2.153 Adaptive Control
Lecture 5
Adaptive Systems: States Accessible-MIMO Plants

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- **Pset #1** out: Thu 19-Feb, **due: Fri 27-Feb**
- **Pset #2** out: Wed 25-Feb, **due: Fri 6-Mar**
- **Pset #3** out: Wed 4-Mar, **due: Fri 13-Mar**
- **Pset #4** out: Wed 11-Mar, **due: Fri 20-Mar**
- **Midterm (take home)** out: Mon 30-Mar, **due: Fri 3-Apr**

Adaptive Control of nth order plants - with single input

$$\begin{aligned} \text{Plant:} \quad \dot{X}_p &= A_p X_p + b_p u, A_p \in \mathbb{R}^{n\times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R} \\ \text{Controller:} \quad u &= \theta_c^T X_p + k_c r \\ \text{Closed-loop:} \quad \dot{X}_p &= \left[A_p + b_p \theta_c^T\right] X_p + b_p k_c r \end{aligned}$$

Matching conditions: $A_p + b_p \theta^{*T} = A_m;\ b_p k^* = b_m$

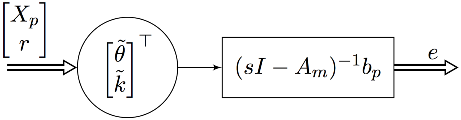
Reference Model $\dot{X}_m = A_m X_m + b_m r$

Solution: $\theta_c = \theta^*, \quad k_c = k^*$

$$A_p, b_p \text{ unknown} \implies \theta^*, k^* \text{ unknown}$$

$$\begin{aligned} \text{Choose Controller:} \quad u &= \theta^T(t) X_p + k(t) r \\ \text{Closed-loop:} \quad \dot{X}_p &= \left[A_p + b_p \theta^T(t)\right] X_p + b_p(k^* + \widetilde{k}) r \\ &= A_m X_p + b_p \left(\widetilde{\theta}^T X_p + \widetilde{k} r\right) + b_m r \end{aligned}$$

Error Model 2 and Stability Analysis



$$\text{Error equation:} \quad \dot{e} = A_m e + b_p \left(\widetilde{\theta}^T X_p + \widetilde{k} r \right)$$

$$\begin{aligned}
V &= e^T P e + |k^*| \left(\widetilde{\theta}^T \widetilde{\theta} + \widetilde{k}^2 \right) \\
\dot{V} &= e^T [A_m^T P + P A_m] e + 2 e^T P b_p \, \widetilde{\theta}^T X_p + 2 |k^*| \widetilde{\theta}^T \dot{\widetilde{\theta}} \\
&\quad + 2 e^T P b_p \, \widetilde{k} r + 2 |k^*| \widetilde{k} \dot{\widetilde{k}} \\
&= -e^T Q e \\
\text{if } \dot{\widetilde{\theta}} &= -sign(k^*) e^T P b_m \, X_p, \; \dot{\widetilde{k}} = -sign(k^*) e^T P b_m \, r
\end{aligned}$$

$$\Rightarrow e(t), \widetilde{\theta}(t), \quad \text{and} \quad \widetilde{k}(t) \;\; \text{are bounded for all } t \geq t_0$$

$$\lim_{t \to \infty} e(t) = 0 \text{ from Barbalat's Lemma}$$

Lyapunov functions and Linear Time-invariant Systems

$$\text{LTI System:} \quad \dot{x} = \quad A_m x$$

Theorem: Given $Q = Q^T > 0$, there exists $P = P^T > 0$ that solves

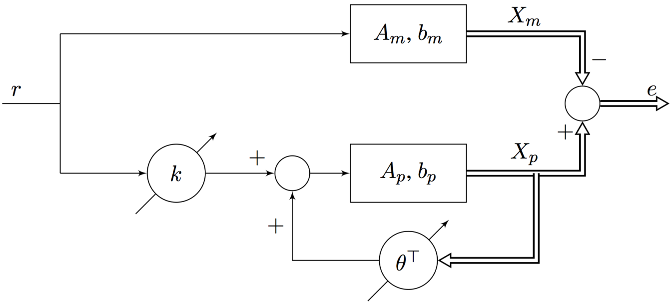
$$A_m^T P + P A_m = -Q$$

if and only if A_m is a Hurwitz matrix.

