Lecture 15: Nonlinear adaptive systems

- Adaptive feedback linearization
- Adaptive backstepping design
- ~~Adaptive forwarding design~~

Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 + f(x_1) \\ \frac{d}{dt}x_2 &= u \end{aligned}$$

Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the nonlinear change of coordinates

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_2 + f(x_1) \end{aligned}$$

Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the change of coordinates as

$$\xi_1 = x_1, \qquad \xi_2 = x_2 + f(x_1)$$

Then

$$\begin{aligned} \frac{d}{dt}\xi_1 &= \xi_2 \\ \frac{d}{dt}\xi_2 &= \frac{d}{dt}x_2 + \frac{d}{dt}f(x_1) = u + \xi_2 f'(\xi_1) \end{aligned}$$

Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the nonlinear change of coordinates

$$\xi_1 = x_1, \qquad \xi_2 = x_2 + f(x_1)$$

Then

$$\frac{d}{dt}\xi_1 = \xi_2, \qquad \frac{d}{dt}\xi_2 = u + \xi_2 f'(\xi_1)$$

Making the feedback transform $(u \longrightarrow v)$

$$u = -a_2\xi_1 - a_1\xi_2 - \xi_2 f'(\xi_1) + v$$

we bring the system dynamics into the linear form

$$\frac{d}{dt}\xi_1 = \xi_2, \qquad \frac{d}{dt}\xi_2 = -a_2\xi_1 - a_1\xi_2 + v$$

Adaptive Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\begin{aligned}\frac{d}{dt}x_1 &= x_2 + \theta_0 \cdot f(x_1)\\ \frac{d}{dt}x_2 &= u\end{aligned}$$

Adaptive Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta_0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the nonlinear change of coordinates

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_2 + \theta \cdot f(x_1) \end{aligned}$$

Adaptive Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta_0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the nonlinear change of coordinates

$$\xi_1 = x_1, \qquad \xi_2 = x_2 + \theta \cdot f(x_1)$$

Then

$$\begin{aligned}
\frac{d}{dt}\xi_1 &= x_2 + \theta_0 \cdot f(x_1) = \xi_2 + (\theta_0 - \theta) f(\xi_1) \\
\frac{d}{dt}\xi_2 &= \frac{d}{dt}x_2 + \frac{d}{dt}\left[\theta \cdot f(x_1)\right] = u + \frac{d}{dt}[\theta] \cdot f(x_1) + \theta \cdot \frac{d}{dt} f(x_1) \\
&= u + \frac{d}{dt}[\theta] \cdot f(x_1) + \theta \cdot f'(x_1)\left[x_2 + \theta_0 \cdot f(x_1)\right]
\end{aligned}$$

Adaptive Feedback Linearization

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta_0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Introduce the nonlinear change of coordinates

$$\xi_1 = x_1, \qquad \xi_2 = x_2 + \theta \cdot f(x_1)$$

Then

$$\begin{aligned} \frac{d}{dt}\xi_1 &= \xi_2 + (\theta_0 - \theta) f(\xi_1) \\ \frac{d}{dt}\xi_2 &= u + \frac{d}{dt}[\theta] \cdot f(x_1) + \theta \cdot f'(x_1)\,[x_2 + \theta_0 \cdot f(x_1)] \end{aligned}$$

Introduce the feedback transform $(u \longrightarrow v)$

$$u = -a_2\xi_1 - a_1\xi_2 - \frac{d}{dt}[\theta] \cdot f(x_1) - \theta \cdot f'(x_1)\,[x_2 + \theta \cdot f(x_1)] + v$$

Adaptive Feedback Linearization

The system dynamics before feedback transform

$$
\begin{aligned}
\frac{d}{dt}\xi_1 &= \xi_2 + (\theta_0 - \theta) f(\xi_1) \\
\frac{d}{dt}\xi_2 &= u + \frac{d}{dt}[\theta] \cdot f(x_1) + \theta \cdot f'(x_1)\,[x_2 + \theta_0 \cdot f(x_1)]
\end{aligned}
$$

Introduce the feedback transform ($u \longrightarrow v$)

$$
u = -a_2\xi_1 - a_1\xi_2 - \frac{d}{dt}[\theta] \cdot f(x_1) - \theta \cdot f'(x_1)\,[x_2 + \theta \cdot f(x_1)] + v
$$

The system dynamics after the feedback transform is

$$
\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} f(\xi_1) \\ \theta f'(x_1) f(x_1) \end{bmatrix} (\theta_0 - \theta) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v
$$

Adaptive Feedback Linearization (Cont'd)

