2.153 Adaptive Control
Lecture 2
Simple Adaptive Systems: Identification and Control

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Tuesday 17-Feb-2015

Introduction

Last time:

- Parameter identification
  - Algebraic (error model 1) and dynamic (error model 3)
  - Scalar and vector parameter systems
  - Non-recursive and recursive
- Stability
  - Introduction to using Lyapunov functions

Today:

- Identification of multiple parameters in first order plant
  - Error model 1 and 3
  - Determining update law using Lyapunov functions
- Adaptive control

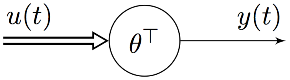
Error Models

See page 273.

- Relate parameter errors (cannot measure) to output/measurement error (which can be measured)
- The stability and performance of an adaptive system is dependent upon the evolution of these errors
- An *error model* is the mathematical model which describes the evolution of these errors
- By using error models, we can understand and solve adaptive control problems more easily, as the error models are independent of the specific adaptive system

Identification of a Vector Parameter in an Algebraic System

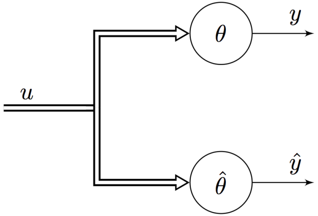
Problem:



$$y(t) = \theta^\top u(t)$$

- θ: unknown
- $y(t), u(t)$: measured

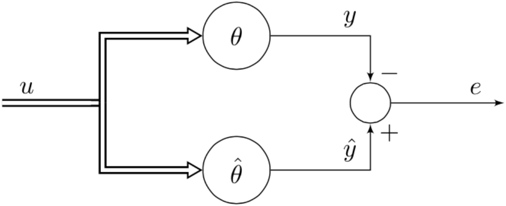
Construct a parameter estimate $\hat{\theta}$



Identification of a Vector Parameter in an Algebraic System

Difference the two output signals as

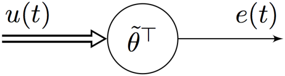
$$e = \hat{y} - y$$



Express the *parameter error* as

$$\widetilde{\theta} = \hat{\theta} - \theta$$

Reduces to Error Model 1:



Identification of a Vector Parameter in an Algebraic System

Error model 1:

$$e = \widetilde{\theta}^\top u$$

Propose the candidate Lyapunov function:

$$V(\widetilde{\theta}) = \widetilde{\theta}^\top \widetilde{\theta}$$

Time differentiating

$$\dot{V} = 2\widetilde{\theta}^\top \dot{\widetilde{\theta}}$$

Adaptive law

$$\dot{\widetilde{\theta}} = -eu$$

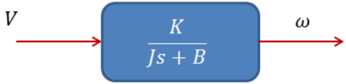
Gives

$$\begin{aligned}\dot{V} &= -2\widetilde{\theta}^\top eu \\ &= -2(\widetilde{\theta}^\top u)^2\end{aligned}$$

Thus $\dot{V} \leq 0 \;\Rightarrow$ stability

Parameter Identification: Motivating Example

Transfer function of a DC motor:



V : Voltage input
ω: Angular Velocity output
K, J, B: *unknown* physical parameters

Express the plant transfer function as:

$$\frac{\omega}{V} = \frac{K}{Js + B} = \frac{a_1}{s + \theta_1}$$

K, J, B unknown $\Rightarrow a_1,\ \theta_1$ unknown

Parameter Identification: Motivating Example

The differential equation describing the DC motor is

$$\dot{\omega} = -\theta_1 \omega + a_1 V$$

- The DC motor is a *first-order* dynamical system
- Recall: last time we assumed a_1 was known
- We looked at two procedures for identification
    - Error model 1
    - Error model 3
- Now we will consider both a_1 and θ_1 to be unknown
- Will go through both procedures (error model 1 and 3) again

