Lecture 12: Lyapunov-based design

- Elements of Lyapunov Theory
- Adaptive design based on Lyapunov functions

Definition of Stability

The solution $x = 0$ of the system

$$\dot{x}(t) = f(x, t), \quad f(0, t) = 0, \; \forall t$$

is stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that

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can be chosen independent on t_0.

The solution is called uniformly asymptotically stable, if it is uniformly stable and for all $x(t_0)$ uniformly in t_0

$$\|x(t,t_0)\| \to 0 \qquad \text{as} \quad t \to +\infty$$

The simplest example

Let us investigate stability of the solution $x = 0$ of the system

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(1) $a \geq 0$

$\Rightarrow \quad -a\,(t - t_0) \leq 0$ for $t \geq t_0$

$\Rightarrow \quad |e^{-a\,(t-t_0)}| \leq 1$ for $t \geq t_0$

$\Rightarrow \quad |x(t, t_0)| \leq 1 \cdot |x(t_0)| \leq \delta$ for $t \geq t_0$.

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(2) $a > 0$

$\Rightarrow \quad |e^{-a\,(t-t_0)}| \to 0$ for $t \to \infty$

$\Rightarrow \quad |x(t, t_0)| \to 0$ as $t \to \infty$.

We conclude uniform asymptotic stability.

Classical Lyapunov's Theorem

A continuous function $\alpha : [0, c) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if

- it is strictly increasing
- $\alpha(0) = 0$

It belongs to class $\mathcal{K}_\infty$, if $c = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$

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Theorem: Let $x = 0$ be a stationary point of $\dot{x} = f(x, t)$.
Let $V(t, x)$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{d}{dt} V(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) \cdot f(x, t) \le -\alpha_3(\|x\|)$$

for $t \ge 0$, $\alpha_1(\cdot)$ and $\alpha_2(\cdot) \in \mathcal{K}$.

(i) If $\alpha_3(\cdot) \ge 0$, then $x = 0$ is *uniformly stable*.

(ii) If $\alpha_3(\cdot) \in \mathcal{K}$, then $x = 0$ is *uniformly asymptotically stable*.

Remarks on Classical Lyapunov's Theory

(1) V is *positive definite*: $\alpha_1(\|x\|) \leq V(t,x)$ means

- $\|x\| \geq r \quad \Rightarrow \quad V(t,x) \geq \alpha_1(r)$
- $\boxed{V\big(t, x(t)\big) \leq C \quad \Rightarrow \quad \|x(t)\| \leq \alpha_1^{-1}(C)}$

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(2) V is *decrescent*: $V(t,x) \le \alpha_2(\|x\|)$ means

- $\boxed{\|x(t_0)\| \le \delta \quad \Rightarrow \quad V\big(t_0, x(t_0)\big) \le \alpha_2(\delta) = C}$
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(3) Negative *Lie derivative*: $\frac{d}{dt}V(t,x) \le 0$ means

- $V\big(t, x(t)\big)$ is not increasing along the trajectories and converges (to a non negative number).
- $\boxed{\|x(t_0)\| \le \delta \quad \Rightarrow \quad \|x(t)\| \le \alpha_1^{-1}\big(\alpha_2(\delta)\big) = \varepsilon}$

Lyapunov equation

Let us investigate stability of the solution $\boldsymbol{x} = \mathbf{0}$ of the system

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\, \boldsymbol{x}(\boldsymbol{t})$$

without explicitly solving it.

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Let us investigate stability of the solution $x = 0$ of the system

$$\dot{x}(t) = A\, x(t)$$

without explicitly solving it. Consider the Lyapunov function

$$V(t, x) = x^T P\, x$$

Note that with a positive definite $P = P^T > 0$

$$\alpha_1(\|x\|) \equiv \lambda_{min}\{P\}\, \|x\|^2 \leq V(t, x) \leq \lambda_{max}\{P\}\, \|x\|^2 \equiv \alpha_2(\|x\|)$$

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Computing derivative along the trajectories we have

$$\frac{d}{dt}V = 0 + \dot{x}^T P\,x + x^T P\,\dot{x} = (A\,x)^T P\,x + x^T P\,A\,x$$

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Computing derivative along the trajectories we have

$$\frac{d}{dt}V = x^T\,\left(A^T\,P + P\,A\right)\,x = -x^T\,Q\,x \equiv -\alpha_3(\|x\|)$$

provided the Lyapunov equation

$$A^T\,P + P\,A = -Q, \qquad Q = Q^T > 0$$

is solved with $P = P^T > 0$.

