Lecture 6: Deterministic Self-Tuning Regulators

- Feedback Control Design for Nominal Plant Model via Pole Placement
- Indirect Self-Tuning Regulators
- Direct Self-Tuning Regulators

Pole Placement for Linear Systems

Consider a single input single output (SISO) system

$$
\begin{aligned}
y(t+n) + a_1 y(t+n-1) + \cdots + a_n y(t) \;=\; & b_0 u(t+n-\boldsymbol{d_0}) + \\
& + b_1 u(t+n-\boldsymbol{d_0}-1) + \cdots + b_m u(t+n-\boldsymbol{d_0}-m)
\end{aligned}
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Here

- $t = 1,\ 2,\ 3,\ \ldots$
- $u(t)$ is the (control) input
- $y(t)$ is the system output
- d_0 is the pole excess (representing a time delay)

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With forward shift operator $q\,\{y(t)\} \to y(t+1)$, the system is

$$q^n y(t) + a_1 q^{n-1} y(t) + \cdots + a_n y(t) \;=\; \\ b_0 q^{n-d_0} u(t) + b_1 q^{n-d_0-1} u(t) + \cdots + q^{n-d_0-m} b_m u(t)$$

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$$\left(q^n + a_1 q^{n-1} + \cdots + a_n\right) y(t) \; = \\ \left(b_0 q^{n-d_0} + b_1 q^{n-d_0-1} + \cdots + q^{n-d_0-m} b_m\right) u(t)$$

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$$A(q)y(t) = \\ \left(b_0 q^{n-d_0} + b_1 q^{n-d_0-1} + \cdots + q^{n-d_0-m} b_m\right) u(t)$$

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$$A(q)y(t) = B(q)u(t)$$

Pole Placement for Linear Systems

A general linear controller is given by

$$
\begin{aligned}
u(t+l) + r_1 u(t+l-1) + \cdots + r_l u(t) &= \\
= \; & p_0 u_c(t+l) + p_1 u_c(t+l-1) + \cdots + p_k u_c(t+l-k) \\
& + s_0 y(t+l) + s_1 y(t+l-1) + \cdots + s_q y(t+l-q)
\end{aligned}
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Pole Placement for Linear Systems

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Here $t = 1, 2, 3, \ldots$

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- $u_c(t)$ is the reference signal

Pole Placement for Linear Systems

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$$\left(q^l + r_1 q^{l-1} + \cdots + r_l\right) u(t) =$$

$$= \left(p_0 q^l + p_1 q^{l-1} + \cdots + p_k q^{l-k}\right) u_c(t)$$

$$+ \left(s_0 q^l + s_1 q^{l-1} + \cdots + s_q q^{l-q}\right) y(t)$$

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Here $t = 1, 2, 3, \ldots$

- $u(t)$ is the (control) input
- $y(t)$ is the system output
- $u_c(t)$ is the reference signal

With forward shift $q\{y(t)\} \to y(t+1)$, the controller is

$$R(q)u(t) = T(q)u_c(t) - S(q)y(t)$$

Pole Placement for Linear Systems

If we solve the equations (with added input disturbance $v(t)$)

$$\begin{aligned} A(q)y(t) &= B(q)\Big(u(t) + v(t)\Big) \\ R(q)u(t) &= T(q)u_c(t) - S(q)y(t) \end{aligned}$$

with respect to $y(t)$ and $u(t)$, we obtain

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with respect to $y(t)$ and $u(t)$, we obtain

$$
\begin{aligned}
y(t) &= \frac{B(q)T(q)}{A(q)R(q) + B(q)S(q)} u_c(t) \\
&\qquad + \frac{B(q)R(q)}{A(q)R(q) + B(q)S(q)} v(t) \\
u(t) &= \frac{A(q)T(q)}{A(q)R(q) + B(q)S(q)} u_c(t) \\
&\qquad + \frac{B(q)S(q)}{A(q)R(q) + B(q)S(q)} v(t)
\end{aligned}
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Pole Placement for Linear Systems

The typical specifications are

- All four transfer functions (from v, u_c to y, u) are stable

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The closed-loop characteristic polynomial

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Pole Placement for Linear Systems

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- All four transfer functions (from v, u_c to y, u) are stable

$$\Updownarrow$$

  The closed-loop characteristic polynomial

$$A(q)R(q) + B(q)S(q) = A_c(q)$$

  is stable

- The response of the close-loop system to the reference signal

$$y(t) = \frac{B(q)T(q)}{A(q)R(q) + B(q)S(q)} u_c(t)$$

  is as close as possible to the response of a nominal model

$$y_m(t) = \frac{B_m(q)}{A_m(q)} u_c(t)$$

Pole Placement for Linear Systems

If it is possible to find polynomials $R(q)$, $T(q)$, $S(q)$ so that

$$\frac{B(q)T(q)}{A(q)R(q)+B(q)S(q)} = \frac{B_m(q)}{A_m(q)}$$

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Naive Step 1: Pick up somehow $A_{ax}(q)$, $(\deg A_{ax}(q) = ??)$ and solve the Diophantine equation

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$$\frac{B(q)T(q)}{A_{ax}(q)A_m(q)} = \frac{B_m(q)}{A_m(q)} \quad \Rightarrow \quad B_m(q) \approx B(q), \quad T(q) \approx A_{ax}(q)$$

Pole Placement for Linear Systems

The relation

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Let us factorize $B(q)$ as

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Pole Placement for Linear Systems

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Let us factorize $B(q)$ as

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Then, we assume

$$B_m(q) = B^-(q) \cdot B_{mp}(q)$$

where

- $B_{mp}(q)$ is new polynomial to choose;
- $B^-(q)$ is the inherited part.