Identification: First order plant, two unknown parameters

Plant Transfer Function:

$$\frac{x_p}{u} = \frac{a_1}{s + \theta_1}$$

Express the plant transfer function as

$$\begin{aligned}
\frac{x_p}{u} &= \frac{1}{s + \theta_1} \\
&= \frac{1}{s + \theta_m} \cdot \frac{s + \theta_m}{s + \theta_1} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{\frac{s+\theta_1}{s+\theta_m}} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{\frac{\theta_m - \theta_m}{s+\theta_m} + \frac{s+\theta_1}{s+\theta_m}} \\
&= \frac{1}{s + \theta_m} \cdot \frac{1}{1 - \frac{\theta_m - \theta_1}{s+\theta_m}}
\end{aligned}$$

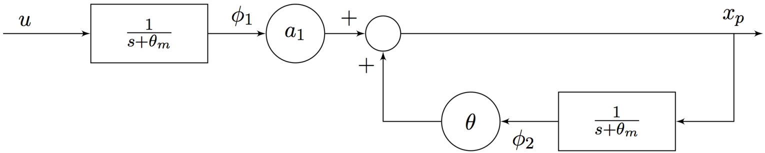
where $\theta_m > 0$ is a known positive parameter selected by control designer.

Identification: First order plant, two unknown parameters

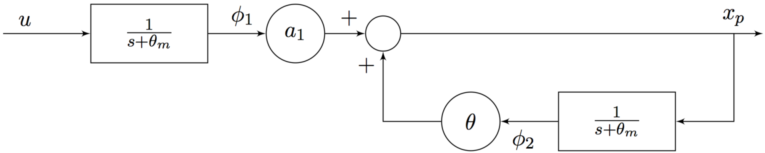
Define $\theta \triangleq \theta_m - \theta_1$ and simplify this transfer function as

$$\frac{\omega}{u} = \frac{1}{s+\theta_m} \cdot \frac{1}{1 + \frac{\theta}{s+\theta_m}}$$

and realize this in the following block diagram representation, where we are using two states to represent a first order system.



Identification: First order plant, two unknown parameters



Recall: when a_1 known

$$y(t) \equiv x_p(t) - a_1\phi(t) = \theta\phi_2(t)$$

$\Rightarrow$Error Model 1, Identify θ, a scalar

When a_1 unknown:

$$y(t) \equiv x_p(t) = a_1\phi(t) + \theta\phi_2(t) = \overline{\theta}^\top \phi(t)$$

$\Rightarrow$Error Model 1, Identify $\overline{\theta}$, a vector

Identification: First order plant, two unknown parameters

$$y(t) = \overline{\theta}^{\top} \phi(t)$$

- We have just seen how to determine an update law for this system on slide 5: error model 1 with vector parameter
- Now we will see an alternate method of identifying the two unknown parameters

Identification: First order plant, two unknown parameters - An Alternate Method

The differential equation describing the plant is given by

$$\dot{x}_p = -\theta_1 \omega + a_1 u$$

Generate an estimate of the plant output as follows, using parameter estimates in place of the unknown parameters

$$\dot{\hat{x}}_p = -\hat{\theta}_1 \hat{x}_p + \hat{a}_1 u$$

Define the following output error and parameter errors

$$\begin{aligned} e &= \hat{x}_p - x_p \\ \tilde{\theta}_1 &= \hat{\theta}_1 - \theta_1 \\ \tilde{a}_1 &= \hat{a}_1 - a_1 \end{aligned}$$

Identification: First order plant, two unknown parameters - An Alternate Method

The output error dynamics are given by

$$\dot{e} = -\hat{\theta}_1 \hat{x}_p + \hat{a}_1 u + \theta_1 x_p - a_1 u$$

Add and subtract $\theta_1 \hat{x}_p$

$$\begin{aligned}
\dot{e} &= -\hat{\theta}_1 \hat{x}_p + \theta_1 \hat{x}_p - \theta_1 \hat{x}_p + \hat{a}_1 u + \theta_1 x_p - a_1 u \\
&= (\theta_1 - \hat{\theta}_1)\hat{x}_p - \theta_1(\hat{x}_p - x_p) + \tilde{a}_1 u \\
&= -\theta_1 e - \tilde{\theta}_1 \hat{x}_p + \tilde{a}_1 u \\
&= -\theta_1 e + \tilde{\theta}^\top \phi
\end{aligned}$$

