$$
$$\eta(\pi(x)) = x \tag{7.31}$$
$$\pi(\eta(x)) = x \tag{7.32}$$
$$\eta(\pi(x)) = \pi(\eta(x)) \tag{7.33}$$

7.7.1 Partial empirical solving Guy type Diophantine equations

For solving these equations we will use the files $\eta.prn$, $\varphi.prn$, $\sigma0.prn$, $\sigma1.prn$ and $s.prn$ which were generated by programs 2.12, 4.3 and 3.4. To solve a Diophantine equation we will read the files as in the next sequence:

$$\eta := READPRN(\text{"}...\backslash\eta.prn\text{"}) \ \ last(\eta) = 10^6 \ .$$

Hence, we will have the vectors η, φ, $\sigma 0$, $\sigma 1$ and s with the values of the functions η, φ, σ_0, σ and s.

Let us consider the search domain

$$D_c = \left\{ 1, 2, \ldots, 10^6 \right\}. \tag{7.34}$$

Equations (7.19), (7.20), (7.25), (7.26) , (7.28) and (7.29) have a sole solution, the trivial solution $x = 1$, equations (7.22) and (7.23) have two trivial solutions $x = 1$ and $x = 2$. Equations (7.21), (7.24), (7.27) and (7.30) have more solutions.

7.7.2   The equation 7.21

The η–φ–Diophantine equation $\eta(\varphi(x)) = \varphi(\eta(x))$ is equivalent with relation $\eta_{\varphi_x} = \varphi_{\eta_x}$ if we take into account the significance of vectors η and φ. In the search domain (7.34) equation (7.21) has 842 solutions. We give the first 60 solutions and the last 60 solutions.

1, 2, 3, 4, 5, 6, 10, 15, 20, 27, 30, 54, 63, 105, 108, 112, 126, 135, 140, 168, 210, 216, 252, 270, 275, 315, 432, 504, 540, 550, 630, 825, 1100 1650 1925, 2200, 2475, 2783, 2816, 3125, 3159, 3300, 3328, 3520, 3850, 4160, 4224, 4400, 4950, 4992, 5280, 5566, 5775, 6240, 6250, 6318, 6600, 6656, 7371, 7425, ...

864864, 866320, 868296, 868725, 870205, 875160, 876645, 881280, 884000, 886464, 890560, 891072, 893142, 895068, 897600, 898909, 900000, 900315, 901689, 901692, 904475, 904932, 905177, 914166, 918750, 919931, 926640, 928200, 933504, 934375 935088, 940032, 941868, 942480, 942761, 943488, 944794, 946220, 950000, 951786, 952000, 954569, 954720, 956250, 959616, 960336, 969570, 969657, 972400, 974050, 975000, 976661, 976833, 979200, 980343, 982800, 990080, 992380, 993531, 994520 .

7.7.3   The equation 7.24

The Diophantine equation $\eta(\sigma_0(x)) = \sigma_0(\eta(x))$ has 82655 solutions. In order to solve empirically the equation, the equivalent relation $\eta_{\sigma 0_x} = \sigma 0_{\eta_x}$ is used , taking into account the significance of vectors η and $\sigma 0$. 120 solutins are listed, the first 60 solutions and the last 60 solutions.

1, 2, 3, 4, 5, 7, 11, 12, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 72,
73, 79, 83, 89, 90, 96, 97, 101, 103, 107, 109, 113, 125, 127, 128, 131, 137, 139,
149, 150, 151, 157, 160, 163, 167, 173, 179, 181, 191, 193, 197, 199, 200, 211,
223, 224, . . .
999101, 999133, 999149, 999169, 999181, 999199, 999217, 999221, 999233,
999239, 999269, 999287, 999307, 999329, 999331, 999359, 999370, 999371,
999377, 999389, 999431, 999433, 999437, 999451, 999491, 999499, 999521,
999529, 999541, 999553 999563, 999599, 999611, 999613, 999623, 999631,
999653, 999667, 999671, 999683, 999721, 999727, 999749, 999763, 999769,
999773, 999809, 999853, 999863, 999883, 999907, 999917, 999931, 999949,
999953, 999959, 999961, 999979, 999983, 999998 .

