Lecture 11: Model-Reference Adaptive Systems

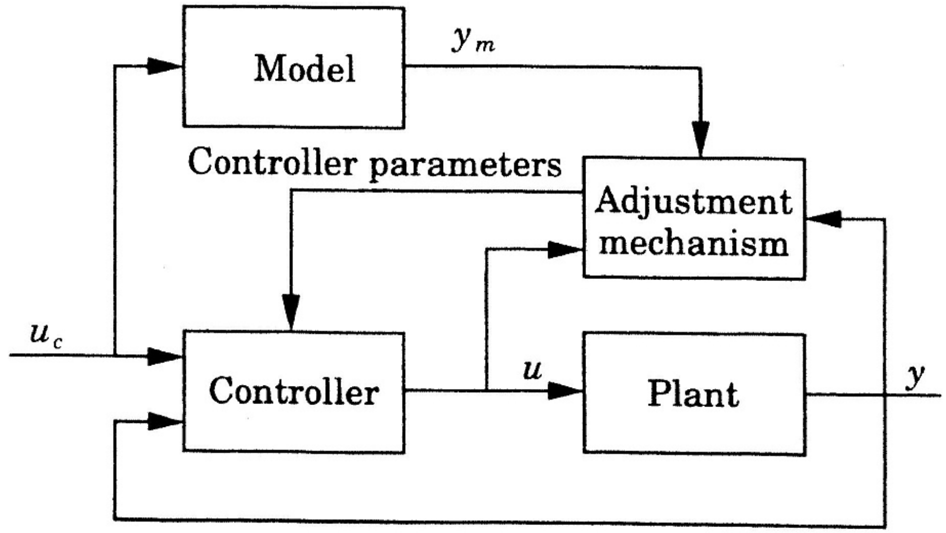

Figure 1.18 Block diagram of a model-reference adaptive system (MRAS).

Given: $$y(t) = G_\theta(p)\, u(t), \qquad y_m(t) = G_m(p)\, u_c(t),$$

Find: $$u(t) = -\frac{S_{\hat\theta}(p)}{R_{\hat\theta}(p)}\, y(t) + \frac{T_{\hat\theta}(p)}{R_{\hat\theta}(p)}\, u_c(t), \quad \frac{d}{dt}\hat\theta = \ldots$$

MIT rule (Example 5.1)

Consider a stable single input single output (SISO) system

$$y(t) = k \cdot G(p)\Big(u(t)\Big)$$

where

- $y(t)$ is the system output,
- $G(s)$ is a known stable transfer function,
- $u(t)$ is a control input,
- k is a constant unknown gain.

MIT rule (Example 5.1)

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where

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- $G(s)$ is a known stable transfer function,
- $u(t)$ is a control input,
- k is a constant unknown gain.

The problem is find a controller $u(t) = \frac{T(p)}{R(p)}\, u_c(t)$ to follow

$$y_m(t) = G_m(p)\left(u_c(t)\right) = k_0 \cdot G(p)\left(u_c(t)\right),$$

where k_0 is a given constant gain.

MIT rule (Example 5.1)

If k were known we can solve the problem

$$y(t) = k \cdot G(p)\Big(u(t)\Big) \quad \longrightarrow \quad y_m(t) = k_0 \cdot G(p)\Big(u_c(t)\Big)$$

using the simple proportional controller

$$u(t) = \theta \cdot u_c(t)$$

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This is, indeed, works, because if θ is chosen as

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then

$$y(t) = k \cdot G(p) \left(\theta \cdot u_c(t) \right) = k \cdot G(p) \left(\frac{k_0}{k} \cdot u_c(t) \right) = y_m(t)$$

MIT rule (Example 5.1)

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Let us consider the error between the real and simulated outputs

$$e(t,\theta) = y(t) - y_m(t) = k \cdot G(p) \left(\theta(t) \cdot u_c(t) \right) - k_0 \cdot G(p) \left(u_c(t) \right)$$

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Let us update $\theta(t)$ so that $e(t,\theta)$ is getting smaller.

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Let us update $\theta(t)$ so that $e(t,\theta)$ is getting smaller.

Consider the loss function that measures the size of $e(t,\theta)$

$$\mathcal{J}(t,\theta) = |e(t,\theta)|^2$$

Its time derivative is given by (chain rule)

$$\frac{d}{dt}\mathcal{J} = \left[\frac{\partial}{\partial t}\mathcal{J}\right] + \left[\frac{\partial}{\partial \theta}\mathcal{J}\right] \cdot \frac{d}{dt}\theta$$

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Consider the function that measures the size of $e(t,\theta)$

$$\mathcal{J}(t,\theta) = |e(t,\theta)|^2$$

Its time derivative should be made negative:

$$\frac{d}{dt}\mathcal{J} = \cdots + \left[\frac{\partial}{\partial\theta}\mathcal{J}\right]\cdot\frac{d}{dt}\theta \quad\Rightarrow\quad \frac{d}{dt}\theta = -\gamma\cdot\left[\frac{\partial}{\partial\theta}\mathcal{J}\right]$$

MIT rule (Example 5.1)

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Let us consider the error between the real and simulated outputs

$$e(t,\theta) = y(t) - y_m(t) = k \cdot G(p) \Big(\theta(t) \cdot u_c(t)\Big) - k_0 \cdot G(p) \Big(u_c(t)\Big)$$

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Consider the function that measures the size of $e(t,\theta)$

$$\mathcal{J}(t,\theta) = |e(t,\theta)|^2$$

Its time derivative should be made negative:

$$\frac{d}{dt}\mathcal{J} = \cdots + \left[2\, e\, \frac{\partial}{\partial\theta} e\right] \cdot \frac{d}{dt}\theta \quad \Rightarrow \quad \boxed{\frac{d}{dt}\theta = -\gamma \cdot \left[2\, e\, \frac{\partial}{\partial\theta} e\right]}$$