Pole Placement for Linear Systems

Then the identity

$$\frac{B(q)T(q)}{A_{ax}(q)A_m(q)} = \frac{B_m(q)}{A_m(q)}$$

becomes

$$\frac{B^+(q) \cdot B^-(q) \cdot T(q)}{A_{ax}(q) \cdot A_m(q)} = \frac{B^-(q) \cdot B_{mp}(q)}{A_m(q)}$$

Pole Placement for Linear Systems

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The only way to satisfy it, is to factorize $A_{ax}(q)$

$$\frac{B^+(q) \cdot B^-(q) \cdot T(q)}{\boxed{B^+(q) \cdot A_o(q)} \cdot A_m(q)} = \frac{B^-(q) \cdot B_{mp}(q)}{A_m(q)}$$

Pole Placement for Linear Systems

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and to factorize $T(q)$

Pole Placement for Linear Systems

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and to factorize $T(q)$

$$\frac{B^-(q) \cdot B_{mp}(q) \cdot A_o(q)}{A_o(q) \cdot A_m(q)} = \frac{B^-(q) \cdot B_{mp}(q)}{A_m(q)}$$

Pole Placement for Linear Systems

Given the polynomial $A_m(q)$

Naive Step 0: Factorize the polynomial $B(q)$ as

$$B(q) = B_{\{good\,to\,cancel\}}(q) \cdot B_{\{bad\,to\,cancel\}}(q) = B^{+}(q) \cdot B^{-}(q)$$

Choose the polynomial $B_m(q) = B^{-}(q) \cdot B_{mp}(q)$

Pole Placement for Linear Systems

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Choose the polynomial $\boldsymbol{B_m(q)} = B^-(q) \cdot \boldsymbol{B_{mp}(q)}$

Naive Step 1: Pick up somehow $A_o(q)$, ($\deg A_o(q) = ??$) and solve the Diophantine equation

$$A(q) \cdot \boldsymbol{R(q)} + B^+(q) \cdot B^-(q) \cdot \boldsymbol{S(q)} = B^+(q) \cdot A_o(q) \cdot \boldsymbol{A_m(q)}$$

Pole Placement for Linear Systems

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$$A(q) \cdot R(q) + B^+(q) \cdot B^-(q) \cdot S(q) = B^+(q) \cdot A_o(q) \cdot A_m(q)$$

Naive Step 2: Define the polynomial $T(q)$ as

$$T(q) = A_o(q) \cdot B_{mp}(q)$$

Pole Placement for Linear Systems

The solution $[\boldsymbol{R},\ \boldsymbol{S}]$ of the Diophantine equation

$$\boldsymbol{A(q) \cdot R(q) + B^{+}(q) \cdot B^{-}(q) \cdot S(q) = B^{+}(q) \cdot A_o(q) \cdot A_m(q)}$$

may exist or not!

Pole Placement for Linear Systems

The solution $[\boldsymbol{R},\ \boldsymbol{S}]$ of the Diophantine equation

$$A(q)\cdot \boldsymbol{R}(q) + B^+(q)\cdot B^-(q)\cdot \boldsymbol{S}(q) = B^+(q)\cdot A_o(q)\cdot \boldsymbol{A_m}(q)$$

may exist or not!

Does it depend on the choices of $A_o(q)$ and $\boldsymbol{A_m}(q)$?

Pole Placement for Linear Systems

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may exist or not!

Does it depend on the choices of $A_o(q)$ and $\boldsymbol{A_m(q)}$?

If the solution exists then

$$\boldsymbol{R(q)} = B^{+}(q) \cdot \boldsymbol{R_p(q)}$$

Pole Placement for Linear Systems

The solution $[\boldsymbol{R},\ \boldsymbol{S}]$ of the Diophantine equation

$$A(q)\cdot \boldsymbol{R(q)} + B^{+}(q)\cdot B^{-}(q)\cdot \boldsymbol{S(q)} = B^{+}(q)\cdot A_o(q)\cdot \boldsymbol{A_m(q)}$$

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Does it depend on the choices of $A_o(q)$ and $\boldsymbol{A_m(q)}$?