7.7.4 The equation 7.27

The Diophantine equation $\eta(\sigma(x)) = \sigma(\eta(x))$ has 648 solutions. To
solve empirically the equation, the equivalent relation is used $\eta_{\sigma 1_x} = \sigma 1_{\eta_x}$,
if we take into account the significance of vectors η and $\sigma 1$. The first 60
solutions were listed:
1, 2, 3, 4, 6, 10, 12, 21, 30, 40, 42, 52, 84, 105, 120, 156, 168, 210, 260, 364, 416,
420, 468, 572, 780, 840, 976, 1092, 1248, 1404, 1525, 1716, 1813, 1820 2080,
2340, 2860, 2912, 2928, 3050, 3125, 3159, 3276, 3626, 3744, 4004, 4477, 4575,
4576, 4880, 5148, 5439, 5460, 6100, 6240, 6250, 6318, 6832, 7020, 7252, . . .
as well as the last 60 solutions:
420616, 425315, 425475, 426512, 436150, 437675, 440176, 440559, 446875,
447811, 452925, 455975, 458689, 459025, 459375, 462315, 470085, 473193,
478125, 486475, 492575, 498575, 501725, 503125, 505827, 507825, 514855,
520025, 523075, 523809 531471, 532763, 542087, 559625, 565775, 571875,
574925, 578347, 581371, 584375, 585599, 589225, 595441, 596275, 614575,
618233, 620675, 629825, 635221, 649165, 653125, 666425, 683501, 687775,
690625, 693935, 708883, 718153, 720797, 730639 .

x	$s(x)$	$\eta(x)$	$s(\eta(x))$	$\eta(s(x))$
4	3	4	3	3
64	63	8	7	7
90	144	6	6	6
224	280	8	7	7
441	300	14	10	10
5145	4455	21	11	11
71148	141120	22	14	14
166012	206388	22	14	14

Table 7.1: The check of the solutions of equation 7.30

7.7.5   The equation 7.30

The Diophantine equation $\eta(s(x)) = s(\eta(x))$ is equivalent with relation

$$\eta_{s_x} = s_{\eta_x} \, , \tag{7.35}$$

if we take into account the significance of vectors η and s. We consider the search domain D_c given by (7.34). The equation has following solutions: the 78498 prime numbers $< 10^6$ and 8 solutions non-prime numbers: 4, 64, 90, 224, 441, 5145, 71148, 166012 . The check of the non-prime solutions is presented in table 7.1.

7.7.6   The equations 7.31–7.32

The η–π–Diophantine equations $\eta(\pi(x)) = x$ and $\pi(\eta(x)) = x$ are equivalent with the relations $\eta_{\pi(x)} = x$ and $\pi(\eta_x) = x$ if we take into account the significance of vector η and formula (1.19). These relations do not have solutions on the search domain $D_c = \{4, 5, \ldots, 10^3\}$.

7.7.7   The equation 7.33

The η–π–Diophantine equation $\eta(\pi(x)) = \pi(\eta(x))$ is equivalent with the relation

$$\eta_{\pi(x)} = \pi(\eta_x) \, , \tag{7.36}$$

x	$\pi(x)$	$\eta(x)$	$\eta(\pi(x))$	$\pi(\eta(x))$
4	2	4	2	2
15	6	5	3	3
16	6	6	3	3
21	8	7	4	4
26	9	13	6	6
65	18	13	6	6
96	24	8	4	4
133	32	19	8	8
156	36	13	6	6
176	40	11	5	5
187	42	17	7	7
232	50	29	10	10
236	51	59	17	17
253	54	23	9	9
364	72	13	6	6
416	80	13	6	6
527	99	31	11	11
598	108	23	9	9
660	120	11	5	5
726	128	22	8	8
738	130	41	13	13
744	132	31	11	11
870	150	29	10	10
885	153	59	17	17
899	154	31	11	11
966	162	23	9	9

Table 7.2: The check of the solutions of equation 7.33

if we take into account the significance of vector η and formula (1.19). Let $D_c = \left\{4, 5, \ldots, 10^3\right\}$ be the search domain, then relation (7.36) has 26 solutions: 4, 15, 16, 21, 26, 65, 96, 133, 156, 176, 187, 232, 236, 253, 364, 416, 527, 598, 660, 726, 738, 744, 870, 885, 899, 966 .

