Lecture 5: Real-Time Parameter Estimation

- Least Squares and Recursive Computations
- Estimating Parameters in Dynamical Systems
- Persistent Excitation and Linear Filtering
- Examples

Theorem (Persistent Excitation for FIR model):

Assume that data are generated by the following FIR model:

$$y(t) = g_1\, u(t-1) + g_2\, u(t-2) + \cdots + g_n\, u(t-n) + e(t)$$

with $\quad E\{e(t)\} = 0, \quad$ and mutually independent $e(t)$.
Then, one can use Recursive Least Square algorithm with

$$\phi(t,n)^T = \Big[u(t-1),\, u(t-2),\, \ldots,\, u(t-n)\Big]$$

to estimate $\qquad \theta^0(n) = \Big[g_1,\, g_2,\, \ldots,\, g_n\Big]^T.$

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If the signal $u(t)$ is persistently exciting of at least order n, the estimate is unbiased. Possible conditions to check is

$$\lim_{N\to\infty} \frac{1}{N} \begin{bmatrix} \phi(n,n)^T \\ \phi(n+1,n)^T \\ \vdots \\ \phi(N,n)^T \end{bmatrix}^T \begin{bmatrix} \phi(n,n)^T \\ \phi(n+1,n)^T \\ \vdots \\ \phi(N,n)^T \end{bmatrix} > 0$$

Persistence of Excitation (Def 1):

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and if

$$C_n = \begin{bmatrix} c(0) & c(1) & \ldots & c(n-1) \\ c(1) & c(0) & \ldots & c(n-2) \\ \vdots & & & \\ c(n-1) & c(n-2) & \ldots & c(0) \end{bmatrix} > 0$$

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It is called persistently exciting of order n if for all t there exists an integer m such that

$$\rho_1 I_n > \sum_{k=t}^{t+m} \phi(k,n)\phi(k,n)^T > \rho_2 I_n$$

where $\rho_1,\ \rho_2 > 0$ and

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It can be shown that conditions of Def. 2 imply the ones of Def. 1.

Polynomial Conditions for Persistent Excitation:

The signal with the property that limits

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exist, is PE of order n if and only if

$$L = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} [A(q)\, u(k)]^2 > 0$$

for all non-zero polynomials of degree $n-1$ or less.

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Proof: with $A(q) = a_0\, q^{n-1} + \cdots + a_{n-1}$

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Corollary: if $u(t)$ satisfies a polynomial equation of degree n for $t \geq t_0$ with some t_0, then it can be at most PE of order n.

Examples of computing order of PE:

Example 1: The STEP signal

$$u(t) = 0 \quad \text{for} \quad t < 0, \qquad\qquad u(t) = 1 \quad \text{for} \quad t \geq 0$$

satisfies: $$(q-1)u(t) = 0 \qquad \text{for} \quad t \geq 0$$

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Example 2: Any m-PERIODIC signal:

$$u(t+m) = u(t) \qquad \text{for} \quad \forall t$$

satisfies $\;(q^m - 1)\, u = 0,\;$ so, it can be at most PE of order m.

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Hence, it can at most be PE of order $\mathbf{2}$.

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Let us compute $c(k)$:

$$c(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \Big[\sin(\omega_0\, i)\, \sin(\omega_0\, i - \omega_0\, k)\Big]$$

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Frequency domain characterization of PE signals

Definition: A signal $u(t)$ is called stationary if the limit

$$R_u(k) = \lim_{t\to\infty} \frac{1}{t} \sum_{i=m}^{t+m-1} u(i)\, u(i-k)$$

exists uniformly in m.

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Definition: The Fourier transform of the covariance, $\Phi_u(\omega)$, is called the frequency spectrum of $u(t)$:

$$R_u(k) = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} e^{j\,\omega\,k}\, \Phi_u(\omega)\, d\omega, \qquad j = \sqrt{-1}$$

$$\Phi_u(\omega) = \sum_{k=-\infty}^{+\infty} R_u(k)\, e^{-j\,\omega\,k} = \left| \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} u(t)\, e^{-j\,\omega\,t}\, dt \right|^2$$

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Example: for the signal

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we have

$$R_u(k) = \frac{1}{2}\cos(\omega_0\, k), \qquad \Phi_u(\omega) = \frac{\pi}{2}\Big[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\Big]$$

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Lemma: A stationary signal is PE of order n if its frequency spectrum is non zero for at least n points.