MIT rule (Example 5.1)

Computing the partial derivative of e wrt θ we have

$$\begin{aligned}
\frac{\partial}{\partial\theta} e &= \frac{\partial}{\partial\theta}\left[k \cdot G(p)\left(\theta(t) \cdot u_c(t)\right)\right] = k \cdot G(p)\left(u_c(t)\right) \\
&= \frac{k}{k_0} \cdot k_0 \cdot G(p)\left(u_c(t)\right) = \frac{k}{k_0}\, y_m(t)
\end{aligned}$$

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&= \frac{k}{k_0} \cdot k_0 \cdot G(p)\left(u_c(t)\right) = \frac{k}{k_0}\, y_m(t)
\end{aligned}
$$

Then the update law for θ becomes

$$
\frac{d}{dt}\theta = -\gamma \cdot \left[2\, e\, \frac{\partial}{\partial \theta} e\right] = -\gamma_n \cdot y_m(t) \cdot e(t, \theta)
$$

where $\gamma_n > 0$ is arbitrary since $\gamma_n = \gamma\, \dfrac{k}{k_0}$ with arbitrary $\gamma > 0$.

MIT rule (Example 5.1)

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Then the update law for θ becomes

$$\frac{d}{dt}\theta = -\gamma\cdot\left[2\,e\,\frac{\partial}{\partial\theta}e\right] = -\gamma_n\cdot y_m(t)\cdot e(t,\theta)$$

where $\gamma_n > 0$ is arbitrary since $\gamma_n = \gamma\,\dfrac{k}{k_0}$ with arbitrary $\gamma > 0$.

Remark: $\mathcal{J}(\cdot) = |e(\cdot)| \Rightarrow \frac{d}{dt}\theta = -\gamma_n\cdot y_m(t)\cdot \mathrm{sign}[e(t,\theta)]$.

MIT rule (Example 5.1)

Suppose that

$$G(s) = \frac{1}{s+1}$$

and

$$k = 1, \qquad k_0 = 2$$

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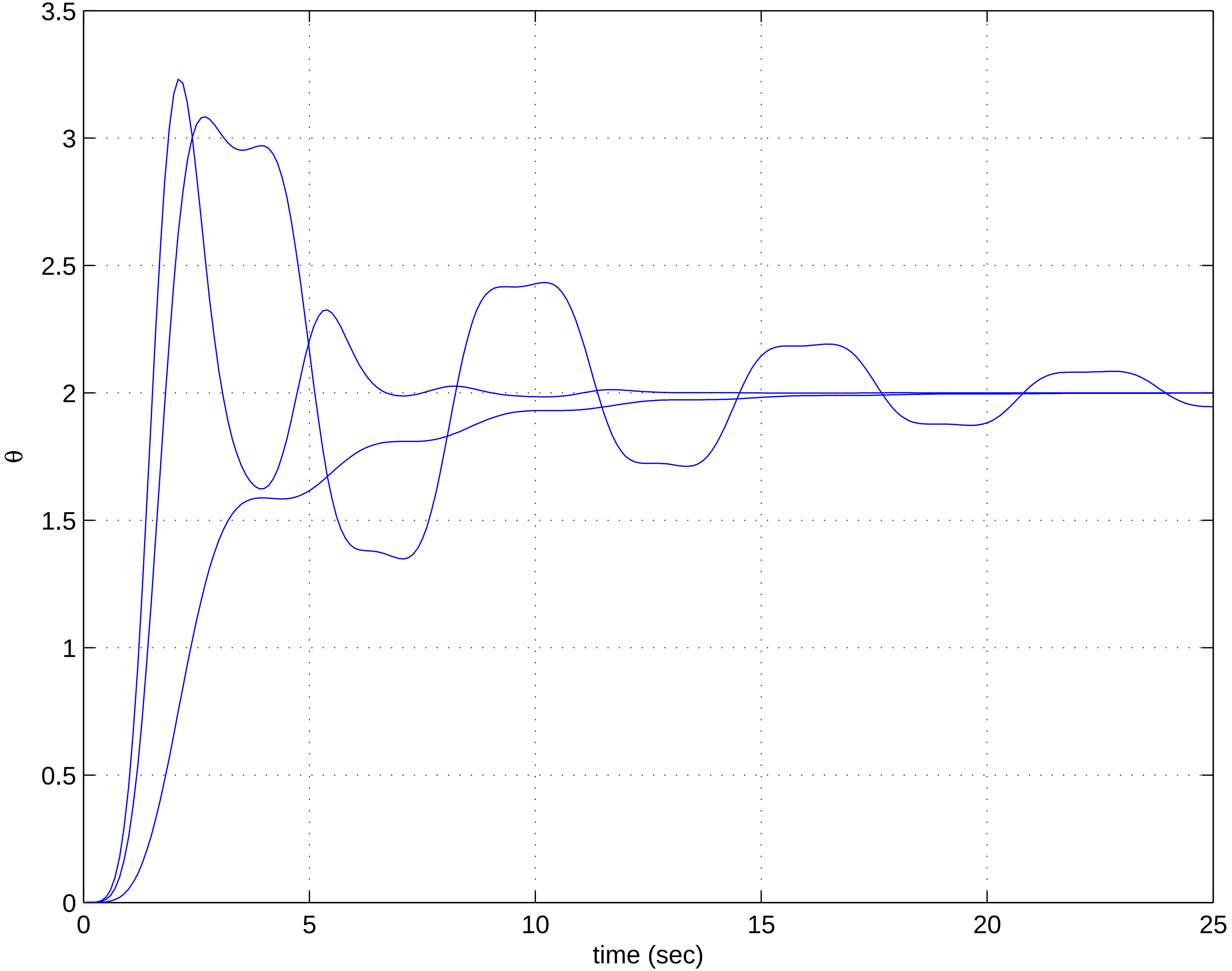
The update law for θ

$$\frac{d}{dt}\theta = -\gamma \cdot \left[2\, e\, \frac{\partial}{\partial \theta} e \right] = -\gamma_n \cdot y_m(t) \cdot e(t)$$