If the solution exists then

$$\boldsymbol{R(q)} = B^{+}(q)\cdot \boldsymbol{R_p(q)}$$

The equation to solve becomes of lower order

$$A(q)\cdot \boldsymbol{R_p(q)} + B^{-}(q)\cdot \boldsymbol{S(q)} = A_o(q)\cdot \boldsymbol{A_m(q)}$$

Pole Placement for Linear Systems

The solution $[\boldsymbol{R}, \ \boldsymbol{S}]$ of the Diophantine equation

$$\boldsymbol{A}(\boldsymbol{q}) \cdot \boldsymbol{R}(\boldsymbol{q}) + \boldsymbol{B}(\boldsymbol{q}) \cdot \boldsymbol{S}(\boldsymbol{q}) = \boldsymbol{A}_{\boldsymbol{c}}(\boldsymbol{q})$$

exists for any $\boldsymbol{A}_{\boldsymbol{c}}(\boldsymbol{q})$, iff $\boldsymbol{A}(\boldsymbol{q})$ and $\boldsymbol{B}(\boldsymbol{q})$ are co-prime!

Pole Placement for Linear Systems

The solution $[\boldsymbol{R},\ \boldsymbol{S}]$ of the Diophantine equation

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exists for any $\boldsymbol{A_c(q)}$, iff $\boldsymbol{A(q)}$ and $\boldsymbol{B(q)}$ are co-prime!

Furthermore, if $[\boldsymbol{R^0(q)},\ \boldsymbol{S^0(q)}]$ is the solution, then

$$\begin{aligned} \boldsymbol{R(q)} &= \boldsymbol{R^0(q)} - \boldsymbol{Q(q)} \cdot \boldsymbol{B(q)} \\ \boldsymbol{S(q)} &= \boldsymbol{S^0(q)} + \boldsymbol{Q(q)} \cdot \boldsymbol{A(q)} \end{aligned}$$

is the solution too, for any polynomial $\boldsymbol{Q(p)}$!

Pole Placement for Linear Systems

The solution $[R,\ S]$ of the Diophantine equation

$$A(q) \cdot R(q) + B(q) \cdot S(q) = A_c(q)$$

exists for any $A_c(q)$, iff $A(q)$ and $B(q)$ are co-prime!

Furthermore, if $[R^0(q),\ S^0(q)]$ is the solution, then

$$\begin{aligned} R(q) &= R^0(q) - Q(q) \cdot B(q) \\ S(q) &= S^0(q) + Q(q) \cdot A(q) \end{aligned}$$

is the solution too, for any polynomial $Q(p)$!

We look for a solution of minimal degree and with the properties

$$\begin{aligned} \deg R(q) &\geq \deg S(q) \\ \deg R(q) &\geq \deg T(q) \quad \left\{ = A_o(q) \cdot B_{mp}(q) \right\} \end{aligned}$$

Pole Placement for Linear Systems

To find a minimum degree solution, observe that if the equation

$$A(q) \cdot R(q) + B(q) \cdot S(q) = A_c(q)$$

has a solution $[R^0(q),\ S^0(q)]$ with

$$\deg S^0(q) = M \geq n = \deg A(q)$$

Pole Placement for Linear Systems

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then there exists a solution $[R^\star(q),\ S^\star(q)]$ with

$$\deg S^\star(q) < \deg A(q) = n$$

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$$\deg S^\star(q) < \deg A(q) = n$$

Indeed, $S^\star(q)$ can be defined as the reminder from the division $S^0(q)$ on $A(q)$

$$S^0(q) = Q(q) \cdot A(q) + S^\star(q)$$

Pole Placement for Linear Systems

Suppose that we have chosen $A_c(q)$ of degree $(2n-1)$, we know that

$$A(q) \cdot R(q) + B(q) \cdot S(q) = A_c(q)$$

has a solution $[R^\star(q),\ S^\star(q)]$ with

$$\deg S^\star(q) \le n-1$$

Pole Placement for Linear Systems

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Our plant is strictly proper

$$\deg A(q) = n > \deg B(q)$$

What is the degree of $R^\star(q)$?

Pole Placement for Linear Systems

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Our plant is strictly proper

$$\deg A(q) = n > \deg B(q)$$

What is the degree of $R^\star(q)$?

It is

$$\begin{aligned} \deg R^\star(q) &= \deg A_c(q) - \deg A(q) = n-1 \\ &\ge \deg S^\star(q) \end{aligned}$$

Pole Placement for Linear Systems

The last condition to satisfy is

$$\deg R(q) \geq \deg T(q) \quad \Big\{ = A_o(q) \cdot B_{mp}(q) \Big\}$$

How to achieve that?

Pole Placement for Linear Systems

The last condition to satisfy is

$$\deg \boldsymbol{R(q)} \geq \deg \boldsymbol{T(q)} \quad \left\{ = \boldsymbol{A_o(q)} \cdot \boldsymbol{B_{mp}(q)} \right\}$$

How to achieve that?

Such condition imposes constraint of choice of $\boldsymbol{A_m(q)}$, $\boldsymbol{B_m(q)}$

Pole Placement for Linear Systems

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As an example, consider the system

$$y(t) = u(t - 100)$$

Pole Placement for Linear Systems

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How to achieve that?

Such condition imposes constraint of choice of $A_m(q)$, $B_m(q)$

As an example, consider the system

$$y(t) = u(t - 100)$$

Would it be possible to design controller that ensures

$$y(t) = u_c(t - 1)$$

i.e. the closed-loop is tracking the reference u_c without delay?