In table 7.2 the check of these solutions is presented.

Conclusions

We, the authors, hope that this book offers a valuable insight into a fascinating range of Number Theory problems. The approaches and algorithms proposed are not intended to just solve proposed problems, but also to inspire the reader to find better, more efficient or more beautiful solutions that further enrich our understanding of this field of mathematics.

The "partial results" of over 62 η–Diophantine equations presented in the last chapter could prove to be an excellent starting point for anyone motivated to explore more solutions. By simply running the proposed programs and algorithms on machines with better hardware capabilities, one can extend the set numbers that verify this equations. The mathematicians interested in analytical solutions are encouraged to explore Chapter 6 where, various analytical approaches into solving Diophantine equations are presented. At the same time, students or researchers unfamiliar with the details of such mathematical problems will discover, in the first five chapters some of the most important concepts, algorithms and tools to help them in their quest to learn about Diophantine equations.

The content of this book is a result of our collective mathematical and computing expertise. While the choice of the problems is a subjective one, our intent was to cover the most interesting Diophantine equations involving Smarandache's function η. We encourage anyone to approach us with comments and observation regarding the content of this book and also with other fresh problems related to it.

Indexes

Index of notations

$\mathbb{N} = \{0, 1, 2, \ldots\}$;

$\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$;

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$;

$I_s = \{1, 2, \ldots, s\}$: the set of indexes;

$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, \ q \neq 0 \right\}$;

$\mathbb{R}$: the real numbers;

$\pi(x)$: the number of prime numbers up to x;

$\pi_m(x)$: the function which is the lower bound of function $\pi(x)$;

$\pi_M(x)$: the function which is the upper bound of function $\pi(x)$;

$[x]$: the integer part of number x;

$\{x\}$: the fractional part of x;

$\sigma_k(n)$: the sum of the powers of order k of the divisors of n;

$\sigma(n)$: the sum of the divisors of n; $\sigma(n) = \sigma_1(n)$;

$s(n)$: the sum of the divisors of n without n; $s(n) = \sigma(n) - n$;

$\lfloor a \rfloor$: the lower integer part of a; the greatest integer, smaller than a;

$\lceil a \rceil$: the upper integer part of a; the smallest integer, greater than a;

$n \mid m : n$ divides m;

$n \nmid m : n$ does not divide m;

(m, n) : the greatest common divisor of m and n; $(m, n) = gcd(m, n)$;

$[m, n]$: the smallest common multiple of m and n; $[m, n] = lcd(m, n)$;

$\{n_1, n_2, \ldots, n_m\}^k$: the cartesian product k *times*

$$\{n_1, n_2, \ldots, n_m\}^k$$
$$= \underbrace{\{n_1, n_2, \ldots, n_m\} \times \{n_1, n_2, \ldots, n_m\} \times \ldots \times \{n_1, n_2, \ldots, n_m\}}_{k \ ori}$$
$$= \left\{ \left(\underbrace{n_1, n_1, \ldots, n_1}_{k \ elemente} \right), \left(\underbrace{n_1, n_1, \ldots, n_2}_{k \ elemente} \right), \ldots, \left(\underbrace{n_1, n_1, \ldots, n_m}_{k \ elemente} \right), \right.$$
$$\left(\underbrace{n_1, n_1, \ldots, n_2, n_1}_{k \ elemente} \right), \left(\underbrace{n_1, n_1, \ldots, n_2, n_2}_{k \ elemente} \right), \ldots$$
$$\left. \ldots, \left(\underbrace{n_m, n_m, \ldots, n_m}_{k \ elemente} \right) \right\} ;$$

$|f(x)|$: the module or the absolute value of $f(x)$;

