Lecture 4: Real-Time Parameter Estimation

- Least Squares and Recursive Computations
- Estimating Parameters in Dynamical Systems
- Experimental Conditions
- Examples

Theorem (Recursive Least Squares):

Assume that

$$\boxed{y(t) = \phi(t-1)^T\,\theta^{\mathbf{0}}} \qquad \text{and} \qquad \boldsymbol{\Phi}(t)^T\,\boldsymbol{\Phi}(t) > \mathbf{0}, \qquad \text{for} \quad t \geq t_{\mathbf{0}}$$

Theorem (Recursive Least Squares):

Assume that

$$\boxed{y(t) = \phi(t-1)^T \theta^0} \qquad \text{and} \qquad \Phi(t)^T \Phi(t) > 0, \qquad \text{for} \quad t \geq t_0$$

With any $\hat{\theta}(t_0)$, $P(t_0) = P_0 = P_0^T > 0$, the recursive procedure

$$\boxed{\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\left(y(t) - \phi(t-1)^T \hat{\theta}(t-1)\right)}$$

$$\boxed{K(t) = P(t-1)\phi(t-1) / \left(1 + \phi(t-1)^T P(t-1)\phi(t-1)\right)}$$

$$\boxed{P(t) = \left(I_m - K(t)\,\phi(t-1)^T\right) P(t-1)}$$

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Assume that

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$$\boxed{P(t) = \left(I_m - K(t)\,\phi(t-1)^T\right)\,P(t-1)}$$

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results in parameter convergence:

$$\lim_{t\to\infty}\left(\hat{\theta}(t) - \theta^0\right) = 0.$$

Informal understanding of excitation?

What can go wrong if

$$\underbrace{\Phi(t)^T \, \Phi(t)}_{n \times n} = \sum_{i=1}^{t} \phi(i-1)\, \phi(i-1)^T \not> 0, \qquad \text{for} \quad t \geq t_0 \quad ?$$

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Suppose

$$\lim_{t\to\infty} \left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\qquad \theta_\infty \neq 0 \quad ?$

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$$\lim_{t\to\infty} \left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\theta_\infty \neq 0$?

Since

$$\hat{\theta}(t) - \hat{\theta}(t-1) = P(t)\phi(t-1)\left[y(t) - \phi(t-1)^T\hat{\theta}(t-1)\right]$$

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Suppose

$$\lim_{t\to\infty}\left(\hat\theta(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\qquad \theta_\infty \neq 0 \quad ?$

Taking the limit, we obtain

$$\lim_{t\to\infty}\left(\hat\theta(t) - \hat\theta(t-1)\right) = \lim_{t\to\infty}\left(P(t)\phi(t-1)\left[y(t) - \phi(t-1)^T\hat\theta(t-1)\right]\right)$$

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Suppose

$$\lim_{t\to\infty}\left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\quad \theta_\infty \neq 0 \quad$?

Since $\quad \lim_{t\to\infty}\left(\hat{\theta}(t) - \hat{\theta}(t-1)\right) = 0,$

$$0 = \lim_{t\to\infty}\left(P(t)\phi(t-1)\left[y(t) - \phi(t-1)^T\hat{\theta}(t-1)\right]\right)$$

Informal understanding of excitation?

What can go wrong if

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Suppose

$$\lim_{t\to\infty} \left(\hat{\theta}(t) - \theta^0 \right) = \theta_\infty$$

Can it happen that $\quad \theta_\infty \neq 0 \quad$?

Since $\quad P(t) > 0$,

$$0 = \lim_{t\to\infty} \left(\phi(t-1) \left[y(t) - \phi(t-1)^T \hat{\theta}(t-1) \right] \right)$$

Informal understanding of excitation?

What can go wrong if

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Suppose

$$\lim_{t\to\infty}\left(\hat\theta(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\qquad \theta_\infty \neq 0 \quad$?

Substituting $\quad y(t) = \phi(t-1)^T\theta^0,$

$$0 = \lim_{t\to\infty}\left(\phi(t-1)\phi(t-1)^T\left[\theta^0 - \hat\theta(t-1)\right]\right)$$

Informal understanding of excitation?

What can go wrong if

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Suppose

$$\lim_{t\to\infty} \left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\quad \theta_\infty \neq 0 \quad$?

Suppose $\quad n = 2, \quad \lim_{t\to\infty} \left(\phi(t-1)^T\right) = [1, \quad 0],$

$$0 = \lim_{t\to\infty} \left(\phi(t-1)\phi(t-1)^T \left[\theta^0 - \hat{\theta}(t-1)\right]\right)$$

Informal understanding of excitation?