Main implication:

$$\begin{aligned} V &= x^T P x \\ \dot{V} &= x^T [A_m^T P + P A_m] x \\ &= -x^T Q x < 0 \end{aligned}$$

Overall Adaptive System - single input



Assumption: θ^* and k^* exist such that

$$b_p k^* = b_m \qquad A_p + b_p {\theta^*}^\top = A_m$$

$$\left.\begin{aligned}\dot{\theta} &= -sign(k^*)e^\top P b_m X_p \\ \dot{k} &= -sign(k^*)e^\top P b_m r\end{aligned}\right\} \Rightarrow \text{Stability, } e(t) \to 0$$

Adaptive Control of nth order plants - with multiple inputs

$$\text{Plant:} \quad \dot{X}_p = A_p X_p + B_p u,$$

$$A_p \in \mathbb{R}^{n\times n}, \qquad B_p \in \mathbb{R}^{n\times m}, \qquad u \in \mathbb{R}^m$$

$$\begin{aligned}
\text{Controller:} \quad u &= \Theta_{Ac} X_p + \Theta_{Bc} r \\
\text{Closed-loop:} \quad \dot{X}_p &= \left[A_p + B_p \Theta_{Ac}\right] X_p + B_p \Theta_{Bc} r
\end{aligned}$$

$$\text{Matching conditions:} \quad A_p + B_p \Theta_A^* = A_m;\ B_p \Theta_B^* = B_m$$

$$\text{Reference Model} \quad \dot{X}_m = A_m X_m + B_m r$$

$$\text{Solution:} \quad \Theta_{Ac} = \Theta_A^*, \ \ \Theta_{Bc} = \Theta_B^*$$

$$\Theta_A^* \in \mathbb{R}^{m\times n}, \qquad \Theta_B^* \in \mathbb{R}^{m\times m}$$

$$A_p, B_p \text{ unknown} \quad \Longrightarrow \quad \Theta_A^*, \Theta_B^* \text{ unknown}$$

Adaptive Control of nth order plants - with multiple inputs; B_p known

$$
\begin{aligned}
\text{Plant:} \quad \dot{X}_p &= A_p X_p + B_p u \\
\text{Choose Controller:} \quad u &= \Theta_A(t) X_p + \Theta_B^* r \\
\text{Closed-loop:} \quad \dot{X}_p &= \left[A_p + B_p \Theta_A(t)\right] X_p + B_p \Theta_B^* r
\end{aligned}
$$

$$
\begin{aligned}
A_p + B_p \Theta_A^* &= A_m; \quad B_p \Theta_B^* = B_m, \quad \widetilde{\Theta}_A = \Theta_A - \Theta_A^* \\
\dot{X}_p &= A_m X_p + B_p \widetilde{\Theta}_A X_p + B_m r
\end{aligned}
$$

Reference Model $\qquad \dot{X}_m = A_m X_m + B_m r$

Error Model $\qquad \dot{e} = A_m e + B_p \widetilde{\Theta}_A X_p$

Error Model 2 and Stability Analysis

$$
\begin{aligned}
\text{Error equation: } \dot{e} &= A_m e + B_p \widetilde{\Theta}_A X_p \\
\widetilde{\Theta}_A &\in \mathbb{R}^{m\times n}
\end{aligned}
$$

Use of $Trace$ operator: converts matrices to scalars.

$$
\begin{aligned}
Trace(ab^T) &= b^T a, \quad a, b \in \mathbb{R}^n \\
V &= e^T P e + Tr\left(\widetilde{\Theta}_A^T \widetilde{\Theta}_A\right) \\
\dot{V} &= e^T [A_m^T P + P A_m] e + 2 e^T P B_p \, \widetilde{\Theta}_A X_p + 2 Tr\left(\widetilde{\Theta}_A^T \dot{\widetilde{\Theta}}_A\right) \\
\text{Choose } \dot{\widetilde{\Theta}}_A &= -B_p^T P e \, X_p^T \\
\dot{V} &= -e^T Q e \leq 0
\end{aligned}
$$