Choose the target dynamics for

$$\frac{d}{dt}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}+\begin{bmatrix}f(\xi_1)\\ \theta f'(x_1)f(x_1)\end{bmatrix}(\theta_0-\theta)+\begin{bmatrix}0\\ 1\end{bmatrix}v$$

as

$$\frac{d}{dt}\begin{bmatrix}x_{m1}\\ x_{2m}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_{m1}\\ x_{2m}\end{bmatrix}+\begin{bmatrix}0\\ a_2\end{bmatrix}v_m$$

Adaptive Feedback Linearization (Cont'd)

Choose the target dynamics for

$$\frac{d}{dt}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}f(\xi_1)\\ \theta f'(x_1)f(x_1)\end{bmatrix}(\theta_0-\theta) + \begin{bmatrix}0\\ 1\end{bmatrix}v$$

as

$$\frac{d}{dt}\begin{bmatrix}x_{m1}\\ x_{2m}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_{m1}\\ x_{2m}\end{bmatrix} + \begin{bmatrix}0\\ a_2\end{bmatrix}v_m$$

Then

$$v = a_2 v_m$$

Adaptive Feedback Linearization (Cont'd)

Introducing the error signal

$$e(t) = \xi(t) - x_m(t)$$

we get the error dynamics

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \underbrace{\begin{bmatrix} f(\xi_1) \\ \theta f'(x_1) f(x_1) \end{bmatrix}}_{B} (\theta_0 - \theta)$$

It is almost as we have before, but B is dependent on states!

The possible choice for θ-dynamics is

$$\frac{d}{dt}\theta = \gamma B(\cdot)^T P e$$

where $\gamma > 0$ and P is a solution for: $A^T P + P A < 0$

Backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 + f(x_1) \\ \frac{d}{dt}x_2 &= u \end{aligned}$$

Backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Treat x_2 as a virtual control signal v_1 for x_1 in the 1st equation

$$\dot{x}_1 = v_1 + f(x_1), \qquad x_2 = v_1$$

Backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Treat x_2 as a virtual control signal v_1 for x_1 in the 1st equation

$$\dot{x}_1 = v_1 + f(x_1), \qquad x_2 = v_1$$

Use Lyapunov-based design with

$$V_1(x_1) = \frac{1}{2}\,x_1^2 \qquad \Rightarrow \qquad \dot{V}_1 = x_1(v_1 + f(x_1)) < 0$$

so that we can take with $c_1 > 0$

$$v_1 = -f(x_1) - c_1\,x_1 \qquad \Rightarrow \qquad \dot{V}_1 = -c_1\,x_1^2 < 0$$

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$$v_1 = -f(x_1) - c_1\,x_1 \qquad \Rightarrow \qquad \dot{V}_1 = -c_1\,x_1^2 < 0$$

Define $\alpha_1(x_1) = v_1$ to be the desired law of change for x_2.

Backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Define $\alpha_1(x_1)$ to be the desired law of change of the variable x_2

$$\alpha_1(x_1) = -f(x_1) - c_1\, x_1$$

and introduce the new variable z_2 to measure the difference:

$$\frac{d}{dt}x_1 = \underbrace{f(x_1) + \alpha_1(x_1)}_{-c_1\, x_1} + \underbrace{(x_2 - \alpha_1(x_1))}_{z_2}, \qquad \frac{d}{dt}x_2 = u$$

Backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

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$$\alpha_1(x_1) = -f(x_1) - c_1\,x_1$$

and introduce the new variable $z_2 = x_2 - \alpha_1(x_1)$:

$$\frac{d}{dt}x_1 = -c_1\,x_1 + z_2, \qquad \frac{d}{dt}z_2 = u - \alpha_1'(x_1)\left(\frac{d}{dt}x_1\right)$$

Backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

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$$\frac{d}{dt}x_1 = -c_1\, x_1 + z_2, \qquad \frac{d}{dt}z_2 = u - \alpha_1'(x_1)(-c_1\, x_1 + z_2)$$

Backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + f(x_1), \qquad \frac{d}{dt}x_2 = u$$

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$$\frac{d}{dt}x_1 = -c_1\, x_1 + z_2, \qquad \frac{d}{dt}z_2 = u - \alpha_1'(x_1)(-c_1\, x_1 + z_2)$$

Use Lyapunov-based design with

$$V_2 = V_1(x_1) + \frac{1}{2}\, z_2^2 \quad \Rightarrow \quad \dot{V}_2 = -c_1\, x_1^2 + x_1\, z_2 + z_2\left(\frac{d}{dt}z_2\right) < 0$$

so that we can take with $c_2 > 0$

$$u = \alpha_2(x_1, z_2) \qquad \Rightarrow \qquad \dot{V}_2 = -c_1\, x_1^2 - c_2\, z_2^2 < 0$$

where $\alpha_2(x_1, z_2) = \alpha_1'(x_1)(-c_1\, x_1 + z_2) - x_1 - c_2\, z_2$.