What can go wrong if

$$\underbrace{\Phi(t)^T \Phi(t)}_{n \times n} = \sum_{i=1}^{t} \phi(i-1)\,\phi(i-1)^T \not> 0, \qquad \text{for} \quad t \geq t_0 \quad ?$$

Suppose

$$\lim_{t\to\infty} \left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\theta_\infty \neq 0$?

Suppose $n = 2$, $\lim_{t\to\infty} \left(\phi(t-1)^T\right) = [1, \quad 0]$, then

$$0 = \lim_{t\to\infty} \left([1, \quad 0]^T [1, \quad 0] \left[\theta^0 - \hat{\theta}(t-1)\right]\right)$$

Informal understanding of excitation?

What can go wrong if

$$\underbrace{\Phi(t)^T\,\Phi(t)}_{n\times n} = \sum_{i=1}^{t} \phi(i-1)\,\phi(i-1)^T \not> 0, \qquad \text{for} \quad t \geq t_0 \quad ?$$

Suppose

$$\lim_{t\to\infty}\left(\hat{\theta}(t) - \theta^0\right) = \theta_\infty$$

Can it happen that $\qquad \theta_\infty \neq 0 \quad ?$

And finally, estimate of the second parameter can be arbitrary!

$$[1, \quad 0]\left[\theta^0 - \theta_\infty\right] = 0$$

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the iterative least square algorithm results in

(i) $P(t)^{-1}\left(\hat{\theta}(t) - \theta^0\right)$ is constant,

(ii) $P(t)$ converges to some matrix P_∞,

(iii) $\hat{\theta}(t)$ converges to some vector θ_∞

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Suppose $$\frac{1}{t}\sum_{i=1}^{t}\phi(i-1)\phi^T(i-1)$$

converges to a positive definite matrix.

Then, $\theta_\infty = 0$, i.e. the estimation error vanishes!

Estimating Parameters in Dynamical Systems

Development up to now was dedicated to recovering LS estimate $\hat{\theta}$ from data generated by the model

$$y(t) = \phi(t)^T \theta^0 + e(t), \quad k = 1, 2, \ldots$$

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Models for control systems are given in different forms

$$y(t) \;=\; b_1 u(t-1) + \cdots + b_m u(t-m) + e(t)$$

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) \;=\; b_1 u(t-1) + \cdots + b_m u(t-m) + e(t)$$

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) \;=\; b_1 u(t-1) + \cdots + b_m u(t-m) + e(t) + c_1 e(t-1) + \cdots + c_p e(t-p)$$

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We need to know how to transform, if possible,

such models into the standard form

Estimating Parameters for FIR Models

Finite Impulse Response model

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \cdots + b_m u(t-m) + e(t)$$

can be immediately rewritten in the standard form

$$y(t) = \phi(t-1)^T \theta^0 + e(t)$$

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can be immediately rewritten in the standard form

$$y(t) = \phi(t-1)^T \theta^0 + e(t)$$

Here

$$\begin{aligned} \phi(t-1)^T &= \Big[u(t-1),\, u(t-2),\, \ldots,\, u(t-m) \Big] \\ \theta^0 &= \Big[b_1,\, b_2,\, \ldots,\, b_m \Big]^T \end{aligned}$$

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FIR models can be used to approximate a class of systems!

But a number of parameters can be large!

Estimating Parameters for ARX Models

Auto-Regressive models with eXternal input

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) =$$

$$= b_1 u(t-1) + \cdots + b_m u(t-m) + e(t)$$

can be also rewritten in the standard form

$$y(t) = \phi(t-1)^T \theta^0 + e(t)$$

Estimating Parameters for ARX Models

Auto-Regressive models with eXternal input

$$
\begin{aligned}
y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) &= \\
&= b_1 u(t-1) + \cdots + b_m u(t-m) + e(t)
\end{aligned}
$$

can be also rewritten in the standard form

$$y(t) = \phi(t-1)^T \theta^0 + e(t)$$

Here

$$
\begin{aligned}
\phi(t-1)^T &= \Big[-y(t-1),\, -y(t-2),\, \ldots,\, -y(t-n), \\
&\qquad u(t-1),\, u(t-2),\, \ldots,\, u(t-m) \Big] \\
\theta^0 &= \Big[a_1,\, a_2,\, \ldots,\, a_n,\, b_1,\, b_2,\, \ldots,\, b_m \Big]^T
\end{aligned}
$$

Estimating Parameters for ARMAX Models

Auto-Regressive model with Moving Average and eXternal input

$$
\begin{aligned}
y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) &= \\
= \; & b_1 u(t-1) + \cdots + b_m u(t-m) + \\
& + e(t) + c_1 e(t-1) + \cdots + c_p e(t-p)
\end{aligned}
$$

cannot be transformed into the standard form!