$\Rightarrow e(t)$ and $\widetilde{\Theta}_A(t)$ are bounded for all $t \geq t_0$

$$\lim_{t\to\infty} e(t) = 0 \text{ from Barbalat's Lemma}$$

Adaptation gain Γ

$$\begin{array}{rcl}
\text{Error equation: } \dot{e} & = & A_m e + B_p \widetilde{\Theta}_A X_p \\
\text{Choose } \dot{\widetilde{\Theta}}_A & = & -\Gamma B_p^T P e \, X_p^T, \qquad \Gamma = \Gamma^T > 0 \\
V & = & e^T P e + Tr\left(\widetilde{\Theta}_A^T \Gamma^{-1} \widetilde{\Theta}_A\right) \\
.. & .. & \\
\dot{V} & = & -e^T Q e \leq 0 \\
\Rightarrow e(t) & \text{and} & \widetilde{\Theta}_A(t) \text{ are bounded for all } t \geq t_0 \\
\lim_{t \to \infty} e(t) & = & 0 \text{ from Barbalat's Lemma}
\end{array}$$

Adaptive Control of nth order plants - with multiple inputs; $B_p = B\Lambda$

$$
\begin{aligned}
\text{Plant: } \dot{X}_p &= A_p X_p + B\Lambda u \\
A_p, \Lambda \text{ unknown, } B \text{ known, } & \quad \Lambda \in \mathbb{R}^{m\times m} \text{ and diagonal} \\
sign(\Lambda) & \quad \text{known} \\
\text{Matching Conditions: } A_m &= A_p + B\Lambda\Theta_A^*, \; B\Lambda\Theta_B^* = B_m \\
\text{Reference Model} & \quad \dot{X}_m = A_m X_m + Br \\
\text{Controller: } u &= \Theta_A(t) X_p + \Theta_B(t) r \\
\text{Choose } \dot{\tilde{\Theta}}_A &= -\Gamma sign(\Lambda) B^T P e \, X_p^T, \qquad \Gamma = \Gamma^T > 0 \\
\text{Choose } \dot{\tilde{\Theta}}_B &= -\Gamma sign(\Lambda) B^T P e \, r^T
\end{aligned}
$$

$\Rightarrow$ Stability. $\lim_{t\to\infty} e(t) = 0.$

Adaptive Control of nth order plants - with multiple inputs; General B_p

$$
\begin{aligned}
\text{Plant: } \dot{X}_p &= A_p X_p + B_p u \\
A_p, B_p & \quad \text{unknown} \\
\text{Matching Conditions: } A_m &= A_p + B_p \Theta_A^*, \; B_p \Theta_B^* = B_m \\
\text{Reference Model} & \quad \dot{X}_m = A_m X_m + B_m r \\
u &= \Theta_A(t) X_p + \Theta_B(t) r - \text{doesn't lead to Error Model 2}
\end{aligned}
$$

Modify the controller structure:

$$
\text{Controller: } u = \Theta_B(t)\left(\Theta_A(t) X_p + r\right)
$$

Leads to

$$
\text{Error equation: } \dot{e} = A_m e + B_m \left(\widetilde{\Theta}_A X_p + \widetilde{\Psi} u\right)
$$

$$
\widetilde{\Theta}_A = \Theta_A - \Theta_A^* \qquad \widetilde{\Psi} = {\Theta_B^*}^{-1} - \Theta_B^{-1}
$$

Local Stabilty: $B_p\Theta_B^* = B_m$

$$
\begin{aligned}
\text{Error equation: } \dot{e} &= A_m e + B_m\left(\widetilde{\Theta}_A X_p + \widetilde{\Psi} u\right) \\
V &= e^T P e + tr\left(\widetilde{\Theta}_A^T \widetilde{\Theta}_A + \widetilde{\Psi}^T \widetilde{\Psi}\right) \\
\dot{V} &= -e^T Q e \leq 0
\end{aligned}
$$

So what's the problem?

$$V(e, \widetilde{\Theta}_a, \widetilde{\Psi}) \not\to \infty \text{ as } \widetilde{\Theta}_B \to \infty$$

$$\Longrightarrow \underline{\text{Local Stability}}$$