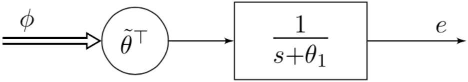
where

$$\theta = \begin{bmatrix} \theta_1 \\ a_1 \end{bmatrix} \qquad \text{and} \qquad \phi = \begin{bmatrix} -\hat{x}_p \\ u \end{bmatrix}$$

Identification: First order plant, two unknown parameters - An Alternate Method

Error Model 3:

$$\dot{e} = -\theta_1 e + \tilde{\theta}^\top \phi$$



- We saw error model 3 last lecture for a system with one unknown
- We will again determine an update law using a Lyapunov function
- Choose a quadratic function V of the dominant errors in the system

$$V(e, \widetilde{\theta}) = \frac{1}{2}\left(e^2 + \widetilde{\theta}^\top \widetilde{\theta}\right)$$

<u>Goal: Choose $\dot{\widetilde{\theta}}$ so that $\dot{V} \leq 0$</u>

Identification: First order plant, two unknown parameters - An Alternate Method

Take the time derivative of V

$$\begin{aligned} \dot{V} &= e\dot{e} + \widetilde{\theta}\dot{\widetilde{\theta}} \\ &= -\theta_1 e^2 + \widetilde{\theta}^\top e\phi + \widetilde{\theta}^\top \dot{\widetilde{\theta}} \end{aligned}$$

Choose

$$\dot{\widetilde{\theta}} = -e\phi$$

$$\Rightarrow \quad \dot{V} = -\theta_1 e^2$$

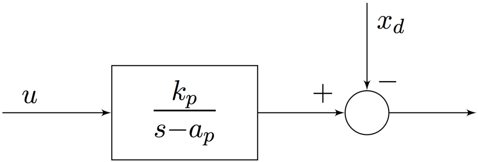
Thus $\dot{V} \leq 0 \Rightarrow$ stability.

Adaptive Control

- Everything we have seen thus far has been identification of unknown parameters
- Now we introduce adaptive control: how to control systems with unknown parameters

Adaptive Control of a First-Order Plant

Problem:



Plant:

$$\dot{x}_p = a_p x_p + k_p u$$

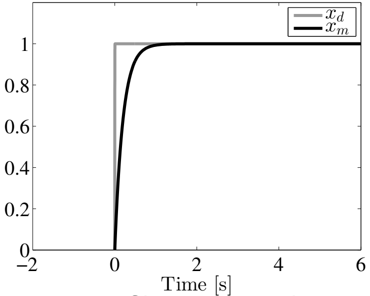
Find u such that x_p follows a desired command.

A Model-reference Approach

- x_p: Output of a first-order system - can only follow 'smooth' signals
- Ensure x_d is a 'smooth' signal essentially by filtering the desired command

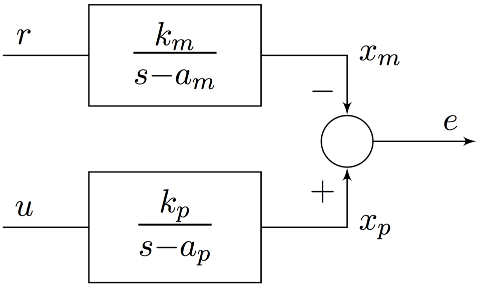
Pose the problem as

$$\dot{x}_m = a_m x_m + k_m r$$



Set $a_m = -5$ and $k_m = 5$, say. Choose r so that $x_m \approx x_d$

Statement of the problem



Choose u so that $e(t) \to 0$ as $t \to \infty$.

- a_p unknown
- k_p unknown, but with known sign

Certainty Equivalence Principle

Step 1: Algebraic Part: Find a solution to the problem when parameters are known.

Step 2: Analytic Part: Replace the unknown parameters by their estimates. Ensure stable and convergent behavior.

The use of the parameter estimates in place of the true parameters is known as the *certainty equivalence principle*.