$f(x) \equiv g(x)$: f is asymptotic to g if

$$\frac{f(x)}{g(x)} \to 1 \ \text{ when } \ x \to \infty ;$$

$f(x) = o\big(g(x)\big)$:

$$\frac{f(x)}{g(x)} \to 0 \ \text{ when } \ x \to \infty ;$$

$f(x) = O\big(g(x)\big)$: there exists a constant c such that $|f(x)| < c \cdot g(x)$, for all x; this property is also denoted $f(x) \ll g(x)$;

Mathcad functions

$augment(M, N)$: concatenates matrices M and N that have the same number of lines;

$ceil(x)$: the upper integer part function;

$cols(M)$: the number of columns of matrix M;

$eigenvals(M)$: the eigenvalues of matrix M;

$eigenvec(M, \lambda)$: the eigenvector of matrix M relative to the eigenvalue λ;

$eigenvecs(M)$: the matrix of the eigenvectors of matrix M;

$n\ factor \rightarrow$: symbolic computation function that factorizes n;

$floor(x)$: the lower integer part function;

$gcd(n_1, n_2, \ldots)$: the function which computes the greatest common divisor of $n_1, n_2, \ldots$;

$last(v)$: the last index of vector v;

$lcm(n_1, n_2, \ldots)$: the function which computes the smallest common multiple of $n_1, n_2, \ldots$;

$max(v)$: the maximum of vector v;

$min(v)$: the minimum of vector v;

$mod(m, n)$: the rest of the division of m by n;

$ORIGIN$: the variable dedicated to the origin of indexes, 0 being an implicit value;

$rref(M)$: determines the matrix *row-reduced echelon form*;

$rows(M)$: the number of lines of matrix M;

$solve$: the function of symbolic solving the equations;

$stack(M, N)$: concatenates matrices M and N that have the same care number of columns;

$submatrix(M, k_r, j_r, k_c, j_c)$: extracts from matrix M, from line k_r to line j_r and from column k_c to column j_c, a submatrix;

$trunc(x)$: the truncation function;

$\sum v$: the function that sums the components of vector v .

Index of name

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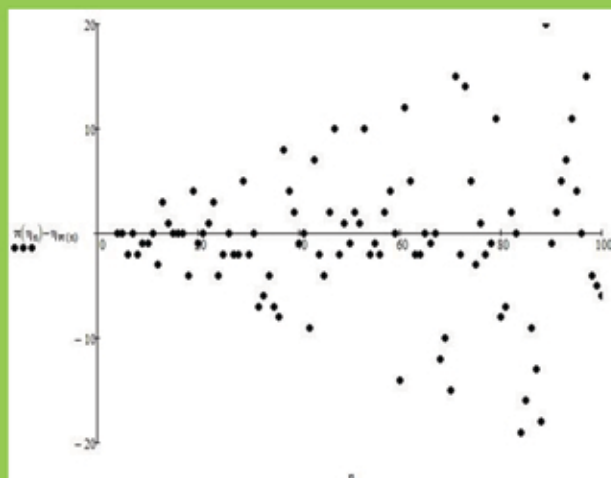
In this book a multitude of Diophantine equations and their partial or complete solutions are presented. How should we solve, for example, the equation $\eta(\pi(x)) = \pi(\eta(x))$, where η is the Smarandache function and π is Riemann function of counting the number of primes up to x, in the set of natural numbers?

If an analytical method is not available, an idea would be to recall the empirical search for solutions. We establish a domain of searching for the solutions and then we check all possible situations, and of course we retain among them only those solutions that verify our equation.

In other words, we say that the equation does not have solutions in the search domain, or the equation has n solutions in this domain. This mode of solving is called partial resolution. Partially solving a Diophantine equation may be a good start for a complete solving of the problem.

The authors have identified 62 Diophantine equations that impose such approach and they partially solved them. For an efficient resolution it was necessarily that they have constructed many useful "tools" for partially solving the Diophantine equations into a reasonable time.

The computer programs as tools were written in Mathcad, because this is a good mathematical software where many mathematical functions are implemented. Transposing the programs into another computer language is facile, and such algorithms can be turned to account on other calculation systems with various processors.