Pole Placement for Linear Systems

The identity

$$A(q) \cdot R(q) + B^{+}(q) \cdot B^{-}(q) \cdot S(q) = B^{+}(q) \cdot A_o(q) \cdot A_m(q)$$

implies that

$$\deg A(q) + \deg R(q) = \deg B^{+}(q) + \deg A_o(q) + \deg A_m(q)$$

Pole Placement for Linear Systems

The identity

$$A(q) \cdot \boldsymbol{R(q)} + B^{+}(q) \cdot B^{-}(q) \cdot \boldsymbol{S(q)} = B^{+}(q) \cdot A_o(q) \cdot \boldsymbol{A_m(q)}$$

implies that

$$\deg A(q) + \deg \boldsymbol{R(q)} = \deg B^{+}(q) + \deg A_o(q) + \deg \boldsymbol{A_m(q)}$$

Combining this with the condition

$$\deg \boldsymbol{R(q)} \geq \deg T(q) = \deg A_o(q) + \deg \boldsymbol{B_{mp}(q)}$$

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$$\deg A(q) - \deg B^{+}(q) \leq \deg \boldsymbol{A_m(q)} - \deg \boldsymbol{B_{mp}(q)}$$

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The identity

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$$\deg A(q)+\deg R(q)=\deg B^{+}(q)+\deg A_o(q)+\deg A_m(q)$$

Combining this with the condition

$$\deg R(q)\geq \deg T(q)=\deg A_o(q)+\deg B_{mp}(q)$$

---

we obtain new inequality

$$\deg A(q)-\deg B(q)\quad\leq\quad \deg A_m(q)-\deg B_m(q)$$

This is a constraint on the relative degree of the target dynamics!

Pole Placement for Linear Systems: Summary

Given co-prime polynomials $A(q)$, $B(q)$ and $A_m(q)$

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Step 0: Factorize the polynomial $B(q)$ as

$$B(q) = B_{\{good\,to\,cancel\}}(q) \cdot B_{\{bad\,to\,cancel\}}(q) = B^+(q) \cdot B^-(q)$$

Choose $\boldsymbol{B_m(q)} = B^-(q) \cdot \boldsymbol{B_{mp}(q)}$ such that the pole access for the target model is the same or larger than for the plant;

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Step 1: Find minimum degree solution for

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when $A_o(q)$ is stable, $\deg A_o = 2n - 1 - \deg B^{+} - \deg A_m$

Step 2: Define the polynomial $T(q)$ as

$$T(q) = A_o(q) \cdot B_{mp}(q)$$

Example 3.1

Given a continuous time system

$$\ddot{y} + \dot{y} = u,$$

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The task is to synthesize a 2-degree-of-freedom controller such that the complementary sensitivity transfer function is

$$\frac{B_m(q)}{A_m(q)} = \frac{0.1761q}{q^2 - 1.3205q - 0.4966}$$

Example 3.1 (Cont'd)

It is clear that polynomials $B(q)$ and $A(q)$ for

$$\frac{B(q)}{A(q)} = \frac{b_1 q + b_0}{q^2 + a_1 q + a_0} = \frac{0.10653(q + 0.8467)}{(q - 1)(q - 0.6065)}$$

are co-prime.

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$$B(q) = B^+(q) \cdot B^-(q) = (q + 0.8467) \cdot 0.10653$$

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The Diophantine equation is then

$$A(q)\underbrace{B^+(q)R_p(q)}_{R(q)} + \underbrace{B^+(q)B^-(q)}_{B(q)}S(q) \quad = \quad B^+(q)A_o(q)A_m(q)$$

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where the polynomial should be of degree $2n - 1 = 3$.

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$\Rightarrow$ The polynomial $A_o(q)$ can be trivial, $A_o(q) = 1$.

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The Diophantine equation is then

$$A(q)R_p(q) + B^-(q)S(q) \quad = \quad A_m(q)$$

or

$$\left(q^2 - 1.607q + 0.6065\right) R_p(q) + 0.10653 \left(s_1 q + s_0\right) =$$

$$= \quad q^2 - 1.3205q - 0.4966$$

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$\Rightarrow R_p(q) = 1$ and s_1, s_2 are readily computed!

Example 3.1 (Cont'd)

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The model-following problem for this case is solvable!

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The model-following problem for this case is solvable!

The polynomials $R(q)$, $S(q)$ and $T(q)$ are

$$\begin{aligned} R(q) &= q + 0.8467 \\ S(q) &= 2.6852 \cdot q - 1.0321 \\ T(q) &= 1.6531 \cdot q \end{aligned}$$

Example 3.1 (Cont'd)

To summarize:

The model-following problem for this case is solvable!