The simplest example (cont'd)

Lyapunov equation for checking stability of the solution $x = 0$ of the system

$$\dot{x}(t) = -a\, x(t)$$

is

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$$(-a)^T P + P\,(-a) = -Q \quad \Leftrightarrow \quad (2\,a)\,P = Q$$

Clearly, for any

$$Q > 0$$

the solution

$$P = Q/(2\,a)$$

is positive if and only if

$$a > 0$$

Barbalat's Lemma (1959)

Lemma: Suppose $g(t)$ is *uniformly continuous* on $[0, \infty)$ and

$$\lim_{t \to \infty} \int_0^t g(\tau)\, d\tau$$ exists and is finite.

Then, $$\lim_{t \to \infty} g(t) = 0.$$

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- $g(t)$ is *uniformly continuous* $\Longleftarrow$ $\dot{g}(t)$ is bounded.

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Remarks:

- $g(t)$ is *uniformly continuous* $\Longleftarrow$ $\dot{g}(t)$ is bounded.
- $g(t)$ and $\dot{g}(t)$ are bounded, $\lim_{t\to\infty} \int_0^t g^2(\tau)\, d\tau < \infty$
  $$\Longrightarrow \quad \lim_{t\to\infty} g(t) = 0.$$

Boundedness and convergence set

Theorem: Consider a system $\dot{x} = f(t,x)$ where $f(x,t)$ is piecewise continuous in t and locally Lipschitz in $x \in D$, and $f(t,0)$ is uniformly bounded for all $t \geq 0$.
Let $V(x,t)$ be a continuously differentiable function such that

$$W_1(x) \leq V(x,t) \leq W_2(x)$$

$$\frac{d}{dt}V = \frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x) \cdot f(x,t) \leq -W(x)$$

for $t \geq 0$, where $W_1(x)$ and $W_2(x)$ are continuous positive defined and radially unbounded, and $W(x)$ is continuous positive semidefinite.

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for $t \geq 0$, where $W_1(x)$ and $W_2(x)$ are continuous positive defined and radially unbounded, and $W(x)$ is continuous positive semidefinite.

Take r: $\{\|x\| \leq r\} \in D$ and define $\rho = \min\limits_{\|x\|=r} \{W_1(x)\}$.

Then, for any $x(t_0) \in \{W_2(x) \leq \rho\}$:

(i) $x(t)$ is bounded for all $t \geq t_0$.

(ii) $W\big(x(t)\big) \to 0$ as $t \to \infty$.

Example 5.7

The system dynamics and the model to match are

$$\dot{y}(t) = -a\,y(t) + b\,u(t), \qquad \dot{y}_m(t) = -a_m\,y_m(t) + b_m\,u_c(t)$$

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$$u(t) = \theta_1\,u_c(t) - \theta_2\,y(t), \qquad e(t) = y(t) - y_m(t)$$

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Taking time derivative of $e(t)$, we obtain

$$\frac{d}{dt}e = -a_m e - (b\theta_2 + a - a_m)\, y + (b\theta_1 - b_m)\, u_c$$

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Introduce a Lyapunov function candidate as

$$V(x) = \frac{1}{2}\left(e^2 + \frac{(b\theta_2 + a - a_m)^2}{b\gamma} + \frac{(b\theta_1 - b_m)^2}{b\gamma}\right)$$

where $\quad x(t) = \left[e(t),\ \ b\,\theta_2(t) + a - a_m,\ \ b\,\theta_1(t) - b_m\right]^T.$

Example 5.7 (cont'd)

The time derivative of

$$V = \frac{1}{2}\left(e^2 + \frac{(b\theta_2 + a - a_m)^2}{b\gamma} + \frac{(b\theta_1 - b_m)^2}{b\gamma}\right)$$

is

$$\frac{d}{dt}V = e\frac{d}{dt}e + \frac{b\theta_2 + a - a_m}{\gamma}\frac{d}{dt}\theta_2 + \frac{b\theta_1 - b_m}{\gamma}\frac{d}{dt}\theta_1$$