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Example: white noise is PE of any order, while the signal

$$u(t) = \sum_{i=1}^{m} A_i\, \sin(\omega_i\, t+\varphi_i), \qquad \text{with} \quad 0 < \omega_i < \pi, \quad \omega_i \neq \omega_j$$

is PE of order $2\, m$.

Theorem (Persistent Excitation for ARMA model):

Assume that data are generated by the following ARMA model:

$$\underbrace{\left(q^n + a_1\, q^{n-1} + \cdots + a_n\right)}_{A(q)}\, y(t) = \underbrace{\left(b_1\, q^{m-1} + b_2\, q^{m-2} + \cdots + b_m\right)}_{B(q)}\, u(t)$$

with $n > m$. Then, one can use RLS algorithm with

$$\phi(t-1)^T = \Big[-y(t-1), \ldots, -y(t-n), u(t-n+m-1), \ldots, u(t-n) \Big]$$

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The estimate is unbiased if the signal $u(t)$ is such that

$$\lim_{t\to\infty} \frac{1}{t} \Big[\Phi(t-1)^T\, \Phi(t-1) \Big] = \lim_{t\to\infty} \frac{1}{t} \begin{bmatrix} \phi(n)^T \\ \phi(n+1)^T \\ \vdots \\ \phi(t-1)^T \end{bmatrix}^T \begin{bmatrix} \phi(n)^T \\ \phi(n+1)^T \\ \vdots \\ \phi(t-1)^T \end{bmatrix} > 0$$

Persistent Excitation for $A(q)y(t) = B(q)u(t)$

$$\phi(t-1) = \begin{bmatrix} -y(t-1) \\ \ldots \\ -y(t-n) \\ u(t-n+m-1) \\ \ldots \\ u(t-n) \end{bmatrix}$$

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where $v(t) = \dfrac{q^m}{A(q)} u(t)$ and S is not singular in the case when $A(q)$ and $B(q)$ are relatively prime.

Persistent Excitation for $A(q)y(t) = B(q)u(t)$

$$\phi(t-1) = \begin{bmatrix} -y(t-1) \\ \dots \\ -y(t-n) \\ u(t-n+m-1) \\ \dots \\ u(t-n) \end{bmatrix} = \frac{q^m}{A(q)} \begin{bmatrix} -q^{-1-m}B(q)u(t) \\ \dots \\ -q^{-n-m}B(q)u(t) \\ q^{-n-1}A(q)u(t) \\ \dots \\ q^{-n-m}A(q)u(t) \end{bmatrix} = S \underbrace{\begin{bmatrix} -v(t-1) \\ \dots \\ -v(t-n) \\ v(t-n-1) \\ \dots \\ v(t-n-m) \end{bmatrix}}_{\psi(t-1)}$$

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where $v(t) = \frac{q^m}{A(q)} u(t)$ and S is not singular in the case when $A(q)$ and $B(q)$ are relatively prime.

$$\lim_{t\to\infty} \frac{1}{t} \left[\Phi(t-1)^T \, \Phi(t-1) \right] = S \left(\lim_{t\to\infty} \frac{1}{t} \left[\Psi(t-1)^T \, \Psi(t-1) \right] \right) S^T > 0$$

The conditions for parameter convergence, formulated in the case of FIR model for the signal $u(t)$, are similar in the case of ARMA model, but should be concerning the signal

$$v(t) = \frac{q^m}{A(q)} u(t)$$

$n > m$, q^m is stable, $q^m / A(q)$ is stable and minimum phase.

Hence, $\boxed{\Phi_v(\omega) = \left| \frac{e^{jm\omega}}{A(e^{j\omega})} \right|^2 \Phi_u(\omega)}$.

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Hence, $\boxed{\Phi_v(\omega) = \left| \frac{e^{jm\omega}}{A(e^{j\omega})} \right|^2 \Phi_u(\omega)}$.

Theorem (Persistent Excitation for ARMA model):

Suppose that

1. $A(q)$ and $B(q)$ are relatively prime,
2. $A(q)$ is stable (i.e. all roots are inside the unite circle),
3. $u(t)$ is a stationary signal such that $\Phi_u(\omega)$ is nonzero for at least $(n+m)$ points.

Then, RLS algorithm converges to the correct value.