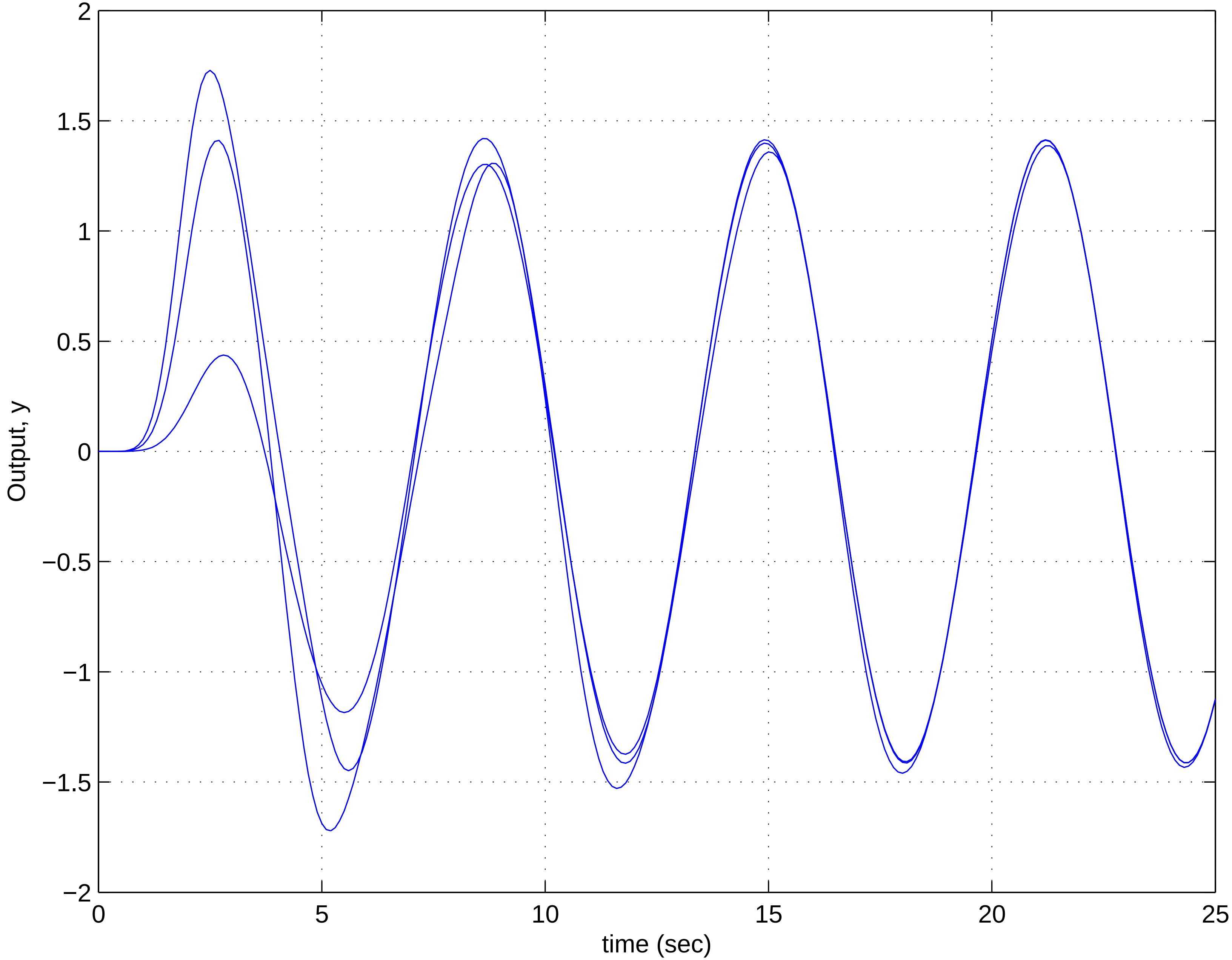
is simulated for various values of

$$\gamma = 0.5, \quad 2.5, \quad 4.5$$

with $u_c(t) = \sin t$.



The behavior of θ for various values of γ.



The behavior of $y(t)$ for various values of γ.