Estimating Parameters for ARMAX Models

Auto-Regressive model with Moving Average and eXternal input

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) =$$

$$= b_1 u(t-1) + \cdots + b_m u(t-m) +$$

$$+ e(t) + c_1 e(t-1) + \cdots + c_p e(t-p)$$

cannot be transformed into the standard form!

Indeed, the signals

$$e(t-1),\ e(t-2),\ \ldots,\ e(t-p)$$

are not measured, and cannot be included in the vector

$$\phi(t-1)^T = \Big[-y(t-1), -y(t-2), \ldots, -y(t-n),$$

$$u(t-1), u(t-2), \ldots, u(t-m), e(t-1), e(t-2), \ldots, e(t-m) \Big]$$

Estimating Parameters for ARMAX Models

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$$u(t-1), u(t-2), \ldots, u(t-m), e(t-1), e(t-2), \ldots, e(t-m) \Big]$$

$$\theta^0 = \Big[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_m \Big]^T$$

Estimating Parameters for ARMAX Models

The way to bring the ARMAX model into the form similar to

$$y(t) = \phi(t)^T \theta^0 + e(t)$$

is to introduce estimates for unmeasured signals

$$e(t-1),\ e(t-2),\ \ldots,\ e(t-p)$$

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Consider the system

$$y(t) \quad = \quad b_1 u(t-1) + e(t) + c_1 e(t-1)$$

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$$\begin{aligned} y(t) &= b_1 u(t-1) + e(t) + c_1 e(t-1) \\ &= \Big[u(t-1),\ {\color{red} e(t-1)}\Big] \begin{bmatrix} b_1 \\ c_1 \end{bmatrix} + e(t) \end{aligned}$$

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$$\begin{aligned} e(t-1) &= y(t-1) - \{b_1 u(t-2) + c_1 e(t-2)\} \\ e(t-2) &= y(t-2) - \{b_1 u(t-3) + c_1 e(t-3)\} \end{aligned}$$

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$$\begin{aligned} e(t-1) &= y(t-1) - \{ b_1 u(t-2) + c_1 e(t-2) \} \\ e(t-2) &= y(t-2) - \{ b_1 u(t-3) + c_1 e(t-3) \} \\ e(t-3) &\approx 0 \end{aligned}$$

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The way to bring the ARMAX model into the form similar to

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$$\begin{aligned} e(t-1) &= y(t-1) - \{b_1 u(t-2) + c_1 e(t-2)\} \\ e(t-2) &\approx y(t-2) - \{b_1 u(t-3) + c_1 0\} \\ e(t-3) &\approx 0 \end{aligned}$$

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$$\begin{aligned} e(t-1) &= y(t-1) - \Big\{ b_1 u(t-2) + c_1 \Big(y(t-2) - b_1 u(t-3) \Big) \Big\} \\ e(t-2) &\approx y(t-2) - \{ b_1 u(t-3) + c_1 0 \} \\ e(t-3) &\approx 0 \end{aligned}$$

Estimating Parameters for ARMAX Models

The way to bring the ARMAX model into the form similar to

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Consider the system

$$\begin{aligned} y(t) &= b_1 u(t-1) + e(t) + c_1 e(t-1) \\ &\approx \phi(t,\theta^0)^T \theta^0 + e(t), \qquad \theta^0 = [b_1,\ c_1]^T \end{aligned}$$

$$\begin{aligned} \phi(t,\theta^0)^T &= \Big[u(t-1), \\ & y(t-1) - \Big\{ b_1 u(t-2) + c_1 \Big(y(t-2) - b_1 u(t-3) \Big) \Big\} \Big] \end{aligned}$$

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Consider the system

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$$\begin{aligned} \phi(t,\theta^0)^T &= \Big[u(t-1), \\ &\qquad y(t-1) - \Big\{b_1 u(t-2) + c_1\Big(y(t-2) - b_1 u(t-3)\Big)\Big\}\Big] \end{aligned}$$

This is called pseudo-linear regression form

Excitation Condition for Dynamical Systems:

As mentioned, a large class of systems can be approximated by FIR models:

$$
\begin{aligned}
y(t) &= G(q)u(t) + e(t) = \sum_{i=1}^{\infty} g_i u(t-i) + e(t) \\
&= g_1 u(t-1) + g_2 u(t-2) + \cdots + g_n u(t-n) + \\
&\qquad + g_{n+1} u(t-n-1) + \cdots + e(t) \\
&\approx g_1 u(t-1) + g_2 u(t-2) + \cdots + g_n u(t-n) + e(t) \\
&= \phi(t)^T \theta^0 + e(t)
\end{aligned}
$$

$$
\begin{aligned}
\phi(t)^T &= \Big[u(t-1),\, u(t-2),\, \ldots,\, u(t-n) \Big] \\
\theta^0 &= \Big[g_1,\, g_2,\, \ldots,\, g_n \Big]^T
\end{aligned}
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&\approx g_1 u(t-1) + g_2 u(t-2) + \cdots + g_n u(t-n) + e(t) \\
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\end{aligned}
$$

LS can be applied for reconstruction of values of θ^0, if excitation condition holds. Fix $\{u(t)\}_1^N \Rightarrow \exists n$: $\{g_i\}_1^n$ can be identified!

Excitation Condition for Dynamical Systems:

For the model

$$
\begin{aligned}
y(t) &= \phi(t, n)^T \theta^0(n) + e(t) \\
\phi(t, n)^T &= \Big[u(t-1),\ u(t-2),\ \ldots,\ u(t-n) \Big] \\
\theta^0(n) &= \Big[g_1,\ g_2,\ \ldots,\ g_n \Big]^T
\end{aligned}
$$

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\begin{aligned}
y(t) &= \phi(t, n)^T \theta^0(n) + e(t) \\
\phi(t, n)^T &= \left[u(t-1),\, u(t-2),\, \ldots,\, u(t-n) \right] \\
\theta^0(n) &= \left[g_1,\, g_2,\, \ldots,\, g_n \right]^T
\end{aligned}
$$

condition of excitation is

$$
\lim_{N\to\infty} \frac{\Phi^T \Phi}{N} = \lim_{N\to\infty} \frac{1}{N} \begin{bmatrix} \phi(n, n)^T \\ \phi(n+1, n)^T \\ \vdots \\ \phi(N, n)^T \end{bmatrix}^T \begin{bmatrix} \phi(n, n)^T \\ \phi(n+1, n)^T \\ \vdots \\ \phi(N, n)^T \end{bmatrix} > 0
$$

Excitation Condition for Dynamical Systems (Example):

For example, if $n = 3$, then

$$\begin{aligned} \phi(t, n)^T &= \Big[u(t-1),\, u(t-2),\, \ldots,\, u(t-n) \Big] \\ &= \Big[u(t-1),\, u(t-2), u(t-3) \Big] \end{aligned}$$

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condition of excitation is $\lim_{N \to \infty} \{\Phi^T \Phi / N\} > 0$

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\Phi^T \Phi = \begin{bmatrix} \phi(1, n)^T \\ \phi(2, n)^T \\ \vdots \\ \phi(N, n)^T \end{bmatrix}^T \begin{bmatrix} \phi(1, n)^T \\ \phi(2, n)^T \\ \vdots \\ \phi(N, n)^T \end{bmatrix} = \begin{bmatrix} \phi(1, 3)^T \\ \phi(2, 3)^T \\ \vdots \\ \phi(N, 3)^T \end{bmatrix}^T \begin{bmatrix} \phi(1, 3)^T \\ \phi(2, 3)^T \\ \vdots \\ \phi(N, 3)^T \end{bmatrix}
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$$\begin{aligned} \Phi^T\Phi &= \phi(1,3)\phi(1,3)^T + \phi(2,3)\phi(2,3)^T + \cdots + \phi(N,3)\phi(N,3)^T \\ &= \begin{bmatrix} u(0) \\ u(-1) \\ u(-2) \end{bmatrix} [u(0),\, u(-1),\, u(-2)] + \\ &\qquad + \begin{bmatrix} u(1) \\ u(0) \\ u(-1) \end{bmatrix} [u(1),\, u(0),\, u(-1)] + \ldots \end{aligned}$$

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$$\approx \left[\begin{array}{c|c|c} \sum_1^N u(i)^2 & \sum_1^N u(i)u(i-1) & \sum_1^N u(i)u(i-2) \\ \sum_1^N u(i)u(i-1) & \sum_1^N u(i)^2 & \sum_1^N u(i)u(i-1) \\ \sum_1^N u(i)u(i-2) & \sum_1^N u(i)u(i-1) & \sum_1^N u(i)^2 \end{array}\right]$$

