Lecture 9: Stochastic / Predictive Self-Tuning Regulators.

- Minimum Variance Control.
- Moving Average Controller.

The stochastic model

Assume the plant is represented by the special ARMAX model

$$A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)$$

where $\{e(t)\}$ is white noise,

$$\begin{aligned} A(q) &= q^n + a_1\,q^{n-1} + \cdots + a_n, && \deg\{A\} = n \\ B(q) &= b_1\,q^{n-d_0} + \cdots + b_n, && \deg\{B\} = n - d_0 \\ C(q) &= q^n + c_1\,q^{n-1} + \cdots + c_n, && \deg\{C\} = n \end{aligned}$$

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Equivalently,

$$\begin{aligned} y(t) &= -a_1\,y(t-1) - \cdots - a_n\,y(t-n) \\ &\quad + b_{d_0}\,u(t-d_0) + \cdots + b_n\,u(t-n) \\ &\quad + e(t) + c_1\,e(t-1) + \cdots + c_n\,e(t-n) \end{aligned}$$

Remarks on modeling noise.

Assuming that C is stable is not restrictive:

Typically, such a model is obtained experimentally from spectrum characteristics. It does not change if the unstable roots are substituted by stable, which are symmetrical to them with respect to the unite circle.

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It can be rewritten as

$$(q + a)\, y(t) = b\, u(t) + c\, e(t)$$

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$$y(t+1) + a\,y(t) = b\,u(t) + c\,e(t)$$

It can be rewritten as

$$(q+a)\,y(t) = b\,u(t) + q\,e_{new}(t)$$

introducing $e_{new(t)} = c\,e(t-1)$.

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$$y(t+1) + a\,y(t) = b\,u(t) + c\,e(t)$$

It can be rewritten as

$$y(t+1) + a\,y(t) = b\,u(t) + e_{new}(t+1)$$

with $e_{new(t)} = c\,e(t-1)$ being white noise with $\mathrm{var}\{e_{new}(t)\} = c^2\ \mathrm{var}\{e(t)\}$.

Minimum-variance control: Example

Consider the model

$$y(t) = -a\,y(t-1) + b\,u(t-1) + c\,e(t-1) + e(t)$$

where $|c| < 1$ and $\{e(t)\}$ is a sequence of random variables with $E\{e(t)\} = 0$ and $\mathrm{var}\{e(t)\} = \sigma^2$.

Our goal is to regulate $y(t)$ to 0 as close as possible.

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Our goal is to regulate $y(t)$ to 0 with minimal variance.

In the absence of noise:

$$y(t) = -a\,y(t-1) + b\,u(t-1)$$

the best strategy (dead beat design) is

$$u(t) = \frac{a}{b}\,y(t) \qquad \Longrightarrow \qquad y(t+1) = 0$$

What should we do when noise is present?

Minimum-variance control: Example

Consider the model

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In the absence of noise or when $c = 0$:

$$y(t) = -a\,y(t-1) + b\,u(t-1) + e(t)$$

the best strategy (dead beat design) is

$$u(t) = \frac{a}{b}\,y(t) \qquad \Longrightarrow \qquad y(t+1) = e(t+1)$$

What should we do when noise dynamics are present?

Minimum-variance control: Example (cont'd)

We have the process

$$y(t+1) + a\,y(t) = b\,u(t) + e(t+1) + c\,e(t)$$

Minimum-variance control: Example (cont'd)

We have the process

$$y(t+1) + a\,y(t) = b\,u(t) + e(t+1) + c\,e(t)$$

Let us try to compute $e(t)$ symbolically

$$(q+a)\,y(t) = b\,u(t) + (q+c)\,e(t)$$

Minimum-variance control: Example (cont'd)

We have the process

$$y(t+1) + a\,y(t) = b\,u(t) + e(t+1) + c\,e(t)$$

Let us try to compute $e(t)$ symbolically

$$e(t) = \frac{q+a}{q+c}\,y(t) - \frac{b}{q+c}\,u(t)$$

Minimum-variance control: Example (cont'd)