MIT rule (Example 5.2)

Suppose that the system dynamics are

$$\frac{d}{dt}y = -a\,y + b\,u, \qquad y = \frac{b}{p+a}\,u$$

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$$\frac{d}{dt}y_m = -a_m\,y_m + b_m\,u_c, \qquad y_m = \frac{b_m}{p+a_m}\,u_c$$

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The proportional controller that solves the problem is given by

$$u(t) = \frac{T(p)}{R(p)}\,u_c(t) - \frac{S(p)}{R(p)}\,y(t) = \theta_1\,u_c(t) - \theta_2\,y(t)$$

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The proportional controller that solves the problem is given by

$$u(t) = \theta_1\,u_c(t) - \theta_2\,y(t)$$

where the gains to ensure the desired system response are

$$\theta_1 = \theta_1^0 = \frac{b_m}{b}, \qquad \theta_2 = \theta_2^0 = \frac{a_m - a}{b}$$

MIT rule (Example 5.2)

Introduce the error signal

$$e(t) = y(t) - y_m(t) = \frac{b\,\theta_1}{p + a + b\,\theta_2}\,u_c(t) - \frac{b_m}{p + a_m}\,u_c$$

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Computing partial derivatives of $e(\cdot)$ w.r.t. θ_1, θ_2, we have

$$\begin{aligned} \frac{\partial e}{\partial \theta_1} &= \frac{b}{p + a + b\,\theta_2}\,u_c(t) \\ \frac{\partial e}{\partial \theta_2} &= -\frac{b\,\theta_1 \cdot b}{(p + a + b\,\theta_2)^2}\,u_c(t) = \frac{-b}{p + a + b\,\theta_2}\,y(t) \end{aligned}$$

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\end{aligned}$$

Such formulas cannot be used for updating θ_1 and θ_2 values.

Indeed, constants a and b are not known!

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Not much can be done, we will assume that we can initialize θ_2 around its nominal value

$$\theta_2 \approx \theta_2^0 = \frac{a_m - a}{b}$$

MIT rule (Example 5.2)

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Not much can be done, we will assume that we can initialize θ_2 around its nominal value

$$\theta_2 \approx \theta_2^0 = \frac{a_m - a}{b} \quad \Rightarrow \quad (a + b\,\theta_2) \approx a_m$$

MIT rule (Example 5.2)

This results in the following relations

$$\begin{aligned} \frac{d}{dt}\theta_1 &= -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_1} \approx -\gamma \cdot e(t) \cdot \left(\frac{b}{s + a_m} u_c(t) \right) \\ &= -\gamma_n \cdot e(t) \cdot \left(\frac{a_m}{s + a_m} u_c(t) \right) \end{aligned}$$

MIT rule (Example 5.2)

This results in the following relations

$$\frac{d}{dt}\theta_1 \approx -\gamma_n \cdot e(t) \cdot \left(\frac{a_m}{s+a_m} u_c(t)\right)$$

$$\begin{aligned}\frac{d}{dt}\theta_2 &= -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_2} \approx -\gamma \cdot e(t) \cdot \left(\frac{-b}{s+a_m} y(t)\right) \\ &= \gamma_n \cdot e(t) \cdot \left(\frac{a_m}{s+a_m} y(t)\right)\end{aligned}$$

where

$$\gamma_n = \gamma \frac{b}{a_m}$$

and should be positive!

MIT rule (Example 5.2)

This results in the following relations

$$\begin{aligned}
\frac{d}{dt}\theta_1 &\approx -\gamma_n \cdot e(t) \cdot \left(\frac{a_m}{s+a_m} u_c(t)\right) \\
\frac{d}{dt}\theta_2 &= -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_2} \approx -\gamma \cdot e(t) \cdot \left(\frac{-b}{s+a_m} y(t)\right) \\
&= \gamma_n \cdot e(t) \cdot \left(\frac{a_m}{s+a_m} y(t)\right)
\end{aligned}$$

where

$$\gamma_n = \gamma \frac{b}{a_m}$$

and should be positive!

To implement this algorithm we need to know the sign of b!

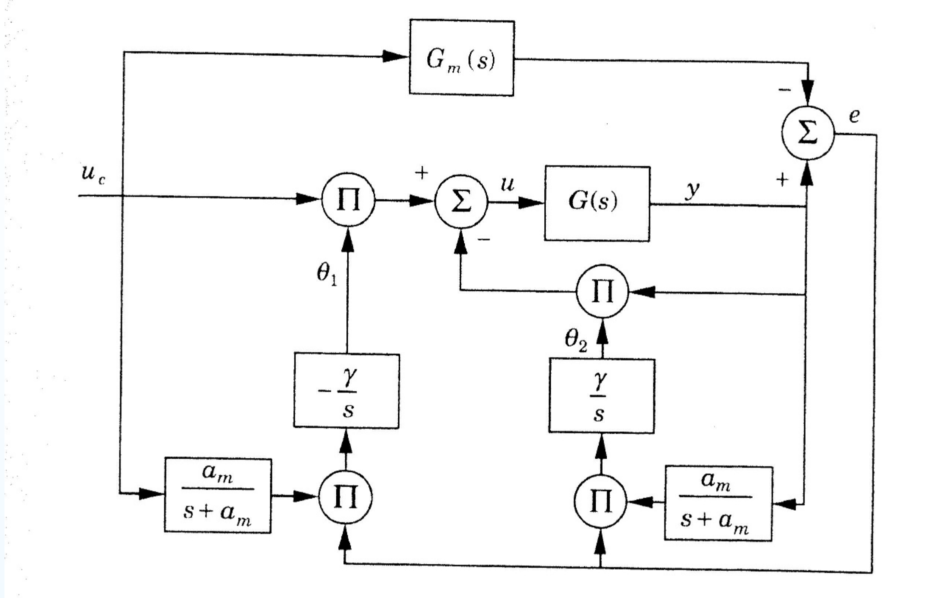

Figure 5.4 Block diagram of a model-reference controller for a first-order process.