Link to Empirical Covariance:

Definition:

Given a sequence of numbers

$$\ldots, r_{-2}, r_{-1}, r_0, r_1, \ldots$$

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The sequence

$$c_0, c_1, c_2, \ldots$$

with elements defined by

$$c(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{N} r(i)r(i-k)$$

if limits exist, is called Empirical Covariance of $\{r_i\}$

Excitation Condition for Dynamical Systems (Example):

In example, condition of excitation is $\lim\limits_{N\to\infty} \{\Phi^T\Phi/N\} > 0$

$$\Phi^T\Phi \approx \left[\begin{array}{c|c|c} \sum_1^N u(i)^2 & \sum_1^N u(i)u(i-1) & \sum_1^N u(i)u(i-2) \\ \sum_1^N u(i)u(i-1) & \sum_1^N u(i)^2 & \sum_1^N u(i)u(i-1) \\ \sum_1^N u(i)u(i-2) & \sum_1^N u(i)u(i-1) & \sum_1^N u(i)^2 \end{array}\right]$$

$$\approx N \begin{bmatrix} c(0) & c(1) & c(2) \\ c(1) & c(0) & c(1) \\ c(2) & c(1) & c(0) \end{bmatrix} > 0$$

where $c(0)$, $c(1)$ $c(2)$ are elements of empirical covariance of $\{u(t)\}$

$$c(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u(i)u(i-k)$$

Persistence of Excitation:

Definition 1: Given a signal $u(t)$, i.e. a sequence of numbers

$$u_0, u_1, u_2 \ldots$$

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and if

$$C_n = \begin{bmatrix} c(0) & c(1) & \ldots & c(n-1) \\ c(1) & c(0) & \ldots & c(n-2) \\ \vdots & & & \\ c(n-1) & c(n-2) & \ldots & c(0) \end{bmatrix} > 0$$

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It is called persistently excited of order n if for all t there exists an integer m such that

$$\rho_1 I_n > \sum_{k=t}^{t+m} \phi(k,n)\phi(k,n)^T > \rho_2 I_n$$

where $\rho_1, \rho_2 > 0$ and

$$\phi(t,n)^T \;=\; \Big[u(t-1),\, u(t-2),\, \ldots,\, u(t-n)\Big]$$

Persistence of Excitation of Signals (Impulse):

For the impulse signal

$$u(0) = a,\ u(1) = u(2) = \cdots = u(m) = \cdots = 0$$

the coefficients

$$c(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u(i)u(i-k) = 0, \quad k = 1,\, 2,\, \ldots$$

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Conclusion: the impulse is not PE of any order!

Persistence of Excitation of Signals (Step):

For the step signal

$$u(-1) = u(-2) \cdots = 0, \quad u(0) = u(1) = \cdots = a \neq 0$$

the coefficients

$$c(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u(i)u(i-k) = \lim_{N\to\infty} \frac{(N-k)\,a^2}{N} = a^2, \; \forall\, k$$

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Therefore, we get matrices

$$C_1 = c(0) = a^2, \quad C_2 = \begin{bmatrix} c(0) & c(1) \\ c(1) & c(0) \end{bmatrix} = \begin{bmatrix} a^2 & a^2 \\ a^2 & a^2 \end{bmatrix}, C_3 = \ldots$$

and we need to find the order n such that

$$C_n > 0, \qquad \det C_{n+1} = 0$$

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Conclusion: the step is PE of the first order!

Next Lecture

Next meeting (April 15, 13:00-15:00, in A206Tekn): Real-Time Parameter Estimation.

Homework problem:

Consider the continuous-time system $y(t) = G(p)\, u(t)$

represented by the transfer function $\quad G(s) = \dfrac{b_1}{s^2 + a_1\, s + a_2}$.

- Write a gradient algorithm of the form

$$\dot{\hat{\theta}} = \frac{\gamma}{1 + \phi(t)^T \phi(t)} \left[\bar{y}(t) - \phi(t)^T \hat{\theta}(t)\right] \phi(t)$$

  to estimate the parameters $\{a_1, a_2, b_1\}$.

- Simulate your algorithm with the true parameters of the system $a_1 = a_2 = 1.5$, $b_1 = 1$. Let $u(t)$ be a square wave. Study performance of the algorithm when varying
  - The initial conditions for the parameter estimates,
  - The adaptation gain γ,
  - The amplitude of the square wave,
  - The period of the square wave.