We have the process

$$y(t+1) + a\,y(t) = b\,u(t) + e(t+1) + c\,e(t)$$

Let us try to compute $e(t)$ symbolically

$$e(t) = \frac{q+a}{q+c}\,y(t) - \frac{b}{q+c}\,u(t)$$

Substituting back into the model

$$y(t{+}1) = -a\,y(t){+}b\,u(t){+}e(t{+}1){+}c\left(\frac{q+a}{q+c}\,y(t) - \frac{b}{q+c}\,u(t)\right)$$

Minimum-variance control: Example (cont'd)

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Let us try to compute $e(t)$ symbolically

$$e(t) = \frac{q+a}{q+c}\,y(t) - \frac{b}{q+c}\,u(t)$$

Substituting back into the model

$$y(t+1) = \underbrace{\frac{(c-a)\,q}{q+c}\,y(t) + \frac{b\,q}{q+c}\,u(t)}_{\text{known and independent from } e(t+1)!} + e(t+1)$$

Minimum-variance control: Example (cont'd)

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Clearly, $\quad \mathrm{var}\{y(t+1)\} \geq \mathrm{var}\{e(t+1)\} = \sigma^2.$

Minimum-variance control: Example (cont'd)

We have the process

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The minimal variance is achieved with: $\quad u(t) = -\dfrac{c-a}{b}\,y(t).$

Minimum-variance control: General case

Assume the plant is represented by the special ARMAX model

$$A(q)\, y(t) = B(q)\, u(t) + C(q)\, e(t)$$

where $\{e(t)\}$ is white noise with $\mathrm{var}\{e(t)\} = \sigma^2$.

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Additional assumptions:

- The polynomials A and B are monic (right rescaling).

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- The polynomials A and B are monic (right rescaling).
- $\deg\{A\} = \deg\{C\} = n \geq 1$ (always can be done).
- $\deg\{A\} - \deg\{B\} = d_0 \geq 1$ – known relative degree.

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- $\deg\{A\} = \deg\{C\} = n \geq 1$ (always can be done).
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- The polynomial C is stable (always can be done).

Minimum-variance control: General case

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where $\{e(t)\}$ is white noise with $\mathrm{var}\{e(t)\} = \sigma^2$.

Additional assumptions:

- The polynomials A and B are monic (right rescaling).
- $\deg\{A\} = \deg\{C\} = n \geq 1$ (always can be done).
- $\deg\{A\} - \deg\{B\} = d_0 \geq 1$ – known relative degree.
- The polynomial C is stable (always can be done).
- The polynomial B is stable – a serious restriction.

Minimum-variance control: General case (cont'd)

We have the process

$$A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)$$

Minimum-variance control: General case (cont'd)

We have the process

$$\boldsymbol{A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)}$$

Let us solve it for $\boldsymbol{y(t)}$

$$\boldsymbol{y(t) = \frac{B(q)}{A(q)}\,u(t) + \frac{C(q)}{A(q)}\,e(t)}$$

Minimum-variance control: General case (cont'd)

We have the process

$$A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)$$

Shifting by the relative degree

$$y(t+d_0) = \frac{B(q)}{A(q)}\,u(t+d_0) + \frac{C(q)}{A(q)}\,e(t+d_0)$$

Minimum-variance control: General case (cont'd)

We have the process

$$A(q)\, y(t) = B(q)\, u(t) + C(q)\, e(t)$$

Shifting by the relative degree and rewriting

$$y(t + d_0) = \frac{B(q)}{A(q)}\, u(t + d_0) + \frac{C(q)\, q^{d_0 - 1}}{A(q)}\, e(t + 1)$$

Minimum-variance control: General case (cont'd)

We have the process

$$A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)$$

Shifting by the relative degree and rewriting

$$y(t+d_0) = \frac{B(q)}{A(q)}\,u(t+d_0) + \frac{C(q)\,q^{d_0-1}}{A(q)}\,e(t+1)$$