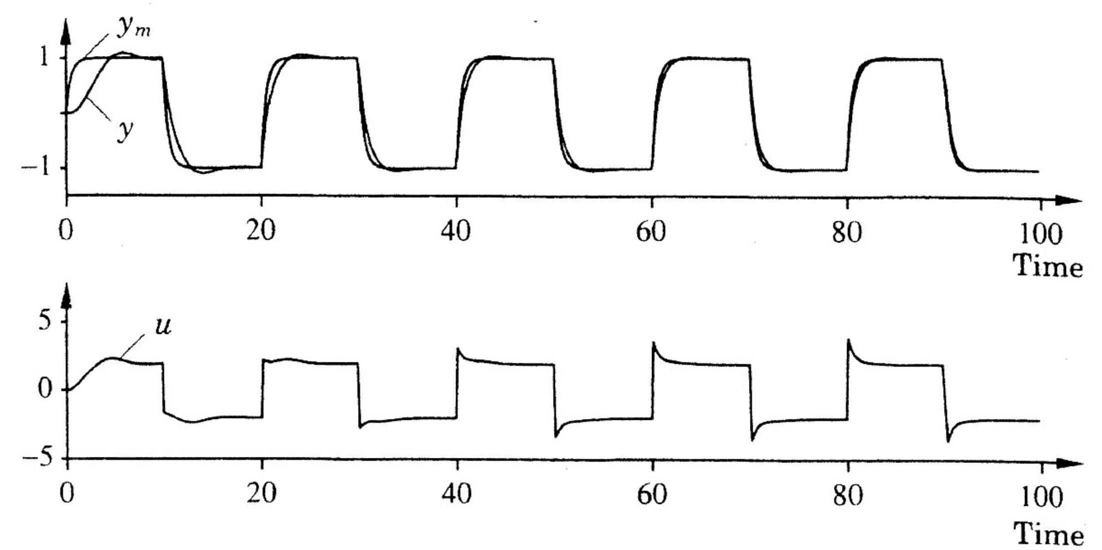

Figure 5.5 Simulation of the system in Example 5.2 using an MRAS. The parameter values are $a = 1$, $b = 0.5$, $a_m = b_m = 2$, and $\gamma = 1$.

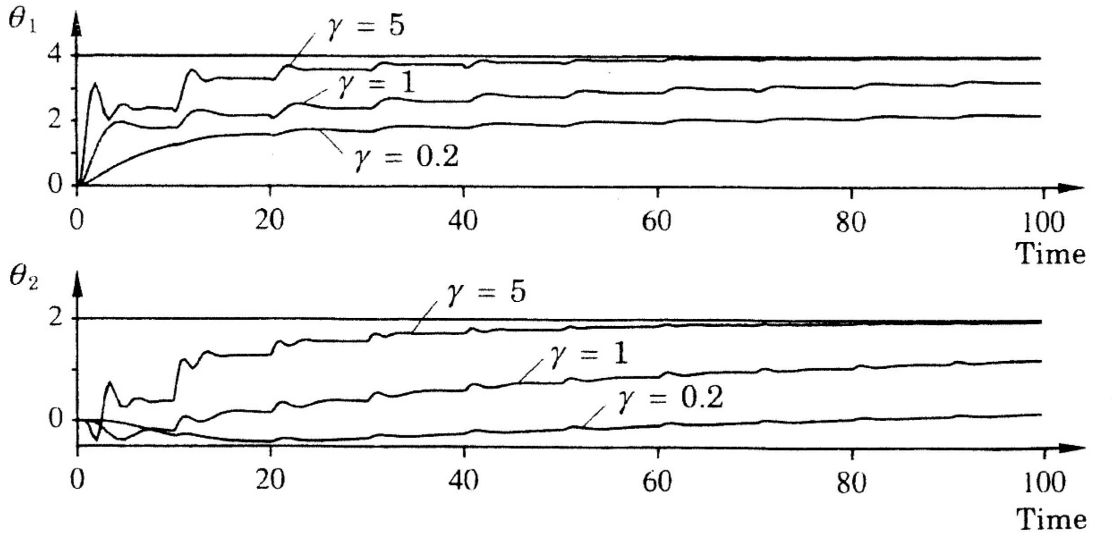

Figure 5.6 Controller parameters θ_1 and θ_2 for the system in Example 5.2 when $\gamma = 0.2$, 1 and 5.

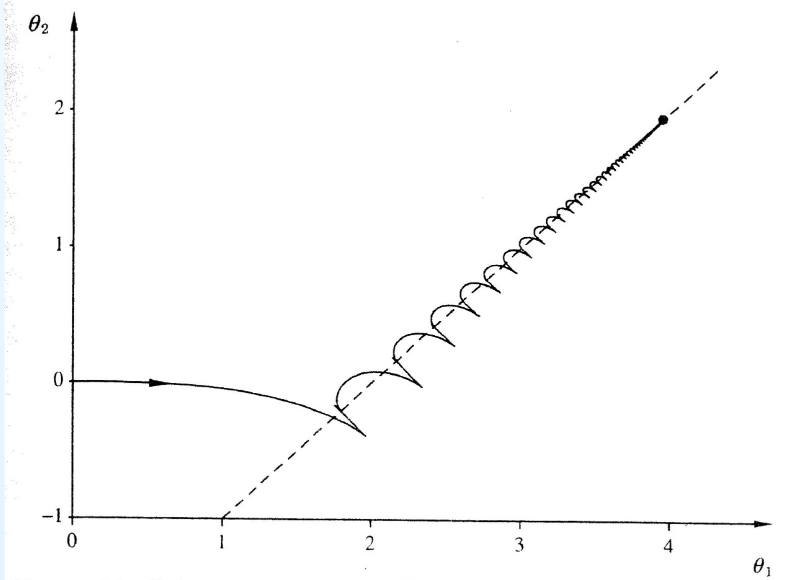

Figure 5.7 Relation between controller parameters θ_1 and θ_2 when the system in Example 5.2 is simulated for 500 time units. The dashed line shows the line $\theta_2 = \theta_1 - a/b$. The dot indicates the convergence point.