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$$\begin{aligned} \dot{V} &= e\dot{e} + \frac{(b\theta_2 + a - a_m)}{\gamma}\dot{\theta}_2 + \frac{(b\theta_1 - b_m)}{\gamma}\dot{\theta}_1 \\ &= e\Big(- a_m e - (b\theta_2 + a - a_m)\, y + (b\theta_1 - b_m)\, u_c\Big) + \\ &\qquad + \frac{(b\theta_2 + a - a_m)}{\gamma}\frac{d\theta_2}{dt} + \frac{(b\theta_1 - b_m)}{\gamma}\frac{d\theta_1}{dt} \end{aligned}$$

Example 5.7 (cont'd)

The time derivative of

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$$\begin{aligned}
\dot{V} &= e\dot{e} + \frac{(b\theta_2 + a - a_m)}{\gamma}\dot{\theta}_2 + \frac{(b\theta_1 - b_m)}{\gamma}\dot{\theta}_1 \\
&= e\left(-a_m e - (b\theta_2 + a - a_m)\, y + (b\theta_1 - b_m)\, u_c\right) + \\
&\qquad + \frac{(b\theta_2 + a - a_m)}{\gamma}\frac{d\theta_2}{dt} + \frac{(b\theta_1 - b_m)}{\gamma}\frac{d\theta_1}{dt} \\
&= -a_m e^2 + \frac{(b\theta_2 + a - a_m)}{\gamma}\left[\frac{d\theta_2}{dt} - \gamma y e\right] + \\
&\qquad + \frac{(b\theta_1 - b_m)}{\gamma}\left[\frac{d\theta_1}{dt} + \gamma u_c e\right]
\end{aligned}$$

Example 5.7 (cont'd)

The time derivative of

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&\qquad + \frac{(b\theta_2 + a - a_m)}{\gamma}\frac{d\theta_2}{dt} + \frac{(\theta_1 - b_m)}{\gamma}\frac{d\theta_1}{dt} \\
&= -a_m e^2 + \frac{(b\theta_2 + a - a_m)}{\gamma}\Big[\, 0 \,\Big] + \\
&\qquad + \frac{(\theta_1 - b_m)}{\gamma}\Big[\, 0 \,\Big]
\end{aligned}$$

Example 5.7 (cont'd)

The choice of the updating laws

$$\frac{d\theta_1}{dt} = -\gamma u_c e, \quad \frac{d\theta_2}{dt} = \gamma y e$$

leads to shaping the time-derivative for V as

$$\frac{d}{dt}V = -a_m\, e^2(t), \qquad \left\{ \leftarrow \text{ this is } W(x)! \right\}$$

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Since $V(x) = x^T P x$ with $P = \mathrm{diag}\{1, 1/(b\gamma), 1/(b\gamma)\}$, we conclude from the Theorem that

(i) $x(t) = \left[e(t),\ b\,\theta_2(t) + a - a_m,\ b\,\theta_1(t) - b_m\right]^T$ is bounded,

(ii) $W\big(x(t)\big) \to 0 \quad \Longrightarrow \quad e(t) \to 0.$

Note that parameter convergence is not guaranteed.

Lyapunov vs. MIT

Lyapunov-based design and MIT rule give:

$$\dot{\theta} = \gamma \, \phi \, e$$

where

Lyapunov: $\phi = \begin{bmatrix} -u_c(t) \\ y(t) \end{bmatrix}$, MIT: $\phi = \dfrac{a_m}{p + a_m} \begin{bmatrix} -u_c(t) \\ y(t) \end{bmatrix}$

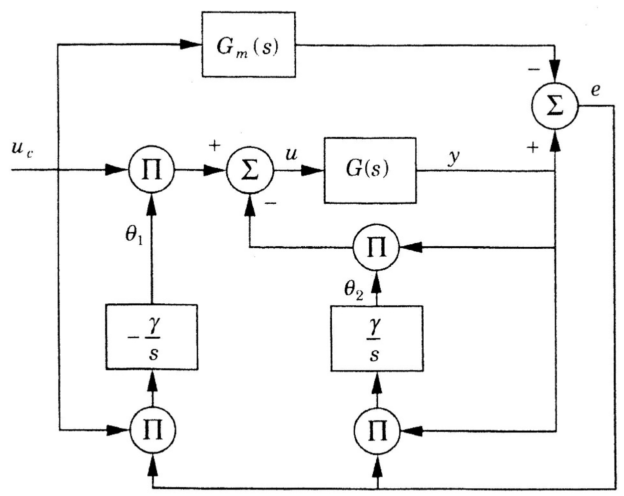

Figure 5.11 Block diagram of an MRAS based on Lyapunov theory for a first-order system. Compare with the controller based on the MIT rule for the same system in Fig. 5.4.