Using polynomial long division

$$\boxed{\frac{C(q)\,q^{d_0-1}}{A(q)} = F(q) + \frac{G(q)}{A(q)}}$$

where $\frac{G(q)}{A(q)}$ is strictly proper and
$\deg\{F\} = \deg\{C(q)\,q^{d_0-1}\} - \deg\{A\} = d_0 - 1.$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \frac{B(q)}{A(q)}\, u(t+d_0) + \left(F(q) + \frac{G(q)}{A(q)} \right) e(t+1)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\, B(q)}{A(q)}}_{\text{proper fraction}}\; u(t) + \underbrace{\frac{q\, G(q)}{A(q)}}_{\text{proper fraction}}\; e(t) + F(q)\, e(t+1)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\, B(q)}{A(q)}}_{\text{proper fraction}}\, u(t) + \underbrace{\frac{q\, G(q)}{A(q)}}_{\text{proper fraction}}\, e(t) + F(q)\, e(t+1)$$

Solving the model for $e(t)$, we have

$$e(t) = \frac{A(q)}{C(q)}\, y(t) - \frac{B(q)}{C(q)}\, u(t)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\, B(q)}{A(q)}}_{\text{proper fraction}}\, u(t) + \underbrace{\frac{q\, G(q)}{A(q)}}_{\text{proper fraction}}\, e(t) + F(q)\, e(t+1)$$

Solving the model for $e(t)$, we have

$$e(t) = \frac{A(q)}{C(q)}\, y(t) - \frac{B(q)}{C(q)}\, u(t)$$

After substituting it back

$$y(t+d_0) = \frac{q^{d_0}\, B}{A}\, u(t) + \frac{q\, G}{A} \left(\frac{A}{C}\, y(t) - \frac{B}{C}\, u(t) \right) + F\, e(t+1)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\,B(q)}{A(q)}}_{\text{proper fraction}}\,u(t) + \underbrace{\frac{q\,G(q)}{A(q)}}_{\text{proper fraction}}\,e(t) + F(q)\,e(t+1)$$

Solving the model for $e(t)$, we have

$$e(t) = \frac{A(q)}{C(q)}\,y(t) - \frac{B(q)}{C(q)}\,u(t)$$

After substituting it back and collecting terms

$$y(t+d_0) = \frac{q\,G}{C}\,y(t) + \frac{q\,B}{A\,C}\left(q^{d_0}\,C - G\right)\,u(t) + F\,e(t+1)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\, B(q)}{A(q)}}_{\text{proper fraction}}\, u(t) + \underbrace{\frac{q\, G(q)}{A(q)}}_{\text{proper fraction}}\, e(t) + F(q)\, e(t+1)$$

Solving the model for $e(t)$, we have

$$e(t) = \frac{A(q)}{C(q)}\, y(t) - \frac{B(q)}{C(q)}\, u(t)$$

After substituting it back and collecting terms

$$y(t+d_0) = \frac{q\, G}{C}\, y(t) + \frac{q\, B}{C}\, \frac{(q^{d_0}\, C - G)}{A}\, u(t) + F\, e(t+1)$$

Minimum-variance control: General case (cont'd)

Finally the process can be represented by

$$y(t+d_0) = \underbrace{\frac{q^{d_0}\,B(q)}{A(q)}}_{\text{proper fraction}}\,u(t) + \underbrace{\frac{q\,G(q)}{A(q)}}_{\text{proper fraction}}\,e(t) + F(q)\,e(t+1)$$

Solving the model for $e(t)$, we have

$$e(t) = \frac{A(q)}{C(q)}\,y(t) - \frac{B(q)}{C(q)}\,u(t)$$

Substituting $F(q)$ from $\frac{C(q)\,q^{d_0-1}}{A(q)} = F(q) + \frac{G(q)}{A(q)}$

$$y(t+d_0) = \frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t) + F(q)\,e(t+1)$$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

Minimum-variance control: General case (cont'd)

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Clearly, $\quad \mathrm{var}\{y(t+d_0)\} \geq \mathrm{var}\{F(q)\,e(t+1)\}$

Minimum-variance control: General case (cont'd)

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$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

$$\mathrm{var}\{y(t+d_0)\} \geq \mathrm{var}\{F(q)\,e(t+1)\} = \left(1 + f_1^2 + \cdots + f_{d_0-1}^2\right)\sigma^2$$

where $F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}$.