MIT rule (Example 5.3)

Consider the static system with unknown gain k

$$y(t) = k \cdot u(t), \qquad G(s) \equiv 1$$

and the problem of amplifying $u_c(t)$ so that we match

$$y_m(t) = k_0 \cdot u_c(t)$$

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$$y_m(t) = k_0 \cdot u_c(t)$$

With $u(t) = \theta\, u_c(t)$ introduce the error

$$e(t) = y(t) - y_m(t) = k \cdot \Big(\theta\, u_c(t)\Big) - k_0 \cdot u_c(t) = k\,\left(\theta - \theta^0\right) u_c(t)$$

with $\theta^0 = k_0/k$.

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with $\theta^0 = k_0/k$.

$$\frac{d}{dt}\theta(t) = -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta} = -\gamma \cdot k\,\left(\theta(t) - \theta^0\right) u_c(t) \cdot k\, u_c(t)$$

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with $\theta^0 = k_0/k$.

$$\frac{d}{dt}\theta(t) = -\gamma \cdot k^2 \cdot \left(u_c(t)\right)^2 \cdot \left(\theta(t) - \theta^0\right)$$

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with $\theta^0 = k_0/k$.

$$\frac{d}{dt}\left(\theta(t) - \theta^0\right) = -\gamma_n \cdot k \cdot \left(u_c(t)\right)^2 \cdot \left(\theta(t) - \theta^0\right)$$

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Consider the static system with unknown gain k

$$y(t) = k \cdot u(t), \qquad G(s) \equiv 1$$

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with $\theta^0 = k_0/k$.

$$\left(\theta(t) - \theta^0\right) = \exp\left\{-\gamma_n \cdot k \cdot \int_0^t \left(u_c(t)\right)^2 d\tau\right\} \cdot \left(\theta(0) - \theta^0\right)$$

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with $\theta^0 = k_0/k$.

$$e(t) = k \cdot \underbrace{\exp\left\{-\gamma_n \cdot k \cdot \int_0^t \left(u_c(t)\right)^2 d\tau\right\} \cdot \left(\theta(0) - \theta^0\right)}_{\theta(t)-\theta^0} \cdot u_c(t)$$

MIT rule (Example 5.3), cont'd

For the system and model given by

$$y(t) = k \cdot u(t), \qquad y_m(t) = k_0 \cdot u_c(t)$$

we define $e(t) = y(t) - y_m(t)$ and take

$$u(t) = \theta(t)\, u_c(t), \qquad \frac{d}{dt}\theta(t) = -\gamma_n \cdot k \cdot \left(u_c(t)\right)^2 \cdot \left(\theta(t) - \theta^0\right)$$

MIT rule (Example 5.3), cont'd

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we define $e(t) = y(t) - y_m(t)$ and take

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MIT rule (Example 5.3), cont'd

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we define $e(t) = y(t) - y_m(t)$ and take

$$u(t) = \theta(t)\, u_c(t), \qquad \frac{d}{dt}\theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t)$$

As the result we obtain

$$\theta(t) = \theta^0 + \sigma(t), \qquad e(t) = k \cdot \sigma(t) \cdot u_c(t)$$

$$\sigma(t) = \exp\Big\{-\gamma_n \cdot k \cdot I_t\Big\}\Big(\theta(0) - \theta^0\Big), \qquad I_t = \int_0^t (u_c(t))^2\, d\tau$$

MIT rule (Example 5.3), cont'd

For the system and model given by

$$y(t) = k \cdot u(t), \qquad y_m(t) = k_0 \cdot u_c(t)$$

we define $e(t) = y(t) - y_m(t)$ and take

$$u(t) = \theta(t)\, u_c(t), \qquad \frac{d}{dt}\theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t)$$

As the result we obtain

$$\theta(t) = \theta^0 + \sigma(t), \qquad e(t) = k \cdot \sigma(t) \cdot u_c(t)$$

$$\sigma(t) = \exp\Big\{-\gamma_n \cdot k \cdot I_t\Big\}\Big(\theta(0) - \theta^0\Big), \qquad I_t = \int_0^t (u_c(t))^2\, d\tau$$

If $\theta(0) \neq \theta^0$, for $e(t) \to 0$ as $t \to \infty$ we need:

$$\exp\Big\{-\gamma_n \cdot k \cdot I_t\Big\} \to 0 \qquad \text{or} \qquad u_c(t) \to 0$$

MIT rule (Example 5.3), cont'd

For the system and model given by

$$y(t) = k \cdot u(t), \qquad y_m(t) = k_0 \cdot u_c(t)$$

we define $e(t) = y(t) - y_m(t)$ and take

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.

Tuning the Gain for MIT rule

Consider again the problem with scaling the reference

$$y = k \cdot G(p)\, u, \qquad y_m = \color{red}{k_0} \cdot G(p)\, \color{red}{u_c}, \qquad u = \theta\, \color{red}{u_c}$$

where $\theta(t)$ is determined by MIT rule:

$$\frac{d}{dt}\theta = -\gamma \cdot y_m \cdot e, \qquad e = y - y_m$$

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The equation for θ can be re-written as follows

$$\frac{d}{dt}\theta = -\gamma \cdot y_m \cdot \Big(y - y_m\Big) = -\gamma \cdot y_m \cdot \Big(k \cdot G(p)\, \theta\, u_c - y_m\Big)$$

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$$\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p)\Big[\theta(t) u_c(t)\Big] = \gamma y_m^2(t)$$

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Here

- the functions $y_m(t)$ and $u_c(t)$ are known!
- the range of the constant gain γ, for which the nominal value θ^0 (its stationary point) is stable, should be determined.