Adaptive backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$
\begin{aligned}
\frac{d}{dt}x_1 &= x_2 + \theta^0 \cdot f(x_1) \\
\frac{d}{dt}x_2 &= u
\end{aligned}
$$

Adaptive backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Treat x_2 as a virtual control signal v_1 for x_1 in the 1st equation

$$\dot{x}_1 = v_1 + \theta^0 \cdot f(x_1), \qquad x_2 = v_1$$

Adaptive backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Treat x_2 as a virtual control signal v_1 for x_1 in the 1st equation

$$\dot{x}_1 = v_1 + \theta^0 \cdot f(x_1), \qquad x_2 = v_1$$

Use Lyapunov-based design with $\tilde{\theta} = \theta - \theta^0$

$$V_1(x_1, \tilde{\theta}) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2\right) \Rightarrow \dot{V}_1 = x_1\left(v_1 + \theta^0 \cdot f(x_1)\right) + \frac{1}{\gamma}\tilde{\theta}\dot{\theta} \leq 0$$

so that we can take with $c_1 > 0$ and $\gamma > 0$

$$v_1 = -\theta \cdot f(x_1) - c_1\, x_1, \quad \dot{\theta} = \gamma\, x_1\, f(x_1) \quad \Rightarrow \quad \dot{V}_1 = -c_1\, x_1^2 \leq 0$$

Adaptive backstepping design

Consider the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Treat x_2 as a virtual control signal v_1 for x_1 in the 1st equation

$$\dot{x}_1 = v_1 + f(x_1), \qquad x_2 = v_1$$

Use Lyapunov-based design with $\tilde{\theta} = \theta - \theta^0$

$$V_1(x_1, \tilde{\theta}) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2\right) \Rightarrow \dot{V}_1 = x_1(v_1 + \theta^0 \cdot f(x_1)) + \frac{1}{\gamma}\tilde{\theta}\dot{\theta} \le 0$$

so that we can take with $c_1 > 0$ and $\gamma > 0$

$$v_1 = -\theta \cdot f(x_1) - c_1 x_1, \quad \dot{\theta} = \gamma x_1 f(x_1) \quad \Rightarrow \quad \dot{V}_1 = -c_1 x_1^2 \le 0$$

Define $\alpha_1(x_1, \theta) = v_1$ to be the desired law of change for x_2.

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Define $\alpha_1(x_1, \theta)$ to be the desired law of change of x_2

$$\alpha_1(x_1, \theta) = -\theta\, f(x_1) - c_1\, x_1$$

and introduce the new variable z_2 to measure the difference:

$$\frac{d}{dt}x_1 = \underbrace{\theta^0 \cdot f(x_1) + \alpha_1(x_1, \theta)}_{-c_1\, x_1 - \tilde{\theta}\, f(x_1)} + \underbrace{(x_2 - \alpha_1(x_1, \theta))}_{z_2}, \qquad \frac{d}{dt}x_2 = u$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Define $\alpha_1(x_1, \theta)$ to be the desired law of change of x_2

$$\alpha_1(x_1, \theta) = -\theta\, f(x_1) - c_1\, x_1$$

and introduce the new variable $z_2 = x_2 - \alpha_1(x_1, \theta)$:

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \qquad \frac{d}{dt}z_2 = u - \frac{\partial \alpha_1}{\partial x_1}\left(\frac{d}{dt}x_1\right) - \frac{\partial \alpha_1}{\partial \theta}\,\dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Define $\alpha_1(x_1, \theta)$ to be the desired law of change of x_2

$$\alpha_1(x_1, \theta) = -\theta\, f(x_1) - c_1\, x_1$$

and introduce the new variable $z_2 = x_2 - \alpha_1(x_1, \theta)$:

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \qquad \frac{d}{dt}z_2 = u - \frac{\partial \alpha_1}{\partial x_1}\left(\frac{d}{dt}x_1\right) + f(x_1)\, \dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

Define $\alpha_1(x_1, \theta)$ to be the desired law of change of x_2

$$\alpha_1(x_1, \theta) = -\theta\, f(x_1) - c_1\, x_1$$

and introduce the new variable $z_2 = x_2 - \alpha_1(x_1, \theta)$:

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \qquad \frac{d}{dt}z_2 = u - \frac{\partial \alpha_1}{\partial x_1}\left(\frac{d}{dt}x_1\right) + f(x_1)\, \dot{\theta}$$

We will use Lyapunov-based design with

$$V_2(x_1, z_2, \theta) = V_1(x_1, \theta) + \frac{1}{2}\, z_2^2 = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2 + z_2^2\right)$$

achieving with $c_2 > 0$

$$\dot{V}_2 = -c_1\, x_1^2 - c_2\, z_2^2 \leq 0$$

with $u = \alpha_2(x_1, z_2, \theta)$.