Example

Consider the system

$$y(t) + a_1\, y(t-1) + a_2\, y(t-2) = b_1\, u(t-1) + b_2\, u(t-2)$$

Choose a nice (as simple as possible) input signal, which is rich enough to ensure parameters convergence.

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Solution:

$$A(q) = q^2 + a_1\, q + a_2, \qquad B(q) = b_1\, q + b_2\, q^2$$

Since $n = 2$ and $m = 2$, we can take

$$u(t) = A_1\, \sin(\omega_1\, t) + A_2\, \sin(\omega_2\, t), \qquad \omega_1 \neq \omega_2, \quad 0 < \omega_{1,2} < \pi.$$

to have exactly 4 non zero points in $\Phi_u(\omega)$.

Example 2.12 (book)

Consider the system $\quad y(t) + a\, y(t-1) = b\, u(t-1) + e(t)$
with $a = -0.8$, $b = 0.5$, $e(t)$ – zero mean white noise with standard deviation $\sigma = 0.5$.

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Consider the system $\quad y(t) + a\,y(t-1) = b\,u(t-1) + e(t)$
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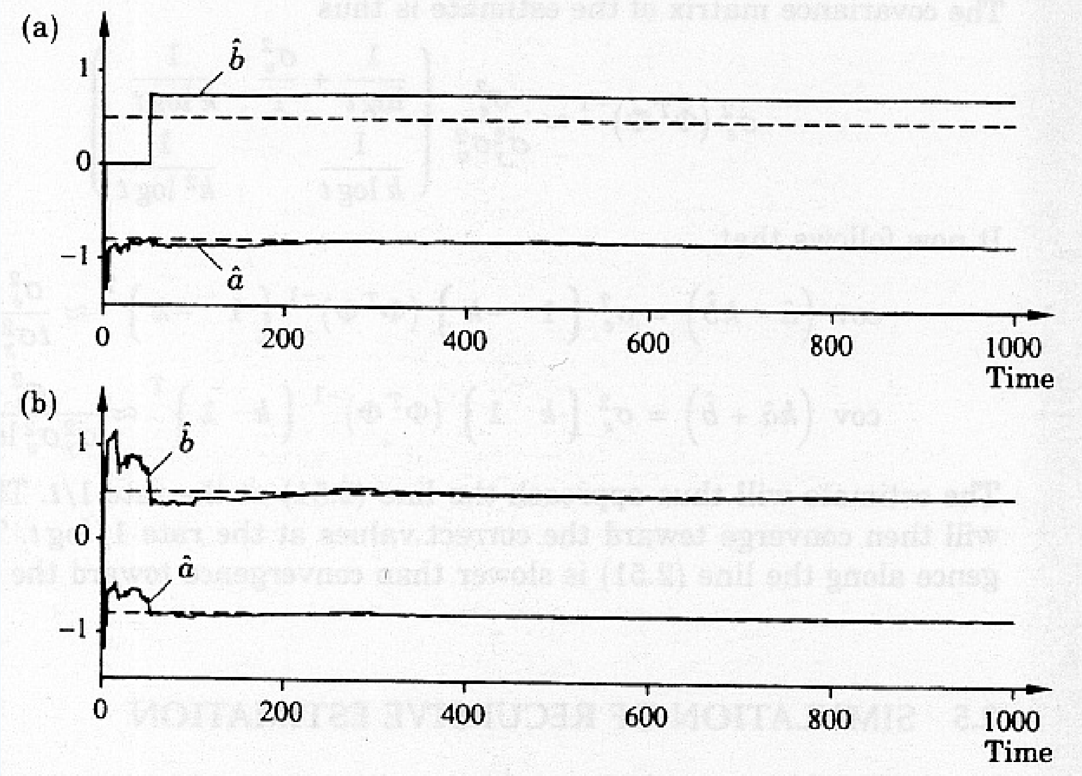

Figure 2.7 The estimated (solid line) and true (dashed line) parameter values in estimating the parameters in the model (2.53). The input signal $u(t)$ is (a) a unit pulse at $t = 50$, (b) a unit amplitude square wave with period 100.

Next Lecture / Assignments:

Next meeting (April 19, 13:00-15:00, in A205Tekn): Recitations

Homework problem: repeat the simulation for Example 2.12 shown above (see pages 71–72), taking $\boldsymbol{P(0) = 100\, I_2}$ and $\boldsymbol{\hat{\theta}(0) = 0}$.