The polynomials $R(q)$, $S(q)$ and $T(q)$ are

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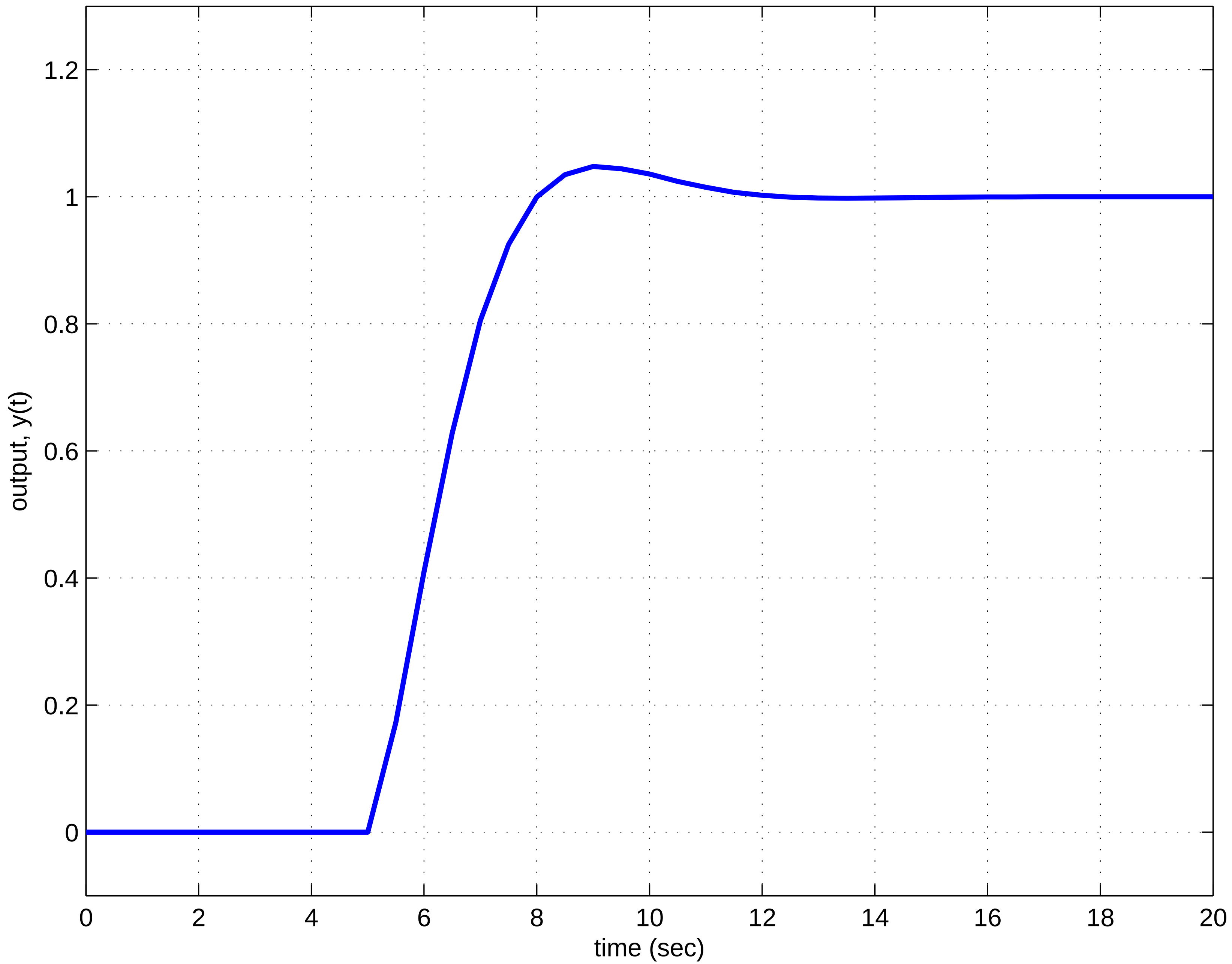
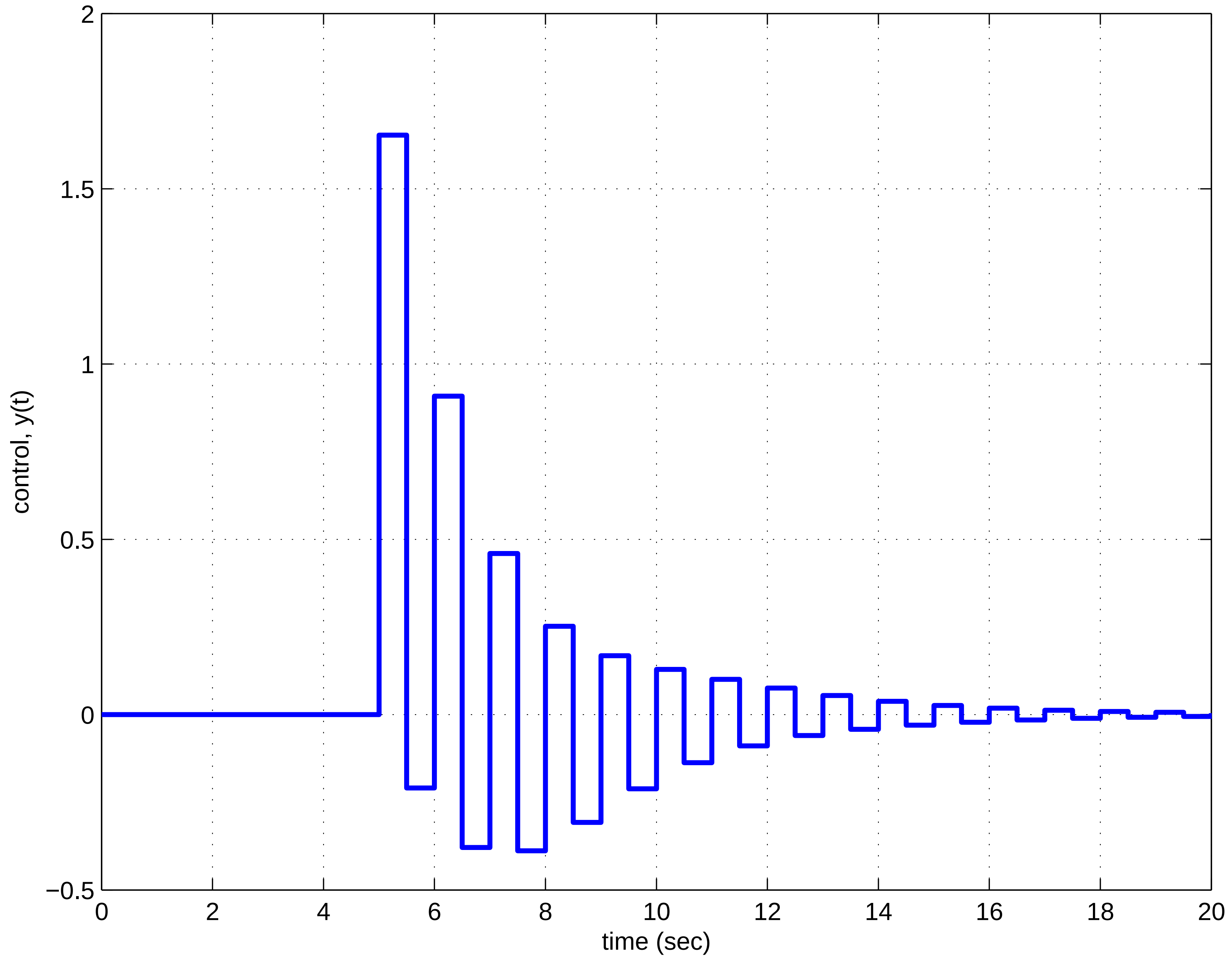
Let us check the step response of the closed-loop system