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt} x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt} x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Dynamics can be rewritten as

$$\frac{d}{dt} x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2,$$

$$\frac{d}{dt} z_2 = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2) + f(x_1)\,\dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Dynamics can be rewritten as

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \quad \frac{d}{dt}z_2 = v - (\theta\, f'(x_1) + c_1)\, \tilde{\theta}\, f(x_1)$$

$$v = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 + z_2) + f(x_1)\, \dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Dynamics can be rewritten as

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \quad \frac{d}{dt}z_2 = v - (\theta\, f'(x_1) + c_1)\, \tilde{\theta}\, f(x_1)$$

$$v = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 + z_2) + f(x_1)\, \dot{\theta}$$

Derivative of the Lyapunov function

$$V_2(x_1, z_2, \theta) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2 + z_2^2\right)$$

is given by

$$\dot{V}_2 = x_1 \frac{d}{dt}x_1 + z_2 \frac{d}{dt}z_2 + \frac{1}{\gamma}\tilde{\theta}\dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Dynamics can be rewritten as

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \quad \frac{d}{dt}z_2 = v - (\theta\, f'(x_1) + c_1)\, \tilde{\theta}\, f(x_1)$$

$$v = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 + z_2) + f(x_1)\,\dot{\theta}$$

Derivative of the Lyapunov function

$$V_2(x_1, z_2, \theta) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2 + z_2^2\right)$$

is given by

$$\dot{V}_2 = x_1(-c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2) + z_2(v - (\theta\, f'(x_1) + c_1)\, \tilde{\theta}\, f(x_1)) + \frac{1}{\gamma}\tilde{\theta}\dot{\theta}$$

Adaptive backstepping design (Cont'd)

For the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

introduce new state variables z_2 and θ

$$z_2 = x_2 - \alpha_1(x_1, \theta) = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Dynamics can be rewritten as

$$\frac{d}{dt}x_1 = -c_1\, x_1 - \tilde{\theta}\, f(x_1) + z_2, \quad \frac{d}{dt}z_2 = v - (\theta\, f'(x_1) + c_1)\, \tilde{\theta}\, f(x_1)$$

$$v = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 + z_2) + f(x_1)\, \dot{\theta}$$

Derivative of the Lyapunov function

$$V_2(x_1, z_2, \theta) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma}\tilde{\theta}^2 + z_2^2\right)$$

is given by

$$\dot{V}_2 = -c_1\, x_1^2 - c_2\, z_2^2 + z_2\,(v + x_1 + c_2\, z_2) + \tilde{\theta}(\cdots + \frac{1}{\gamma}\dot{\theta})$$

Adaptive backstepping design (Cont'd)

Finally, for the nonlinear system with $f(\cdot) \in C^1$

$$\frac{d}{dt}x_1 = x_2 + \theta^0 \cdot f(x_1), \qquad \frac{d}{dt}x_2 = u$$

and

$$z_2 = x_2 + \theta\, f(x_1) + c_1\, x_1$$

Adaptive backstepping design (Cont'd)

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provided

$$-x_1 - c_2\, z_2 = u + (\theta\, f'(x_1) + c_1)\,(-c_1\, x_1 + z_2) + f(x_1)\,\dot{\theta}$$

and

$$\dot{\theta} = \gamma\left(x_1 + z_2\,(\theta\, f'(x_1) + c_1)\right) f(x_1)$$

Next Lecture / Assignments:

Last meetings: May 23, 13:00-15:00, in A206Tekn – Recitations;
June 9, 10:00-12:00, in A206Tekn: – Project reports / EXAM

Homework problem: Consider the nonlinear system

$$\dot{x}_1 = x_2 + a\, x_1^2, \qquad \dot{x}_2 = b\, x_1\, x_2 + u$$

where a and b are unknown parameters.

- Design an adaptive state feedback controller using the backstepping method.
- Argue why $x_1(t) \to 0$ and $x_2(t) \to 0$.
- Simulate the closed-loop system with $a = b = 1$.
- How to modify the design to ensure that the output $y = x_1$ follows the model
$$G_m(s) = \frac{1}{s^2 + 1.4\, s + 1}$$
driven by a unit step signal $u_c(t)$?