Tuning the Gain for MIT rule (cont'd)

$$\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p)\Big[\theta(t)\, u_c(t)\Big] = \gamma y_m^2(t)$$

In general the analysis of stability is difficult!

Tuning the Gain for MIT rule (cont'd)

$$\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p)\Big[\theta(t)\, u_c(t)\Big] = \gamma y_m^2(t)$$

In general the analysis of stability is difficult!

Consider the case when $y_m(t) \equiv y_m^o$, $u_c(t) = u_c^o$, then ODE

$$\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m^o \cdot u_c^0 \cdot G(p)\Big[\theta(t)\Big] = \gamma (y_m^o)^2$$

is linear and time-invariant!

Tuning the Gain for MIT rule (cont'd)

$$\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p)\Big[\theta(t)\, u_c(t)\Big] = \gamma y_m^2(t)$$

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Stability is determined by the roots of the algebraic equation

$$s + \mu \cdot G(s) = 0, \qquad \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0$$

Tuning the Gain for MIT rule (cont'd)

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is linear and time-invariant!

Stability is determined by the roots of the algebraic equation

$$s + \mu \cdot G(s) = 0, \qquad \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0$$

Root locus analysis (variation of zeros with μ) can be used.
A reasonable value for γ can be obtained from this analysis and might work for slowly varying signals.

Example 5.4

Let, as in Example 5.1

$$G(s) = \frac{1}{s+1}, \qquad k = 1, \qquad k_0 = 2$$

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The characteristic equation

$$s + \mu \, \frac{1}{s+1} = 0 \quad \Leftrightarrow \quad s^2 + s + \mu = 0$$

has stable zeros if and only if

$$\mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0 = \gamma \cdot \Big(k_0 \, G(0) \, u_c^0\Big) \cdot u_c^0 = 2\,\gamma\,(u_c^0)^2 > 0$$

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So, $\gamma > 0$ will work.

Note, however, that the transient depends on u_c^0!

The relative damping is $\zeta = \frac{1}{2\sqrt{\mu}} = \frac{1}{2\sqrt{2\gamma}\,|u_c^0|}$.

$\mu \approx 1$ is reasonable $\Leftarrow$ take $\gamma \approx 0.5$ for $u_c^0 \approx 1$ in average.

Example 5.5

Consider the stable system with relative degree 2:

$$G(s) = \frac{1}{s^2 + a_1\, s + a_2}, \qquad a_1 > 0, \qquad a_2 > 0$$

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The characteristic equation

$$s + \mu \frac{1}{s^2 + a_1\, s + a_2} = 0 \quad \Leftrightarrow \quad s^3 + a_1\, s^2 + a_2\, s + \mu = 0$$

has stable zeros if and only if

$$\mu > 0 \qquad \text{and} \qquad a_1\, a_2 > \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0$$

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$$s + \mu\, \frac{1}{s^2 + a_1\, s + a_2} = 0 \quad \Leftrightarrow \quad s^3 + a_1\, s^2 + a_2\, s + \mu = 0$$

has stable zeros if and only if

$$\mu > 0 \qquad \text{and} \qquad a_1\, a_2 > \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0$$

Conclusion: with any choice of $\gamma > 0$, stability is lost for sufficiently large magnitudes of the reference signal u_c^0!