The unit step response of the closed-loop system



The control signal for the step response

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Given a continuous time system

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with

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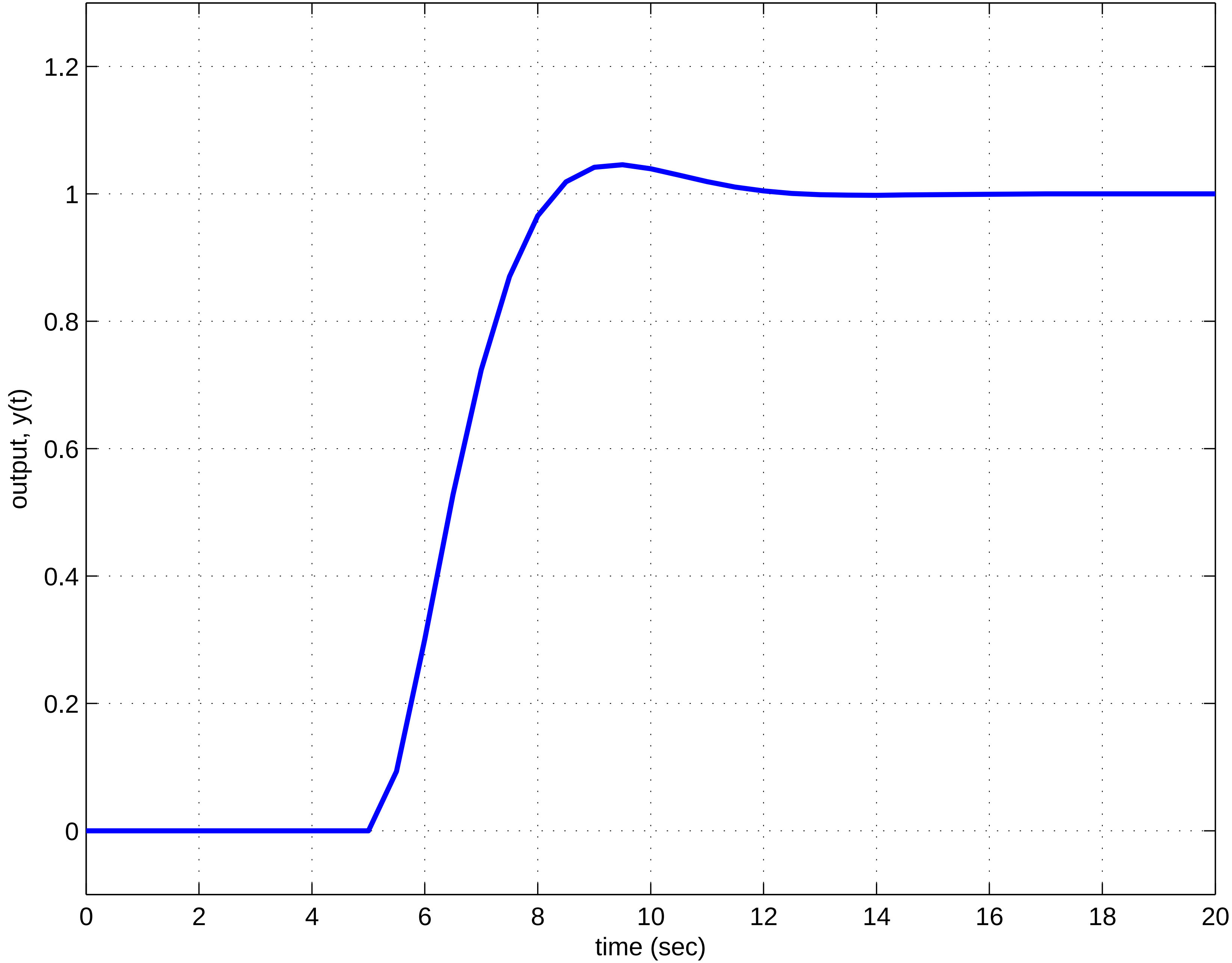