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

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where $F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}$.

The minimal variance is achieved with: $\boxed{u(t) = -\dfrac{G(q)}{B(q)\,F(q)}\,y(t)}.$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

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where $F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}$.

The minimal variance is achieved with: $\boxed{u(t) = -\dfrac{G(q)}{B(q)\,F(q)}\,y(t)}$.

The closed-loop system is

$$A(q)\,y(t) = B(q)\,u(t) + C(q)\,e(t)$$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

$$\mathrm{var}\{y(t+d_0)\} \geq \mathrm{var}\{F(q)\,e(t+1)\} = \left(1 + f_1^2 + \cdots + f_{d_0-1}^2\right)\sigma^2$$

where $F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}$.

The minimal variance is achieved with: $\boxed{u(t) = -\dfrac{G(q)}{B(q)\,F(q)}\,y(t)}$.

The closed-loop system is

$$A(q)\,y(t) = B(q)\left(-\frac{G(q)}{B(q)\,F(q)}\,y(t)\right) + C(q)\,e(t)$$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

$$\mathrm{var}\{y(t+d_0)\} \geq \mathrm{var}\{F(q)\,e(t+1)\} = \left(1 + f_1^2 + \cdots + f_{d_0-1}^2\right)\,\sigma^2$$

where $F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}$.

The minimal variance is achieved with: $\boxed{u(t) = -\dfrac{G(q)}{B(q)\,F(q)}\,y(t)}$.

The closed-loop system is

$$B(q)\ \left(A(q)\,F(q) + G(q)\right)\ y(t) = B(q)\,F(q)\,C(q)\,e(t)$$

Minimum-variance control: General case (cont'd)

We have rewritten the process as

$$y(t+d_0) = \underbrace{\frac{q\,G(q)}{C(q)}\,y(t) + \frac{q\,B(q)\,F(q)}{C(q)}\,u(t)}_{\text{known and independent from } F(q)\,e(t+1)} + F(q)\,e(t+1)$$

$$\mathrm{var}\{y(t+d_0)\} \geq \mathrm{var}\{F(q)\,e(t+1)\} = \left(1 + f_1^2 + \cdots + f_{d_0-1}^2\right)\sigma^2$$

where $\quad F(q) = q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}.$

The minimal variance is achieved with: $\boxed{u(t) = -\dfrac{G(q)}{B(q)\,F(q)}\,y(t)}.$

The closed-loop system (using: $q^{d_0-1}\,C = A\,F + G$) is

$$q^{d_0-1}\,B(q)\,C(q)\,y(t) = B(q)\,F(q)\,C(q)\,e(t)$$

Minimum-variance control: Remarks

(1) The closed-loop system

$$q^{d_0-1}\,B(q)\,C(q)\,y(t) = B(q)\,F(q)\,C(q)\,e(t)$$

defines the noise-to-output relation

$$y(t) = \frac{B(q)\,F(q)\,C(q)}{q^{d_0-1}\,B(q)\,C(q)}\,e(t)$$

Minimum-variance control: Remarks

(1) The closed-loop system

$$q^{d_0-1}\,B(q)\,C(q)\,y(t) = B(q)\,F(q)\,C(q)\,e(t)$$

defines the noise-to-output relation

$$y(t) = \frac{F(q)}{q^{d_0-1}}\,e(t) = \frac{q^{d_0-1} + f_1\,q^{d_0-2} + \cdots + f_{d_0-1}}{q^{d_0-1}}\,e(t)$$

Minimum-variance control: Remarks

(1) The closed-loop system

$$q^{d_0-1}\, B(q)\, C(q)\, y(t) = B(q)\, F(q)\, C(q)\, e(t)$$

defines the noise-to-output relation

$$y(t) = (1 + f_1\, q^{-1} + \cdots + f_{-d_0+1})\, e(t)$$

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(3) The minimum variance controller

$$u(t) = -\frac{G(q)}{B(q)\, F(q)}\, y(t)$$

can be interpreted as a pole-placement controller.