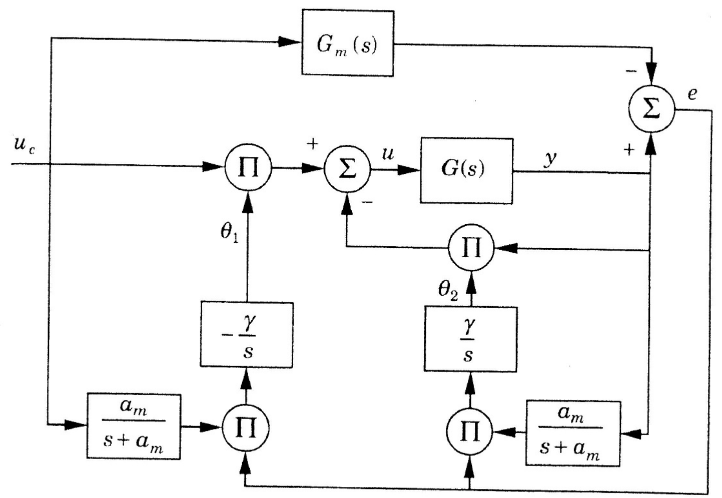

Figure 5.4 Block diagram of a model-reference controller for a first-order process.

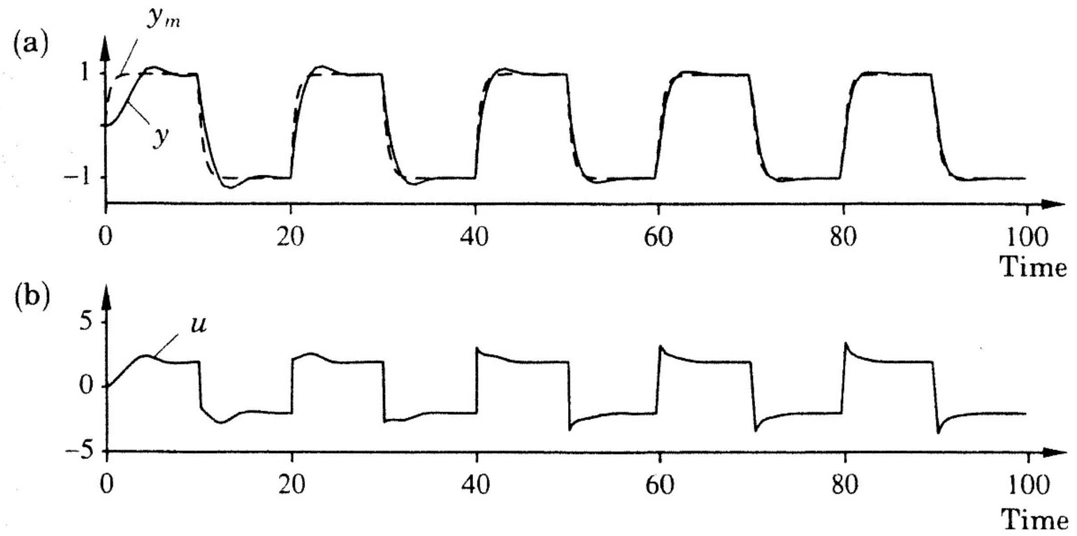

Figure 5.12 Simulation of the system in Example 5.7 using an adaptive controller based on Lyapunov theory. The parameter values are $a = 1, b = 0.5$, $a_m = b_m = 2$, and $\gamma = 1$. (a) Process (solid line) and model (dashed line) outputs. (b) Control signal.