Certainty Equivalence Principle- Step 1

Step 1: Algebraic Part: Propose the control law

$$u(t) = \theta_c x_p + k_c r$$

and choose θ_c, k_c so that closed-loop transfer function matches the reference model transfer function.

$$\begin{aligned}\dot{x}_p &= a_p x_p + k_p(\theta_c x_p + k_c r) \\ &= (a_p + k_p\theta_c)x_p + k_p k_c r\end{aligned}$$

Now compare this to the reference model equation

$$\dot{x}_m = a_m x_m + k_m r$$

Desired Parameters: $\theta_c = \theta^*$ and $k_c = k^*$ must satisfy

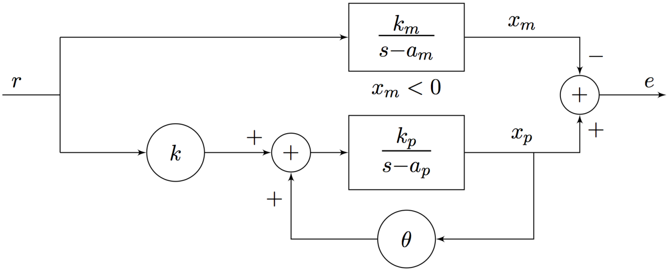
$$a_p + k_p\theta^* = a_m \qquad \text{and} \qquad k_p k^* = k_m$$

Certainty Equivalence Principle- Step 1

Solving for the nominal or ideal parameters

$$\theta^* = \frac{a_m - a_p}{k_p} \qquad \text{and} \qquad k^* = \frac{k_m}{k_p}$$

This is represented with the following block diagram



Certainty Equivalence Principle - Step 2

Step 2: Analytic Part: Replace the unknown parameters by their estimates. Ensure stable and convergent behavior. From Step 1, we have

$$u(t) = \theta^* x_p + k^* r, \qquad \theta^* = \frac{a_m - a_p}{k_p}, \qquad k^* = \frac{k_m}{k_p}$$

Replace θ^* and k^* by their estimates $\theta(t)$ and $k(t)$.

$$u(t) = \theta(t) x_p + k(t) r$$

$$\dot{\theta}(t) = ?? \qquad \dot{k}(t) = ??$$

Certainty Equivalence Principle - Step 2

Adaptive control input:

$$u(t) = \theta(t)x_p + k(t)r$$

Define the parameter errors as

$$\widetilde{\theta}(t) = \theta(t) - \theta^*$$
$$\widetilde{k}(t) = k(t) - k^*$$

Plug the control law into the plant equation:

$$\begin{aligned}
\dot{x}_p &= a_p x_p + k_p u(t) \\
&= a_p x_p + k_p\big[\theta(t)x_p + k(t)r\big] \\
&= a_p x_p + k_p\big[\widetilde{\theta}(t)x_p + \theta^* x_p + \widetilde{k}(t)r + k^* r\big] \\
&= \big[a_p + k_p\theta^*\big]x_p + k_p\widetilde{\theta}(t)x_p + k_p k^* r + k_p\widetilde{k}(t)r \\
&= a_m x_p + k_p\widetilde{\theta}(t)x_p + k_m r + k_p\tilde{k}(t)r
\end{aligned}$$

Certainty Equivalence Principle - Step 2

Reference Model:

$$\dot{x}_m = a_m x_m + k_m r$$

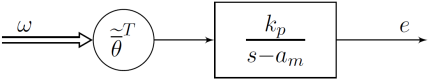
Define the tracking error as

$$e = x_p - x_m$$

Error model:

$$\begin{aligned} \dot{e} &= a_m e + k_p \widetilde{\theta}(t) x_p + k_p \tilde{k}(t) r \\ &= a_m e + k_p \tilde{\bar{\theta}}^T(t)\omega \end{aligned}$$

$$\omega = \begin{bmatrix} x_p \\ r \end{bmatrix}$$



$$\tilde{\bar{\theta}} = \begin{bmatrix} \widetilde{\theta}(t) \\ \tilde{k}(t) \end{bmatrix}$$

Certainty Equivalence Principle - Step 2

$$\dot{e} = a_m e + k_p \overline{\tilde{\theta}}^\top (t) \omega$$

This is again error model 3!