Minimum-variance control as pole-placement

Since

$$q^{d_0-1}\,C(q) = A(q)\,F(q) + G(q)$$

the closed-loop characteristic polynomial

$$A_c(q) = q^{d_0-1}\,B(q)\,C(q) = A(q)\,\underbrace{B(q)\,F(q)}_{R(q)} + B(q)\,\underbrace{G(q)}_{S(q)}$$

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Note that for the Diophantine equation $A_c = A\,R + B\,S$

$$\deg\{S(q)\} = \deg\{G(q)\} = n - 1$$

and S/R is proper since

$$\deg\{R\} = \deg\{B\} + \deg\{F\} = (n - d_0) + (d_0 - 1) = n - 1.$$

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Can we use an analogous pole-placement technique for the case when $B(q)$ has unstable zeros?

Moving Average Controller

Design the controller

$$u(t) = -\frac{S(q)}{R(q)} y(t) \qquad \text{for} \quad A(q)\, y(t) = B(q)\, u(t) + C(q)\, e(t)$$

as follows.

Moving Average Controller

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as follows.

- Factor B as $B = B^+ B^-$, where B^+ is monic and stable.

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$$u(t) = -\frac{S(q)}{R(q)} y(t) \qquad \text{for} \quad A(q)\, y(t) = B(q)\, u(t) + C(q)\, e(t)$$

as follows.

- Factor B as $B = B^{+} B^{-}$, where B^{+} is monic and stable.
- Set $d = \deg\{A\} - \deg\{B^{+}\}$.

Moving Average Controller

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$$u(t) = -\frac{S(q)}{R(q)} y(t) \qquad \text{for} \quad A(q)\, y(t) = B(q)\, u(t) + C(q)\, e(t)$$

as follows.

- Factor B as $B = B^+ B^-$, where B^+ is monic and stable.
- Set $d = \deg\{A\} - \deg\{B^+\}$.
- Find R_p and S solving the Diophantine equation

$$q^{d-1} C = A\, R_p + B^- S, \qquad d \geq d_0$$

Moving Average Controller

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- Let $R = R_p\, B^+$, where $\deg\{R_p\} = d - 1$.

The obtained transfer function for the closed-loop system is

$$y(t) = q^{1-d} R_p(q)\, e(t) = e(t) + r_1\, e(t-1) + \cdots + r_{d-1}\, e(t-d+1)$$

Example 4.3

Consider the plant described by

$$(q^2 + a_1\, q + a_2)\, y(t) = (b_0\, q + b_1)\, u(t) + (q^2 + c_1\, q + c_2)\, e(t)$$

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1. If $|b_1/b_0| < 1$ minimum variance (MV) controller can be designed noticing that

$$\frac{q^{1-1}\,(q^2 + c_1\, q + c_2)}{q^2 + a_1\, q + a_2} = 1 + \frac{(c_1 - a_1)\, q + (c_2 - a_2)}{q^2 + a_1\, q + a_2}$$

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2. If $|b_1/b_0| > 1$ minimum variance (MV) controller cannot be designed but we can apply moving average controller (MA) with $d = 2$ using the solution of the Diophantine equation

$$q\,(q^2 + c_1\, q + c_2) = (q^2 + a_1\, q + a_2)(q + r_1) + (b_0\, q + b_1)(s_0\, q + s_1)$$

Next Lecture / Assignments:

Next meeting (May 10, 13:00-15:00, in A208Tekn): Recitations.

Homework problems: Consider the process in Example 4.3 with $a_1 = -1.5$, $a_2 = 0.7$, $b_0 = 1$, $c_1 = -1$, and $c_2 = 0.2$.

Determine the variance of the output in the closed-loop system as a function of b_1 when the Moving Average controller is used. Compare with the lowest achievable variance.