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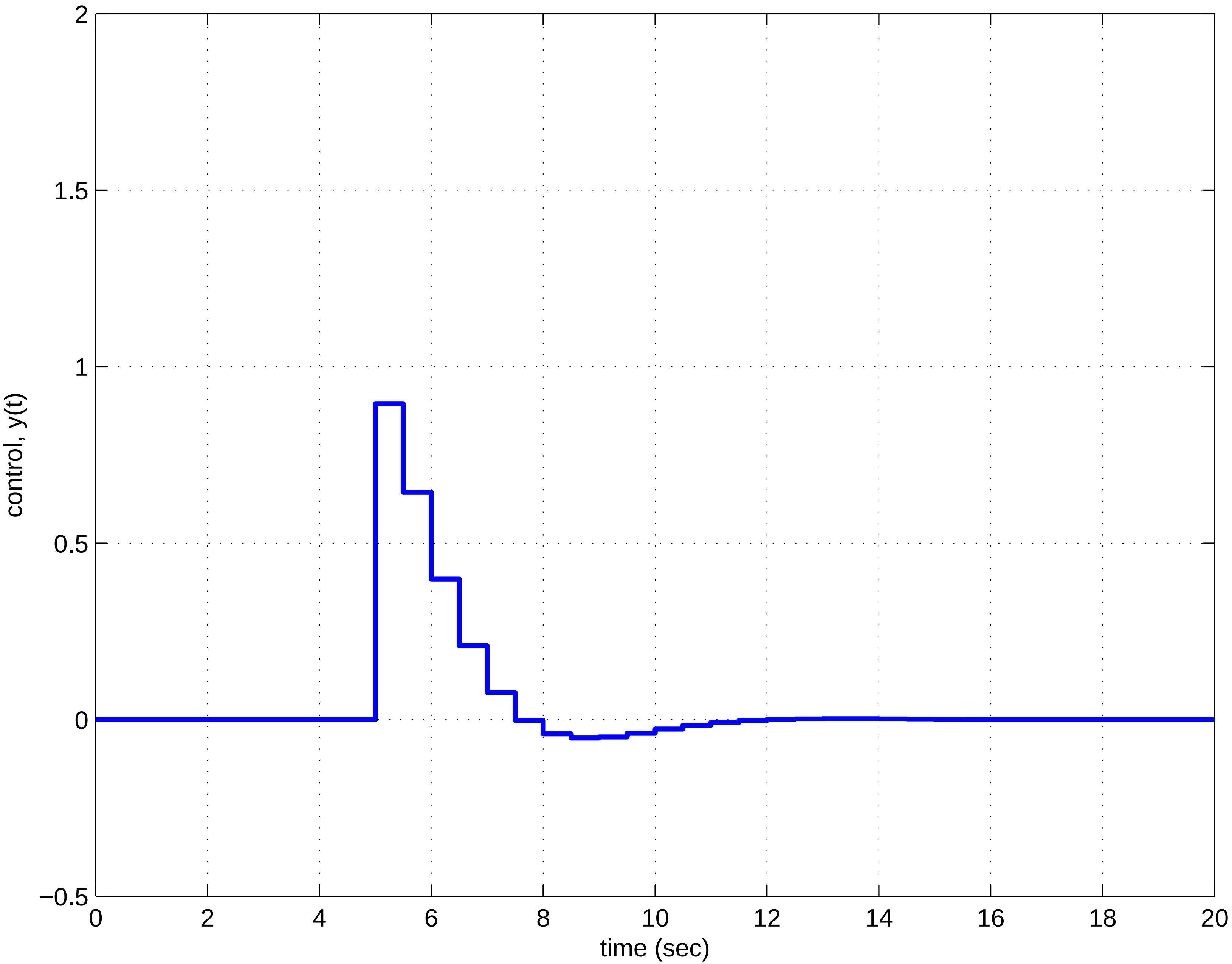