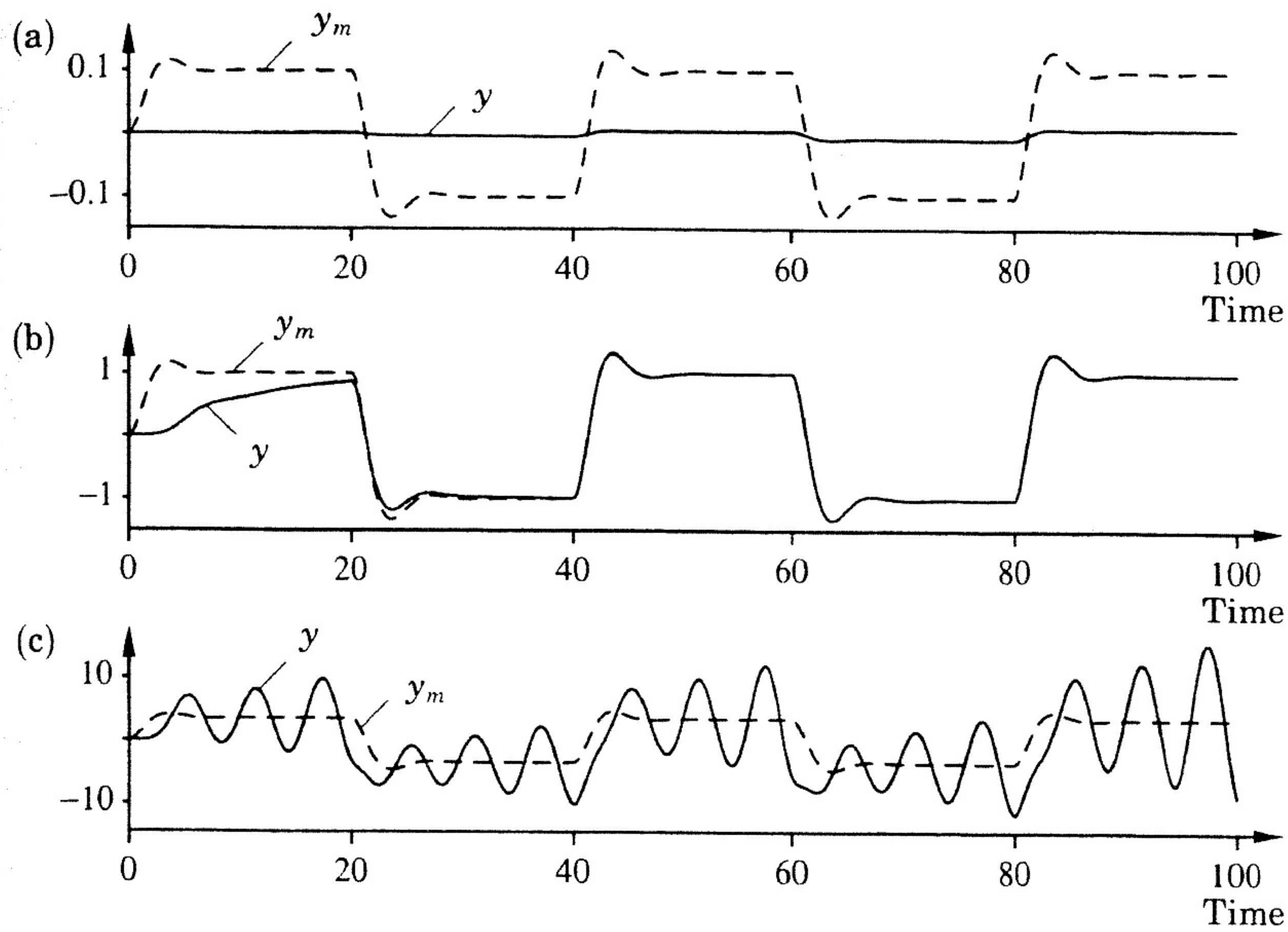

Figure 5.8 Simulation of the MRAS in Example 5.5. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. The model output y_m is a dashed line; the process output is a solid line. The following parameters are used: $k = a_1 = a_2 = \theta^0 = 1$, and $\gamma = 0.1$.

Normalized MIT rule

$$\frac{d}{dt}\theta = -\gamma \cdot e(t,\theta) \cdot \frac{\phi}{\alpha + \phi^T \phi}, \qquad \phi = \frac{\partial}{\partial \theta} e(t,\theta), \qquad \alpha > 0$$

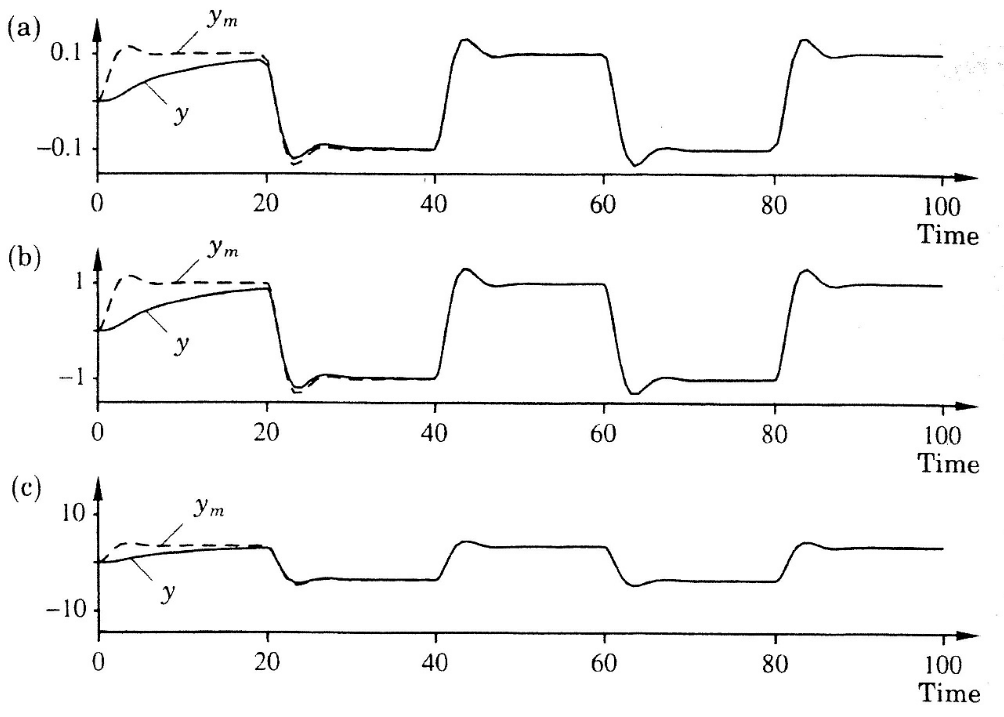

Figure 5.9 Simulation of the MRAS in Example 5.5 with the normalized MIT rule. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. Compare with Fig. 5.8. The model output y_m is a dashed line; the process output is a solid line. The parameters used are $k = a_1 = a_2 = \theta^0 = 1$, $\alpha = 0.001$, and $\gamma = 0.1$.

Next Lecture / Assignments:

Next meeting (May 24, 13:00-15:00, in A208Tekn):
Lyapunov-based design.

Homework problem: The process and model are described by

$$G(s) = \frac{1}{s}, \qquad G_m(s) = \frac{2}{s+2}$$

For the control law

$$u(t) = \theta_1 \, u_c(t) - \theta_2 \, y(t)$$

design an MIT-like adaptation law such that

$$\theta_i \approx -\left(\gamma_1 + \gamma_2 \frac{1}{p}\right) \left[e \, \frac{\partial}{\partial \theta_i} e \right], \qquad i \in \{1, 2\}.$$

Simulate the MRAS with various gains.

Consider $\gamma_{1,2} \in \{0, 1, 5\}$ and a unit square wave for $u_c(t)$.
Compare performance for different combinations.