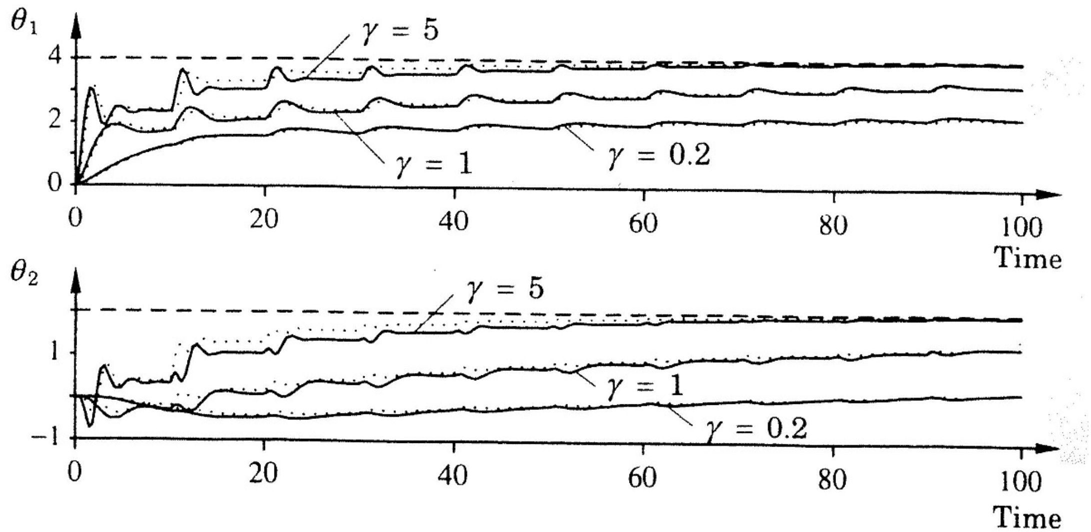

Figure 5.13 Controller parameters θ_1 and θ_2 for the system in Example 5.7 when $\gamma = 0.2$, 1, and 5. The dotted lines are the parameters obtained with the MIT rule. Compare Fig. 5.6.

General Case

Given a continuous time system and the target dynamics

$$\dot{x} = Ax + Bu, \qquad \dot{x}_m = A_m x_m + B_m u_c$$

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If the model-matching problem is solvable, then the error dynamics is

$$\begin{aligned} \frac{d}{dt} e &= Ax + Bu - A_m x_m - B_m u_c \\ &= A_m e + \Big(A - A_m - BL \Big) x + \Big(BM - B_m \Big) u_c \\ &= A_m e + \Psi (\theta - \theta^0) \end{aligned}$$

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Consider a Lyapunov function candidate

$$V = \frac{1}{2}\left[e^T P e + \frac{1}{\gamma}(\theta - \theta^0)^T(\theta - \theta^0)\right]$$

General Case (cont'd)

The time-derivative of $V = \frac{1}{2}\left[e^T P e + \frac{1}{\gamma}(\theta - \theta^0)^T(\theta - \theta^0)\right]$ is

$$\dot{V} = \tfrac{1}{2} e^T \left[P A_m + A_m{}^T P\right] e + (\theta - \theta^0)^T \Psi^T P e + \frac{1}{\gamma}(\theta - \theta^0)^T \dot{\theta} e$$

General Case (cont'd)

The time-derivative of $V = \frac{1}{2}\left[e^T P e + \frac{1}{\gamma}(\theta - \theta^0)^T(\theta - \theta^0)\right]$ is

$$\dot{V} = \tfrac{1}{2} e^T \left[P A_m + A_m{}^T P\right] e + (\theta - \theta^0)^T \Psi^T P e + \frac{1}{\gamma}(\theta - \theta^0)^T \dot{\theta} e$$

If we solve the Lyapunov equation for P

$$P\, A_m + A_m{}^T P = -Q, \quad Q > 0$$

and choose the update law as

$$\dot{\theta} = -\gamma\, \Psi^T\, P\, e = -\gamma\, \Psi^T\, P\, (x - x_m)$$

General Case (cont'd)

The time-derivative of $V = \frac{1}{2}\left[e^T P e + \frac{1}{\gamma}(\theta-\theta^0)^T(\theta-\theta^0)\right]$ is

$$\dot V = \tfrac{1}{2} e^T \left[P A_m + A_m{}^T P\right] e + (\theta-\theta^0)^T \Psi^T P e + \frac{1}{\gamma}(\theta-\theta^0)^T \dot\theta e$$

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then

$$\dot V = -\frac{1}{2}\, e^T(t)\, Q\, e(t)$$

and we conclude that $e(t) \to 0$.

Next Lecture / Assignments:

Next meeting (May 26, 10:00-12:00, in A206Tekn):
Adaptation of a feedforward gain.

Homework problem: Consider the process

$$G(s) = \frac{1}{s(s+a)}$$

where a is an unknown parameter.
Determine a controller that can give the closed-loop system

$$G_m(s) = \frac{\omega^2}{s^2 + 2\zeta\omega\, s + \omega^2}$$

Determine model-reference adaptive controllers based on MIT-rule and on Lyapunov theory.