The control signal for the step response

Lecture 5: Deterministic Self-Tuning Regulators

- Feedback Control Design for Nominal Plant Model via Pole Placement
- Indirect Self-Tuning Regulators
- Direct Self-Tuning Regulators

Indirect Self-Tuning Regulators

Consider a single input single output (SISO) system

$$
\begin{aligned}
y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) \;&=\; b_0 u(t-d_0) + \\
&+ b_1 u(t-d_0-1) + \cdots + b_m u(t-d_0-m)
\end{aligned}
$$

Indirect Self-Tuning Regulators

Consider a single input single output (SISO) system

$$y(t)+a_1y(t-1)+\cdots+a_ny(t-n) = b_0u(t-d_0)+ \\ +b_1u(t-d_0-1)+\cdots+b_mu(t-d_0-m)$$

Rewrite it in the form

$$\begin{aligned} y(t) &= -a_1y(t-1)-\cdots-a_ny(t-n)+b_0u(t-d_0)+ \\ &\quad +b_1u(t-d_0-1)+\cdots+b_mu(t-d_0-m) \\ &= \phi(t-1)^T\theta \end{aligned}$$

Indirect Self-Tuning Regulators

Consider a single input single output (SISO) system

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Rewrite it in the form

$$\begin{aligned} y(t) &= -a_1 y(t-1) - \cdots - a_n y(t-n) + b_0 u(t-d_0) + \\ &\quad + b_1 u(t-d_0-1) + \cdots + b_m u(t-d_0-m) \\ &= \phi(t-1)^T \theta \end{aligned}$$

where $\theta = \Big[a_1, \ldots, a_n, b_0, \ldots, b_m\Big]^T$

$$\begin{aligned} \phi(t-1) = \Big[&-y(t-1), \ldots, -y(t-n), \\ & u(t-d_0), \ldots, u(t-d_0-m)\Big]^T \end{aligned}$$

Recursive LS Algorithm

Given the data $\left\{y(t), \phi(t-1)\right\}_1^N$ defined by the model

$$y(t) = \phi(t-1)^T \theta^0, \quad t = 1, 2, \ldots, N$$

Recursive LS Algorithm

Given the data $\left\{y(t), \phi(t-1)\right\}_1^N$ defined by the model

$$y(t) = \phi(t-1)^T \theta^0, \quad t = 1,\, 2,\, \ldots,\, N$$

Given $\hat{\theta}(t_0)$ and $P(t_0) = \left(\Phi(t_0)^T \Phi(t_0)\right)^{-1}$, the LS estimate satisfies the recursive equations

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\left(y(t) - \phi(t)^T \hat{\theta}(t-1)\right) \\
K(t) &= P(t)\phi(t) = P(t-1)\phi(t)\left(1 + \phi^T P(t-1)\phi(t)\right)^{-1} \\
P(t) &= P(t-1) - \\
&\quad - P(t-1)\phi(t)\left(1 + \phi^T P(t-1)\phi(t)\right)^{-1} \phi^T P(t-1)
\end{aligned}$$

Algorithm using RLS and MD Pole Placement

Off-line Parameters: Given polynomials $B_m(q)$, $A_m(q)$, $A_o(q)$

Algorithm using RLS and MD Pole Placement

Off-line Parameters: Given polynomials $B_m(q)$, $A_m(q)$, $A_o(q)$

Step 1: Estimate the coefficients of $A(q)$ and $B(q)$, i.e. θ, using the Recursive Least Squares algorithm.

Algorithm using RLS and MD Pole Placement

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Step 1: Estimate the coefficients of $A(q)$ and $B(q)$, i.e. θ, using the Recursive Least Squares algorithm.

Step 2: Apply the Minimum Degree Pole Placement algorithm to compute

$$R(q), \quad T(q), \quad S(q)$$

with $A(q)$, $B(q)$ taken from the previous Step.

Algorithm using RLS and MD Pole Placement

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Step 3: Compute the control variable by

$$R(q)u(t) = T(q)u_c(t) - S(q)y(t)$$

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$$R(q)u(t) = T(q)u_c(t) - S(q)y(t)$$

Repeat Steps 1, 2, 3

Next Lecture / Assignments:

Next meeting (April 22, 15:00-17:00, in A206Tekn)

Homework problems: (see also:

`http://www.engin.umich.edu/group/ctm/digital/digital.html`)

- Show that if $z(t) = \frac{1}{(p+a)}\, u(t)$ and $u(t) = u[k]$ for $k\,h \le t < (k+1)\,h$, then

$$z[k] = \frac{1 - e^{-a\,h}}{a\,(q - e^{-a\,h})}\, u[k]$$

.

- Show that if $y(t) = \frac{1}{p\,(p+a)}\, u(t)$ and $u(t) = u[k]$ for $k\,h \le t < (k+1)\,h$, then

$$y[k] = \frac{\left[h - \frac{1}{a}(1 - e^{-a\,h})\right]\, q + \left[-h\,e^{-a\,h} + \frac{1}{a}(1 - e^{-a\,h})\right]}{a\,\left(q^2 + (-1 - e^{-a\,h})\, q + e^{-a\,h}\right)}\, u[k].$$